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H-measures and system of Maxwell's

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We are interested in the homogenization of energy like quantities in electromagnetism. We prove a general propagation Theorem for H-measures associated to Maxwell's system, in the full space $\Omega = \mathbb{R}^3$, without boundary conditions. We shall distinguish between two cases: constant coefficient case, and non coefficient-scalar case. In the two cases we give the behaviour of the H-measures associated to this system.

Keywords: Electromagnetism, homogenization of energy, H-measures, Maxwell's system.

Mathematics Subject Classification 2000: 35BXX, 35B27

1. Introduction

Herein, we are interested in the homogenization of energy like quantities in electromagnetism, and more particularly in Maxwell's equations, without boundary conditions. We use the notion of H-measures, introduced by Gérard and Tartar [5], [23]. We prove a general propagation Theorem for H-measures associated to Maxwell's system. This result, combined with the localisation property, is then used to obtain more precise results on the behaviour of H-measures associated to this system.

As known, an H-measure is a (possibly matrix of) Radon measures on the product space $\Omega \times S^{n-1}$, where $\Omega \subseteq \mathbb{R}^n$ is an open domain and S^{n-1} is the unit sphere in \mathbb{R}^n . In order to apply Fourier transform, functions defined on the whole of \mathbb{R}^n should be considered and this can be achieved by extending them by zero outside the domain. For this reason, we consider Maxwell's system in the full space \mathbb{R}^3 , which means without boundary conditions. Let us mention that similar works already exist, see in particular [2]. However, in [2], computations are far from being complete.

If one is interested in coupling with boundary conditions, the usual pseudodifferential calculus behind the notion of H-measures is not sufficient, and one should use much more technical tools. In the case of semi-classical measures, this is now well known, see for instance [6], [8], [12].

In the context of H-measures, and in particular without a typical scale, then one should use tools similar to those developed in recent works, see for instance [1]. However, the results presented here will be important for the full Maxwell problem, with suitable boundary conditions.

Let Ω be an open set of \mathbb{R}^3 . We will consider Maxwell's system in the material Ω with electric permeability $\check{\epsilon}$, conductivity $\check{\sigma}$ and magnetic susceptibility $\check{\eta}$ given by

$$\left\{ \begin{array}{l} i) \quad \partial_t D^\varepsilon(x, t) + J^\varepsilon(x, t) = \quad \text{rot} H^\varepsilon(x, t) + F^\varepsilon, \\ ii) \quad \partial_t B^\varepsilon(x, t) = -\text{rot} E^\varepsilon(x, t) + G^\varepsilon(x, t), \\ iii) \quad \text{div} B^\varepsilon(x, t) = 0, \\ iv) \quad \text{div} D^\varepsilon(x, t) = \varrho^\varepsilon(x, t), \end{array} \right. \quad (1.1)$$

where $x \in \Omega$ and $t \in (0, T)$, E^ε , H^ε , D^ε , J^ε and B^ε are the electric, magnetic, induced electric, current density and induced magnetic fields, respectively.

Moreover, ρ^ε (the charge density), F^ε and G^ε are given, and we have the three constitutive relations

$$\left\{ \begin{array}{l} 1) \quad D^\varepsilon(x, t) = \check{\epsilon}(x)E^\varepsilon(x, t), \\ 2) \quad J^\varepsilon(x, t) = \check{\sigma}(x)E^\varepsilon(x, t), \\ 3) \quad B^\varepsilon(x, t) = \check{\eta}(x)H^\varepsilon(x, t), \end{array} \right. \quad (1.2)$$

where $\check{\epsilon}$, $\check{\sigma}$ and $\check{\eta}$ are 3×3 matrix valued functions and ε is a typical length going to 0.

We shall consider this system in the full space $\Omega = \mathbb{R}^3$, without boundary conditions. Since we are not taking into account the initial data, we will also assume that the time variable t belongs to \mathbb{R} .

We shall use the notion of H-measure to compute for instance energy quantities in the following cases :

i) Constant coefficient case: here, we assume that the electric permittivity $\check{\epsilon}$, conductivity $\check{\sigma}$ and magnetic susceptibility $\check{\eta}$ are 3×3 identity matrices, i.e.

$$\check{\epsilon} = \check{\sigma} = \check{\eta} = (Id)_{3 \times 3}. \quad (1.3)$$

ii) Non constant coefficient-scalar case: in this case, we consider that the matrix $\check{\epsilon}$, $\check{\sigma}$, $\check{\eta}$ are scalar 3×3 matrix valued smooth functions, i.e.

$$\check{\epsilon} = \epsilon(Id)_{3 \times 3}, \quad \check{\sigma} = \sigma(Id)_{3 \times 3}, \quad \check{\eta} = \eta(Id)_{3 \times 3} \quad (1.4)$$

where ϵ , σ , η are smooth functions, given in $C_b^1(\mathbb{R}^3)$, bounded from below.

We will also assume that

$$f^\varepsilon = (F^\varepsilon, G^\varepsilon)^t, \quad \varrho^\varepsilon \rightharpoonup (0, 0, 0) \text{ in } [L^2(\mathbb{R} \times \mathbb{R}^3)]^6 \times L^2(\mathbb{R} \times \mathbb{R}^3) \text{ weakly} \quad (1.5)$$

and

$$u^\varepsilon \equiv (E^\varepsilon, H^\varepsilon)^t \rightharpoonup 0 \text{ in } L^2(\mathbb{R} \times \mathbb{R}^3)^6 \text{ weakly.} \quad (1.6)$$

After some prerequisites on H-measures, see [6] and [23], presented in Section I, we use this notion in Section II to prove

Theorem 1.1. Constant coefficient case

Assume (1.2), (1.5), (1.6) and (1.3). Then, up to a suitable extraction, the H-measure $\mathfrak{M}_\zeta = \mathfrak{M}_\zeta(t, x, \zeta)$, $\zeta = (\zeta_0, \zeta')$, $\zeta' = (\zeta_1, \zeta_2, \zeta_3)$, associated to (u^ε) , can be expressed as follows

$$\mathfrak{M}_\zeta = \begin{pmatrix} \zeta' \otimes \zeta' a(t, x, \zeta) & \zeta' \otimes \zeta' c(t, x, \zeta) \\ \zeta' \otimes \zeta' d(t, x, \zeta) & \zeta' \otimes \zeta' b(t, x, \zeta) \end{pmatrix}. \quad (1.7)$$

Here $a(t, x, \zeta)$ and $b(t, x, \zeta)$ are positive measures, while $c(t, x, \zeta)$ and $d(t, x, \zeta)$ are complex measures such that $c = \bar{d}$, all supported in $[\{\zeta_0 = 0\} \cup \{\zeta' = 0\}] \cap \{\zeta_1 \zeta_2 \zeta_3 = 0\}$.

They satisfy the transport system (propagation property)

$$\begin{cases} |\zeta'|^2 \left(-\frac{\partial a}{\partial t} - 2a\right) - \sum_{l'=1}^3 \partial_{l'} \cdot [Tr(\zeta' \otimes \zeta' \partial^{l'} \mathbf{E})c] = 2Re Tr \mu_{uf_{11}}, \\ \sum_{l'=1}^3 \partial_{l'} \cdot [Tr(\zeta' \otimes \zeta' \partial^{l'} \mathbf{E})a] - |\zeta'|^2 \frac{\partial c}{\partial t} = 2Re Tr \mu_{uf_{12}}, \\ |\zeta'|^2 \left(-\frac{\partial b}{\partial t}\right) + \sum_{l'=1}^3 \partial_{l'} \cdot [Tr(\zeta' \otimes \zeta' \partial^{l'} \mathbf{E})d] = 2Re Tr \mu_{uf_{22}}, \\ -\sum_{l'=1}^3 \partial_{l'} \cdot [Tr(\zeta' \otimes \zeta' \partial^{l'} \mathbf{E})b] + |\zeta'|^2 \left(\frac{\partial d}{\partial t} - 2d\right) = 2Re Tr \mu_{uf_{21}}. \end{cases} \quad (1.8)$$

Above, a derivative with an upper (resp. lower) index denotes a derivative wrt. variable ζ' (resp. x). $\mathbf{E} = \mathbf{E}(\zeta')$ is the constant 3×3 matrix whose action is given by $E \cdot \alpha = \zeta' \wedge \alpha$, for all $\alpha \in \mathbb{R}^3$. Finally, μ_{uf} is the 6×6 matrix correlating the sequences u^ε and f^ε , written with blocks of size 3×3 . In particular, it is zero if at least f^ε is strongly convergent to 0.

Finally, one has the following constraint

$$\zeta_1^2 \partial_{x_1} a + \zeta_2^2 \partial_{x_2} a + \zeta_3^2 \partial_{x_3} a = 2Re Tr \mu_{u\tilde{e}_{11}}$$

and similarly for b , c and d , where $\mu_{u\tilde{e}}$ is the 6×6 matrix correlating the sequence u^ε with the sequence $\tilde{e}^\varepsilon \equiv (\varrho^\varepsilon, 0, 0, 0, 0, 0)^t$.

Theorem 1.2. Non constant coefficient-scalar case

Assume (1.2), (1.5), (1.6) and (1.4). Let the dispersion matrix (see formula (3.35)

below) be $L(x, \zeta) = \sum_{j=1}^3 A_0^{-1}(x) \zeta_j A^j$, which has three eigenvalues, each with fixed multiplicity two, for $\zeta' \neq 0$ and given by

$$\omega_0 = 0, \quad \omega_+ = v|\zeta'|, \quad \omega_- = -v|\zeta'|.$$

Then the matrix $P(x, \zeta) = A_0(\zeta_0 \mathbf{Id} + L(x, \zeta))$ has also the following three eigenvalues

$$\omega_0 = \zeta_0, \quad \omega_+ = \zeta_0 + v|\zeta'|, \quad \omega_- = \zeta_0 - v|\zeta'|,$$

each with fixed multiplicity two, where $v(x) = \frac{1}{\sqrt{\epsilon(x)\eta(x)}}$ is the propagation speed.

Using the propagation basis and the eigenvector basis introduced in (3.51) and (3.52), it follows that the H-measure \mathfrak{M}_ν associated to a suitable subsequence of u^ϵ can be expressed as:

$$\mathfrak{M}_\nu = \begin{pmatrix} \mathfrak{M}_{\nu 11} & \mathfrak{M}_{\nu 12} \\ \mathfrak{M}_{\nu 21} & \mathfrak{M}_{\nu 22} \end{pmatrix}$$

where $\mathfrak{M}_{\nu ij}$ are 3×3 matrix valued measures. Furthermore, one has

$$\left\{ \begin{array}{l} \mathfrak{M}_{\nu 11} = \frac{1}{\epsilon} [(\hat{\zeta}' \otimes \hat{\zeta}')a_0 + \frac{1}{2}(z^1 \otimes z^1)a_+ + \frac{1}{2}(z^2 \otimes z^2)b_+ + \frac{1}{2}(z^1 \otimes z^1)a_- + \frac{1}{2}(z^2 \otimes z^2)b_-] \\ \mathfrak{M}_{\nu 12} = \frac{v}{2} [(z^1 \otimes z^2)a_+ - (z^2 \otimes z^1)b_+ - (z^1 \otimes z^2)a_- + (z^2 \otimes z^1)b_-] \\ \mathfrak{M}_{\nu 21} = \frac{v}{2} [(z^2 \otimes z^1)a_+ - (z^1 \otimes z^2)b_+ - (z^2 \otimes z^1)a_- + (z^1 \otimes z^2)b_-] \\ \mathfrak{M}_{\nu 22} = \frac{1}{\mu} [(\hat{\zeta}' \otimes \hat{\zeta}')b_0 + \frac{1}{2}(z^2 \otimes z^2)a_+ + \frac{1}{2}(z^1 \otimes z^1)b_+ + \frac{1}{2}(z^2 \otimes z^2)a_- + \frac{1}{2}(z^1 \otimes z^1)b_-] \end{array} \right.$$

using notations given by (3.51). Above a_0 , b_0 , a_\pm and b_\pm are all scalar positive measures supported in the set $[\{\zeta_0 = 0\} \cup \{\zeta_0 = \pm v \mid \zeta' \mid\}] \cap \{\zeta_1 \zeta_2 \zeta_3 = 0\}$. Finally, one has the following propagation type system

$$\left\{ \begin{array}{l} -\epsilon(x)\partial_t \mathfrak{M}_{\nu 11} + \zeta_0 \sum_{l'=1}^3 \partial_{l'} \epsilon(x) \partial^{l'} \mathfrak{M}_{\nu 11} - 2\sigma \mathfrak{M}_{\nu 11} - \sum_{l'=1}^3 \partial^{l'} \mathbf{E} \cdot \partial_{l'} \mathfrak{M}_{\nu 12} = 2Re\mu_{uf_{11}} \\ -\eta(x)\mathfrak{M}_{\nu 12} + \zeta_0 \sum_{l'=1}^3 \partial_{l'} \eta(x) \partial^{l'} \mathfrak{M}_{\nu 12} + \sum_{l'=1}^3 \partial^{l'} \mathbf{E} \partial_{l'} \mathfrak{M}_{\nu 11} = 2Re\mu_{uf_{12}} \\ -\epsilon(x)\partial_t \mathfrak{M}_{\nu 21} + \zeta_0 \sum_{l'=1}^3 \partial_{l'} \epsilon(x) \partial^{l'} \mathfrak{M}_{\nu 21} - 2\sigma \mathfrak{M}_{\nu 21} - \sum_{l'=1}^3 \partial^{l'} \mathbf{E} \cdot \partial_{l'} \mathfrak{M}_{\nu 22} = 2Re\mu_{uf_{21}} \\ -\eta(x)\mathfrak{M}_{\nu 22} + \zeta_0 \sum_{l'=1}^3 \partial_{l'} \eta(x) \partial^{l'} \mathfrak{M}_{\nu 22} + \sum_{l'=1}^3 \partial^{l'} \mathbf{E} \partial_{l'} \mathfrak{M}_{\nu 21} = 2Re\mu_{uf_{22}} \end{array} \right.$$

where we are using same notations as in Theorem 1.1 for the right hand side.

2. Some basic facts on H-measures

In this Section, we recall some results from the H-measures theory, taking the presentation of Tartar [23]. However, this is also similar to the exposition of Gérard [5,7], relying upon Hormander [10], [11].

Definition 1. Let Ω be an open set of \mathbb{R}^n and let u^ϵ be a sequence of functions defined in \mathbb{R}^n with values in \mathbb{R}^p . We assume that u^ϵ converges weakly to zero in $(L^2(\mathbb{R}^n))^p$. Then after extracting a subsequence (still denoted by ϵ), there exists a family of complex-valued Radon measures $(\mathfrak{M}_{\nu ij}(x, \zeta))_{1 \leq i, j \leq p}$ on $\mathbb{R}^n \times S^{n-1}$, such that for every functions ϕ_1, ϕ_2 in $C_0(\mathbb{R}^n)$, the space of continuous functions

converging to zero at infinity, and for every function ψ in $C(S^{n-1})$, the space of continuous functions on the unit sphere S^{n-1} in \mathbb{R}^n , one has

$$\begin{cases} \langle \mathfrak{M}_{ij}, \phi_1 \bar{\phi}_2 \otimes \psi \rangle = \int_{\mathbb{R}^n} \int_{S^{n-1}} \phi_1 \bar{\phi}_2 \psi(\zeta/|\zeta|) \mathfrak{M}_{ij}(x, \zeta) dx d\zeta \\ = \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^n} [F(\phi_1 u_i^\varepsilon)(\zeta)] \overline{[F(\phi_2 u_j^\varepsilon)(\zeta)]} \psi(\zeta/|\zeta|) d\zeta. \end{cases} \quad (2.1)$$

Above, F denotes the Fourier transform operator defined in $L^2(\mathbb{R}^n)$, for an integrable function f as $[F(f)](\zeta) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \zeta} dx$, while \bar{F} is the inverse Fourier transform defined as $[\bar{F}(f)](x) = \int_{\mathbb{R}^n} f(\zeta) e^{2\pi i x \cdot \zeta} d\zeta$. \bar{z} denotes the complex conjugate of the complex number z .

The matrix valued measure $\mathfrak{M} = (\mathfrak{M}_{ij})_{1 \leq i, j \leq p}$ is called the H-measure associated with the extracted subsequence u^ε .

Remark 1. H-measures are hermitian and non-negative matrices in the following sense

$$\begin{cases} \mathfrak{M}_{ij} = \overline{\mathfrak{M}_{ji}} & \text{and} \\ \sum_{i,j=1}^p \mathfrak{M}_{ij} \phi_i \bar{\phi}_j \geq 0 \text{ for all } \phi_1 \phi_2 \dots \phi_n \in C_0(\mathbb{R}^n), \end{cases} \quad (2.2)$$

and it is clear that the H-measure for a strongly convergent sequence is zero.

Although we consider the scalar case for all properties of H-measures, all the following facts are easily extended to the vectorial case.

Definition 2. Let $a \in C(S^{n-1})$, $b \in C_0(\mathbb{R}^n)$. We associate with a the linear continuous operator A on $L^2(\mathbb{R}^n)$ defined by

$$F(Au)(\zeta) = a(\zeta/|\zeta|) F(u)(\zeta) \quad a.e. \quad \zeta \in \mathbb{R}^n \quad (2.3)$$

and with b we associate the operator

$$Bu(x) = b(x)u(x) \quad a.e. \quad x \in \mathbb{R}^n. \quad (2.4)$$

A continuous function P on $\mathbb{R}^n \times S^{n-1}$ with values in \mathbb{R} is called an admissible symbol if it can be written as

$$P(x, \zeta) = \sum_{n=1}^{+\infty} b_n(x) \otimes a_n(\zeta) = \sum_{n=1}^{+\infty} b_n(x) a_n(\zeta) \quad (2.5)$$

where a_n are continuous functions on S^{n-1} and b_n are continuous bounded functions converging to zero at infinity on \mathbb{R}^n with

$$\sum_{n=1}^{+\infty} \max_{\zeta} |a_n(\zeta)| \max_x |b_n(x)| < \infty. \quad (2.6)$$

An operator L with symbol P is defined by:

- 1) L is linear continuous on $L^2(\mathbb{R}^n)$.
- 2) P is an admissible symbol with a decomposition (2.5), satisfying (2.6).
- 3) L can be written as the following form

$$L = \sum_{n=1}^n A_n B_n + \text{compact operator}$$

where A_n, B_n are the operators associated with a_n, b_n as in (2.3) and (2.4).

With notations as in (2.3) and (2.4), one can show that the operator $C := AB - BA$ is a compact operator from $L^2(\mathbb{R}^n)$ into itself. Denote by $X^m(\mathbb{R}^n)$ the space of functions v with derivatives up to order m belonging to the image by the Fourier transform of the space $L^1(\mathbb{R}^n)$ i.e. $(F(L^1(\mathbb{R}^n)))$, equipped with the norm

$$\|v\|_{X^m} = \int_{\mathbb{R}^n} (1 + |2\pi\zeta|^m) |F(v)(\zeta)| d\zeta.$$

Then, if A and B are operators with symbols a and b as in (2.3) and (2.4), satisfying one of the following conditions

- 1) $a \in C^1(S^{n-1})$ and $b \in X^1(\mathbb{R}^n)$,
- 2) $a \in X_{loc}^1(\mathbb{R}^n \setminus \{0\})$ and $b \in C_0^1(\mathbb{R}^n)$,

it follows that the operator $C = AB - BA$ is a continuous operator from $L^2(\mathbb{R}^n)$ into $H^1(\mathbb{R}^n)$ and extending a to be homogeneous of degree zero on \mathbb{R}^n , then $\nabla C = \frac{\partial}{\partial x_i}(AB - BA)$ has the symbol

$$(\nabla_{\zeta} a \cdot \nabla_x b) \zeta = \zeta_i \sum_{k=1}^n \frac{\partial a}{\partial \zeta_k} \frac{\partial b}{\partial x_k}. \quad (2.7)$$

The main results of H-measures theory are given by the next two results

Theorem 2.1. Localisation property Let u^ε be a sequence converging weakly to zero in $(L^2(\mathbb{R}^n))^p$ and let \mathfrak{M}_ε be the H-measure associated to u^ε . Assume that one has the balance relation

$$\sum_{k=1}^n \frac{\partial}{\partial x_k} (A^k u^\varepsilon) \longrightarrow 0 \quad (H_{loc}^{-1}(\Omega))^p \quad \text{strongly,}$$

where A^k are continuous matrix valued functions on $\Omega \subset \mathbb{R}^n$. Then, on $\Omega \times S^{n-1}$, one has

$$P(x, \zeta) \mathfrak{M}_\varepsilon \equiv \left(\sum_{k=1}^n \zeta_k A^k(x) \right) \mathfrak{M}_\varepsilon = 0. \quad (2.8)$$

This result shows that the support of the H-measure \mathfrak{M}_ε is contained in the (characteristic) set

$$\{(x, \zeta) \in \Omega \times S^{n-1}, \det P(x, \zeta) = 0\}.$$

Theorem 2.2. Propagation property for symmetric systems *Let be given matrix valued functions A^k in the class $C_0^1(\Omega)$. Assume that the pair of sequences $(u^\varepsilon, f^\varepsilon)$ satisfies the symmetric system*

$$\sum_{k=1}^n A^k \frac{\partial u^\varepsilon}{\partial x_k} + B u^\varepsilon = f^\varepsilon \quad (2.9)$$

and that both sequences $(u^\varepsilon), (f^\varepsilon)$ converge weakly to zero in $L^2(\Omega)^p$. Then the H -measure μ associated to the sequence $(u^\varepsilon, f^\varepsilon)$ and given under the form

$$\mu = \begin{pmatrix} \mu_{11} & \mu_{12} \\ \mu_{21} & \mu_{22} \end{pmatrix} \quad (2.10)$$

satisfies the equation

$$\langle \mu_{11}, \{P, \psi\} + \psi \sum_{k=1}^n \partial_k A^k - 2\psi S \rangle = \langle 2\Re(\text{Tr}\mu_{12}), \psi \rangle \quad (2.11)$$

for all smooth functions $\psi(x, \zeta)$. Here $S := 1/2(B + B^)$ is the hermitian part of the matrix B and $\{P, \psi\}$ is the Poisson bracket of P and ψ , i.e.*

$$\{P, \psi\} = \partial^l P \partial_l \psi - \partial^l \psi \partial_l P \equiv \sum_{l=1}^n \left(\frac{\partial P}{\partial \zeta_l} \frac{\partial \psi}{\partial x_l} - \frac{\partial \psi}{\partial \zeta_l} \frac{\partial P}{\partial x_l} \right). \quad (2.12)$$

3. Applications to Maxwell's system

This section is devoted to the proofs of our main results stated in the Introduction.

3.1. Proof of Theorem 1.1: Constant coefficient case

This case corresponds to the assumption (1.3), that is all the matrices $\check{\epsilon}$, $\check{\eta}$, and $\check{\sigma}$ are the identity matrix, i.e

$$\check{\epsilon} = \check{\sigma} = \check{\eta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = (\text{Id})_{3 \times 3}. \quad (3.1)$$

In this case, system (1.1) can be rewritten as

$$\begin{cases} i) & \frac{\partial D^\varepsilon}{\partial t}(x, t) + E^\varepsilon(x, t) = \text{rot}H^\varepsilon(x, t) + F^\varepsilon(x, t), \\ ii) & \frac{\partial H^\varepsilon}{\partial t}(x, t) = -\text{rot}E^\varepsilon(x, t) + G^\varepsilon(x, t), \\ iii) & \text{div}H^\varepsilon(x, t) = 0, \\ iv) & \text{div}E^\varepsilon(x, t) = \rho^\varepsilon(x, t). \end{cases} \quad (3.2)$$

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Recalling the notation of the Introduction, it follows that Maxwell's system (3.2) can be written as

$$\sum_{i=0}^3 A^i \frac{\partial u^\varepsilon}{\partial x_i} + C u^\varepsilon = f^\varepsilon \quad (3.3)$$

and

$$\sum_{i=1}^3 B^i \frac{\partial u^\varepsilon}{\partial x_i} = \tilde{\varrho}^\varepsilon. \quad (3.4)$$

Here

$$A^0 = \begin{pmatrix} \ddot{\epsilon} & \mathbf{0} \\ \mathbf{0} & \ddot{\eta} \end{pmatrix} = \begin{pmatrix} \mathbf{Id} & \mathbf{0} \\ \mathbf{0} & \mathbf{Id} \end{pmatrix} \quad (3.5)$$

and

$$A^1 = \begin{pmatrix} \mathbf{0} & \mathbf{Q}_1^t \\ \mathbf{Q}_1 & \mathbf{0} \end{pmatrix}, A^2 = \begin{pmatrix} \mathbf{0} & \mathbf{Q}_2^t \\ \mathbf{Q}_2 & \mathbf{0} \end{pmatrix}, A^3 = \begin{pmatrix} \mathbf{0} & \mathbf{Q}_3^t \\ \mathbf{Q}_3 & \mathbf{0} \end{pmatrix}. \quad (3.6)$$

The constant antisymmetric matrices \mathbf{Q}_k , $1 \leq k \leq 3$ are given by

$$\mathbf{Q}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \mathbf{Q}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \mathbf{Q}_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (3.7)$$

the matrix C and f^ε by

$$C = \begin{pmatrix} \ddot{\sigma} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{Id} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, f^\varepsilon = \begin{pmatrix} F^\varepsilon \\ G^\varepsilon \end{pmatrix}. \quad (3.8)$$

Matrices B^i , $i = 1, 2, 3$, are given by

$$B^i = \begin{pmatrix} \beta^i & \mathbf{0} \\ \mathbf{0} & \beta^i \end{pmatrix}$$

where the 3×3 matrices β^i are such that

$$\beta_{kl}^i = 0 \text{ except for } \beta_{ii}^i = 1$$

Finally we have denoted $\tilde{\varrho}^\varepsilon \equiv (\varrho^\varepsilon, 0, 0, 0, 0, 0)^t$.

Denote the H-measure corresponding to (a subsequence of) the sequence u^ε by

$$\mathfrak{M}_\varepsilon = \begin{pmatrix} \nu_{\mathbf{e}} & \nu_{\mathbf{em}} \\ \nu_{\mathbf{me}} & \nu_{\mathbf{m}} \end{pmatrix}. \quad (3.9)$$

The measure \mathfrak{M}_ε is a 2×2 block matrix measure, each block being of size 3×3 .

In the following, let $x_0 = t$, $\tilde{x} = (x_0, x)$, $x = (x_1, x_2, x_3)$. We let ζ denote the dual variable to \tilde{x} , with $\zeta = (\zeta_0, \zeta')$, $\zeta' = (\zeta_1, \zeta_2, \zeta_3)$.

To state the localisation property (2.1), we need first to express the symbol of the differential operator appearing in (3.3), for which one has

$$P(x, \zeta) \equiv \sum_{j=0}^3 \zeta_j A^j(x)$$

and thus

$$\begin{aligned} P(x, \zeta) &= \zeta_0 A^0(x) + \zeta_1 A^1(x) + \zeta_2 A^2(x) + \zeta_3 A^3(x) = \\ &= \zeta_0 \begin{pmatrix} \mathbf{Id} & \mathbf{0} \\ \mathbf{0} & \mathbf{Id} \end{pmatrix} + \zeta_1 \begin{pmatrix} \mathbf{0} & \mathbf{Q}_1^t \\ \mathbf{Q}_1 & \mathbf{0} \end{pmatrix} + \zeta_2 \begin{pmatrix} \mathbf{0} & \mathbf{Q}_2^t \\ \mathbf{Q}_2 & \mathbf{0} \end{pmatrix} \\ &\quad + \zeta_3 \begin{pmatrix} \mathbf{0} & \mathbf{Q}_3^t \\ \mathbf{Q}_3 & \mathbf{0} \end{pmatrix} = \begin{pmatrix} \zeta_0 \mathbf{Id} & -\mathbf{E} \\ \mathbf{E} & \zeta_0 \mathbf{Id} \end{pmatrix} \end{aligned}$$

where

$$\mathbf{E} \equiv \begin{pmatrix} 0 & -\zeta_3 & \zeta_2 \\ \zeta_3 & 0 & -\zeta_1 \\ -\zeta_2 & \zeta_1 & 0 \end{pmatrix}. \quad (3.10)$$

Clearly \mathbf{E} is antisymmetric (i.e. $\mathbf{E}^t = -\mathbf{E}$), so that P is a symmetric matrix. Using the localisation property, it follows

$$P \mathfrak{M}_\zeta = \begin{pmatrix} \zeta_0 \mathbf{Id} & -\mathbf{E} \\ \mathbf{E} & \zeta_0 \mathbf{Id} \end{pmatrix} \begin{pmatrix} \nu_e & \nu_{em} \\ \nu_{me} & \nu_m \end{pmatrix} = \mathbf{0} \quad (3.11)$$

and thus

$$\begin{cases} 1) & \zeta_0 \mathbf{Id} \cdot \nu_e + \mathbf{E}^t \cdot \nu_{me} = 0, \\ 2) & \zeta_0 \mathbf{Id} \cdot \nu_{em} + \mathbf{E}^t \cdot \nu_m = 0, \\ 3) & \mathbf{E} \cdot \nu_e + \zeta_0 \mathbf{Id} \cdot \nu_{em} = 0, \\ 4) & \mathbf{E} \cdot \nu_{em} + \zeta_0 \mathbf{Id} \cdot \nu_m = 0. \end{cases} \quad (3.12)$$

First note that from (3.11), since (see also next subsection) P has $\zeta_0, \pm |\zeta'|$ as eigenvalues, that \mathfrak{M}_ζ is supported in $\{\zeta_0 = 0\} \cup \{\zeta' = 0\}$. Then, one has

Lemma 1. *The H-measure \mathfrak{M}_ζ can be written under the form*

$$\mathfrak{M}_\zeta = \begin{pmatrix} \zeta' \otimes \zeta' a(t, x, \zeta) & \zeta' \otimes \zeta' c(t, x, \zeta) \\ \zeta' \otimes \zeta' d(t, x, \zeta) & \zeta' \otimes \zeta' b(t, x, \zeta) \end{pmatrix} \quad (3.13)$$

where a, b are scalar positive measures, c and d are scalar complex measures such that $\bar{c} = d$, all supported in $\{\zeta_0 = 0\} \cup \{\zeta' = 0\}$.

Proof of Lemma 1

Multiplying (3.12-1) par ζ_0 , (3.12-3) par \mathbf{E}^t and subtracting the results, one has

$$(\zeta_0^2 \mathbf{Id} + \mathbf{E}^2) \cdot \nu_{\mathbf{e}} = \mathbf{0}. \quad (3.14)$$

We discuss the following distinct cases

i) case $\zeta_0 = 0$. Then note that $\zeta' \neq 0$ since ζ belongs to the unit sphere of \mathbb{R}^4 . From (3.12), we have $\mathbf{E} \cdot \nu_{\mathbf{e}} = \mathbf{0}$.

Then, we use the following Lemma

Lemma 2.

If $\mathbf{E} \cdot A = 0$, then the matrix A has the form $A = \zeta' \otimes a$, for some vector $a \in \mathbb{R}^3$.

Proof of Lemma 2

We denote the columns of the matrix A by the vectors

$$A = [\vec{a}_1 \ \vec{a}_2 \ \vec{a}_3]. \quad (3.15)$$

But as $\mathbf{E} \cdot A = 0$, we get that

$$[E\vec{a}_1 \ E\vec{a}_2 \ E\vec{a}_3] = 0 \quad (3.16)$$

or

$$E\vec{a}_i = 0, \quad i = 1, 2, 3. \quad (3.17)$$

For $i = 1$, and similarly for the other cases, $a_1 = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \in \mathbb{R}^3$. Then from (3.10),

(3.17), we get that

$$\begin{pmatrix} 0 & -\zeta_3 & \zeta_2 \\ \zeta_3 & 0 & -\zeta_1 \\ -\zeta_2 & \zeta_1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} -v_2\zeta_3 + v_3\zeta_2 \\ v_1\zeta_3 - v_3\zeta_1 \\ -v_1\zeta_2 + v_2\zeta_1 \end{pmatrix} = \zeta' \otimes \vec{a}_1 = 0 \quad (3.18)$$

which implies that $\zeta' // \vec{a}_1$ and thus $\vec{a}_1 = c_1 \zeta'$, for some constant $c_1 \in \mathbb{R}$.

Thus all in all, and for $i = 1, 2, 3$, all the columns of the matrix A are parallel to the vector $\zeta' = (\zeta_1, \zeta_2, \zeta_3)$, so we can write that $\vec{a}_i = c_i \zeta'$, and by arranging these numbers c_i as components of the vector a , we get that

$$A = \vec{a} \otimes \zeta'. \quad (3.19)$$

End of the proof of Lemma 1

Using Lemma 2, we can conclude that $\nu_{\mathbf{e}} = \zeta' \otimes \zeta' a(t, x, \zeta')$ and thus the blocks of the matrix H -measure, which satisfy system (3.12), are such that

$$\left\{ \begin{array}{l} \nu_{\mathbf{e}} = \zeta' \otimes \zeta' a(t, x, \zeta), \\ \nu_{\mathbf{m}} = \zeta' \otimes \zeta' b(t, x, \zeta), \\ \nu_{\mathbf{em}} = \zeta' \otimes \zeta' c(t, x, \zeta), \\ \nu_{\mathbf{me}} = \zeta' \otimes \zeta' d(t, x, \zeta) = \bar{\nu}_{\mathbf{em}} = \zeta' \otimes \zeta' \bar{c}(t, x, \zeta) \end{array} \right. \quad (3.20)$$

where $a(t, x, \zeta)$ and $b(t, x, \zeta)$ are real positive measures, while $d(t, x, \zeta)$ and $c(t, x, \zeta)$ are scalar complex measures, such that $c = \bar{d}$.

ii) case $\zeta_0 \neq 0$ but $\zeta' = (\zeta_1, \zeta_2, \zeta_3) = 0$.

From (3.14) one has $\zeta_0^2 \mathbf{Id} \cdot \nu_e = \mathbf{0}$ (recall that $\zeta' = 0$ implies that $\mathbf{E} = \mathbf{0}$), and in this case we have $\zeta_0^2 \cdot \nu_e = \mathbf{0}$. As $\zeta_0 \neq 0$, we get $\nu_e = \mathbf{0}$. Repeating the same steps, for the other equations of the system (3.12), we get that $\nu_e = \nu_m = \nu_{em} = \overline{\nu_{me}} = 0$ and finally, one has

$$\mathfrak{M}_\zeta = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}. \quad (3.21)$$

iii) case $\zeta_0 \neq 0$ and $\zeta' = (\zeta_1, \zeta_2, \zeta_3) \neq 0$. Using (3.14), one has

$$(\zeta_0^2 \mathbf{Id} + \mathbf{E} \cdot \mathbf{E}) \cdot \nu_e = \mathbf{0} \iff \zeta_0^2 \cdot \nu_e + \mathbf{E} \cdot (\mathbf{E} \cdot \nu_e) = \mathbf{0}. \quad (3.22)$$

But as $(\mathbf{E} \cdot \nu_e) = \vec{\zeta}' \otimes \nu_e$, one has that $\mathbf{E} \cdot (\vec{\zeta}' \otimes \nu_e) = \vec{\zeta}' \otimes (\vec{\zeta}' \otimes \nu_e)$ and from (3.22), one gets

$$\zeta_0^2 \cdot \nu_e + \vec{\zeta}' \otimes (\vec{\zeta}' \otimes \nu_e) = \mathbf{0}. \quad (3.23)$$

Now we shall show that $\nu_e = 0$.

By contradiction, we assume that $\vec{\nu}_e \neq 0$. Using (3.23), one has $(1 - \zeta'^2) \cdot \nu_e = -\vec{\zeta}' \otimes (\vec{\zeta}' \otimes \nu_e)$, thus either $\zeta'^2 = 1$ or $\zeta'^2 < 1$ or $\zeta'^2 > 1$.

If $\zeta'^2 = 1$, one has $\zeta_0 = 0$, which is a contradiction with $\zeta_0 \neq 0$. If $\zeta'^2 > 1$, then one have $\zeta_0^2 = 1 - \zeta'^2 < 0$ which is not possible, because $\zeta_0^2 > 0$, and if $\zeta'^2 < 1$ then we get again $\zeta_0^2 = \zeta'^2 - 1 < 0$, which is a contradiction.

Thus all in all, we conclude that $\nu_e = 0$ and similarly for $\nu_m = \nu_{em} = \overline{\nu_{me}} = 0$, and finally we have also, in this case

$$\mathfrak{M}_\zeta = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

ending the proof of Lemma 3.11.

Let us now turn to the localisation property associated with equation (3.4). In this case, it follows that \mathfrak{M}_ζ satisfies

$$\left(\sum_{j=1}^3 B^j \zeta_j \right) \mathfrak{M}_\zeta = 0.$$

Setting $B(\zeta) = \sum_{j=1}^3 B^j \zeta_j$, since $\det B(\zeta) = \zeta_1 \zeta_2 \zeta_3$, it follows that \mathfrak{M}_ζ is supported in the set $\{\zeta_1 \zeta_2 \zeta_3 = 0\}$.

All in all, the scalar measures a , b , c and d are all (also) supported in the set $\{\zeta_1 \zeta_2 \zeta_3 = 0\}$.

Now, we wish to write down the propagation property, and for this purpose, we need to compute the Poisson bracket, associated with equations (3.3) and (3.4).

Letting $\psi = \psi(\tilde{x}, \zeta)$ be an arbitrary smooth function, recall first that the Poisson bracket is given by

$$\begin{cases} \{P, \psi\} = \sum_{l=0}^3 \partial^l P \partial_l \psi - \partial^l \psi \partial_l P \\ = \sum_{l=0}^3 \left(\frac{\partial P}{\partial \zeta_1} \frac{\partial \psi}{\partial \tilde{x}_l} - \frac{\partial \psi}{\partial \zeta_1} \frac{\partial P}{\partial \tilde{x}_l} \right). \end{cases} \quad (3.24)$$

Recall that a derivative with an upper (resp. lower) index denotes a derivative wrt. variable ζ (resp. \tilde{x}).

In our case, starting with (3.3), we have

$$\{P, \psi\} = \begin{pmatrix} \mathbf{Id} \partial_t \psi & -\left(\sum_{l'=1}^3 \partial^{l'} \mathbf{E} \cdot \partial_{l'} \psi \right) \\ \left(\sum_{l'=1}^3 \partial^{l'} \mathbf{E} \cdot \partial_{l'} \psi \right) & \mathbf{Id} \partial_t \psi \end{pmatrix}. \quad (3.25)$$

Next, we compute the term $\psi \sum_{l=0}^3 \partial_l A^l - 2\psi S$, where $S = 1/2(C + C^*) = C$. Note also that as A^l , $0 \leq l \leq 3$, are constant matrices, one has $\psi \partial_l A^l = 0$. Adding all the terms, we obtain

$$\{P, \psi\} + \sum_{l=0}^3 \psi \partial_l A^l - 2\psi S = \begin{pmatrix} \mathbf{Id} \partial_t \psi & -\left(\sum_{l'=1}^3 \partial^{l'} \mathbf{E} \cdot \partial_{l'} \psi \right) \\ \left(\sum_{l'=1}^3 \partial^{l'} \mathbf{E} \cdot \partial_{l'} \psi \right) & \mathbf{Id} \partial_t \psi \end{pmatrix} - 2 \begin{pmatrix} \psi \mathbf{Id} & 0 \\ 0 & 0 \end{pmatrix}$$

and thus

$$\{P, \psi\} + \sum_{l=0}^3 \psi \partial_l A^l - 2\psi S = \begin{pmatrix} (\partial_t \psi - 2\psi) \mathbf{Id} & -\left(\sum_{l'=1}^3 \partial^{l'} \mathbf{E} \cdot \partial_{l'} \psi \right) \\ \left(\sum_{l'=1}^3 \partial^{l'} \mathbf{E} \cdot \partial_{l'} \psi \right) & \mathbf{Id} \partial_t \psi \end{pmatrix}. \quad (3.26)$$

Taking into account the form of \mathfrak{M}_ζ deduced from the localisation property (3.11), we get

$$\left\{ \begin{array}{l} \langle \mathfrak{M}_\zeta, \{P, \psi\} + \sum_{l=0}^3 \psi \partial_l A^l - 2\psi S \rangle = \langle \begin{pmatrix} \zeta' \otimes \zeta' a(t, x, \zeta) & \zeta' \otimes \zeta' c(t, x, \zeta) \\ \zeta' \otimes \zeta' d(t, x, \zeta) & \zeta' \otimes \zeta' b(t, x, \zeta) \end{pmatrix}, \\ \begin{pmatrix} (\partial_t \psi - 2\psi) \mathbf{Id} & -(\sum_{l'=1}^3 \partial^{l'} \mathbf{E} \cdot \partial_{l'} \psi) \\ (\sum_{l'=1}^3 \partial^{l'} \mathbf{E} \cdot \partial_{l'} \psi) & \mathbf{Id} \partial_t \psi \end{pmatrix} \rangle \end{array} \right. \quad (3.27)$$

so that the propagation property reads as

$$\left\{ \begin{array}{l} \langle \zeta' \otimes \zeta' a(t, x, \zeta) (\partial_t \psi - 2\psi) \mathbf{Id} + \zeta' \otimes \zeta' c(t, x, \zeta) (\sum_{l'=1}^3 \partial^{l'} \mathbf{E} \cdot \partial_{l'} \psi) \rangle = 2Re\mu_{uf_{11}}, \\ \langle -\zeta' \otimes \zeta' a(t, x, \zeta) (\sum_{l'=1}^3 \partial^{l'} \mathbf{E} \cdot \partial_{l'} \psi) + \zeta' \otimes \zeta' c(t, x, \zeta) \mathbf{Id} \partial_t \psi \rangle = 2Re\mu_{uf_{12}}, \\ \langle \zeta' \otimes \zeta' d(t, x, \zeta) (\partial_t \psi - 2\psi) \mathbf{Id} + \zeta' \otimes \zeta' b(t, x, \zeta) (\sum_{l'=1}^3 \partial^{l'} \mathbf{E} \cdot \partial_{l'} \psi) \rangle = 2Re\mu_{uf_{21}}, \\ \langle -\zeta' \otimes \zeta' d(t, x, \zeta) (\sum_{l'=1}^3 \partial^{l'} \mathbf{E} \cdot \partial_{l'} \psi) + \zeta' \otimes \zeta' b(t, x, \zeta) \mathbf{Id} \partial_t \psi \rangle = 2Re\mu_{uf_{22}} \end{array} \right. \quad (3.28)$$

with the notations explained in the statement of Theorem 1.1.

Writing these equations in \mathcal{D}' and then taking the trace of each equation, we get finally (1.8).

Now, it remains to take into account the propagation property coming from (3.4).

Since in this case, the characteristic polynomial is given by $B(\zeta) = \sum_{j=1}^3 B^j \zeta_j$, it

follows with a small computation that one has the following propagation property

$$\left\{ \left\langle \begin{pmatrix} \zeta' \otimes \zeta' a(t, x, \zeta) & \zeta' \otimes \zeta' c(t, x, \zeta) \\ \zeta' \otimes \zeta' d(t, x, \zeta) & \zeta' \otimes \zeta' b(t, x, \zeta) \end{pmatrix}, \begin{pmatrix} \Gamma(\tilde{x}, \zeta) & \Gamma(\tilde{x}, \zeta) \\ \Gamma(\tilde{x}, \zeta) & \Gamma(\tilde{x}, \zeta) \end{pmatrix} \right\rangle = 2Re\mu_{u\bar{e}} \right.$$

again using the notations in the statement of Theorem 1.1. Here $\Gamma(\tilde{x}, \zeta)$ is the 3×3 matrix given by

$$\Gamma(\tilde{x}, \zeta) = \text{diag}(\partial_{x_1} \psi, \partial_{x_2} \psi, \partial_{x_3} \psi).$$

Writing each equation and taking the trace, this gives the last constraint mentioned in the statement of Theorem 1.1.

3.2. Proof of Theorem 1.2: Non constant coefficient-scalar case

Let us recall that we assume (1.4), that is $\check{\epsilon}$, $\check{\eta}$ and $\check{\sigma}$ are 3×3 scalar matrix valued functions given by

$$\check{\epsilon} = \epsilon(\mathbf{Id})_{3 \times 3} \equiv \begin{pmatrix} \epsilon(x) & 0 & 0 \\ 0 & \epsilon(x) & 0 \\ 0 & 0 & \epsilon(x) \end{pmatrix} \quad (3.29)$$

and

$$\check{\eta} = \eta(\mathbf{Id})_{3 \times 3} \equiv \begin{pmatrix} \eta(x) & 0 & 0 \\ 0 & \eta(x) & 0 \\ 0 & 0 & \eta(x) \end{pmatrix}, \check{\sigma} = \sigma(\mathbf{Id})_{3 \times 3} \equiv \begin{pmatrix} \sigma(x) & 0 & 0 \\ 0 & \sigma(x) & 0 \\ 0 & 0 & \sigma(x) \end{pmatrix} \quad (3.30)$$

where ϵ , η and σ are smooth functions in $C^1(\mathbb{R}^3)$, bounded from below.

The first equation of Maxwell's system can then again be written in the form of a symmetric system

$$\sum_{i=0}^3 A^i(x) \frac{\partial u^\epsilon}{\partial x_i} + C(x)u^\epsilon = f^\epsilon \quad (3.31)$$

where

$$A^0 = \begin{pmatrix} \epsilon(x)\mathbf{Id} & \mathbf{0} \\ \mathbf{0} & \eta(x)\mathbf{Id} \end{pmatrix} \quad (3.32)$$

is a (2×2) block matrix with 3×3 blocks, the constant antisymmetric \mathbf{Q}_k , $1 \leq k \leq 3$ and the matrix A^i , $1 \leq i \leq 3$ being the same as in (3.6) and C now given by

$$C(x) = \begin{pmatrix} \sigma(x)\mathbf{Id} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \quad (3.33)$$

is a (2×2) block matrix, each block being of size 3×3 .

As the sequence u^ϵ converges weakly to zero in $L^2(\Omega)^6$, again up to a subsequence, it defines an H- measure \mathfrak{M}_ϵ , which is a 6×6 matrix valued measure.

As in the preceding constant case, to express the localisation property linked with (3.31), we compute the associated symbol $P(x, \zeta)$ which is here given by

$$P(x, \zeta) = \begin{pmatrix} \zeta_0 \epsilon(x)\mathbf{Id} & -\mathbf{E} \\ \mathbf{E} & \zeta_0 \eta(x)\mathbf{Id} \end{pmatrix} \quad (3.34)$$

with \mathbf{E} is still given by (3.10). The localisation property then states that $P\mathfrak{M}_\epsilon = 0$. Let us first show the following

Lemma 3. Assume that $\zeta' \neq 0$ and let $L(x, \zeta) = \sum_{j=1}^3 A_0^{-1}(x) \zeta_j A^j$ be the dispersion matrix. Then L has three eigenvalues, each with constant multiplicity two, given by

$$\omega_0 = 0, \quad \omega_+ = \zeta_0 + v|\zeta'|, \quad \omega_- = \zeta_0 - v|\zeta'|.$$

The matrix $P'(x, \zeta) \equiv \zeta_0 \mathbf{Id} + L(x, \zeta')$ has also three eigenvalues, given by

$$\omega_0 = \zeta_0, \quad \omega_+ = \zeta_0 + v|\zeta'|, \quad \omega_- = \zeta_0 - v|\zeta'|,$$

each with constant multiplicity two, where $v(x) = \frac{1}{\sqrt{\epsilon(x)\eta(x)}}$ is the propagation speed.

Proof of Lemma 3

$L(x, \zeta')$ can be rewritten as

$$L(x, \zeta') = \sum_{j=1}^3 A_0^{-1}(x) \zeta_j A^j = - \begin{pmatrix} 0 & 0 & 0 & 0 & -\zeta_3/\epsilon & \zeta_2/\epsilon \\ 0 & 0 & 0 & \zeta_3/\epsilon & 0 & -\zeta_1/\epsilon \\ 0 & 0 & 0 & -\zeta_2/\epsilon & \zeta_1/\epsilon & 0 \\ 0 & \zeta_3/\mu & -\zeta_2/\mu & 0 & 0 & 0 \\ -\zeta_3/\mu & 0 & \zeta_1/\mu & 0 & 0 & 0 \\ \zeta_2/\mu & -\zeta_1/\mu & 0 & 0 & 0 & 0 \end{pmatrix} \quad (3.35)$$

or in block form

$$L(x, \zeta') = \begin{pmatrix} \mathbf{0} & -1/\epsilon \mathbf{E} \\ 1/\mu \mathbf{E} & \mathbf{0} \end{pmatrix}. \quad (3.36)$$

The action of the matrix $\mathbf{E} = \mathbf{E}(\zeta')$ is also given as

$$\mathbf{E}(\zeta')[\mathbf{p}] = \zeta' \wedge \mathbf{p} \quad (3.37)$$

where $\mathbf{E}(\zeta')$ is given in (3.10), for all $p \in \mathbb{R}^3$. Letting ω be an eigenvalue of $L(x, \zeta')$ corresponding to an eigenvector \mathbf{X} , one has

$$\begin{pmatrix} \mathbf{0} & -1/\epsilon \mathbf{E} \\ 1/\mu \mathbf{E} & \mathbf{0} \end{pmatrix} \mathbf{X} = \omega \mathbf{X} \quad (3.38)$$

or

$$\begin{pmatrix} \mathbf{0} & -1/\epsilon \mathbf{E} \\ 1/\mu \mathbf{E} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} = \omega \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix}. \quad (3.39)$$

Let us check that $\omega = 0$ is an eigenvalue. In order to see this, we have to solve the algebraic system

$$\begin{cases} -1/\epsilon \mathbf{E}(\zeta') \vec{\mathbf{v}} = \mathbf{0}, \\ 1/\mu \mathbf{E}(\zeta') \vec{\mathbf{u}} = \mathbf{0} \end{cases} \quad (3.40)$$

which is equivalent to

$$\begin{cases} \vec{\zeta} \wedge \vec{v} = 0, \\ \vec{\zeta} \wedge \vec{u} = 0. \end{cases} \quad (3.41)$$

It follows that \vec{u} and \vec{v} are colinear to ζ' , and thus that $(\vec{0}, \vec{\zeta}')$ and $(\vec{\zeta}', \vec{0})$ is a basis of the eigenspace corresponding to the eigenvalue 0 of L . It shows also that 0 is indeed an eigenvalue of multiplicity two.

It remains to find the eigenvalues $\omega \neq 0$ of L . For this purpose, set

$$D(\zeta') := \sum_{j=1}^3 \frac{\zeta_j}{|\zeta'|} A^j = \left(\frac{1}{|\zeta'|} L(x, \zeta') \right).$$

We can as well assume that $\zeta' \in S^2$, the unit sphere in \mathbb{R}^3 . Then $\omega = \mp 1$ are eigenvalues of the matrix $D(\zeta')$.

Indeed, we have to solve the system of equation

$$\begin{cases} -\vec{\zeta}' \wedge \vec{v} = \omega \vec{u}, \\ \vec{\zeta}' \wedge \vec{u} = \omega \vec{v}. \end{cases} \quad (3.42)$$

For $\omega = 1$ to be an eigenvalue of the matrix $D(\zeta')$, we have to solve the system

$$\begin{cases} -\vec{\zeta}' \wedge \vec{v} = \vec{u}, \\ \vec{\zeta}' \wedge \vec{u} = \vec{v} \end{cases} \quad (3.43)$$

which admits a $2D$ space of solutions, and for $\omega = -1$ to be an eigenvalue of the matrix $D(\zeta')$, we have to solve

$$\begin{cases} -\vec{\zeta}' \wedge \vec{v} = -\vec{u}, \\ \vec{\zeta}' \wedge \vec{u} = -\vec{v} \end{cases} \quad (3.44)$$

which again admits a $2D$ space of solutions. Thus, all in all, we have obtained that

$$\omega_1 = -1, \omega_3 = 1$$

are the eigenvalues of multiplicity $m = 2$ of the matrix $D(\zeta')$. Recall that $\omega_2 = 0$ is also an eigenvalue of the matrix $D(\zeta')$ with multiplicity $m = 2$.

Next, note that if λ is an eigenvalue of a matrix A then $c\lambda$ is an eigenvalue of the matrix cA , where $c > 0$ is a constant. Thus we conclude that

$$-|\zeta'|, 0, |\zeta'| \quad (3.45)$$

are the eigenvalues of the matrix $B(\zeta') = L(x, \zeta')$ with multiplicity $m = 2$, for all $\zeta' \neq 0$.

Set $\mu(x, \zeta) = \lambda(x, \zeta') - \zeta_0$ and recall that if $\lambda(x, \zeta)$ is an eigenvalue of the matrix P corresponding to an eigenvector $\vec{u} = \vec{u}(x, \zeta)$, then $\mu(x, \zeta) = \lambda(x, \zeta') - \zeta_0$ is an eigenvalue of the matrix $L(x, \zeta')$ corresponding to an eigenvector $\vec{X} = \vec{X}(x, \zeta')$.

If $\mu(x, \zeta) = \lambda(x, \zeta) - \zeta_0$ is an eigenvalue of the matrix $L(x, \zeta')$ corresponding to an eigenvector $\vec{\mathbf{X}} = \vec{\mathbf{X}}(x, \zeta')$, then $\lambda(x, \zeta) = \mu(x, \zeta) + \zeta_0$ is an eigenvalue of the matrix $P'(x, \zeta)$ corresponding to an eigenvector $\vec{\mathbf{X}} = \vec{\mathbf{X}}(x, \zeta)$. As the eigenvalues of the matrix $L(x, \zeta')$ are given in (3.45), we can conclude that

$$\zeta_0 - v|\zeta'|, \zeta_0, \zeta_0 + v|\zeta'| \quad (3.46)$$

are the eigenvalues of the matrix $P'(x, \zeta)$ with multiplicity $m = 2$. Now we shall show that the propagation speed v is given by

$$v(x) = \frac{1}{\sqrt{\epsilon(x)\mu(x)}}.$$

Indeed, from (3.39) and for $\omega \neq 0$, one has

$$\begin{cases} -1/\epsilon \mathbf{E}(\zeta') \vec{\mathbf{v}} = \omega \vec{\mathbf{u}} \\ 1/\mu \mathbf{E}(\zeta') \vec{\mathbf{u}} = \omega \vec{\mathbf{v}} \end{cases} \quad (3.47)$$

or

$$\begin{cases} \vec{\zeta}' \wedge \vec{\mathbf{v}} = -\epsilon \omega \vec{\mathbf{u}} \\ \vec{\zeta}' \wedge \vec{\mathbf{u}} = \mu \omega \vec{\mathbf{v}} \end{cases} \quad (3.48)$$

and thus

$$\vec{\mathbf{v}} = \frac{1}{\omega \mu} \vec{\zeta}' \wedge \vec{\mathbf{u}}.$$

Thus

$$\begin{cases} \vec{\zeta}' \wedge (\frac{1}{\mu \omega} \vec{\zeta}' \wedge \vec{\mathbf{u}}) = -\epsilon \omega \vec{\mathbf{u}} \Rightarrow \\ \vec{\zeta}' \wedge (\vec{\zeta}' \wedge \vec{\mathbf{u}}) = -\epsilon \mu \omega^2 \vec{\mathbf{u}} \Rightarrow \\ \{\vec{\zeta}' \wedge (\vec{\zeta}' \wedge \vec{\mathbf{u}})\} \cdot \vec{\mathbf{u}} = -\epsilon \mu \omega^2 \|\vec{\mathbf{u}}\|^2 \Rightarrow \\ (\vec{\zeta}' \wedge \vec{\mathbf{u}}) \cdot (\vec{\zeta}' \wedge \vec{\mathbf{u}}) = -\epsilon \mu \omega^2 \|\vec{\mathbf{u}}\|^2 \Rightarrow \\ \|\vec{\zeta}' \wedge \vec{\mathbf{u}}\|^2 = -\epsilon \mu \omega^2 \|\vec{\mathbf{u}}\|^2 \end{cases} \quad (3.49)$$

but $\vec{\zeta}'$ is orthogonal on the vector $\vec{\mathbf{u}}$ and thus

$$\omega^2 = \frac{\|\vec{\zeta}'\|^2 \|\vec{\mathbf{u}}\|^2}{\epsilon \mu \|\vec{\mathbf{u}}\|^2} \implies \omega = \pm \frac{\|\vec{\zeta}'\|}{\sqrt{\epsilon \mu}} = \pm \frac{|\zeta'|}{\sqrt{\epsilon \mu}} = v|\zeta'|. \quad (3.50)$$

This ends the proof of Lemma 3.

For the eigenvectors of the matrix $P'(x, \zeta)$, one first chooses an orthonormal basis of \mathbb{R}^3 . This basis consists of the propagation triple in the direction of propagation $\hat{\zeta}'$ and of the two transverse unit vectors $z^{(1)}(\zeta'), z^{(2)}(\zeta')$.

Let $(\hat{\zeta}', z^{(1)}(\zeta'), z^{(2)}(\zeta')) \in \mathbb{R}^3$ be this basis of propagation. In polar coordinates they are given by, see for more details [15], [16]

$$\hat{\zeta}' = \frac{\zeta'}{|\zeta'|} = \begin{pmatrix} \sin \theta \cos \phi \\ \sin \theta \sin \phi \\ \cos \theta \end{pmatrix}, \quad z^{(1)}(\zeta') = \begin{pmatrix} \cos \theta \cos \phi \\ \cos \theta \sin \phi \\ -\sin \theta \end{pmatrix}, \quad z^{(2)}(\zeta') = \begin{pmatrix} -\sin \phi \\ \cos \phi \\ 0 \end{pmatrix} \quad (3.51)$$

where $|\zeta'| = (\zeta_1^2 + \zeta_2^2 + \zeta_3^2)^{1/2}$.

Then, one can show that the eigenvectors of the matrix $P'(x, \zeta)$ are given by

$$\begin{cases} b_0^1 = \frac{1}{\sqrt{\epsilon}}(\hat{\zeta}', 0), & b_0^2 = \frac{1}{\sqrt{\mu}}(0, \hat{\zeta}') \\ b_+^1 = \left(\frac{1}{\sqrt{2\epsilon}}z^1, \frac{1}{\sqrt{2\mu}}z^2\right), & b_+^2 = \left(\frac{1}{\sqrt{2\epsilon}}z^2, -\frac{1}{\sqrt{2\mu}}z^1\right) \\ b_-^1 = \left(\frac{1}{\sqrt{2\epsilon}}z^1, -\frac{1}{\sqrt{2\mu}}z^2\right), & b_-^2 = \left(\frac{1}{\sqrt{2\epsilon}}z^2, \frac{1}{\sqrt{2\mu}}z^1\right) \end{cases} \quad (3.52)$$

The eigenvectors b_0^1 and b_0^2 represent the non-propagating longitudinal and the other eigenvectors correspond to transverse modes of propagation with respect the propagation speed v .

Lemma 4. *The H-measure \mathfrak{M}_ζ has the form*

$$\mathfrak{M}_\zeta = b_0^1 \otimes b_0^1 a_0 + b_0^2 \otimes b_0^2 b_0 + b_+^1 \otimes b_+^1 a_+ + b_+^2 \otimes b_+^2 b_+ + b_-^1 \otimes b_-^1 a_- + b_-^2 \otimes b_-^2 b_- \quad (3.53)$$

where a_0, b_0 are two positives measures supported in the set $\{\zeta_0 = 0\}$, a_+, b_+ are two positives measures supported in the set $\{\zeta_0 = -v \mid \zeta' \mid\}$, and a_-, b_- are two positives measures supported in the set $\{\zeta_0 = +v \mid \zeta' \mid\}$. Here $b_0^1, b_0^2, b_+^1, b_+^2, b_-^1$, and b_-^2 are the eigenvectors of the matrix P' given by (3.51) and (3.52).

Proof of Lemma (4) Since $P\mathfrak{M}_\zeta = 0$ by the localization property, it follows that

$$(A_0)^{-1}P\mathfrak{M}_\zeta = 0. \quad (3.54)$$

Since $P'(x, \zeta) = (A_0)^{-1}P(x, \zeta)$, thus (3.54) becomes

$$P'(x, \zeta)\mathfrak{M}_\zeta = 0. \quad (3.55)$$

It follows that the support of the H-measure \mathfrak{M}_ζ is included in the set

$$U = \{(x, \zeta) \in \mathbb{R}^4 \times S^3, \det P' = 0\}. \quad (3.56)$$

Note that, for every $t = \zeta_0, x, \zeta$ fixed, the matrix P' is diagonalizable. In fact, we will discuss the following cases :

a) Case of equal eigenvalues

i) If $\omega_0 = \omega_+$, (resp. $\omega_0 = \omega_-$), then from Lemma 3), one has $\zeta' = 0$, (resp. $\omega_0 = \omega_-$), and thus $\zeta_0 = \pm 1$, and $P' = \zeta_0 Id$, which is diagonal.

ii) If $\omega_0 \neq \omega_+, \omega_0 \neq \omega_-$, but $\omega_+ = \omega_-$. In this case, again from Lemma 3, one has $\zeta' = 0$, and thus $\zeta_0 = \pm 1$ and $P' = \zeta_0 Id$, which is diagonal.

b) Case of distinct eigenvalue

If $\omega_0 \neq \omega_+ \neq \omega_-$, then again from Lemma (3), using the basis of eigenvectors corresponding to an the eigenvalues $\omega_0, \omega_+, \omega_-$, given by (3.51) and (3.52), $P' = \zeta_0 Id$ is diagonal.

Next, as the support of the H-measure \mathfrak{M}_ζ is contained in the set of points (3.56), and as the matrix P' is diagonalizable, recalling that the determinant of the matrix P' is given by (eventually with powers)

$$\det P' \sim \omega_0 \omega_+ \omega_- .$$

It follows that the support of the H-measure \mathfrak{M}_ζ is included in the set

$$U = \{(x, \zeta) \in \mathbb{R}^4 \times S^3, \omega_0 = 0 \cup \omega_+ = 0 \cup \omega_- = 0\} .$$

Next, one has

$$P' \mathfrak{M}_{\zeta_j} = 0 \quad j = 1, 2, \dots, 6$$

where \mathfrak{M}_{ζ_j} is the j -th column vector of the matrix \mathfrak{M}_ζ . Using Lemma (3), we get that, for $\omega_0 = 0$, there exist two scalars α_j, β_j such that

$$\mathfrak{M}_{\zeta_j} = \alpha_j b_0^1 + \beta_j b_0^2 .$$

Repeating the same steps for the other eigenvalues ω_+, ω_- , and using the hermitian property of the H-measure, this **ends the proof of Lemma 4**.

In order to write down the propagation property for equation (3.31), we first need to compute the Poisson bracket, which in this case, is given by

$$\{P, \psi\} = \begin{pmatrix} \epsilon(x) \mathbf{Id} \partial_t \psi - \zeta_0 \sum_{l'=1}^3 \partial^{l'} \psi \partial_{l'} \epsilon(x) \mathbf{Id} & -(\sum_{l'=1}^3 \partial^{l'} \mathbf{E} \cdot \partial_{l'} \psi) \\ (\sum_{l'=1}^3 \partial^{l'} \mathbf{E} \cdot \partial_{l'} \psi) & \eta(x) \mathbf{Id} \partial_t \psi - \zeta_0 \sum_{l'=1}^3 \partial^{l'} \psi \partial_{l'} \eta(x) \mathbf{Id} \end{pmatrix} . \quad (3.57)$$

It follows that, since $\sum_{k=0}^3 \partial_k A^k = 0$

$$\left\{ \begin{array}{l} \{P, \psi\} + \psi \sum_{k=0}^3 \partial_k A^k - 2\psi S = \\ = \begin{pmatrix} \epsilon(x) \mathbf{Id} \partial_t \psi - \zeta_0 \sum_{l'=1}^3 \partial^{l'} \psi \partial_{l'} \epsilon(x) \mathbf{Id} - 2\psi \sigma \mathbf{Id} & -(\sum_{l'=1}^3 \partial^{l'} \mathbf{E} \cdot \partial_{l'} \psi) \\ (\sum_{l'=1}^3 \partial^{l'} \mathbf{E} \cdot \partial_{l'} \psi) & \eta(x) \mathbf{Id} \partial_t \psi - \zeta_0 \sum_{l'=1}^3 \partial^{l'} \psi \partial_{l'} \eta(x) \mathbf{Id} \end{pmatrix} \end{array} \right. \quad (3.58)$$

Writing the H-measure \mathfrak{M}_ζ as

$$\mathfrak{M}_\zeta = \begin{pmatrix} \mathfrak{M}_{\zeta_{11}} & \mathfrak{M}_{\zeta_{12}} \\ \mathfrak{M}_{\zeta_{21}} & \mathfrak{M}_{\zeta_{22}} \end{pmatrix} \quad (3.59)$$

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where \mathfrak{M}_{ij} are 3×3 matrix valued measures, it follows that one has

$$\left\{ \begin{array}{l} -\varepsilon(x)\partial_t \mathfrak{M}_{11} + \zeta_0 \sum_{l'=1}^3 \partial_{l'} \varepsilon(x) \partial^{l'} \mathfrak{M}_{11} - 2\sigma \mathfrak{M}_{11} - \sum_{l'=1}^3 \partial^{l'} \mathbf{E} \cdot \partial_{l'} \mathfrak{M}_{12} = 2Re\mu_{uf_{11}}, \\ -\eta(x)\mathfrak{M}_{12} + \zeta_0 \sum_{l'=1}^3 \partial_{l'} \eta(x) \partial^{l'} \mathfrak{M}_{12} + \sum_{l'=1}^3 \partial^{l'} \mathbf{E} \partial_{l'} \mathfrak{M}_{11} = 2Re\mu_{uf_{12}}, \\ -\varepsilon(x)\partial_t \mathfrak{M}_{21} + \zeta_0 \sum_{l'=1}^3 \partial_{l'} \varepsilon(x) \partial^{l'} \mathfrak{M}_{11} - 2\sigma \mathfrak{M}_{21} - \sum_{l'=1}^3 \partial^{l'} \mathbf{E} \cdot \partial_{l'} \mathfrak{M}_{22} = 2Re\mu_{uf_{21}}, \\ -\eta(x)\mathfrak{M}_{22} + \zeta_0 \sum_{l'=1}^3 \partial_{l'} \eta(x) \partial^{l'} \mathfrak{M}_{22} + \sum_{l'=1}^3 \partial^{l'} \mathbf{E} \partial_{l'} \mathfrak{M}_{21} = 2Re\mu_{uf_{22}} \end{array} \right. \quad (3.60)$$

which is the form given in Theorem 1.2.

Using the eigenvector basis (3.51) and (3.52), with the decomposition given by Lemma 4, we note that the elements of the H-measure \mathfrak{M} defined in (3.59), can be expressed as

$$\left\{ \begin{array}{l} \mathfrak{M}_{11} = \frac{1}{\epsilon} [(\hat{\zeta}' \otimes \hat{\zeta}') a_0 + \frac{1}{2}(z^1 \otimes z^1) a_+ + \frac{1}{2}(z^2 \otimes z^2) b_+ + \frac{1}{2}(z^1 \otimes z^1) a_- \\ \quad + \frac{1}{2}(z^2 \otimes z^2) b_-], \\ \mathfrak{M}_{12} = \frac{\nu}{2} [(z^1 \otimes z^2) a_+ - (z^2 \otimes z^1) b_+ - (z^1 \otimes z^2) a_- + (z^2 \otimes z^1) b_-], \\ \mathfrak{M}_{21} = \frac{\nu}{2} [(z^2 \otimes z^1) a_+ - (z^1 \otimes z^2) b_+ - (z^2 \otimes z^1) a_- + (z^1 \otimes z^2) b_+], \\ \mathfrak{M}_{22} = \frac{1}{\mu} [(\hat{\zeta}' \otimes \hat{\zeta}') b_0 + \frac{1}{2}(z^2 \otimes z^2) a_+ + \frac{1}{2}(z^1 \otimes z^1) b_+ \\ \quad + \frac{1}{2}(z^2 \otimes z^2) a_- + \frac{1}{2}(z^1 \otimes z^1) b_-]. \end{array} \right. \quad (3.61)$$

If one wants to find an equation for a_0 for instance, one can proceed as follows. Recalling the form of \mathfrak{M}_{11} just given above, we take the equation for it in (3.60) and apply it to the vector ζ' .

Finally, because the divergence constraint in Maxwell's system only involves scalar valued functions, similar statement as in the end of Theorem 1.1 holds true again.

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