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Ruelle's probability cascades seen as a fragmentation process

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Abstract

In this paper, we study Ruelle's probability cascades [22] in the framework of time-inhomogeneous fragmentation processes. We describe Ruelle's cascades mechanism exhibiting a family of measures $(\nu_t, t \in [0, 1])$ that characterizes its infinitesimal evolution. To this end, we will first extend the time-homogeneous fragmentation theory to the inhomogeneous case. In the last section, we will study the behavior for small and large times of Ruelle's fragmentation process.

Key Words. Fragmentation, exchangeable partition, Ruelle's cascades.

A.M.S. Classification. 60 J 25, 60 G 09.

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1 Introduction

Ruelle [22] introduced a cascade of random probability measures in order to study Derrida's GREM model in statistical mechanics. This approach was further developed by Bolthausen and Sznitman [8], who pointed out that an exponential time-reversal transforms Ruelle's probability cascades into a remarkable coalescent process. Previously Neveu (unpublished) observed that Ruelle's probability cascades were also related to the genealogy of some continuous state branching process ; we refer to [5] for precise statements and the connexion with Bolthausen-Sznitman coalescent. Furthermore, Pitman [20] obtained a number of explicit formulas on the law of Ruelle's cascades ; in particular he showed that the latter can be viewed as a fragmentation process and specified its semi-group in terms of certain Poisson-Dirichlet distributions. Returning to applications to Derrida's GREM model, we mention the important works by Bovier and Kurkova [9, 10, 11] who established in particular properties of the limiting Gibbs measure.

The purpose of this paper is to dwell on Pitman's observation that Ruelle's cascade can be viewed as a time-inhomogeneous fragmentation process. The theory of time-homogeneous fragmentation processes was developed recently (see eg [1, 2, 3]), and we shall briefly show how it can be extended to the time-inhomogeneous setting. Roughly the basic result is that the distribution of a time-inhomogeneous fragmentation can be characterized by a so-called instantaneous rate of erosion (which is a non-negative real number that depends on the time parameter), and an instantaneous dislocation measure (which specifies the rate of sudden dislocation). We shall establish that for Ruelle's probability cascades, the instantaneous erosion is zero, and we will provide several descriptions of the instantaneous dislocation measure. Specifically, the latter is related to the well-known Poisson-Dirichlet distributions, in particular we shall establish a stick-breaking construction, compute the corresponding exchangeable partition probability function, and derive some relations of absolute continuity. In this direction, we mention that related (but somewhat less precise) results have been proven independently by Marchal [16]. Finally, as examples of applications, we shall prove some asymptotic results for Ruelle's probability cascades at small and large times.

The rest of this work is organized as follows. The next section is devoted to preliminaries, then we briefly present the extension of the theory of fragmentation processes to the time-inhomogeneous setting. The main results on Ruelle's probability cascades are established in section 4, and finally section deals with applications to the asymptotic behavior.

2 Preliminaries

2.1 Ruelle's cascades and their representation with stable subordinators

Let us briefly recall the construction of Ruelle's cascades [5, 8, 22]. Let $p > 1$ be an integer and let $0 < x_1 < \dots < x_p < 1$ be a finite sequence of real numbers. For $k \in \{1, \dots, p\}$, $(\eta_{i_1, \dots, i_k}, i_1 \dots i_k \in \mathbb{N})$ denotes a family of random variables such that :

- for $k \in \{1, \dots, p\}$, $i_1, \dots, i_{k-1} \geq 1$ fixed, the distribution of $(\eta_{i_1, \dots, i_{k-1}, j}, j \in \mathbb{N})$ is that of the sequence of atoms of a Poisson measure on $]0, \infty[$ with intensity $x_k r^{-1-x_k} dr$, arranged according to the decreasing order of their sizes,
- the families $(\eta_{i_1, \dots, i_{k-1}, j}, j \in \mathbb{N})$ for $k \in \{1, \dots, p\}$, $i_1, \dots, i_{k-1} \geq 1$ are independent.

Set $\theta_{i_1, \dots, i_k} = \eta_{i_1} \dots \eta_{i_1, \dots, i_k}$. We can easily show that $C = \sum_{i_1 \dots i_p} \theta_{i_1, \dots, i_p}$ is almost surely finite. Next we define Ruelle's cascades :

$$\bar{\theta}_{i_1, \dots, i_p} = \frac{\theta_{i_1, \dots, i_p}}{C} \quad \text{and recursively} \quad \bar{\theta}_{i_1, \dots, i_{k-1}} = \sum_{j=1}^{\infty} \bar{\theta}_{i_1, \dots, i_{k-1}, j}.$$

Bertoin and Le Gall [5] have proved we can relate this process to the genealogy of Neveu's CSBP (continuous-state branching process). Precisely, they have proved that there exists a process $(S^{(s,t)}(a), 0 \leq s < t, a \geq 0)$ such that :

- $\forall 0 \leq s < t$, $S^{(s,t)} = (S^{(s,t)}(a), a \geq 0)$ is a stable subordinator with index $e^{-(t-s)}$,
- $\forall p \geq 2$, $0 \leq t_1 \leq \dots \leq t_p$, $S^{(t_1, t_2)}, \dots, S^{(t_{p-1}, t_p)}$ are independent and $S^{(t_1, t_p)}(a) = S^{(t_{p-1}, t_p)} \circ \dots \circ S^{(t_1, t_2)}(a)$.

Set $0 < t_1 < \dots < t_p$ such that

$$x_1 = e^{-t_p} \text{ and } x_k = e^{-(t_p - t_{k-1})}, k = 2, \dots, p.$$

Let us fix $a > 0$. We define recursively, for $k = 1, \dots, p$, random intervals $D_{i_1, \dots, i_k}^{(t_1, \dots, t_k, a)}$ in the following way :

$$D_{i_1, \dots, i_k}^{(a)} =]0, a[.$$

Let $k \geq 1$, $i_1, \dots, i_{k-1} \in \mathbb{N}$. Let $(b_{i_1, \dots, i_k}, i_k \in \mathbb{N})$ be the jump times of $S^{(t_{k-1}, t_k)}$ on the interval $D_{i_1, \dots, i_{k-1}}^{(t_1, \dots, t_{k-1}, a)}$ listed in the decreasing order of sizes. We set

$$D_{i_1, \dots, i_k}^{(t_1, \dots, t_k, a)} =]S^{(t_{k-1}, t_k)}(b_{i_1, \dots, i_k} -), S^{(t_{k-1}, t_k)}(b_{i_1, \dots, i_k})[\text{ and } \xi_{i_1, \dots, i_k}^{(t_1, \dots, t_k, a)} = |D_{i_1, \dots, i_k}^{(t_1, \dots, t_k, a)}|. \quad (1)$$

Bertoin et Le Gall have proved that the families

$$\left(\left(S^{(0, t_p)}(a) \right)^{-1} \xi_{i_1, \dots, i_p}; i_1, \dots, i_p \in \mathbb{N} \right) \text{ and } (\bar{\theta}_{i_1, \dots, i_p}; i_1, \dots, i_p \in \mathbb{N})$$

have the same law.

2.2 Ruelle's cascades as fragmentation processes

Using this representation of Ruelle's cascades in terms of stable subordinators, we can exhibit a link with fragmentation processes.

Recall that the law $\beta(a, b)$ has density

$$\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1} (1-x)^{b-1} \mathbf{1}_{[0,1]} dx,$$

and let us introduce some definition :

Definition 2.1 [21] For $0 \leq \alpha \leq 1$, $\theta > -\alpha$, let $(Y_n)_{n \geq 1}$ be a sequence of independent random variables with respective laws $\beta(1 - \alpha, \theta + n\alpha)$. Set

$$\hat{f}_1 = Y_1 \quad \hat{f}_n = (1 - Y_1) \dots (1 - Y_{n-1}) Y_n \quad \hat{f} = \left(\hat{f}_n \right)_{n \geq 1}.$$

Then $\sum_i \hat{f}_i = 1$. Let $f = (f_n)_{n \geq 0}$ be the decreasing rearrangement of the sequence $(\hat{f}_n)_{n \geq 1}$. We define the Poisson-Dirichlet law with parameter (α, θ) , denoted $PD(\alpha, \theta)$, as the distribution of f .

Thereafter \mathcal{S} stands for the set of decreasing sequences of non-negative numbers with sum equal to 1. \mathcal{S} is endowed with the uniform distance. $\bar{\mathcal{S}}$ denotes its closure, it is the set of decreasing sequences of non-negative real numbers whose sum is less than or equal to 1 and is called the set of mass-partitions. Notice that $\bar{\mathcal{S}}$ is a compact set.

Definition 2.2 Let $s = (s_i, i \in \mathbb{N})$ be an element of $\bar{\mathcal{S}}$ and $s^{(\cdot)} = (s^{(i)}, i \in \mathbb{N})$ a sequence in $\bar{\mathcal{S}}$. Consider the fragmentation of s_i by $s^{(i)}$, i.e. the sequence $\tilde{s}^{(i)} = (s_i s_j^{(i)}, j \in \mathbb{N})$. The decreasing rearrangement of all the terms of the sequences $\tilde{s}^{(i)}$ as i describes \mathbb{N} is called fragmentation of s by $s^{(\cdot)}$. If \mathbb{P} is a probability on $\bar{\mathcal{S}}$, we define the transition kernel $\mathbb{P} - \text{FRAG}(s, \cdot)$ as the distribution of a fragmentation of s by $s^{(\cdot)}$, where $s^{(\cdot)}$ is an iid sequence of random mass-partition with law \mathbb{P} .

A Markov process $(F(t), t \in [0, 1])$ with values in $\overline{\mathcal{S}}$ is called a fragmentation process if the following properties are fulfilled :

- $F(t)$ is continuous in probability.
- Its semi-group has the following form :
for all $t, t' \in [0, 1[$ such that $t + t' \in [0, 1]$, the conditional law of $F(t + t')$ given $F(t) = s$ is the law of $\mathbb{P}_{t, t+t'} - FRAG(s, \cdot)$ where $\mathbb{P}_{t, t+t'}$ is a probability on \mathcal{S} .

The fragmentation is said homogeneous (in time) if $\mathbb{P}_{t, t+t'}$ depends only on t' . Besides, $(F(t), t \in [0, 1])$ is called a standard fragmentation process if $F(0)$ is almost surely equal to the sequence $\mathbf{1} = (1, 0, \dots)$.

In the case of Ruelle's cascades, using the work of Bertoin et Pitman [6] (Lemma 9), we know that for any integer $2 \leq k \leq p$, $(\overline{\theta}_{i_1, \dots, i_k}, i_1, \dots, i_k \geq 1)$ is a $PD(x_k, -x_{k-1})$ -fragmentation of $(\overline{\theta}_{i_1, \dots, i_{k-1}}, i_1, \dots, i_{k-1} \geq 1)$. More precisely we have :

Proposition 2.3 *There exists a time-inhomogeneous fragmentation $(F(t), t \in [0, 1])$ with semi-group $\mathbb{P}_{t, t+t'} = PD(t + t', -t)$ such that*

$$\left((\overline{\theta}_{i_1}; i_1 \in \mathbb{N}), \dots, (\overline{\theta}_{i_1, \dots, i_p}; i_1, \dots, i_p \in \mathbb{N}) \right) \stackrel{law}{=} (F(x_1), \dots, F(x_p)).$$

In the sequel, we call F , Ruelle's fragmentation. To study Ruelle's cascade, it should be possible to use the fragmentation process theory developed for example in [4], but first, we must extend this theory to time-inhomogeneous fragmentations.

2.3 Exchangeable random partitions

In this section, we recall the connections between exchangeable random partitions and mass-partitions. Let us first introduce some useful notation :

we denote by \mathbb{N} the set of positive integers. For $n \in \mathbb{N}$, $[n]$ denotes the set $\{1, \dots, n\}$ and \mathcal{P}_n denotes the set of partitions of $[n]$, \mathcal{P}_∞ the set of partitions of \mathbb{N} . For all $n < m$, for all $\pi \in \mathcal{P}_m$, $\pi|_n$ denotes the restriction of π to \mathcal{P}_n . We endow \mathcal{P}_∞ with the distance $d(\pi, \pi') = \frac{1}{\sup\{n \in \mathbb{N} \mid \pi|_n = \pi'|_n\}}$. The partition with a single block is denoted by $\mathbf{1}$. We always label the blocks of a partition according to the increasing order of their smallest element.

A random partition of \mathbb{N} is called exchangeable if its distribution is invariant by the action of the group of finite permutations of \mathbb{N} . Kingman [13] has proved that each block of an exchangeable random partition has a frequency, i.e. if $\pi = (\pi_1, \pi_2, \dots)$ is an exchangeable random partition, then

$$\forall i \in \mathbb{N} \quad f_i = \lim_{n \rightarrow \infty} \frac{\#\{\pi_i \cap [n]\}}{n} \quad \text{exists a.s.}$$

One calls f_i the frequency of the block π_i . Therefore, for all exchangeable random partitions, we can associate a probability on $\overline{\mathcal{S}}$ which will be the law of the decreasing rearrangement of the sequence of the partition frequencies.

Conversely, given a law \mathbb{P} on $\overline{\mathcal{S}}$, we can construct an exchangeable random partition whose law of its frequency sequence is \mathbb{P} (cf. [13]). Let us specify this construction : we pick $s \in \overline{\mathcal{S}}$ with law \mathbb{P} and we draw a sequence of independent random variables U_i with uniform law on $[0, 1]$. Conditionally on s , two integers i and j are in the same block of Π iff there exists an

integer k such that $\sum_{l=1}^k s_l \leq U_i < \sum_{l=1}^{k+1} s_l$ and $\sum_{l=1}^k s_l \leq U_j < \sum_{l=1}^{k+1} s_l$. This construction of a law on the set of partitions from a law on $\overline{\mathcal{S}}$ is often called “paint-box process”.

Kingman’s representation Theorem states that any exchangeable random partition can be constructed in this way. Therefore, we have a natural bijection between the laws on $\overline{\mathcal{S}}$ and the laws on the exchangeable random partitions.

We also define an exchangeable measure ρ_ν on \mathcal{P}_∞ from a measure ν on $\overline{\mathcal{S}}$ by :

$$\rho_\nu(\cdot) = \int_{\overline{\mathcal{S}}} \rho_u(\cdot) \nu(du)$$

where ρ_u is the law on \mathcal{P}_∞ obtained by the paint-box based on the mass-partition u .

For any exchangeable random partition Π , we define a symmetric function p on finite sequences of \mathbb{N} such that, for every n, n_1, \dots, n_k integers with $n = n_1 + \dots + n_k$,

$$p(n_1, \dots, n_k) = \mathbb{P}(\Pi|_n = \pi),$$

where π is a partition of $[n]$ with k blocks of size n_1, \dots, n_k . The fact that $\mathbb{P}(\Pi|_n = \pi)$ depends only on n_1, \dots, n_k stems from the exchangeability of Π . One calls p the EPPF (exchangeable partition probability function) of Π .

Proposition 2.4 [18, 19] *Let $\widehat{f} = (\widehat{f}_n)_{n \in \mathbb{N}}$ be a sequence of random variables of $[0, 1]$ defined as in Definition 2.1 . Then there exists an exchangeable random partition with frequency distribution \widehat{f} , where \widehat{f}_i is the i -th block frequency and where the blocks are listed order of their smallest element. It is a (α, θ) -partition. Besides the EPPF of this partition is*

$$p_{\alpha, \theta}(n_1, \dots, n_k) = \frac{[\frac{\theta}{\alpha}]_k}{[\theta]_n} \prod_{i=1}^k -[-\alpha]_{n_i} \quad \text{for } \theta \neq 0 \quad (\text{Ewens-Pitman's formula}) \quad (2)$$

where $[x]_n = \prod_{i=1}^n (x + i - 1)$ and $n = \sum_{i=1}^k n_i$.

For $\theta = 0$, the formula is extended by continuity.

This proposition also proves that the law of the sequence \widehat{f} is invariant by size-biased rearrangement.

In the case of Ruelle’s fragmentation, we know that, at time t , $F(t)$ has the $PD(t, 0)$ law. So we have the following proposition :

Proposition 2.5 *The EPPF q_t of the random partition associated with Ruelle’s fragmentation at time t , $F(t)$, is :*

$$q_t(n_1, \dots, n_k) = \frac{(k-1)!}{(n-1)!} t^{k-1} \prod_{i=1}^k [1-t]_{n_i-1} \quad (3)$$

where $n = \sum_{i=1}^k n_i$.

Remark 2.6 *We can also construct a random partition with distribution $p_{\alpha, \theta}$ recursively (Chinese restaurant construction):*

First, the integer 1 necessarily belongs to the first block, denoted B_1 . Suppose the n first integers

split up in b blocks : $\Pi_n = (B_1, \dots, B_b)$, where block B_i has cardinal n_i . We now define Π_{n+1} with the following rule :

$$\mathbb{P}(\Pi_{n+1} = (B_1, \dots, B_i \cup \{n+1\}, \dots, B_b)) = \frac{n_i - \alpha}{n + \theta}$$

$$\mathbb{P}(\Pi_{n+1} = (B_1, \dots, B_b, \{n+1\})) = \frac{b\alpha + \theta}{n + \theta}.$$

Then Π is a (α, θ) -partition (cf. [17]).

We can also define a notion of fragmentation process of exchangeable partitions such that there is still a bijection with fragmentation processes of mass-partitions :

Set $A \subseteq B \subseteq \mathbb{N}$ and $\pi \in \mathcal{P}_A$ with $\#\pi = n$. Let $\pi^{(\cdot)} = (\pi^{(i)}, i \in \{1, \dots, n\})$, $\pi^{(i)} \in \mathcal{P}_B$ for all i . Consider the partition of the i -th block of π, π_i , induced by $\pi^{(i)}$, i.e. $\pi_{|\pi_i}^{(i)} = \tilde{\pi}^{(i)}$.

As i describes $\{1, \dots, n\}$, the blocks of $\tilde{\pi}^{(i)}$ form the blocks of a partition $\tilde{\pi}$ of A . This partition is denoted $FRAG(\pi, \pi^{(\cdot)})$. This is the fragmentation of π by $\pi^{(\cdot)}$.

If \mathbb{P} is a probability on \mathcal{P}_B , define the transition kernel $\mathbb{P} - FRAG(\pi, \cdot)$ as the distribution of a fragmentation of π by $\pi^{(\cdot)}$, where $\pi^{(\cdot)}$ is a sequence of iid partition with law \mathbb{P} .

Let $(\Pi(t), t \in [0, 1])$ be a Markov process on \mathcal{P}_∞ . We call $(\Pi(t), t \in [0, 1])$ an exchangeable fragmentation process if the following properties are fulfilled :

- $\Pi(t)$ is continuous in probability.
- Its semi-group has the following form :
for all $t, t' \geq 0$ such that $t + t' < 1$, the conditional law of $\Pi(t + t')$ given $\Pi(t) = \pi$ is $\mathbb{P}_{t, t'} - FRAG(\pi, \cdot)$, where $\mathbb{P}_{t, t'}$ is an exchangeable probability on \mathcal{P}_∞ .

The fragmentation is homogeneous if $\mathbb{P}_{t, t'}$ depends only on t' . Furthermore, $(\Pi(t), t \in [0, 1])$ is a standard fragmentation process if $\Pi(0)$ is equal to $\mathbf{1}$.

We can check that, with these definitions, if $(\Pi(t), t \in [0, 1])$ is a fragmentation process on partitions, then $(F(t), t \in [0, 1])$ the frequency process of Π , is a fragmentation process on mass-partitions. Furthermore, the converse is true, i.e., if one takes a fragmentation process on mass-partitions, then one can construct a fragmentation process on partitions Π such that the frequency process of Π is equal to the initial fragmentation process (cf. [1]).

We also remark that if we consider a fragmentation $(\Pi(t), t \in [0, 1])$ with semi-group $\mathbb{P}_{t, t'} - FRAG$, then its restriction to \mathcal{P}_n , $(\Pi|_n(t), t \in [0, 1])$, is a Markov process with semi-group $\mathbb{P}_{t, t'}^n - FRAG$ where $\mathbb{P}_{t, t'}^n$ is equal to $\mathbb{P}_{t, t'}$ restricted to \mathcal{P}_n (cf. [2]).

For Ruelle's fragmentation, we have an explicit construction of its corresponding fragmentation on the partitions. Indeed, recall the representation of Ruelle's cascades with the jumps of a family of subordinators (cf. Section 2.1). Let $(\sigma_t^*, t \in [0, 1])$ be a family of stable subordinators such that for every $0 \leq t_p < \dots < t_1 < 1$, the joint distribution of $\sigma_{t_1}^*, \dots, \sigma_{t_p}^*$ is the same as that of $\sigma_{t_1}, \dots, \sigma_{t_p}$ with $\sigma_{t_i} = \tau_{\beta_1} \dots \tau_{\beta_i}$ where $t_i = \beta_1 \dots \beta_i$ and $\tau_{\beta_1}, \dots, \tau_{\beta_p}$ are independent stable subordinators with indices β_1, \dots, β_p . For $t \in]0, 1[$, let M_t be the closure of $\{\sigma_t^*(u), u \geq 0\}$. Consider then the family of open subsets of $[0, 1[: G(t) = [0, 1[\setminus M_t$, for $t \in [0, 1[$. Then $(G(t), t \in [0, 1])$ is a nested family, i.e. $G(t) \subset G(s)$ for $0 < s < t < 1$ and furthermore, if $F(t)$ is the sequence of ranked lengths of component intervals of $G(t)$, then $(F(t), t \in [0, 1])$ has the law of Ruelle's fragmentation ; see [6].

Set $0 \leq t_1 < \dots < t_p < 1$. Let us now draw $(U_i)_{i \in \mathbb{N}}$, uniform and independent random variables on $]0, 1[$. For $1 \leq k \leq p$, we construct a partition $\Pi(k)$ of \mathbb{N} with the rule :

$$i \stackrel{\Pi(k)}{\sim} j \Leftrightarrow U_i \text{ and } U_j \text{ are in the same component interval of } G(t_i).$$

Then $(\Pi(1), \dots, \Pi(k))$ has the law of a Ruelle's fragmentation on the partition at times (t_1, \dots, t_p) .

2.4 Connexion with Bolthausen-Sznitman's coalescent

Bolthausen et Sznitman [8] have shown that it is possible to formulate Ruelle's fragmentation as a coalescent process if we reverse time. Moreover, for a good choice for the time reversal, the coalescent process is time-homogeneous [8]. Let us first recall the definition of a coalescent process.

Set $s \in \overline{\mathcal{S}}$ and let $\Pi = \{B_1, B_2, \dots\}$ be a partition of \mathbb{N} . Set $\tilde{s}_i = \sum_{j \in B_i} s_j$. The Π -coagulation of s , denoted $COAG(s, \Pi)$ is the decreasing rearrangement of the sequence $(\tilde{s}_i, i \in \mathbb{N})$. If \mathbb{P} is a probability on \mathcal{S} , we define the transition kernel $\mathbb{P} - COAG(s, \cdot)$ as the distribution of a Π -coagulation of s , where Π has the law on \mathcal{P}_∞ obtained from \mathbb{P} by the paint-box construction. Let $(C(t), t \geq 0)$ be a Markov process on $\overline{\mathcal{S}}$. $(C(t), t \geq 0)$ is a time-homogeneous mass-coalescent process if the following properties are fulfilled :

- $C(t)$ is continuous in probability.
- Its semi-group has the following form :
for all $t, t' \geq 0$, the conditional law of $C(t+t')$ given $C(t) = s$ is the law of $\mathbb{P}_{t'} - COAG(s, \cdot)$ where $\mathbb{P}_{t'}$ is a probability on \mathcal{S} .

To see that Ruelle's fragmentation reversed in time is a time-homogeneous coalescent process, we use the following property :

Proposition 2.7 [19] *Set $\alpha \in]0, 1[$, $\beta \in [0, 1[$ and $\theta > -\alpha\beta$. The following assertions are equivalent :*

- s has $PD(\alpha, \theta)$ distribution and s' is a $PD(\beta, \theta/\alpha)$ -coagulation of s .
- s' has $PD(\alpha\beta, \theta)$ distribution and s is a $PD(\alpha, -\alpha\beta)$ -fragmentation of s' .

Thus, if we define $C(t) = F(e^{-t})$ where $(F(t), t \in [0, 1])$ is Ruelle's fragmentation, then $(C(t), t \geq 0)$ is a homogeneous coalescent process with semi-group $PD(e^{-t}, 0)$ -COAG. This process is called the Bolthausen and Sznitman's coalescent.

Just like the case of the fragmentation processes, we can associate a coalescent process on exchangeable partition to any mass-coalescent process. For the Bolthausen-Sznitman's coalescent process on partitions, we have an explicit construction [19]. It is a simple exchangeable coalescent process, i.e, at each jump-time of the process $\Pi_n(t)$, only one block can be formed. The jump rates of this process can be explicitly written. If we start from a partition with b blocks, each k -uplet of blocks coagulates with rate $\lambda_{b,k}$ that depends only on b and k and which is equal to :

$$\lambda_{b,k} = \frac{(k-2)!(b-k)!}{(b-1)!} = \int_0^1 x^{k-2} (1-x)^{b-k} dx.$$

Remark 2.8 *One can be surprised that an homogeneous Markov process becomes an inhomogeneous Markov process after time-reversal. In fact, Ruelle's fragmentation can also be seen as a homogeneous Markov process, but, if one takes this point of view, it is no longer a fragmentation process since the evolution of a particle depends on the other particles. Actually, it is known that if a random variable $x = (x_1, x_2, \dots) \in \overline{\mathcal{S}}$ has the PD($\alpha, 0$) law, then $\lim_{n \rightarrow \infty} \frac{\alpha \ln x_n}{\ln n} = -1$ (cf. [21]).*

In particular, in the case of Ruelle's fragmentation, $F(t)$ has law PD($t, 0$), therefore

$$t = - \lim_{n \rightarrow \infty} \frac{\ln n}{\ln x_n(t)}.$$

Let $(p_{t,t+s})_{t,s>0}$ be the transition probabilities of F . Suppose that the process is in state $x \in \overline{\mathcal{S}}$. For all $t \in [0, 1[$, the process F has a Poisson-Dirichlet law, so $T(x) = - \lim_{n \rightarrow \infty} \frac{\ln n}{\ln x_n}$ exists and $T(x)$ determines the considered time. For $y \in \overline{\mathcal{S}}$. We define

$$q_s(x, y) = p_{T(x), T(x)+s}(x, y).$$

Then $(q_s)_{s \in [0, 1[}$ is a homogeneous transition kernel for F . However, remark that to determine $T(x)$, we must know the other particles state and the branching property is lost.

3 General theory of time-inhomogeneous fragmentation processes

In this section, we extend the theory of time-homogeneous fragmentations to time-inhomogeneous fragmentations. For this, we will first work on fragmentations of partition and next on mass-fragmentations.

3.1 Measure of an inhomogeneous fragmentation

Let us first define precisely the class of fragmentations we consider (which includes Ruelle's fragmentation). We denote $\mathcal{P}_n \setminus \{\mathbf{1}\}$ by \mathcal{P}_n^* .

Hypothesis 3.1 *In the sequel, we always suppose that $(\Pi(t), t \in [0, 1[)$ is a standard time-inhomogeneous exchangeable fragmentation for which the following properties are fulfilled :*

- *for all $n \in \mathbb{N}$, let τ_n be the time of the first jump of $\Pi|_n$ and λ_n be its law. We have*

$$\forall t \in [0, 1[, \bar{\lambda}_n(t) := \lambda_n([t, 1]) > 0$$

and λ_n is absolutely continuous with respect to Lebesgue measure with continuous density $g_n(t)$.

- *for all $\pi \in \mathcal{P}_n^*$, $h_\pi^n(t) = \mathbb{P}(\Pi|_n(t) = \pi \mid \tau_n = t)$ is a continuous function of t .*

Let us now define an instantaneous jump rate for a fragmentation fulfilling Hypothesis 3.1. Set $\pi \in \mathcal{P}_n^*$. Set $h_\pi^n(t) = \mathbb{P}(\Pi_{|n}(t) = \pi \mid \tau_n = t)$. It is the law of the jump given τ_n . We set

$$f_n(t) = \lim_{s \rightarrow 0} \frac{1}{s} \mathbb{P}(\tau_n \in [t, t+s] \mid \tau_n \geq t) = \frac{g_n(t)}{\lambda_n(t)},$$

and

$$q_{\pi,t} = h_\pi^n(t) f_n(t) = \lim_{s \rightarrow 0} \frac{1}{s} \mathbb{P}(\Pi_{|n}(\tau_n) = \pi \ \& \ \tau_n \in [t, t+s] \mid \tau_n \geq t).$$

It is the probability density that the process $\Pi_{|n}$ jumps at time t from the state $\mathbf{1}$ to the state π given that $\Pi_{|n}$ has not jump before.

Proposition 3.2 *For $\pi \in \mathcal{P}_n$, $n' \geq n$, set $Q_{n',\pi} = \{\pi' \in \mathcal{P}_{n'}, \pi'_{|n} = \pi\}$. For each $t \in [0, 1[$, there exists a unique measure μ_t on \mathcal{P}_∞ such that*

$$\forall n \in \mathbb{N} \ \forall \pi \in \mathcal{P}_n^* \ \mu_t(Q_{\infty,\pi}) = q_{\pi,t} \ \text{and} \ \mu_t(\mathbf{1}) = 0.$$

The family of measures $(\mu_t, t \geq 0)$ characterizes the law of the fragmentation.

Proof. We have

$$\forall n' > n, \forall \pi \in \mathcal{P}_n^* \ \sum_{\pi' \in Q_{n',\pi}} q_{\pi',t} = q_{\pi,t}. \quad (4)$$

In fact, at time t , if the process $\Pi_{|n'}$ has not jumped yet, it will jump between time t and time $t + dt$ to the state such that $\Pi_{|n} = \pi$ with probability $\sum_{\pi' \in Q_{n',\pi}} q_{\pi',t} dt$. Besides, we have the following equality :

$$\mathbb{P}(\Pi_{|n}(\tau_n) = \pi \ \& \ \tau_n \in [t, t+dt] \mid \tau_n \geq t) = \mathbb{P}(\Pi_{|n}(\tau_n) = \pi \ \& \ \tau_n \in [t, t+dt] \mid \tau_n \geq t),$$

because the event that the block $[n']$ has already split, does not affect the process $\Pi_{|n}$. In fact, as $(\Pi_{|n}(t), t \in [0, 1[)$ is a Markov process, the law of the process $(\Pi_{|n}(t), t \in [t_0, 1[)$ depends only on $\Pi_{|n}(t_0)$. Therefore, we have equality (4).

Let us now define $\mu_t(Q_{\infty,\pi}) = q_{\pi,t}$. By (4), this application can be extended to an additive application. By Caratheodory's Theorem, μ can be extended to an unique measure on \mathcal{P}_∞ .

We have now to prove that this family of measures determines the fragmentation law. To this end we just have to prove that the family of measures $(\mu_t, t \in [0, 1[)$ characterizes every jump rate of $\Pi_{|n}(t)$. Set $\pi, \pi' \in \mathcal{P}_n$, $t_0 \in [0, 1[$. Let τ'_n be the time of the first jump of $\Pi_{|n}(t)$ after t_0 . We must express $\lim_{s \rightarrow 0} \frac{1}{s} \mathbb{P}(\Pi_{|n}(\tau'_n) = \pi' \ \& \ \tau'_n \in [t, t+s] \mid \Pi_{|n}(t_0) = \pi)$ in terms of $(\mu_t, t \in [0, 1[)$. If π' can not be obtain from the fragmentation of one block of π , we clearly have :

$$\mathbb{P}(\Pi_{|n}(\tau'_n) = \pi' \ \& \ \tau'_n \in [t, t+dt] \mid \Pi_{|n}(t_0) = \pi) = 0.$$

Permuting the indices (which does not change the law by exchangeability), we can suppose $\pi = (A_1, \dots, A_N)$ and $\pi' = (B_1, \dots, B_k, A_2, \dots, A_N)$ where $\pi'' = (B_1, \dots, B_k) \in \mathcal{P}_{A_1}$. Let $\tau'_{[A_i]}$ be the first jump time of $\Pi_{|A_i}$. Then, by the branching property :

$$\mathbb{P}(\Pi_{|n}(\tau'_n) = \pi' \ \& \ \tau'_n \in [t, t+dt] \mid \Pi_{|n}(t_0) = \pi)$$

$$\begin{aligned}
&= \mathbb{P}(\Pi_{|A_1|}(\tau_{[A_1]}) = \pi'' \ \& \ \tau_{[A_1]} \in [t, t + dt] \mid \Pi_{|A_1|}(t_0) = \mathbf{1}) \prod_{i=2}^N \mathbb{P}(\tau_{[A_i]} > t \mid \tau_{[A_i]} > t_0) \\
&= \mathbb{P}(\Pi_{|A_1|}(\tau_{[A_1]}) = \pi'' \ \& \ \tau_{[A_1]} \in [t, t + dt] \mid \tau_{[A_1]} > t_0) \prod_{i=2}^N \mathbb{P}(\tau_{[A_i]} > t \mid \tau_{[A_i]} > t_0) \\
&= \mathbb{P}(\Pi_{|A_1|}(\tau_{[A_1]}) = \pi'' \ \& \ \tau_{[A_1]} \in [t, t + dt] \mid \tau_{[A_1]} > t_0) \prod_{i=1}^N \frac{\bar{\lambda}_{|A_i|}(t)}{\bar{\lambda}_{|A_i|}(t_0)}.
\end{aligned}$$

Thus we have :

$$\lim_{s \rightarrow 0} \frac{1}{s} \mathbb{P}(\Pi_{|n|}(\tau'_n) = \pi' \ \& \ \tau'_n \in [t, t + s] \mid \Pi_{|n|}(t_0) = \pi) = \mu_t(Q_{\infty, \pi'}) \prod_{i=1}^N \frac{\bar{\lambda}_{|A_i|}(t)}{\bar{\lambda}_{|A_i|}(t_0)}$$

and $\bar{\lambda}_n$ is easily expressed as a function of μ_t (cf. below).

Proposition 3.3 *The application from $[0, 1[$ to the set of measures on \mathcal{P}_{∞} which at t associates μ_t constructed according to the proposition above, verifies :*

- μ_t is an exchangeable measure such that $\mu_t\{\mathbf{1}\} = 0$ and $\forall n \in \mathbb{N} \ \mu_t(\{\pi \in \mathcal{P}_{\infty}, \pi_{|n|} \neq \mathbf{1}\}) < \infty$,
- $\forall n \in \mathbb{N} \ \forall t \in [0, 1[$ we have $\int_0^t \mu_u(\{\pi \in \mathcal{P}_{\infty}, \pi_{|n|} \neq \mathbf{1}\}) du < \infty$.

Proof. The exchangeability is clear and we have

$$\mu_t(\{\pi \in \mathcal{P}_{\infty}, \pi_{|n|} \neq \mathbf{1}\}) = f_n(t)$$

and

$$\int_0^t \mu_u(\{\pi \in \mathcal{P}_{\infty}, \pi_{|n|} \neq \mathbf{1}\}) du = -\ln(\lambda_n([t, 1]))$$

which is finite by Hypothesis 3.1. \square

Set $\varepsilon_i = \{\{i\}, \{\mathbb{N} \setminus \{i\}\}\}$ and $\varepsilon = \sum_i \delta_{\varepsilon_i}$. So ε is a measure on \mathcal{P}_{∞} . According to Bertoin [2], we know that for each exchangeable measure μ such that $\mu\{\mathbf{1}\} = 0$ and $\mu(\{\pi \in \mathcal{P}_{\infty}, \pi_{|n|} \neq \mathbf{1}\}) < \infty$, we can find a measure ν on $\bar{\mathcal{S}}$ (dislocation measure) verifying $\nu(\mathbf{1}) = 0$ and $\int_{\bar{\mathcal{S}}} (1 - s_1) \nu(ds) < \infty$, and a constant $c \geq 0$ (erosion coefficient) such that

$$\mu = \rho_{\nu} + c\varepsilon$$

where ρ_{ν} denotes the measure on \mathcal{P}_{∞} associated to ν by the paint-box process.

So for $t \in [0, 1[$ fixed, we can write $\mu_t = \rho_{\nu_t} + c_t \varepsilon$ where ν_t and c_t are the instantaneous dislocation and erosion rates of the fragmentation.

Proposition 3.4 *We have $\mu_t = \rho_{\nu_t} + c_t \varepsilon$ where ν_t and c_t fulfill the following properties :*

•

$$\forall t \in [0, 1[\ \nu_t(\mathbf{1}) = 0 \ \text{and} \ \int_{\bar{\mathcal{S}}} (1 - s_1) \nu_t(ds) < \infty, \tag{5}$$

$$\forall u \in [0, 1[\int_0^u \int_{\overline{\mathcal{S}}} (1 - s_1) \nu_t(ds) dt < \infty \text{ and } \int_0^u c_t dt < \infty. \quad (6)$$

Proof : The property (5) is clear. For the formula (6), we shall look at the proof of the theorem in the time-homogeneous case (cf. [2]). During the proof, we obtain the following upper bound :

$$\int_{\overline{\mathcal{S}}} (1 - s_1) \nu_t(ds) \leq \mu_t(\{\pi \in \mathcal{P}_\infty, \pi|_2 \neq \mathbf{1}\}).$$

Then use Proposition 3.3.

For the upper bound concerning c_t we remark :

$$c_t = \mu_t(\{1\}, \mathbb{N} \setminus \{1\}) - \rho_{\nu_t}(\{1\}, \mathbb{N} \setminus \{1\}) = \mu_t(\{1\}, \mathbb{N} \setminus \{1\}). \square$$

Hence the law of a time-inhomogeneous fragmentation is characterized by a family $(\nu_t, c_t)_{0 \leq t < 1}$ where $(\nu_t)_{0 \leq t < 1}$ and $(c_t)_{0 \leq t < 1}$ fulfill (5) and (6). One calls ν_t the instantaneous dislocation rate and c_t the instantaneous erosion rate at time t of the fragmentation. We will next give a probabilistic interpretation of this family.

As for the time-homogeneous fragmentations, we can construct a fragmentation with measure $(\mu_t, t \in [0, 1[)$ considering a Poisson measure M on $[0, 1[\times \mathcal{P}_\infty \times \mathbb{N}$ with intensity $\mu_t(d\pi)dt \otimes \sharp$ where \sharp is the counting measure. Let M^n be the restriction of M to $[0, 1[\times \mathcal{P}_n^* \times \{1, \dots, n\}$. According to Proposition 3.3, the intensity of the measure is finite on the interval $[0, t]$. Then, we are in a similar case as a time-homogeneous fragmentation (refer to [2] for a proof in the homogeneous case). Let us rearrange the atoms of M^n according to their first coordinate. For $n \in \mathbb{N}$, $(\pi, k) \in \mathcal{P}_n \times \mathbb{N}$, let $\Delta_n^{(\cdot)}(\pi, k)$ be the following sequence of partition of $[n]$:

$$\Delta_n^{(i)}(\pi, k) = \mathbf{1} \text{ if } i \neq k \quad \text{and} \quad \Delta_n^{(k)}(\pi, k) = \pi|_n.$$

We construct the process $(\Pi|_n(t), t \geq 0)$ in \mathcal{P}_n with the following rules :

$$\Pi|_n(0) = \mathbf{1}.$$

$(\Pi|_n(t), t \geq 0)$ is a jump process which jumps at times s , atoms of M^n . More precisely, if (s, π, k) is an atom of M^n , we have $\Pi|_n(s) = \text{FRAG}(\Pi|_n(s^-), \Delta_n^{(\cdot)}(\pi, k))$. We can then check that this construction is compatible with the restriction and the constructed process is a fragmentation with measure μ_t .

We have also a Poissonian construction for a mass-fragmentation (cf. [1]). First we use that if $F = (F(t), t \in [0, 1[)$ is a mass-fragmentation with parameters $(\nu_t, 0)_{0 \leq t < 1}$, then $\tilde{F} = (e^{-\int_0^t c_s ds} F(t), t \in [0, 1[)$ is a mass-fragmentation with parameters $(\nu_t, c_t)_{0 \leq t < 1}$. So, we remark that the family of instantaneous erosion coefficients plays only a deterministic role in the fragmentation. To find a Poissonian construction for the mass-fragmentations $(F(t), t \in [0, 1[)$ with parameters $(\nu_t, 0)_{0 \leq t < 1}$, consider then a fragmentation on partitions $(\Pi(t), t \in [0, 1[)$ such that $F = \Lambda(\Pi)$ where Λ is the application which associates to a partition its frequency sequence. So Π can be constructed from a Poisson measure M . Consider K , image of M by the application

$$\begin{aligned} \mathcal{P}_\infty \times \mathbb{N} &\longrightarrow \mathcal{S} \times \mathbb{N} \cup \infty \\ (\Delta(\cdot), k(\cdot)) &\longmapsto (\Lambda(\Delta(\cdot)), f(\cdot, k(\cdot))), \end{aligned}$$

where f is the function which associates to k the frequency rank of the block $B_k(t^-)$.

Berestycki [1] then proves that K is a Poisson measure on $[0, 1[\times \mathcal{S} \times \mathbb{N}$ with intensity measure

$\nu(ds)dt \otimes \#$.

Set

$$K = (t, S(t), k(t))_{t \in [0,1[} = (t, (s_1(t), s_2(t), \dots), k(t))_{t \in [0,1[}.$$

Then, if $(t, S(t), k(t))$ is an atom of K , then at time t , the $k(t)$ -th largest block of the fragmentation at time t^- will be fragmented according to $S(t)$.

Let us now determine the effects of a deterministic change-time on a fragmentation.

Proposition 3.5 *Let $(\Pi(t), t \in [0, 1[)$ be a fragmentation with parameter (c_t, ν_t) . Set $\Pi'(t) = \Pi(\beta(t))$ where $\beta : [0, 1[\rightarrow \mathbb{R}_+$ is a strictly increasing derivable function. Let J be the image of $[0, 1[$ by β (J is thus an interval of \mathbb{R}_+).*

Then $(\Pi'(t), t \in J)$ is a fragmentation with parameter $(c'_t; \nu'_t)_{t \in J}$ where

$$c'_t = \beta'(t)c_{\beta(t)} \quad \nu'_t = \beta'(t)\nu_{\beta(t)}.$$

Proof. A Markov process remains a Markov process after a deterministic time-change. The law of $\Pi'(t + t')$ given $\Pi'(t) = \pi$, is $FRAG(\pi, \pi^{(\cdot)})$, where $\pi^{(\cdot)}$ is an iid sequence with law $\mathbb{P}_{\beta(t), \beta(t+t') - \beta(t)}$. Thus Π' is a fragmentation.

Let us calculate its jump rates $q'_{\pi, t}$.

$$\begin{aligned} q'_{\pi, t} dt &= \mathbb{P} \left(\Pi'_{|n}(\tau'_n) = \pi \ \& \ \tau'_n \in [t, t + dt] \mid \tau'_n \geq t \right) \\ &= \mathbb{P} \left(\Pi_{|n}(\beta(\tau'_n)) = \pi \ \& \ \tau'_n \in [t, t + dt] \mid \tau'_n \geq t \right) \\ &= \mathbb{P} \left(\Pi_{|n}(\tau_n) = \pi \ \& \ \tau_n \in [\beta(t), \beta(t + dt)] \mid \tau_n \geq \beta(t) \right) \\ &\sim \mathbb{P} \left(\Pi_{|n}(\tau_n) = \pi \ \& \ \tau_n \in [\beta(t), \beta(t) + \beta'(t)dt] \mid \tau_n \geq \beta(t) \right) \\ &\sim \beta'(t)q_{\pi, \beta(t)} dt. \end{aligned}$$

So $q'_{\pi, t} = \beta'(t)q_{\pi, \beta(t)}$. We thus deduce similar relations between ν_t and ν'_t and between c_t and c'_t . \square

3.2 Law of the tagged fragment

An application of the above decomposition is for example to calculate the law of the frequency of the block containing 1, $|\Pi_1(t)|$, for an exchangeable standard fragmentation. This quantity is interesting because it represents the law of a size-biased picked block. We have the following theorem :

Theorem 3.6 *There exists a process $(\xi(t), t \in [0, 1[)$ with independent increments such that $|\Pi_1(t)| = \exp(-\xi_t)$. Its law is characterized by the identity :*

$$\mathbb{E} \left(|\Pi_1(t)|^q \right) = \mathbb{E} \left(\exp(-q\xi_t) \right) = \exp \left(- \int_0^t \phi_u(q) du \right), \quad q > 0$$

$$\text{where } \phi_t(q) = c_t(q+1) + \int_S \left(1 - \sum_{i=1}^{\infty} s_i^{q+1} \right) \nu_t(ds).$$

In the sequel, we will also use the notation $\psi(t, q) = \int_0^t \phi_u(q) du$.

This result is very close to the corresponding result in the homogeneous case. We just loose the stationarity of the increments of $\xi(t)$. The demonstration itself is similar to the homogeneous case and we just sketch the proof here. For more details, refer to [2].

We use the equality :

$$\mathbb{P}[|\Pi_{|k+1}(t) = \mathbf{1}] = \mathbb{E}[|\Pi_1(t)|^k],$$

which we get by conditioning on $|\Pi_1(t)|$. Then remark the event $\{|\Pi_{|k+1}(t) = \mathbf{1}\}$ corresponds, looking at the Poissonian construction, to an absence of Poisson atom in the subset $[0, t] \times \{\pi \in \mathcal{P}_\infty, \pi_{|k+1}(t) \neq \mathbf{1}\} \times \{1\}$.

So the formula is true for every positive integer. Besides, we remark that the law of $|\Pi_1(t)|$ is characterized by its moments, thanks to the independence of the increments (when you take the logarithm) and because the process takes values in $[0, 1]$.

By uniqueness of the analytic continuation, we deduce that the formula is true for every $q > 0$. And by the monotone convergence theorem, $\psi(t, q)$ is continuous in q at 0. \square

Thanks to this formula, we can characterize the processes which have proper frequencies, i.e. with $\sum_{i=1}^\infty |\pi_i| = 1$.

Proposition 3.7 *We have :*

$$\mathbb{P}(\Pi(t) \text{ is proper}) = 1 \Leftrightarrow \left(c_u = 0 \text{ and } \nu_u \left(\sum_i s_i < 1 \right) = 0 \text{ for } 0 \leq u \leq t \text{ a.e.} \right).$$

Proof. First remark

$$\lim_{k \rightarrow 0} \mathbb{E}[|\Pi_1(t)|^k] = \lim_{k \rightarrow 0} \mathbb{E}[|\Pi_1(t)|^k \mathbf{1}_{|\Pi_1(t)| \neq 0}] = \mathbb{E}[\mathbf{1}_{|\Pi_1(t)| \neq 0}] = 1 - \mathbb{P}(|\Pi_1(t)| = 0).$$

Then we have :

$$\begin{aligned} \mathbb{P}(\Pi(t) \text{ is proper}) = 1 &\Leftrightarrow \mathbb{P}(|\Pi_1(t)| = 0) = 0 \\ &\Leftrightarrow \exp(-\psi(t, 0)) = 1 \\ &\Leftrightarrow \psi(t, 0) = 0 \\ &\Leftrightarrow \phi_u(0) = 0 \text{ for } 0 \leq u \leq t \text{ a.e.} \square \end{aligned}$$

Recall from [4] that if $(X(t), t \in [0, 1])$ is a time-homogeneous mass-fragmentation, ϕ the Laplace exponent associated to the tagged fragment and $\mathcal{F}_t = \sigma(X(s), s \leq t)$, then

$$\exp(t\phi(p)) \sum_{i=1}^\infty X_i^{p+1}(t) \text{ is a } \mathcal{F}_t\text{-martingale.}$$

We can obtain a similar theorem in the time-inhomogeneous case.

Proposition 3.8 *Consider $(\Pi(t), t \in [0, 1])$ a time-inhomogeneous fragmentation on partitions.*

Let $X(t) = (X_i(t)) \in \bar{\mathcal{S}}$ be its decreasing sequence of frequencies. Set $\mathcal{F}_t = \sigma(X(u); u \leq t)$.

Let ϕ_u be its instantaneous Laplace exponent and $\psi(t, p) = \int_0^t \phi_u(p) du$. Then

$$M(t, p) = \exp(\psi(t, p)) \sum_{i=1}^\infty X_i^{p+1}(t) \text{ is a } \mathcal{F}_t\text{-martingale.}$$

Proof. It is the same idea as in the time-homogeneous case. Set $\mathcal{G}_t = \sigma(\Pi(u), u \leq t)$. Then $\mathcal{E}(t, p) = \exp(-p\xi_t + \psi(t, p))$ is an \mathcal{G}_t -martingale and we remark that $M(t, p)$ is the projection of $\mathcal{E}(t, p)$ on \mathcal{F}_t . \square

4 Application to Ruelle's cascades

4.1 Jump rates of Ruelle's fragmentation

Let $(\Pi(t), t \in [0, 1])$ be Ruelle's fragmentation with values in partitions. For each integer n , $(\Pi|_n, t \in [0, 1])$ is a Markov process in the finite space of partition of $[n]$. The law of such a process is entirely determined by its jump rates from one state to another.

Let us calculate its jump rates. Set $\pi = (\pi_1, \dots, \pi_k) \in \mathcal{P}_n^*$. Fix $t \in [0, 1]$. Let $q_t(n_1, \dots, n_k)$ be the probability that $\Pi|_n(t)$ has blocks with size (n_1, \dots, n_k) . Recall (cf. Proposition 2.5) that

$$q_t(n_1, \dots, n_k) = \frac{(k-1)!}{(n-1)!} t^{k-1} \prod_{i=1}^k [1-t]_{n_i-1}.$$

So from Proposition 2.3 and 2.4

$$\begin{aligned} \mathbb{P}(\tau_n \in [t, t+s], \widehat{\Pi}|_n(\tau_n) = \pi \mid \tau_n \geq t) &= \mathbb{P}(\widehat{\Pi}|_n(t+s) = \pi \mid \widehat{\Pi}|_n(t) = \mathbf{1}) \\ &= p_{t+s, -t}(n_1, \dots, n_k) \\ &= \frac{\left[\frac{-t}{t+s} \right]_k}{[-t]_n} \prod_{i=1}^k -[-t]_{n_i} \\ &\sim s \frac{(-1)^{k+1} (k-2)! \prod_{i=1}^k [-t]_{n_i}}{t[-t]_n}. \end{aligned}$$

Remark that we could also have calculated this quantity using Proposition 2.7 and Bayes' Formula. So we obtain the following proposition :

Proposition 4.1 *For $\pi = (\pi_1, \dots, \pi_k) \in \mathcal{P}_n^*$ and for $t \in [0, 1[$ we have :*

$$q_{\pi, t} = \frac{q_t(n_1, \dots, n_k)}{t(k-1)q_t(n)}.$$

4.2 Instantaneous erosion coefficient and dislocation measure

It is well known that the Bolthausen-Sznitman's coalescent is a process with proper frequencies (cf. Proposition 3.7). So, the erosion coefficient c_t should be identically zero. We can check this with a short calculation. In fact, consider $\pi = \varepsilon_1 = \left\{ \{1\}, \mathbb{N} \setminus \{1\} \right\}$ and $\pi_n = \pi|_n$.

According to Proposition 4.1, we have $q_{\pi_n, t} = \frac{q_t(1, n-1)}{tq_t(n)} = \frac{1}{n-1-t}$. And $c_t = \lim_{n \rightarrow \infty} q_{\pi_n, t} = 0$. Thus $c_t = 0$ for all $t \in [0, 1[$.

Let us denote by $\tilde{\mathcal{S}}$ the set of the positive sequence with sum 1. From a measure η on $\tilde{\mathcal{S}}$, we can define a measure p on \mathcal{P}_∞ (cf. [20] p. 61) :

Conditionally on a sequence $(s_i, i \geq 1)$ drawn with respect to the measure η , we construct the following law on partitions :

1 is in the first block. Fix $n \geq 1$. Suppose Π_n has k blocks. The integer $n+1$ will be :

- in the block j with probability s_j (for $j \leq k$),
- in a new block with probability $1 - \sum_{i=1}^k s_i$.

So we have

$$p(\pi) = \mathbb{E}^\eta \left(\prod_{i=1}^k s_i^{n_i-1} \prod_{i=1}^{k-1} \left(1 - \sum_{j=1}^i s_j\right) \right), \quad (7)$$

where $\pi = (\pi_1, \dots, \pi_k)$ et $|\pi_i| = n_i$.

If the measure η is a dislocation measure (i.e verifies $\int_{\mathcal{S}} (1 - s_1) \eta(ds) < \infty$), then p is finite on \mathcal{P}_n^* . In fact, for all $k \geq 2$, we have

$$\prod_{i=1}^k s_i^{n_i-1} \prod_{i=1}^{k-1} \left(1 - \sum_{j=1}^i s_j\right) \leq 1 - s_1.$$

Let us now look at the dislocation measure. In this direction, let us introduce the following measure :

Definition 4.2 Fix $\alpha \in]0, 1[$. Consider the measure η_α defined as follows on $\tilde{\mathcal{S}}$: first,

$$\eta_\alpha(s_1 \in dx) = \alpha x^{-\alpha} (1-x)^{-1} \mathbf{1}_{0 < x < 1} dx,$$

and second, conditionally on $s_1 = x$, the sequence $(s_{i+1}/(1-x), i \in \mathbb{N})$ has the law of a random variable with law $PD(\alpha, 0)$ of which the terms have been size-biased rearranged. We denote $PD(\alpha, -\alpha)$ the image of η_α by ranking the s_i in the decreasing order. $PD(\alpha, -\alpha)$ is then an infinite measure on \mathcal{S} .

Remark that the construction of the measure $PD(\alpha, -\alpha)$ is similar, except for the normalization, to the construction of a Poisson-Dirichlet measure with the forbidden parameter $\theta = -\alpha$.

Proposition 4.3 Define p_α as the measure on \mathcal{P}_∞ associated to η_α as above. Then p_α is an exchangeable measure on \mathcal{P}_∞ . Its EPPF for the partitions non-reduced to one block is :

$$p_\alpha(n_1, \dots, n_k) = \frac{(k-2)!}{-[-\alpha]_n} \prod_{i=1}^k -[-\alpha]_{n_i} \text{ for all } k \geq 2. \quad (8)$$

Proof. Let us first check $\int_{\mathcal{S}} (1 - s_1) \eta_\alpha(ds) < \infty$.

$$\int_{\mathcal{S}} (1 - s_1) \eta_\alpha(ds) = \int_0^1 (1 - s_1) \alpha s_1^{-\alpha} (1 - s_1)^{-1} ds_1 = \frac{\alpha}{1 - \alpha}. \quad (9)$$

Using formula (7) and the definition of η_α , we have :

$$\begin{aligned} p_\alpha(\pi) &= \left(\int_0^1 x^{n_1-1} (1-x)^{\sum_{i=2}^k n_i} \eta_\alpha(s_1 \in dx) \right) p_{\alpha,0}(n_2, \dots, n_k) \\ &= \alpha \left(\int_0^1 x^{n_1-1-\alpha} (1-x)^{n-n_1-1} dx \right) p_{\alpha,0}(n_2, \dots, n_k) \\ &= \alpha \frac{\Gamma(n_1 - \alpha) \Gamma(n - n_1)}{\Gamma(n - \alpha)} \frac{(k-2)!}{\alpha(n - n_1 - 1)!} \prod_{i=2}^k -[-\alpha]_{n_i} \quad \text{according to (2)} \\ &= \frac{[-\alpha]_{n_1}}{[-\alpha]_n} (k-2)! \prod_{i=2}^k -[-\alpha]_{n_i} \\ &= \frac{(k-2)!}{-[-\alpha]_n} \prod_{i=1}^k -[-\alpha]_{n_i}. \end{aligned}$$

So, we find the foretold formula and this one is symmetric in the variables (n_1, \dots, n_k) , thus the measure is an exchangeable measure (cf. [20] Theorem 24). We also deduce that η_α is the image of $PD(\alpha, -\alpha)$ by a size-biased reordering and $p_\alpha = \rho_{PD(\alpha, -\alpha)}$ (where $\rho_{PD(\alpha, -\alpha)}$ is the measure on \mathcal{P}_∞ obtained from $PD(\alpha, -\alpha)$ by the paint-box construction.) \square

Next, we observe that for every partition π not reduced to one block, we have

$$q_{\pi,t} = \frac{1}{t} p_t(\pi).$$

Indeed, this follows from Proposition 4.1 and formula (3) of Pitman. In conclusion, we may now state the following theorem :

Theorem 4.4 *The instantaneous dislocation measure ν_t of Ruelle's fragmentation at time t is given by :*

$$\nu_t = \frac{1}{t} PD(t, -t).$$

4.3 Absolute continuity of the dislocation measure and $PD(\alpha, 0)$

Let us recall that, if Π is a random partition with law $p_{\alpha,0}$ and K_n the number of block of $\Pi|_n$, then the limit of K_n/n^α exists almost surely and has the Mittag-Leffler law with index α (cf. [20] Theorem 31)

Proposition 4.5 *For each $\alpha \in]0, 1[$ the measure p_α is absolutely continuous with respect to the measure $p_{\alpha,0}$. More precisely, we have :*

$$p_\alpha(d\pi) = \Gamma(1 - \alpha) S_\alpha^{-1} p_{\alpha,0}(d\pi) \quad \text{where } S_\alpha = \lim_{n \rightarrow \infty} \frac{K_n}{n^\alpha}.$$

Proof. Let $(\mathcal{F}_n)_{n \geq 1}$ be the filtration of $\Pi|_n$.

Fix $k \geq 2$. Set $p_\alpha^k = p_\alpha \mathbf{1}_{\mathcal{P}_k^*}$. We consider

$$M_{\alpha,n}^k = \frac{dp_\alpha^k}{dp_{\alpha,0}} \Big|_{\mathcal{F}_n}.$$

Using formula (3) and (8), we have :

$$M_{\alpha,n}^k = \frac{\Gamma(1 - \alpha)\Gamma(n)}{\Gamma(n - \alpha)(K_n - 1)} \mathbf{1}_{\mathcal{P}_k^*} \quad \text{for } n \geq k,$$

where K_n denotes the number of block of $\Pi|_n$. $M_{\alpha,n}^k$ is a positive martingale, thus it converges almost surely to a random variable M_α^k .

Let now use

$$\frac{K_n}{n^\alpha} \rightarrow S_\alpha \quad \mathbb{P}_{\alpha,0} - \text{a.s.} \quad \text{and} \quad \frac{\Gamma(1 - \alpha)\Gamma(n)}{\Gamma(n - \alpha)(K_n - 1)} \sim \frac{\Gamma(1 - \alpha)n^\alpha}{K_n}.$$

We deduce

$$M_\alpha^k = \frac{dp_\alpha^k}{dp_{\alpha,0}} = \Gamma(1 - \alpha) S_\alpha^{-1} \mathbf{1}_{\mathcal{P}_k^*} \quad \mathbb{P}_{\alpha,0} - \text{a.s.}$$

So, according to martingale theory (cf. [12] p.210), for all $A \subset \mathcal{P}_k^*$, we have :

$$p_\alpha(A) = \mathbb{E}_{\alpha,0}(\Gamma(1-\alpha)S_\alpha^{-1}\mathbf{1}_A) + p_\alpha(A \cap \{S=0\}),$$

where $S = \limsup \frac{K_n}{n^\alpha}$.

Set $x \in]0, 1[$. Let us define $q_\alpha(\cdot) = cp_\alpha(\cdot \mid |\Pi_1| = x)$ where c is chosen such that q_α is a probability. Let $s = (s_1, \dots) \in \overline{\mathcal{S}}$ be the frequency sequence of a partition with law q_α . According to the construction of p_α , we have

$$(s_{i+1})_{i \in \mathbb{N}} \stackrel{\text{law}}{=} (1-x)(p_i)_{i \in \mathbb{N}},$$

where $(p_i)_{i \in \mathbb{N}}$ has the $PD(\alpha, 0)$ law.

According to Lemma 34 of Pitman's course [20], for a random partition, S exists and belongs almost surely to $]0, \infty[$ iff there exists Z random variable on $]0, \infty[$ such that $P_i \sim Zi^{-1/\alpha}$, where P_i is the decreasing sequence of the frequencies. Here we know the existence of such a random variable $Z \in]0, \infty[$ for a $PD(\alpha, 0)$ law. Set $Y = (1-x)Z$ then

$$s_i \sim Yi^{-1/\alpha}.$$

So we have

$$p_\alpha(S=0 \mid |\Pi_1| = x) = 0.$$

Thus

$$p_\alpha(S=0) = 0.$$

We conclude that

$$\forall A \in \mathcal{P}_\infty \text{ such that } \mathbf{1} \notin \overline{A} \quad p_\alpha(A) = \mathbb{E}_{\alpha,0}(\Gamma(1-\alpha)S_\alpha^{-1}\mathbf{1}_A). \square$$

Theorem 4.6 *The dislocation measure of Ruelle's fragmentation at time t is absolutely continuous with respect to the measure $PD(t, 0)$. More precisely, we have for all continuous function f on $\overline{\mathcal{S}}$:*

$$\nu_t(f) = \frac{1}{t} \mathbb{E}_{(t,0)}(L_t^{-1}f(V))$$

where $L_\alpha = \lim_{n \rightarrow \infty} nV_n^\alpha$.

Proof. We use that if $(s_i)_{i \geq 1} \in \mathcal{S}$ is the frequency sequence of an $(\alpha, 0)$ -partition Π_∞ , then $\Gamma(1-\alpha)L_\alpha$ exists almost surely and it is equal almost surely to $S_\alpha = \lim_{n \rightarrow \infty} \frac{K_n}{n^\alpha}$ (cf. [20] Theorem 36).

Use Theorem 4.4 to finish the proof. \square

Remark 4.7 L_α is not a continuous function on \mathcal{S} .

4.4 Law of the tagged fragment

In this section, we determine the law of the tagged fragment. Actually, its law has already been determined by Pitman [19]. He proves that $|\Pi_1(t)|$ has a $\beta(1-t, t)$ law. So we check that we find the same result.

Hence, according to Section 3.2, we shall calculate $\phi_t(k) = \int_{\mathcal{S}} \left(1 - \sum_{i=1}^{\infty} s_i^{k+1}\right) \nu_t(ds)$. Recall that p_t denotes the measure on \mathcal{P}_{∞} associated to the measure $PD(t, -t)$. We have

$$\mathbb{E}[|\Pi_1(t)|^k] = \exp\left(-\int_0^t \phi_u(k) du\right).$$

Thus

$$\begin{aligned} \phi_t(k) &= \mathbb{E}_{v_t}(\rho_s(\Pi_{|k+1} \neq \mathbf{1}) | s) \\ &= \frac{1}{t} p_t(\Pi_{|k+1} \neq \mathbf{1}). \end{aligned}$$

So we must calculate $p_t(\Pi_{|k+1} \neq \mathbf{1})$. We will do this recursively. For $k = 1$, we have

$$p_t(\Pi_{|2} \neq \mathbf{1}) = \frac{[-t]_1^2}{-[-t]_2} = \frac{t}{1-t},$$

and for $k \geq 2$

$$p_t(\Pi_{|k+1} \neq \mathbf{1}) = p_t(\Pi_{|k} \neq \mathbf{1}) + p_t\left(\Pi_{|k+1} = \left\{\{1, \dots, k\}, \{k+1\}\right\}\right) = p_t(\Pi_{|k} \neq \mathbf{1}) + \frac{t}{k-t}.$$

Thus we have :

$$p_t(\Pi_{|k+1} \neq \mathbf{1}) = \sum_{i=1}^k \frac{t}{i-t} \quad \text{and so} \quad \int_0^t \phi_u(k) du = \ln\left(\prod_{i=1}^k \frac{i}{i-t}\right).$$

So we deduce

$$\mathbb{E}[|\Pi_1(t)|^k] = \prod_{i=1}^k \frac{i-t}{i}.$$

The right-hand side coincides with the k -th moment of a $\beta(1-t, t)$ law. So $|\Pi_1(t)|$ has a $\beta(1-t, t)$ law and we deduce :

$$\forall k > 0, \mathbb{E}[|\Pi_1(t)|^k] = \frac{\Gamma(k+1-t)}{\Gamma(1-t)\Gamma(k+1)}.$$

More generally, we can determine the law of the process $(|\Pi_1(t)|, t \in [0, 1])$. By the homogeneous property of fragmentation in space, the process $\left(\frac{|\Pi_1(t+s)|}{|\Pi_1(t)|}, s \in [0, 1-t]\right)$ is independent of $|\Pi_1(t)|$ (cf. Theorem 3.6). So we can calculate the finite dimensional law of the process $(|\Pi_1(t)|, t \in [0, 1])$ and we deduce that the process has the same law as the process $\left(\frac{\gamma(1-t)}{\gamma(1)}, t \in [0, 1]\right)$ (result already proved by Pitman [19]).

Remark 4.8 We have also an expression for $\psi(t, p)$:

$$\psi(t, p) = \ln\left(\frac{\Gamma(1-t)\Gamma(k+1)}{\Gamma(k+1-t)}\right).$$

5 Behavior of the fragmentation at large and small times

5.1 Convergence of the empirical measure

Let $(\Pi(t), t \in [0, 1])$ be a Ruelle's fragmentation on the partitions. Let $(X(t), t \in [0, 1])$, $X(t) = (X_i(t))_{i \geq 1} \in \mathcal{S}$ be its process of ranked frequencies.

We are interested in the empirical measure ρ_t defined by :

$$\rho_t = \sum_{i=1}^{\infty} X_i(t) \delta_{(t-1) \ln X_i(t)}.$$

Proposition 5.1 *For every bounded continuous function f on \mathbb{R}_+ :*

$$\lim_{t \rightarrow 1} \int f(y) \rho_t(dy) = \int_0^{\infty} f(y) e^{-y} dy \text{ in } L^2.$$

We split the proof in two parts. We will successively prove the following two points:

$$\lim_{t \rightarrow 1} \mathbb{E} \left(\int f(y) \rho_t(dy) \right) = \int_0^{\infty} f(y) e^{-y} dy, \quad (10)$$

$$\lim_{t \rightarrow 1} \mathbb{E} \left[\left(\int f(y) \rho_t(dy) \right)^2 \right] = \left(\int_0^{\infty} f(y) e^{-y} dy \right)^2. \quad (11)$$

Set $\xi_t = -\ln |\Pi_1(t)|$. Let us recall

$$|\Pi_1(t)| \sim \beta(1-t),$$

and observe :

$$\mathbb{E} \left(\int f(y) \rho_t(dy) \right) = \mathbb{E} \left(f((1-t)\xi_t) \right).$$

The following lemma clearly implies (10).

Lemma 5.2 *Set $\xi_t = -\ln |\Pi_1(t)|$ where $\Pi(t)$ is the Ruelle's fragmentation. Then*

$$\lim_{t \rightarrow 1} (1-t)\xi_t = \mathbf{e} \text{ in distribution}$$

where \mathbf{e} denotes the exponential law with parameter 1.

Proof. Let us calculate the Laplace transform of $(1-t)\xi_t$.

$$\begin{aligned} \mathbb{E} \left(e^{-q(1-t)\xi_t} \right) &= \mathbb{E} \left(|\Pi_1(t)|^{q(1-t)} \right) \\ &= \frac{\Gamma(q(1-t) + 1 - t)}{\Gamma(1-t) \Gamma(q(1-t) + 1)} \\ &\xrightarrow{t \rightarrow 1} \frac{1}{q+1}. \end{aligned}$$

Since $\frac{1}{q+1}$ is the Laplace transform of the exponential law, by Lévy's Theorem, $(1-t)\xi_t$ converges in law to \mathbf{e} . \square

To prove (11), we consider $\xi'_t = -\ln |\Pi_2(t)|$ where $\Pi_2(t)$ is the block containing the integer 2. Observe that ξ_t and ξ'_t have the same law but are not independent, and that

$$\mathbb{E} \left[\left(\int f(y) \rho_t(dy) \right)^2 \right] = \mathbb{E} \left[f((1-t)\xi_t) f((1-t)\xi'_t) \right].$$

Set $T = \inf \{t > 0, \Pi_1(t) \neq \Pi_2(t)\}$, so T is almost surely finite and conditionally on T , ξ_T and ξ'_T , the processes $(\xi_t, t \geq T)$ and $(\xi'_t, t \geq T)$ are independent. From this, we deduce (11) and then the L^2 -convergence of $\int f(y) \rho_t(dy)$ (refer to [4] for details). \square

So, informally, this proposition proves that, if we consider the size of a typical fragment $X(t)$, then, as t tends to 1, we have

$$|\log X(t)| \sim \frac{C}{1-t}$$

where C is a random factor.

5.2 Additive martingale

In this section, we aim at studying the convergence of the martingale $M(t, p)$ defined in Section 3.2 and we follow the ideas of Bertoin and Rouault [7] who introduce a new probability to prove the convergence.

Recall the following notation :

$\mathcal{F}_t = \sigma(X_i(u), u \leq t)$ is the filtration of the frequency sequence.

$\mathcal{G}_t = \sigma(\Pi(u), u \leq t)$ is the filtration of the fragmentation process on the partitions.

So we have $\mathcal{F}_t \subseteq \mathcal{G}_t$.

Set $\xi_t = -\ln(|\Pi_1(t)|)$. It is an increasing process with independent increments.

$M(t, p) = \exp(\psi(t, p)) \sum_{i=1}^{\infty} |X_i(t)|^{p+1}$. $M(\cdot, p)$ is then a \mathcal{F}_t -martingale.

$\mathcal{E}(t, p) = \exp(\psi(t, p) - p\xi_t)$. $\mathcal{E}(\cdot, p)$ is a \mathcal{G}_t -martingale.

As $\mathbb{E}(|\Pi_1(t)|^p | X(t)) = \sum_i X_i(t)^{p+1}$, we have $\mathbb{E}(\mathcal{E}(t, p) | \mathcal{F}_t) = M(t, p)$.

We denote \mathbb{Q} the probability on \mathcal{G} defined by :

$$d\mathbb{Q}|_{\mathcal{G}_t} = \mathcal{E}(t, p) d\mathbb{P}|_{\mathcal{G}_t}. \text{ So we have also } d\mathbb{Q}|_{\mathcal{F}_t} = M(t, p) d\mathbb{P}|_{\mathcal{F}_t}.$$

Proposition 5.3 *Fix $p > 0$. We have :*

$$\lim_{t \rightarrow 1} M(t, p) = 0 \quad \mathbb{P}\text{-a.s.}$$

Proof. A martingale theorem (cf. [12] p.210) asserts that if $\limsup M(t, p) = \infty$ \mathbb{Q} -a.s., then $\lim M(t, p) = 0$ \mathbb{P} -a.s.

We have

$$M(t, p) \geq \exp(\psi(t, p)) |\Pi_1(t)|^{p+1} = \exp(\psi(t, p) - (p+1)\xi_t).$$

Set $N_t = \psi(t, p) - (p+1)\xi_t$. We will prove that $\limsup N_t = \infty$ \mathbb{Q} -a.s.

Let us recall that, under \mathbb{P} , $|\Pi_1(t)|$ has $\beta(1-t, t)$ law. So for all $\lambda \geq 0$ we have :

$$\mathbb{Q}(\xi_t \geq \lambda) = \mathbb{E}^{\mathbb{P}}(\mathcal{E}(t, p) \mathbf{1}_{\{\xi_t \geq \lambda\}}) = \frac{\Gamma(p+1)}{\Gamma(p+1-t)\Gamma(t)} \int_0^{e^{-\lambda}} x^{p-t} (1-x)^{t-1} dx.$$

So for $A \leq \psi(t, p)$,

$$\mathbb{Q}(N_t \leq A) = \mathbb{Q}\left(\xi_t \geq \frac{\psi(t, p) - A}{p+1}\right) = \frac{\Gamma(p+1)}{\Gamma(p+1-t)\Gamma(t)} \int_0^{e^{-\frac{\psi(t, p) - A}{p+1}}} x^{p-t} (1-x)^{t-1} dx.$$

Recall $\psi(t, p) \sim -\ln(1-t)$ as $t \uparrow 1$. Choose $A(t) = -\frac{1}{3}\ln(1-t)$. So for t large enough, we have $\psi(t, p) - A(t) \geq -\frac{1}{3}\ln(1-t)$.

Set $g(t) = (1-t)^{\frac{1}{3(p+1)}}$. We have :

$$\begin{aligned} \mathbb{Q}(N_t \leq A(t)) &\leq \frac{\Gamma(p+1)}{\Gamma(p+1-t)\Gamma(t)} \int_0^{g(t)} x^{p-t} (1-x)^{t-1} dx \\ &\leq \frac{\Gamma(p+1)}{\Gamma(p+1-t)\Gamma(t)} (1-g(t))^{t-1} \frac{1}{p+1-t} g(t)^{p+1-t} \\ &\leq \varepsilon_p(t), \end{aligned}$$

where $\varepsilon_p(t)$ is a function with limit 0 at $t = 1$.

So $\lim_{t \rightarrow 1} \mathbb{Q}(N_t \geq A(t)) = 1$ and then $\mathbb{Q}(\limsup N_t < \infty) = 0$. We deduce :

$$\lim_{t \rightarrow 1} \sup M(p, t) = \lim_{t \rightarrow 1} \sup N(p, t) = \infty \text{ } \mathbb{Q}\text{-a.s. and so } \lim_{t \rightarrow 1} M(p, t) = 0 \text{ } \mathbb{P}\text{-a.s. } \square$$

Remark 5.4 *In the case $p = 0$, as the process has proper frequencies, we have $M(0, t) = 1$ \mathbb{P} -as for all $t \in [0, 1[$.*

5.3 Small times behavior

In this section, we obtain information on the behavior of the two largest blocks of Ruelle's fragmentation at small times. In this direction, we use the following results due to Berestycki [1].

Let $X_k(t)$ be the frequency of the k -th largest block at time t of Ruelle's fragmentation. Recall that Ruelle's fragmentation can be constructed from a Poisson measure K on $[0, 1[\times \mathcal{S} \times \mathbb{N}$ with intensity $(\nu_t(ds)dt) \otimes \sharp$. Set

$$K = (t, S(t), k(t))_{t \in [0, 1[} = (t, (s_1(t), s_2(t), \dots), k(t))_{t \in [0, 1[}.$$

Let $(S^{(i)}(t), t \in [0, 1[) = (s_1^{(i)}(t), s_2^{(i)}(t), \dots, t \in [0, 1[)$ be the Poisson measure obtained from K restricted to the atoms such that $k(t) = i$. So, it is a Poisson measure with intensity $\nu_t(ds)dt$.

Set

$$R(t) = \max_{s \leq t} s_2^{(1)}(s).$$

Lemma 5.5 • *For t small enough, we have $X_1(t) = \exp(-\xi_t)$ a.s. where ξ_t is an increasing process with independent increments and such that :*

$$\forall k > 0, \mathbb{E}[\exp(-k\xi_t)] = \frac{\Gamma(k+1-t)}{\Gamma(1-t)\Gamma(k+1)}.$$

•

$$X_2(t) \sim R(t) \text{ as } t \rightarrow 0^+ \text{ a.s.}$$

Proof. The proof is the same as in Berestycki [1], since there, time-homogeneity of the fragmentation plays no role. \square

Let us now determine the behavior of $R(t)$.

Proposition 5.6 *Fix $T_0 \in]0, 1/2[$. Then there exists three strictly positive constants C_1, C_2, C_3 such that for all $\lambda > 0$ and for all $t \in]0, T_0[$,*

$$\exp(-C_1\lambda - C_3t) \leq \mathbb{P}\left(R(t) \leq \exp\left(-\frac{\lambda}{t}\right)\right) \leq \exp(-C_2\lambda + C_3t).$$

To estimate the distribution of $R(t)$, we study $\nu_t(s_2 \geq \varepsilon)$ for a fixed ε . Indeed,

$$\mathbb{P}(R(t) \leq \varepsilon) = \exp\left(-\int_0^t \nu_u(s_2 \geq \varepsilon) du\right),$$

and Proposition 5.6 follows from the following lemma :

Lemma 5.7 *Fix $T_0 \in]0, 1/2[$. Then there exists three strictly positive constants C_1, C_2, C_3 such that for all $\varepsilon \in]0, 1[$ and for all $t \in]0, T_0[$,*

$$-t(C_2 \ln \varepsilon + C_3) \leq \int_0^t \nu_u(s_2 \geq \varepsilon) du \leq -t(C_1 \ln \varepsilon - C_3).$$

Proof. We begin with the upper bound. If $(s_i)_{i \geq 1}$ is an element of \mathcal{S} , we denote $(\tilde{s}_i)_{i \geq 1}$ a size-biased rearrangement. We have :

$$s_2 \geq \varepsilon \Rightarrow s_1 \leq 1 - \varepsilon \Rightarrow \tilde{s}_1 \leq 1 - \varepsilon,$$

so

$$\nu_t(s_2 \geq \varepsilon) \leq \nu_t(s_1 \leq 1 - \varepsilon) \leq \nu_t(\tilde{s}_1 \leq 1 - \varepsilon).$$

According to Theorem 4.4, we know the law of \tilde{s}_1 under ν_t :

$$\begin{aligned} \nu_t(\tilde{s}_1 \leq 1 - \varepsilon) &= \int_0^{1-\varepsilon} (1-y)^{-1} y^{-t} dy \\ &\leq \left(\int_0^{1/2} 2y^{-t} dy + \int_{1/2}^{1-\varepsilon} 2^t (1-y)^{-1} dy \right) \\ &\leq \left(-2^t \ln \varepsilon + \frac{2^t}{1-t} \right) \\ &\leq 2(-\ln \varepsilon + 2) \quad \text{for } t \leq \frac{1}{2}. \end{aligned}$$

So we obtain

$$\int_0^t \nu_u(s_2 \geq \varepsilon) du \leq -t(2 \ln \varepsilon - 4).$$

Let us now prove the lower bound. First, we will find a lower bound for $\int_0^t \nu_u(\tilde{s}_2 \geq \varepsilon) du$ and then we will deduce the lemma.

$$\begin{aligned}
\nu_t(\tilde{s}_2 \in dx) &= \int_0^{1-x} \nu_t(\tilde{s}_1 \in dy) \nu_t(\tilde{s}_2 \in dx \mid \tilde{s}_1 \in dy) \\
&= \frac{1}{\Gamma(1-t)\Gamma(t)} \int_0^{1-x} (1-y)^{-1} y^{-t} \left(\frac{x}{1-y}\right)^{-t} \left(1 - \frac{x}{1-y}\right)^{t-1} \frac{dx}{1-y} dy \\
&= \frac{x^{-t} dx}{\Gamma(1-t)\Gamma(t)} \int_0^{1-x} (1-y)^{-1} y^{-t} (1-y-x)^{t-1} dy.
\end{aligned}$$

Set

$$A = \int_{\varepsilon}^1 \int_0^{1-x} x^{-t} (1-y)^{-1} y^{-t} (1-y-x)^{t-1} dy dx,$$

so

$$\nu_t(\tilde{s}_2 \geq \varepsilon) = \frac{1}{\Gamma(1-t)\Gamma(t)} A.$$

We now calculate a lower bound for A :

$$\begin{aligned}
A &= \int_0^{1-\varepsilon} \int_{\varepsilon}^{1-y} x^{-t} (1-y)^{-1} y^{-t} (1-y-x)^{t-1} dx dy \\
&= \int_0^{1-\varepsilon} \left(\int_{\frac{\varepsilon}{1-y}}^1 z^{-t} (1-z)^{t-1} dz \right) (1-y)^{-1} y^{-t} dy \\
&= \int_{\varepsilon}^1 \left(\int_{\frac{\varepsilon}{y}}^1 z^{-t} (1-z)^{t-1} dz \right) y^{-1} (1-y)^{-t} dy \\
&\geq \int_{\varepsilon}^1 \left(\int_{\frac{\varepsilon}{y}}^1 (1-z)^{t-1} dz \right) y^{-1} (1-y)^{-t} dy \\
&\geq \frac{1}{t} \int_{\varepsilon}^1 \left(1 - \frac{\varepsilon}{y}\right) y^{-1} (1-y)^{-t} dy \\
&\geq \frac{1}{t} \int_{\varepsilon}^1 \left(1 - \frac{\varepsilon}{y}\right) y^{-1} dy \\
&\geq \frac{1}{t} (-\ln \varepsilon - 1).
\end{aligned}$$

So

$$\nu_t(\tilde{s}_2 \geq \varepsilon) \geq \frac{1}{\Gamma(1-t)\Gamma(t)t} (-\ln \varepsilon - 1).$$

As $\Gamma(1-t)\Gamma(t)t = \frac{\pi t}{\sin(\pi t)}$ is a positive function which is bounded on $]0, T_0[$, let $1/C_2$ be its maximum. By integration, we obtain :

$$\int_0^t \nu_u(\tilde{s}_2 \geq \varepsilon) du \geq t C_2 (-\ln \varepsilon - 1).$$

We would like now to deduce the lower bound for $\int_0^t \nu_u(s_2 \geq \varepsilon) du$. We use

$$\nu_u(s_2 \geq \varepsilon) \geq \nu_u(\tilde{s}_2 \geq \varepsilon) - \nu_u(\tilde{s}_2 > s_2),$$

and

$$\nu_u(\tilde{s}_2 > s_2) = \nu_u(\tilde{s}_2 = s_1) \leq \nu_u(\tilde{s}_1 \neq s_1) = \int_{\mathcal{S}} (1 - s_1) \nu_u(ds) \leq \int_{\mathcal{S}} (1 - \tilde{s}_1) \nu_u(ds).$$

We have already seen that

$$\int_{\mathcal{S}} (1 - \tilde{s}_1) \nu_u(ds) = \frac{1}{1-u} \quad (\text{cf. Formula (9)}).$$

So, for all $t \leq T_0$, we have

$$\int_0^t \nu_u(\tilde{s}_2 > s_2) du \leq -\ln(1-t) \leq \frac{1}{1-T_0} t.$$

Hence

$$\int_0^t \nu_u(s_2 \geq \varepsilon) \geq t(-C_2 \ln \varepsilon - C_3). \square$$

We can then deduce the lower-asymptotic behavior of $X_2(t)$ from this theorem.

Proposition 5.8 *There exists a constant $\delta > 0$ such that almost surely*

$$\begin{cases} \liminf_{t \rightarrow 0} |\ln t|^{\gamma/t} X_2(t) = 0 & \text{if } \gamma < \delta \\ \liminf_{t \rightarrow 0} |\ln t|^{\gamma/t} X_2(t) = \infty & \text{if } \gamma > \delta. \end{cases}$$

Proof. According to Theorem 5.5, we just have to prove the proposition replacing $X_2(t)$ by $R(t)$. Set $\gamma > \frac{1}{C_2}$. Choose $\beta > 0$ such that $\gamma > \frac{e^\beta}{C_2}$. Set $t_i = e^{-i\beta}$ and $f(t) = \gamma \ln(-\ln t)$. For $t \in [0, e^{-1}[$, $f(t)$ is a decreasing positive function.

For $t \in [t_{i+1}, t_i]$, we have

$$R(t) \geq R(t_{i+1}) \text{ and } \exp\left(-\frac{f(t_i)}{t_i}\right) \geq \exp\left(-\frac{f(t)}{t}\right).$$

So if we prove

$$R(t_{i+1}) \geq \exp\left(-\frac{f(t_i)}{t_i}\right) \tag{12}$$

almost surely for i large enough, then we will deduce

$$\forall \gamma > \frac{1}{C_2} \liminf_{t \rightarrow 0} (\ln \frac{1}{t})^{\gamma/t} R(t) \geq 1 \text{ a.s.} \quad \text{and so } \forall \gamma > \frac{1}{C_2} \liminf_{t \rightarrow 0} (\ln \frac{1}{t})^{\gamma/t} R(t) = \infty \text{ a.s.}$$

To prove (12), we apply Borel-Cantelli's Lemma. Using Proposition 5.6, we obtain :

$$\mathbb{P}\left(R(t_{i+1}) \leq \exp\left(-\frac{f(t_i)}{t_i}\right)\right) \leq K(\beta i)^{-C_2 \gamma e^{-\beta}}.$$

Thanks to the choice of γ and β , the serie converges.

For the second part of the proposition, we use an extension of Borel-Cantelli's Lemma when the sum diverges but the events are not independent (cf. [14]) :

Let $(H_i)_{i \geq 1}$ be a sequence of events such that $\sum \mathbb{P}(H_i)$ diverges and

$$\forall N \geq 1, \frac{\sum_{i,j=1}^N \mathbb{P}(H_i \cap H_j)}{\left(\sum_{i=1}^N \mathbb{P}(H_i)\right)^2} \leq M. \tag{13}$$

Then the set $\{i, \omega \in H_i\}$ is infinite with a probability larger than $1/M$.

In our case, we fix a $\gamma < 1/C_1$ and a $\varepsilon > 0$ such that $(1 + \varepsilon)\gamma C_1 < 1$. Set $t_i = e^{-i^{1+\varepsilon}}$ and $H_i = \{R(t_i) \leq (\ln(1/t_i))^{\gamma/t_i}\}$. Fix $i, j \geq 1$. Recall $R(t)$ is the record process of a point Poisson process. So we have

$$\begin{aligned}\mathbb{P}(H_i \cap H_{j+i}) &= \mathbb{P}(H_i)\mathbb{P}(H_{i+j}) \exp\left(\int_0^{t_{i+j}} \nu_u(s_2 \geq (\ln(1/t_i))^{\gamma/t_i}) du\right) \\ &\leq K\mathbb{P}(H_i)\mathbb{P}(H_{i+j}) \exp\left((1 + \varepsilon)C_1\gamma \ln i e^{-(1+\varepsilon)i^\varepsilon}\right) \\ &\leq K'\mathbb{P}(H_i)\mathbb{P}(H_{i+j}).\end{aligned}$$

(We have used $(i + j)^{1+\varepsilon} - i^{1+\varepsilon} \geq (1 + \varepsilon)i^\varepsilon$ for all $i, j \geq 1$). With this upper bound, we deduce that the sequence H_i verifies (13). We now have to prove that the sum of probabilities diverges. Using Proposition 5.6, we obtain :

$$\sum_i \mathbb{P}(H_i) \geq K \sum_i i^{-C_1\gamma(1+\varepsilon)}.$$

Thus this series diverges thanks to our choice of γ and ε . We now apply the 0-1 law to prove that the probability that the set $\{i, \omega \in H_i\}$ is infinite equal to 1.

So we have proved

$$\begin{cases} \liminf_{t \rightarrow 0} (\ln \frac{1}{t})^{\gamma/t} R(t) = 0 & \text{a.s.} \quad \forall \gamma < \frac{1}{C_1} \\ \liminf_{t \rightarrow 0} (\ln \frac{1}{t})^{\gamma/t} R(t) = \infty & \text{a.s.} \quad \forall \gamma > \frac{1}{C_2}. \end{cases}$$

Thus we deduce that there exists almost surely a (random) critical $\gamma_c \in]1/C_1, 1/C_2[$ such that

$$\begin{cases} \liminf_{t \rightarrow 0} (\ln \frac{1}{t})^{\gamma/t} R(t) = 0 & \forall \gamma < \gamma_c \\ \liminf_{t \rightarrow 0} (\ln \frac{1}{t})^{\gamma/t} R(t) = \infty & \forall \gamma > \gamma_c. \end{cases}$$

By the 0-1 law, the law of γ_c is trivial, i.e. it exists δ verifying Proposition 5.8 \square

We can also determine the upper asymptotic behavior of $X_2(t)$:

Proposition 5.9 *We have almost surely*

$$\begin{cases} \limsup_{t \rightarrow 0} \exp(\frac{1}{t}(-\ln(t))^{-\beta}) X_2(t) = \infty & \text{if } \beta > 1 \\ \limsup_{t \rightarrow 0} \exp(\frac{1}{t}(-\ln(t))^{-\beta}) X_2(t) = 0 & \text{if } \beta \leq 1. \end{cases}$$

Proof. We use the same approach as for the infimum. Fix $\beta > 1$. Set $t_i = e^{-i}$ and $f(t) = \exp(-\frac{1}{t}(-\ln(t))^{-\beta})$. We want to prove that $R(t) \leq f(t)$ almost surely for t small enough. As f is a decreasing function and $R(t)$ an increasing process, we have $R(t) \leq R(t_i)$ and $f(t_{i+1}) \leq f(t)$. So we just have to prove that $R(t_i) \leq f(t_{i+1})$ almost surely for i large enough.

We have

$$\begin{aligned}\mathbb{P}(R(t_i) \geq f(t_{i+1})) &\leq 1 - \exp\left(-C_3 e^{-i} - C_1 e(i+1)^{-\beta}\right) \\ &\leq C_1 e i^{-\beta} + o(i^{-\beta}).\end{aligned}$$

This serie converges. So, thanks to Borel-Cantelli's Lemma, we can conclude.

Let us now prove the case $\beta \leq 1$. Set $t_i = e^{-i}$ and $f(t) = \exp(-\frac{1}{t}(-\ln(t))^{-\beta})$. Set $H_i = \{R(t_i) \geq f(t_i)\}$. Then we have

$$\sum_{i=1}^N \mathbb{P}(H_i) \geq \sum_{i=1}^N \left(1 - \exp\left(C_3 e^{-i} - C_2 i^{-\beta}\right)\right).$$

The right term is equivalent to $\sum_{i=1}^N C_2 i^{-\beta}$, so it diverges.

We have now to check the condition (13) to apply the generalized Borel-Cantelli's Lemma.

$$\begin{aligned} \mathbb{P}(H_i \cap H_{i+j}) &= 1 - \mathbb{P}(\overline{H_i}) - \mathbb{P}(\overline{H_{i+j}}) + \mathbb{P}(\overline{H_i} \cap \overline{H_{i+j}}) \\ &= 1 - \mathbb{P}(\overline{H_i}) - \mathbb{P}(\overline{H_{i+j}}) + \mathbb{P}(\overline{H_i})\mathbb{P}(\overline{H_{i+j}}) \exp\left(\int_0^{t_{i+j}} \nu_u(s_2 \geq f(t_i)) du\right) \\ &\leq \mathbb{P}(H_i)\mathbb{P}(H_{i+j}) + \exp\left(\int_0^{t_{i+j}} \nu_u(s_2 \geq f(t_i)) du\right) - 1. \end{aligned}$$

Then remark

$$\exp\left(\int_0^{t_{i+j}} \nu_u(s_2 \geq f(t_i)) du\right) \leq \exp\left(C_3 e^{-i-j} + C_1 i^{-\beta} e^{-j}\right).$$

Hence we deduce

$$\sum_{i,j=1}^N \left(\exp\left(\int_0^{t_{i+j}} \nu_u(s_2 \geq f(t_i)) du\right) - 1 \right) \leq K \sum_{i=1}^N i^{-\beta}.$$

So

$$\frac{\sum_{i,j=1}^N \left(\exp\left(\int_0^{t_{i+j}} \nu_u(s_2 \geq f(t_i)) du\right) - 1 \right)}{\sum_{i=1}^N \mathbb{P}(H_i)}$$

is bounded and thus the condition (13) is true.

So, we can conclude for the case $\beta < 1$. For $\beta = 1$, we just have

$$\limsup_{t \rightarrow 0} R(t) \exp\left(-\frac{1}{t \ln t}\right) \leq 1 \quad \text{a.s.}$$

Remark then that the same demonstration works with $\gamma f(t)$ instead of $f(t)$ with γ positive constant. So, we have

$$\limsup_{t \rightarrow 0} R(t) \exp\left(-\frac{1}{t \ln t}\right) \leq \gamma \quad \text{a.s.}$$

and thus

$$\limsup_{t \rightarrow 0} R(t) \exp\left(-\frac{1}{t \ln t}\right) = 0 \quad \text{a.s.} \quad \square$$

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