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# Discrete and continuous Yang-Mills measure for non-trivial bundles over compact surfaces

Thierry LÉVY\*

January 6, 2005

## Abstract

In this paper, we construct one Yang-Mills measure on a compact surface for each isomorphism class of principal bundles over this surface. For this, we refine the discretization procedure used in a previous construction [8] and define a new discrete theory which is essentially a covering of the usual one. We prove that the measures corresponding to different isomorphism classes of bundles or to different total areas of the base space are mutually singular. We give also a combinatorial computation of the partition functions which relies on the formalism of fat graphs.

## Introduction

The Yang-Mills measure is the law of a group-valued random process indexed by a family of paths on some manifold. It is usually thought of as the random holonomy induced by a probability measure on the space of connections on a principal bundle over this manifold. We consider in this paper the case where the base manifold is an oriented compact surface and the structure group is a compact connected Lie group. In this case, the measure has been studied at various levels of rigor by several authors. In particular, the origin of its mathematical study is a paper by the physicist A. Migdal [12]. Other important contributions are those of B. Driver [3, 2] and, with different motivations, E. Witten [19]. The first rigorous construction has been given by A. Sengupta [15], by conditioning an infinite-dimensional noise. A second construction has been given by the author in [8], where the random holonomy process is built by passing discrete approximations to the limit. This leads essentially to the same object, though perhaps in a way that gives a better grip on it (see for example [10]).

An important feature of the Yang-Mills measure in two dimensions is its almost invariance by diffeomorphisms (it is actually invariant by those which preserve a volume form on the surface). Still, a diffeomorphism-invariant random holonomy process on a principal bundle should depend not only on the structure group of this bundle but also on its isomorphism class. This is especially clear if one thinks that the characteristic classes of a bundle can be computed using connections on it. However, the construction proposed in [8] does not depend on the particular topological type of the principal bundle one considers. We shall see that the measure constructed there corresponds to an average over all possible isomorphism classes. On the other hand, A. Sengupta's construction does allow one to take a specific topological type of bundle into account.

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The aim of this paper is to fill this gap. More precisely, we propose a construction of the Yang-Mills measure by passing a discrete approximation to the continuous limit *in a way that keeps track of the topology of the bundle*.

This may sound paradoxical for the following reason. Discrete approximations of the measures are usually built by first considering graphs on the base manifold and restricting the bundle to these one-dimensional complexes. But, as long as the structure group is connected, the restricted bundle is always trivial and any topological information about the full bundle is lost in this operation. This is why the construction of [8] produces a single probability measure, which is associated to no particular isomorphism class of fiber bundles, but rather to a random bundle for some natural measure on the set of isomorphism classes. In the case of an Abelian structure group, this was to some extent already understood, because in this case it is easier to compare A. Sengupta's construction and ours (see the informal remarks in [8, Sections 1.9.3 and 3.2]).

The first section of this paper is devoted to the construction of a new discrete theory, namely a finer way of discretizing the differential geometric objects involved than the naive one commonly used in discrete gauge theory. This new discretization is fine enough to capture the topology of the bundle. In fact, what we do is replacing the usual configuration space of discrete gauge theory by a singular covering of it, which happens to be finite when the structure group is semi-simple. In all cases, the singular set is negligible and plays no role at the level of measure theory. A partial study of the topological structure of this singular covering is presented in the last section.

In the second and third sections, we construct the Yang-Mills measures associated to a specific isomorphism class of principal bundles. For this, we consider first semi-simple structure groups, because in this case, as explained above, the new configuration space is a singular *finite* covering of the usual one. Then, although the method used to construct the discrete measures does not extend to general structure groups, the formulæ derived in the semi-simple case make sense without the semi-simplicity assumption. This allows us to construct the discrete measures for general compact connected structure groups. In the case of Abelian groups, we check that our construction is consistent with the remarks of [8] mentioned above. Finally, in the third section, we pass these discrete measures to the continuous limit, following step by step the construction presented in the second chapter of [8]. However, balancing the fact that it carries more geometric informations, the new discretization does not produce a nice projective family of probability spaces as the usual procedure does. It is thus necessary to go back to the usual configuration spaces before passing to the limit. As a consequence, we construct probability measures on the same space as in [8], not on some covering of it.

At the end of the third section, we prove that the Yang-Mills measures corresponding to different isomorphism classes of principal bundles or to different temperatures (one should say, different total areas of the base manifold) are mutually singular. This shows that the canonical probability space of the random holonomy process is a disjoint union of infinitely many sectors, each one corresponding to a specific isomorphism class of bundles and a specific temperature.

The fact that the discrete partition functions do not depend on the graph one considers plays an important role. The proof of this fact given in [8] relies on rather painful approximations. We present in the fourth section a much more natural proof of this invariance, based on the formalism of fat graphs, which seems very well suited to discrete gauge theory. For example, it enables us to compute directly these partition functions in a quite intuitive combinatorial way.

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## 1 The configuration space

Let  $M$  be a compact oriented surface without boundary. Let  $\sigma$  be a volume 2-form on  $M$  consistent with the orientation. For practical purposes, let us endow  $M$  with a Riemannian metric and let us assume that the corresponding Riemannian volume is the density of  $\sigma$ . Let  $G$  be a compact connected Lie group. Let  $P \rightarrow M$  be a principal  $G$ -bundle over  $M$ .

If  $C$  is a closed subset of  $M$ , we call smooth mapping (resp. smooth cross-section, ...) on  $C$  the restriction to  $C$  of a smooth mapping (resp. smooth cross-section, ...) defined on an open neighbourhood of  $C$ . In particular, we denote by  $\Gamma(C, P)$  the set of smooth cross-sections of  $P$  over  $C$ .

### 1.1 Graphs

By an *edge* on  $M$  we mean a segment of a smooth oriented 1-dimensional submanifold. If  $e$  is an edge, we call *inverse* of  $e$  and denote by  $e^{-1}$  the edge obtained by reversing the orientation of  $e$ . We also denote respectively by  $\underline{e}$  and  $\bar{e}$  the starting and finishing point of  $e$ . Let  $e_1, \dots, e_n$  be  $n$  edges. If, for each  $i$  between 1 and  $n-1$ , one has  $\bar{e}_i = \underline{e}_{i+1}$ , then one can form the concatenation  $e_1 \dots e_n$ . If moreover  $f_1, \dots, f_m$  are also edges which can be concatenated, we declare  $e_1 \dots e_n$  equivalent to  $f_1 \dots f_m$  if and only if there exists a continuous mapping  $c: [0, 1] \rightarrow M$  and two finite sequences  $0 = t_0 < t_1 < \dots < t_n = 1$  and  $0 = s_0 < s_1 < \dots < s_m = 1$  of real numbers such that, for each  $i = 1 \dots n$ , the restriction of  $c$  to the interval  $[t_{i-1}, t_i]$  is a smooth embedding of image  $e_i$  and, for each  $j = 1 \dots m$ , the restriction of  $c$  to the interval  $[s_{j-1}, s_j]$  is a smooth embedding of image  $f_j$ . By a *path* we mean an equivalence class of finite concatenations of edges. We denote the set of paths by  $PM$ . Loops, simple loops, starting and finishing points of paths, their concatenation, are defined in the obvious way. Let  $l_1$  and  $l_2$  be two loops. We say that  $l_1$  and  $l_2$  are cyclically equivalent if there exist two paths  $c$  and  $d$  in  $PM$  such that  $l_1 = cd$  and  $l_2 = dc$ . We call *cycle* an equivalence class of loops for this relation. Informally, a cycle is a loop on which one has forgotten the starting point. We say that a cycle is *simple* if its representatives are simple loops.

**Definition 1.1** *A graph is a triple  $\mathbb{G} = (\mathbb{V}, \mathbb{E}, \mathbb{F})$ , where*

1.  $\mathbb{E}$  is a finite collection of edges stable by inversion and such that two distinct edges are either inverse of each other or intersect, if at all, only at some of their endpoints.
2.  $\mathbb{V}$  is the set of endpoints of the elements of  $\mathbb{E}$ .
3.  $\mathbb{F}$  is the set of the closures of the connected components of  $M \setminus \bigcup_{e \in \mathbb{E}} e$ .
4. Each open connected component of  $M \setminus \bigcup_{e \in \mathbb{E}} e$  is diffeomorphic to the open unit disk of  $\mathbb{R}^2$ .

The elements of  $\mathbb{V}, \mathbb{E}, \mathbb{F}$  are respectively called vertices, edges and faces of  $\mathbb{G}$ . We call *open faces* the connected components of  $M \setminus \bigcup_{e \in \mathbb{E}} e$ . Beware that an open face may be strictly contained in the interior of its closure.

We denote by  $\mathbb{E}^*$  the set of paths that can be represented by a concatenation of elements of  $\mathbb{E}$ .

This definition of a graph is that of [8]. In the first part of this paper, we are going to work with a slightly more restrictive notion of a graph. Before defining it, let us explain what we mean by the boundary of a face.

Let  $\mathbb{G}$  be a graph. The orientation of  $M$  determines, at each vertex, a cyclic order on the set of incoming edges. It is defined as the order of the intersection points of these edges with a small geodesic circle around the vertex. This means that  $\mathbb{G}$  induces a structure of *fat graph* on  $\mathbb{E}$  (see [7] or Section 4.2 for a presentation of fat graphs). This fat graph has faces, which are defined as the cycles of some permutation on  $\mathbb{E}$ . Each such cycle is of the form  $(e_1, \dots, e_m)$  with  $\overline{e_i} = e_{i+1}$  for all  $i = 1 \dots m - 1$  and  $\overline{e_m} = e_1$ . Hence, it defines a cycle in  $\mathbb{E}^*$ .

Let now  $L : \mathbb{E} \rightarrow \mathbb{F}$  be the mapping defined by the fact that, for each edge  $e$ ,  $L(e)$  is the face of  $\mathbb{G}$  located on the left of  $e$ , that is, the closure of the unique open face of  $\mathbb{G}$  which  $e$  bounds with positive orientation. The function  $L$  is constant on the cycles of edges corresponding to the faces of the fat graph induced by  $\mathbb{G}$  and, since the open faces are diffeomorphic to disks, this sets up a one-to-one correspondence between the faces of the fat graph induced by  $\mathbb{G}$  and the elements of  $\mathbb{F}$ . For each  $F \in \mathbb{F}$ , the cycle associated in this way with  $F$  is called the boundary of  $F$  and it is denoted by  $\partial F$ . Recall that its origin is ill-defined. However, the following definition makes sense.

**Definition 1.2** *We say that a graph is simple if the boundary of each one of its faces is a simple cycle.*

Let  $\mathbb{G}$  be a graph. We call *unoriented edge* of  $\mathbb{G}$  a pair  $\{e, e^{-1}\}$  where  $e \in \mathbb{E}$ . We call *orientation* of  $\mathbb{G}$  a subset  $\mathbb{E}^+$  of  $\mathbb{E}$  which contains exactly one element of each unoriented edge.

## 1.2 Discretization of a connection on $P$

Let  $\omega$  be a connection 1-form on  $P$ . Let  $\mathbb{G} = (\mathbb{V}, \mathbb{E}, \mathbb{F})$  be a simple graph on  $M$ .

Let  $s_{\mathbb{V}} \in \Gamma(\mathbb{V}, P)$  be a cross-section of  $P$  over  $\mathbb{V}$ . Let  $F$  be a face of  $\mathbb{G}$ . Since the group  $G$  is connected, there exists a smooth cross-section  $s_F \in \Gamma(F, P)$  of  $P$  over  $F$  such that  $s_F|_{F \cap \mathbb{V}} = s_{\mathbb{V}}|_{F \cap \mathbb{V}}$ . Let us choose such a section for each  $F$  and set  $s = (s_F)_{F \in \mathbb{F}}$ . This is an element of the set  $\mathcal{S}(\mathbb{G}, P)$  defined by

$$\mathcal{S}(\mathbb{G}, P) = \left\{ s \in \prod_{F \in \mathbb{F}} \Gamma(F, P) \mid \forall F, F' \in \mathbb{F}, s_F|_{F \cap F' \cap \mathbb{V}} = s_{F'}|_{F \cap F' \cap \mathbb{V}} \right\}.$$

Let  $\pi : \tilde{G} \rightarrow G$  be a universal cover of  $G$ . We shall explain now how the choice of  $s$  allows us to associate one element of  $\tilde{G}$  to each edge of  $\mathbb{G}$ . Let  $\mathfrak{g}$  be the Lie algebra of  $G$ . The covering map determines an isomorphism of Lie algebras through which we identify  $\mathfrak{g}$  with the Lie algebra of  $\tilde{G}$ .

Take  $e$  in  $\mathbb{E}$ . Set  $F = L(e)$ . Let us parametrize  $e$  smoothly as  $e : [0, 1] \rightarrow M$ . Now the ordinary differential equation

$$\begin{cases} h_0 = 1 \\ \dot{h}_t h_t^{-1} = -(s_F^* \omega)(\dot{e}_t) \quad , \quad t \in [0, 1] \end{cases} \quad (1)$$

which one usually solves in  $C^\infty([0, 1], G)$  in order to determine the holonomy of  $\omega$  along  $e$  can just as well be solved in  $C^\infty([0, 1], \tilde{G})$ . This is what we do and we denote by  $\tilde{g}(e)$  the element  $h_1$  of  $\tilde{G}$ .

**Lemma 1.3** For all  $e \in \mathbb{E}$ , one has  $\pi(\tilde{g}(e)) = \pi(\tilde{g}(e^{-1}))^{-1}$ .

**Proof** – The function  $t \mapsto \pi(h_t)$  is the solution of (1) in  $C^\infty([0, 1], G)$ . Hence,  $\pi(\tilde{g}(e))$  and  $\pi(\tilde{g}(e^{-1}))^{-1}$  are both equal to the unique element  $x$  of  $G$  such that the  $\omega$ -horizontal lift of  $e$  starting at  $s_{\mathbb{V}}(\underline{e})$  ends at  $s_{\mathbb{V}}(\bar{e})x$ .  $\square$

Thus, the choice of  $s$  allows us to define an element of what we take as the new configuration space for the discrete Yang-Mills theory on  $\mathbb{G}$  with structure group  $G$ , namely

$$\tilde{G}^{(\mathbb{E})} = \{\tilde{g} \in \tilde{G}^{\mathbb{E}} \mid \forall e \in \mathbb{E}, \pi(\tilde{g}(e)) = \pi(\tilde{g}(e^{-1}))^{-1}\}.$$

More precisely, if  $\mathcal{A}$  denotes the set of smooth<sup>1</sup> connection 1-forms on  $P$ , we have defined a mapping

$$\begin{aligned} \tilde{H}_{\mathbb{G}} : \mathcal{S}(\mathbb{G}, P) \times \mathcal{A} &\longrightarrow \tilde{G}^{(\mathbb{E})} \\ (s, \omega) &\longmapsto \tilde{H}_{\mathbb{G}}(s, \omega) = (\tilde{g}_{\omega}^s(e))_{e \in \mathbb{E}} \end{aligned} \quad (2)$$

### 1.3 Gauge transformations

Let  $\Pi \subset \tilde{G}$  be the kernel of  $\pi : \tilde{G} \longrightarrow G$ . It is a discrete central subgroup of  $\tilde{G}$  and it is finite if and only if  $G$  is semi-simple. We use the generic notation  $z$  for the elements of  $\Pi$ , reminding us in this way that they are central.

Consider the homomorphism

$$\begin{aligned} \Pi^{\mathbb{E}} &\longrightarrow \Pi^{\mathbb{F}} \\ (z_e)_{e \in \mathbb{E}} &\longmapsto \left( \prod_{L(e)=F} z_e \right)_{F \in \mathbb{F}} \end{aligned}$$

and let  $J_{\Pi}$  denote its kernel.

**Definition 1.4** We call discrete gauge group the group  $\tilde{\mathcal{J}}_{\mathbb{G}} = \tilde{G}^{\mathbb{V}} \times J_{\Pi}$ . This group acts on  $\tilde{G}^{(\mathbb{E})}$  as follows: given  $\tilde{g} = (\tilde{g}(e))_{e \in \mathbb{E}} \in \tilde{G}^{(\mathbb{E})}$  and  $j = ((j_v)_{v \in \mathbb{V}}, (z_e)_{e \in \mathbb{E}}) \in \tilde{\mathcal{J}}_{\mathbb{G}}$ ,  $j \cdot \tilde{g}$  is defined by

$$\forall e \in \mathbb{E}, (j \cdot \tilde{g})(e) = j_{\bar{e}}^{-1} \tilde{g}(e) j_{\underline{e}}. \quad (3)$$

Let  $Z(\tilde{G})$  denote the center of  $\tilde{G}$ . Consider the homomorphism

$$\begin{aligned} Z(\tilde{G}) \times \Pi^{\mathbb{V}} &\longrightarrow \tilde{\mathcal{J}}_{\mathbb{G}} \\ (\tilde{x}, (k_v)_{v \in \mathbb{V}}) &\longmapsto ((\tilde{x}k_v)_{v \in \mathbb{V}}, (k_{\bar{e}}k_{\underline{e}}^{-1})_{e \in \mathbb{E}}) \end{aligned}$$

and let  $\mathcal{K}_{\mathbb{G}}$  denote its image.

**Proposition 1.5** The group  $\tilde{\mathcal{J}}_{\mathbb{G}}$  acts on  $\tilde{G}^{(\mathbb{E})}$  with kernel  $\mathcal{K}_{\mathbb{G}}$  and its orbits satisfy the following property: if  $s$  belongs to  $\mathcal{S}(\mathbb{G}, P)$  and  $\omega$  to  $\mathcal{A}$ , then

$$\bigcup_{s' \in \mathcal{S}(\mathbb{G}, P)} \{\tilde{H}_{\mathbb{G}}(s', \omega)\} = \tilde{\mathcal{J}}_{\mathbb{G}} \cdot \tilde{H}_{\mathbb{G}}(s, \omega). \quad (4)$$

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<sup>1</sup>It is enough to consider connections in the Sobolev space  $H^1$  in order to be able to solve (1) and define a holonomy, see [10] for more details.

**Proof** – That the kernel of the action is  $\mathcal{K}_{\mathbb{G}}$  follows easily from the definitions of the action and  $\mathcal{K}_{\mathbb{G}}$ . In order to prove (4), fix  $s$  in  $\mathcal{S}(\mathbb{G}, P)$  and  $\omega$  in  $\mathcal{A}$ . Take  $s'$  in  $\mathcal{S}(\mathbb{G}, P)$ . There exists  $u \in \mathcal{S}(\mathbb{G}, M \times G)$  such that  $s' = su$ , that is, for each  $F$ ,  $s'_F = s_F u_F$ . Since  $\mathbb{G}$  is a simple graph, its faces are contractible and it is possible for each  $F \in \mathbb{F}$  to lift  $u_F$  to a mapping  $\tilde{u}_F : F \rightarrow \tilde{G}$ . Observe however that this does not define an element of  $\mathcal{S}(\mathbb{G}, M \times \tilde{G})$  as there is no guarantee that the lifts coincide over the vertices of the graph. Nevertheless, these lifts allow us to express  $\tilde{H}_{\mathbb{G}}(s', \omega) = \tilde{g}_{\omega}^{s'}$  in function of  $\tilde{H}_{\mathbb{G}}(s, \omega) = g_{\omega}^s$ . Indeed, for each  $e \in \mathbb{E}$ ,

$$\tilde{g}_{\omega}^{s'}(e) = \tilde{u}_{L(e)}(\bar{e})^{-1} \tilde{g}_{\omega}^s(e) \tilde{u}_{L(e)}(\underline{e}), \quad (5)$$

which does not depend on the choice of the lift.

Let  $u_{\mathbb{V}}$  be the cross-section of  $M \times G$  over  $\mathbb{V}$  determined by  $u$ . Let  $\tilde{u}_{\mathbb{V}}$  be a lift of  $u_{\mathbb{V}}$  to a section of  $M \times \tilde{G}$ . Then (5) can be rewritten as

$$\tilde{g}_{\omega}^{s'}(e) = \tilde{u}_{\mathbb{V}}(\bar{e})^{-1} \tilde{g}_{\omega}^s(e) \tilde{u}_{\mathbb{V}}(\underline{e}) [\tilde{u}_{L(e)}(\bar{e})^{-1} \tilde{u}_{\mathbb{V}}(\bar{e})] [\tilde{u}_{L(e)}(\underline{e})^{-1} \tilde{u}_{\mathbb{V}}(\underline{e})]^{-1}.$$

Set, for each  $v \in \mathbb{V}$ ,  $j_v = \tilde{u}_{\mathbb{V}}(v)$  and, for each  $e \in \mathbb{E}$ ,

$$z_e = [\tilde{u}_{L(e)}(\bar{e})^{-1} \tilde{u}_{\mathbb{V}}(\bar{e})] [\tilde{u}_{L(e)}(\underline{e})^{-1} \tilde{u}_{\mathbb{V}}(\underline{e})]^{-1}.$$

Then,  $j = ((j_v)_{v \in \mathbb{V}}, (z_e)_{e \in \mathbb{E}})$  belongs to  $\tilde{\mathcal{J}}_{\mathbb{G}}$  and satisfies  $j \cdot \tilde{H}(s, \omega) = \tilde{H}(s', \omega)$ .

Now let  $j = ((j_v)_{v \in \mathbb{V}}, (z_e)_{e \in \mathbb{E}})$  be an element of  $\tilde{\mathcal{J}}_{\mathbb{G}}$ . Let  $\tilde{u}_{\mathbb{V}} \in \Gamma(\mathbb{V}, M \times \tilde{G})$  be defined by  $\tilde{u}_{\mathbb{V}}(v) = j_v$ . Let  $F$  be a face of the graph. Let  $e_1 \dots e_n$  be a simple loop which represents  $\partial F$ . We construct a cross-section  $\tilde{u}_F$  of  $M \times \tilde{G}$  over  $F$ . The conditions  $\tilde{u}_F(\underline{e}_1) = \tilde{u}_{\mathbb{V}}(\underline{e}_1)$  and

$$z_{e_i} = [\tilde{u}_F(\bar{e}_i)^{-1} \tilde{u}_{\mathbb{V}}(\bar{e}_i)] [\tilde{u}_F(\underline{e}_i)^{-1} \tilde{u}_{\mathbb{V}}(\underline{e}_i)]^{-1}$$

for each  $i = 1 \dots n$  determine the values of  $\tilde{u}_F$  over the vertices located on the boundary of  $F$ . Observe that  $\tilde{u}_F(\underline{e}_1)$  is well-defined because  $e_1 \dots e_n$  is a simple loop and  $z_{e_1} \dots z_{e_n} = 1$ . Now, since  $\tilde{G}$  is connected,  $\tilde{u}_F$  can be extended to the boundary of  $F$  and even, since  $\tilde{G}$  is simply connected, to  $F$  itself. Let  $u_F$  be the projection on  $G$  of  $\tilde{u}_F$ . The value at a vertex  $v$  of  $u_F$  is  $\pi(j_v)$ .

Doing this for each face produces an element  $u \in \mathcal{S}(\mathbb{G}, M \times G)$ . It follows now from (5) that this element satisfies  $\tilde{H}(su, \omega) = j \cdot \tilde{H}(s, \omega)$ .  $\square$

Let  $\mathcal{J}$  denote the smooth gauge group<sup>2</sup> of  $P$ . It acts on  $\mathcal{A}$  by pull-back.

**Corollary 1.6** *Let  $j \in \mathcal{J}$  and  $\omega \in \mathcal{A}$ . Then, for all  $s \in \mathcal{S}(\mathbb{G}, P)$ , one has  $\tilde{H}_{\mathbb{G}}(s, j \cdot \omega) \in \tilde{\mathcal{J}}_{\mathbb{G}} \cdot \tilde{H}_{\mathbb{G}}(s, \omega)$ .*

**Proof** – Since  $\tilde{H}_{\mathbb{G}}(s, j \cdot \omega)$  is built from pull-backs of  $j \cdot \omega$  by  $s$ , which are the same as the pull-backs of  $\omega$  by  $j(s)$ , one has  $\tilde{H}_{\mathbb{G}}(s, j \cdot \omega) = \tilde{H}_{\mathbb{G}}(j(s), \omega)$  and the result follows.  $\square$

Finally, Proposition 1.5 and Corollary 1.6 show that (2) induces a mapping  $\tilde{H}_{\mathbb{G}} : \mathcal{A}/\mathcal{J} \rightarrow \tilde{G}^{(\mathbb{E})}/\tilde{\mathcal{J}}_{\mathbb{G}}$ .

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<sup>2</sup>If one considers  $H^1$  connections rather than smooth ones, then one should consider  $H^2$  gauge transformations, see [17, 18].

## 1.4 Comparison with the classical formalism of discrete gauge theory

Let  $\mathbb{E}^+$  be an orientation of  $\mathbb{G}$ . The usual configuration space in discrete Yang-Mills theory is  $G^{\mathbb{E}^+}$ . There is a mapping

$$H_{\mathbb{G}} : \Gamma(\mathbb{V}, P) \times \mathcal{A} \longrightarrow G^{\mathbb{E}^+}$$

defined as follows. Pick  $s_{\mathbb{V}} \in \Gamma(\mathbb{V}, P)$  and  $\omega \in \mathcal{A}$ . Then  $H_{\mathbb{G}}(s_{\mathbb{V}}, \omega) = (g(e))_{e \in \mathbb{E}^+}$ , where, for each  $e \in \mathbb{E}^+$ , the  $\omega$ -horizontal lift of  $e$  starting at  $s_{\mathbb{V}}(\underline{e})$  finishes at  $s_{\mathbb{V}}(\bar{e})g(e)$ .

The discrete gauge group in this setting is  $\mathcal{J}_{\mathbb{G}} = G^{\mathbb{V}}/Z(G)$ , where  $Z(G)$  is embedded diagonally in  $G^{\mathbb{V}}$ . It acts faithfully on  $G^{\mathbb{E}^+}$  and satisfies a property similar to (4). Hence,  $H_{\mathbb{G}}$  induces a mapping  $\mathcal{A} \longrightarrow G^{\mathbb{E}^+}/\mathcal{J}_{\mathbb{G}}$  and even, by the same argument as Corollary 1.6, a mapping  $H_{\mathbb{G}} : \mathcal{A}/\mathcal{J} \longrightarrow G^{\mathbb{E}^+}/\mathcal{J}_{\mathbb{G}}$ .

Of course, this construction depends on the orientation  $\mathbb{E}^+$ , but Lemma 1.3 ensures that  $\pi : \tilde{G} \longrightarrow G$  induces a covering  $\pi : \tilde{G}^{(\mathbb{E})} \longrightarrow G^{\mathbb{E}^+}$  which is consistent with the choice of orientation. The covering of  $G$  induces also a covering map  $\pi : \tilde{\mathcal{J}}_{\mathbb{G}} \longrightarrow \mathcal{J}_{\mathbb{G}}$  in such a way that, for all  $j \in \tilde{\mathcal{J}}_{\mathbb{G}}$ ,  $\tilde{g} \in \tilde{G}^{(\mathbb{E})}$ ,  $\pi(j \cdot \tilde{g}) = \pi(j)\pi(\tilde{g})$ . This equivariance property implies that  $\pi : \tilde{G}^{(\mathbb{E})} \longrightarrow G^{\mathbb{E}^+}$  maps each orbit of  $\tilde{\mathcal{J}}_{\mathbb{G}}$  onto an orbit of  $\mathcal{J}_{\mathbb{G}}$  and induces a mapping between the topological quotient spaces  $\pi : \tilde{G}^{(\mathbb{E})}/\tilde{\mathcal{J}}_{\mathbb{G}} \longrightarrow G^{\mathbb{E}^+}/\mathcal{J}_{\mathbb{G}}$ . Finally, the following diagram commutes.

$$\begin{array}{ccc} & \tilde{G}^{(\mathbb{E})}/\tilde{\mathcal{J}}_{\mathbb{G}} & \\ \tilde{H}_{\mathbb{G}} \nearrow & \downarrow \pi & \\ \mathcal{A}/\mathcal{J} & & G^{\mathbb{E}^+}/\mathcal{J}_{\mathbb{G}} \\ H_{\mathbb{G}} \searrow & & \end{array}$$

(6)

We prove in the last section that this mapping  $\pi$  is, outside a negligible singular set, a covering with fiber isomorphic to  $\Pi^{\mathbb{F}}$ .

The new discrete theory that we present in this paper is thus essentially a lift of the usual discrete theory. We are now going to show that this lift contains a geometric information which is not present at the level of the usual discrete gauge theory.

## 1.5 The obstruction class of the bundle

Let us fix a connection  $\omega \in \mathcal{A}$  and an element  $s$  of  $\mathcal{S}(\mathbb{G}, P)$ .

The principal  $G$ -bundle  $P$  is classified up to isomorphism by a cohomology class of  $H^2(M; \pi_1(G))$  (see [16]). Using the orientation of  $M$ , we identify this class with an element of  $\pi_1(M)$  which in turn we identify with an element of  $\Pi$ , which we denote by  $\mathfrak{o}(P)$ . It turns out that, once a connection  $\omega$  on  $P$  is chosen,  $\mathfrak{o}(P)$  can be extracted from  $\tilde{H}_{\mathbb{G}}(\omega) = (\tilde{g}(e))_{e \in \mathbb{E}}$  in a very simple way.

**Lemma 1.7** *Let  $\mathbb{E}^+$  be an orientation of  $\mathbb{G}$ . The following equality holds:*

$$\mathfrak{o}(P) = \prod_{e \in \mathbb{E}^+} \tilde{g}(e)\tilde{g}(e^{-1}). \quad (7)$$



**Proof** – The element  $\mathfrak{o}(P)$  of  $\pi_1(G)$  can be defined as follows (see [16] and [13] in the case  $G = U(1)$ ). Choose an element  $s$  of  $\mathcal{S}(\mathbb{G}, P)$ . Let  $e$  be an edge. There exists a unique smooth mapping  $\delta_e : e \rightarrow G$  such that the equality  $s_{L(e^{-1})} = s_{L(e)}\delta$  holds identically over  $e$ . Actually,  $\delta_e$  maps  $e$  to a loop in  $G$  based at the unit element and whose homotopy class is denoted by  $[\delta_e]$ . Then,  $\prod_{e \in \mathbb{E}} [\delta_e]$  does not depend on the choice of  $s$  and it is denoted by  $\mathfrak{o}(P)$ .

On the other hand, it is easy to check that, via the identification  $\pi_1(G) \simeq \Pi$ , the homotopy class  $[\delta_e]$  corresponds to  $\tilde{g}(e)\tilde{g}(e^{-1})$ . The result follows.  $\square$

We denote by  $\mathfrak{o} : \tilde{G}^{(\mathbb{E})} \rightarrow \Pi$  the mapping defined by the right hand side of (7), which in fact does not depend on  $\mathbb{E}^+$ .

**Lemma 1.8** *The mapping  $\mathfrak{o} : \tilde{G}^{(\mathbb{E})} \rightarrow \Pi$  is invariant under the action of  $\tilde{\mathcal{J}}_{\mathbb{G}}$ .*

**Proof** – Let  $j = ((j_v)_{v \in \mathbb{V}}, (z_e)_{e \in \mathbb{E}})$  be an element of  $\tilde{\mathcal{J}}_{\mathbb{G}}$  and  $\tilde{g}$  an element of  $\tilde{G}^{(\mathbb{E})}$ . Let  $\mathbb{E}^+$  be an orientation of  $\mathbb{G}$ . One has

$$\begin{aligned} \mathfrak{o}(j \cdot \tilde{g}) &= \prod_{e \in \mathbb{E}^+} \text{Ad}(j_{\bar{e}}^{-1})(\tilde{g}(e)\tilde{g}(e^{-1}))_{z_e z_{e^{-1}}} \\ &= \mathfrak{o}(\tilde{g}) \prod_{e \in \mathbb{E}} z_e \\ &= \mathfrak{o}(\tilde{g}). \end{aligned}$$

The result is proved.  $\square$

According to this lemma, we may regard  $\mathfrak{o}$  as a function on  $\tilde{G}^{(\mathbb{E})}/\tilde{\mathcal{J}}_{\mathbb{G}}$ .

**Corollary 1.9** *Let  $P$  be a principal  $G$ -bundle over  $M$  and  $\omega$  a connection on  $P$ . Let  $\mathbb{G}$  be a simple graph on  $M$ . Then*

$$\mathfrak{o}(P) = \mathfrak{o}(\tilde{H}_{\mathbb{G}}(\omega)).$$

We finish by explaining what amount of information about a pair  $(P, \omega)$  is encoded in the class  $\tilde{H}_{\mathbb{G}}(\omega) \in \tilde{G}^{(\mathbb{E})}/\tilde{\mathcal{J}}_{\mathbb{G}}$ .

**Proposition 1.10** *Let  $P$  and  $Q$  be two principal  $G$ -bundles over  $M$  equipped respectively with two connections  $\omega$  and  $\eta$ . Let  $\mathbb{G}$  be a simple graph on  $M$ . Let  $i : \cup_{e \in \mathbb{E}} e \hookrightarrow M$  denote the inclusion map. The following propositions are equivalent:*

(i)  $\tilde{H}_{\mathbb{G}}(\omega) = \tilde{H}_{\mathbb{G}}(\eta)$ .

(ii) *There exists a bundle isomorphism  $\varphi : P \rightarrow Q$  such that  $i^*\varphi^*\eta = i^*\omega$ .*

**Proof** – (i)  $\Rightarrow$  (ii) By Corollary 1.9,  $\mathfrak{o}(P) = \mathfrak{o}(Q)$ , so that  $P$  and  $Q$  are isomorphic. We may thus assume that  $P = Q$ . Now,  $\omega$  and  $\eta$  determine the same class in the usual configuration space  $G^{\mathbb{E}^+}/G^{\mathbb{V}}$  and the result follows by [8, Lemma 1.11].

(ii)  $\Rightarrow$  (i) Let us consider  $\tilde{H}_{\mathbb{G}}$  as defined by (2). Let  $s$  be an element of  $\mathcal{S}(\mathbb{G}, P)$ . Then the assumption implies  $\tilde{H}_{\mathbb{G}}(s, \omega) = \tilde{H}_{\mathbb{G}}(s, \varphi^*\eta)$ . Now,  $\varphi(s) \in \mathcal{S}(\mathbb{G}, Q)$  and  $\tilde{H}_{\mathbb{G}}(s, \varphi^*\eta) = \tilde{H}_{\mathbb{G}}(\varphi(s), \eta)$ . Hence,  $\tilde{H}_{\mathbb{G}}(\omega) = \tilde{H}_{\mathbb{G}}(\eta)$ .  $\square$

## 2 The discrete measures

In the first section, we have built a singular covering of the usual configuration space of discrete Yang-Mills theory. We have explained how the covering space  $\tilde{G}^{(\mathbb{E})}$  is partitioned into several sectors, each of which corresponds to an isomorphism class of principal  $G$ -bundles:  $\tilde{G}^{(\mathbb{E})} = \bigsqcup_{z \in \Pi} \mathfrak{o}^{-1}(z)$ .

In order to associate a probability measure on  $G^{\mathbb{E}^+}$  to each element  $z$  of  $\Pi$ , we proceed as follows. First, we construct a lift on  $\tilde{G}^{(\mathbb{E})}$  of the usual discrete Yang-Mills measure on  $G^{\mathbb{E}^+}$ . Then, for each  $z$ , we renormalize and project on  $G^{\mathbb{E}^+}$  the restriction of this lift to  $\mathfrak{o}^{-1}(z)$ .

However, this strategy breaks down when  $\tilde{G} \rightarrow G$  is not a finite covering, that is, when  $G$  is not semi-simple. In fact, in this case, the lift of the discrete measure does not exist as a probability measure, but as an infinite measure on  $\tilde{G}^{(\mathbb{E})}$  and it is not possible to renormalize properly its restriction to a given sector  $\mathfrak{o}^{-1}(z)$ . Thus, we start by studying the semi-simple case.

From now on, we do not assume anymore that the graphs we consider on  $M$  are simple.

### 2.1 The case of a semi-simple structure group

Assume that  $G$  is semi-simple and endow it with its unit-volume bi-invariant Riemannian metric  $\gamma$ . Let  $dg$  be the corresponding Riemannian volume. Let also  $p : \mathbb{R}_+^* \times G \rightarrow \mathbb{R}_+^*$  be the fundamental solution on  $G$  of the heat equation  $\frac{1}{2}\Delta - \partial_t$ .

Let  $\mathbb{G} = (\mathbb{V}, \mathbb{E}, \mathbb{F})$  be a graph on  $M$ . Choose an orientation  $\mathbb{E}^+ = \{e_1, \dots, e_r\}$  of  $\mathbb{G}$ . Let  $F$  be a face of this graph and  $e_{i_1}^{\varepsilon_1} \dots e_{i_n}^{\varepsilon_n}$  a loop which represents  $\partial F$ , where  $\varepsilon_1, \dots, \varepsilon_n = \pm 1$ .

Given an arbitrary group  $X$ , we define the mapping  $h_{\partial F}^X : X^{\mathbb{E}^+} \rightarrow X/\text{Ad}$  by setting  $h_{\partial F}^X(x_1, \dots, x_r) = [x_{i_n}^{\varepsilon_n} \dots x_{i_1}^{\varepsilon_1}]$ , where  $\text{Ad}$  is the adjoint action of  $X$  on itself and  $[x]$  denotes the conjugacy class of  $x$ .

We define also, for each path  $c = e_{i_1}^{\varepsilon_1} \dots e_{i_n}^{\varepsilon_n}$  in  $\mathbb{E}^*$ , the discrete holonomy along  $c$  as the mapping  $h_c : X^{\mathbb{E}^+} \rightarrow X$  defined by  $h_c(x_1, \dots, x_r) = x_{i_n}^{\varepsilon_n} \dots x_{i_1}^{\varepsilon_1}$ .

We shall soon need another mapping, namely  $\tilde{h}_{\partial F} : \tilde{G}^{(\mathbb{E})} \rightarrow \tilde{G}/\text{Ad}$  which is defined by setting  $\tilde{h}_{\partial F}(\tilde{g}) = [\tilde{g}_{e_{i_n}^{\varepsilon_n}} \dots \tilde{g}_{e_{i_1}^{\varepsilon_1}}]$ . With this definition,  $\tilde{h}_{\partial F}(\tilde{H}_{\mathbb{G}}(s, \omega))$  is the conjugacy class of the holonomy of  $s_F^* \omega$  along  $\partial F$ .

Recall that the heat kernel is constant on conjugacy classes. The usual discrete Yang-Mills measure on  $G^{\mathbb{E}^+}$  at temperature  $T > 0$  is the Borel probability measure  $P_T^{\mathbb{G}}$  defined by

$$dP_T^{\mathbb{G}} = \frac{1}{Z_T^{\mathbb{G}}} \prod_{F \in \mathbb{F}} p_{T\sigma(F)} \circ h_{\partial F}^G dg^{\mathbb{E}^+}, \quad (8)$$

where  $Z_T^{\mathbb{G}}$  is the normalization constant (see for example [8]).

Endow  $\tilde{G}$  with the Riemannian metric  $\pi^* \gamma$ , where  $\pi : \tilde{G} \rightarrow G$  is the covering map. Let  $d\tilde{g}$  be the corresponding Riemannian volume. Observe that  $\int_{\tilde{G}} d\tilde{g} = |\Pi|$ .

The configuration space  $\tilde{G}^{(\mathbb{E})}$  is a closed subgroup of the compact Lie group  $\tilde{G}^{\mathbb{E}}$ . It is not connected unless  $G$  is simply connected, in which case the present work is pointless. In any case, it carries bi-invariant measures with finite total mass. Let  $\lambda$  denote the one such that  $\lambda(\tilde{G}^{(\mathbb{E})}) = |\Pi^{\mathbb{E}}|$ . Let us identify  $\tilde{G}^{(\mathbb{E})}$  with a subgroup of  $\tilde{G}^{2r}$ , according to  $\tilde{g} = (\tilde{g}_{e_1}, \tilde{g}_{e_1^{-1}}, \dots, \tilde{g}_{e_r}, \tilde{g}_{e_r^{-1}})$ .

**Lemma 2.1** *Let  $f$  be a continuous function on  $\tilde{G}^{(\mathbb{E})}$ . Then*

$$\int_{\tilde{G}^{(\mathbb{E})}} f \, d\lambda = \sum_{z_1, \dots, z_r \in \Pi} \int_{\tilde{G}^r} f(\tilde{g}_1, \tilde{g}_1^{-1}z_1, \dots, \tilde{g}_r, \tilde{g}_r^{-1}z_r) \, d\tilde{g}_1 \dots d\tilde{g}_r. \quad (9)$$

**Proof** – The right-hand side of the expression above defines a bi-invariant measure with total mass  $|\Pi|^{2r} = |\Pi^{\mathbb{E}}|$  on  $\tilde{G}^{(\mathbb{E})}$ , which then must be  $\lambda$ .  $\square$

Let  $\tilde{p} : \mathbb{R}_+^* \times \tilde{G} \rightarrow \mathbb{R}_+^*$  be the fundamental solution of the heat equation on  $\tilde{G}$ . It is related to the heat kernel on  $G$  as follows.

**Lemma 2.2** *For all  $t > 0$  and all  $\tilde{g} \in \tilde{G}$ , the following relation holds:*

$$\sum_{z \in \Pi} \tilde{p}_t(\tilde{g}z) = p_t(\pi(\tilde{g})).$$

**Proof** – This follows immediately from the fact that both  $G$  and  $\tilde{G}$  are endowed with their Riemannian volumes and  $\pi : \tilde{G} \rightarrow G$  is a local isometry.  $\square$

Most of the following proposition consists in definitions.

**Proposition 2.3** *Let  $T > 0$  be a positive number. Define the discrete Yang-Mills measure on  $\tilde{G}^{(\mathbb{E})}$  at temperature  $T$  as the Borel probability measure  $\tilde{P}_T^{\mathbb{G}}$  given by*

$$\frac{d\tilde{P}_T^{\mathbb{G}}}{d\lambda} = \frac{1}{\tilde{Z}_T^{\mathbb{G}}} \prod_{F \in \mathbb{F}} \tilde{p}_{T\sigma(F)} \circ \tilde{h}_{\partial F}. \quad (10)$$

Choose  $T > 0$  and  $z \in \Pi$ . One has  $\tilde{P}_T^{\mathbb{G}}(\{\tilde{g} \in \tilde{G}^{(\mathbb{E})} \mid \mathfrak{o}(\tilde{g}) = z\}) > 0$ . Define the probability  $\tilde{P}_{T,z}^{\mathbb{G}}$  on  $\tilde{G}^{(\mathbb{E})}$  by

$$\tilde{P}_{T,z}^{\mathbb{G}} = \frac{1}{\tilde{Z}_{T,z}^{\mathbb{G}}} \mathbb{1}_{\mathfrak{o}^{-1}(z)} \tilde{Z}_T^{\mathbb{G}} \tilde{P}_T^{\mathbb{G}}. \quad (11)$$

The following relation holds:

$$\sum_{z \in \Pi} \tilde{Z}_{T,z}^{\mathbb{G}} \tilde{P}_{T,z}^{\mathbb{G}} = \tilde{Z}_T^{\mathbb{G}} \tilde{P}_T^{\mathbb{G}}. \quad (12)$$

Let  $z$  be an element of  $\Pi \simeq \pi_1(G)$ . The discrete Yang-Mills measure on  $\mathbb{G}$  at temperature  $T > 0$  and for a bundle of type  $z$  is the Borel probability measure on  $G^{\mathbb{E}^+}$  defined by

$$P_{T,z}^{\mathbb{G}} = \pi_* \tilde{P}_{T,z}^{\mathbb{G}}, \quad (13)$$

where  $\pi : \tilde{G}^{(\mathbb{E})} \rightarrow G^{\mathbb{E}^+}$  is the natural projection. For all  $T > 0$  and  $z \in \Pi$ , the measure  $\tilde{P}_{T,z}^{\mathbb{G}}$  (resp.  $P_{T,z}^{\mathbb{G}}$ ) is invariant under the action of  $\tilde{\mathcal{J}}_{\mathbb{G}}$  (resp.  $\mathcal{J}_{\mathbb{G}}$ ).

**Proof** – Let us prove that  $\tilde{P}_T^{\mathbb{G}}(\{\tilde{g} \in \tilde{G}^{(\mathbb{E})} \mid \mathfrak{o}(\tilde{g}) = z\}) > 0$ . The density of  $\tilde{P}_T^{\mathbb{G}}$  is a smooth positive function on  $\tilde{G}^{(\mathbb{E})}$ . It is thus enough to prove that  $\lambda(\mathfrak{o}^{-1}(z))$  is positive. This follows from the fact that  $\mathfrak{o}^{-1}(z)$  is the non-empty union of some of the finitely many connected components of  $\tilde{G}^{(\mathbb{E})}$ .

That the decomposition (12) is true is a straightforward consequence of the definitions. Observe in fact that it is a decomposition into mutually singular parts.

There remains to prove the claimed gauge-invariance of the measures. Choose  $T > 0$  and  $z \in \Pi$ . For each face  $F$  of the graph, one checks easily that the action of  $\tilde{\mathcal{J}}_{\mathbb{G}}$  on  $\tilde{G}^{(\mathbb{E})}$  leaves the mapping  $\tilde{h}_{\partial F} : \tilde{G}^{(\mathbb{E})} \rightarrow \tilde{G}/\text{Ad}$  invariant. According to Lemma 1.8, it preserves also the mapping  $\mathfrak{o} : \tilde{G}^{(\mathbb{E})} \rightarrow \Pi$ . Hence, the measures  $\tilde{P}_T^{\mathbb{G}}$  and  $\tilde{P}_{T,z}^{\mathbb{G}}$  are left invariant by  $\tilde{\mathcal{J}}_{\mathbb{G}}$ . One deduces the corresponding assertions for  $P_T^{\mathbb{G}}$  and  $P_{T,z}^{\mathbb{G}}$  by using the equivariance properties of  $\pi : \tilde{G}^{(\mathbb{E})} \rightarrow G^{\mathbb{E}^+}$ .  $\square$

We shall now give an expression of the measures  $P_{T,z}^{\mathbb{G}}$  at the level of  $G^{\mathbb{E}^+}$ .

**Proposition 2.4** *Let  $g = (g_1, \dots, g_r)$  be an element of  $G^{\mathbb{E}^+}$ . Let  $z$  be an element of  $\Pi$ . Let  $\tilde{g} = (\tilde{g}_1, \dots, \tilde{g}_r)$  be a lift of  $g$  to  $\tilde{G}^{\mathbb{E}^+}$ . Then the number*

$$\sum_{\substack{(z_F)_{F \in \Pi^{\mathbb{F}}} \\ \prod z_F = z}} \prod_{F \in \mathbb{F}} \tilde{p}_{T\sigma(F)}(h_{\partial F}^{\tilde{G}}(\tilde{g})z_F) \quad (14)$$

does not depend on the choice of  $\tilde{g}$ . We denote it by  $D_{T,z}^{\mathbb{G}}(g)$ .

Moreover, the measure  $P_{T,z}^{\mathbb{G}}$  satisfies

$$dP_{T,z}^{\mathbb{G}} = \frac{1}{Z_{T,z}^{\mathbb{G}}} D_{T,z}^{\mathbb{G}} dg^{\otimes \mathbb{E}^+}. \quad (15)$$

Finally, one has  $\tilde{Z}_{T,z}^{\mathbb{G}} = |\Pi|^{|\mathbb{E}| - |\mathbb{F}|} Z_{T,z}^{\mathbb{G}}$ .

**Proof** – Let  $w$  be an element of  $\Pi$ . Let us look at the effect of replacing for example  $\tilde{g}_1$  by  $\tilde{g}_1 w$  in (14). Set  $F_1 = L(e_1)$  and  $F_2 = L(e_1^{-1})$ . Then this is equivalent to replacing  $(z_{F_1}, z_{F_2})$  by  $(z_{F_1} w, z_{F_2} w^{-1})$  and it does not change the value of the sum. This proves the first assertion.

To prove the second one, consider a continuous function  $f$  on  $G^{\mathbb{E}^+}$ . Then, by definition of  $P_{T,z}^{\mathbb{G}}$ ,

$$\int_{G^{\mathbb{E}^+}} f(g) dP_{T,z}^{\mathbb{G}} = \frac{1}{Z_T^{\mathbb{G}}} \sum_{\substack{z_1, \dots, z_r \in \Pi \\ z_1 \dots z_r = z}} \int_{\tilde{G}^r} f(\pi(\tilde{g})) \prod_{F \in \mathbb{F}} \tilde{p}_{T\sigma(F)} \circ \tilde{h}_{\partial F}(\tilde{g}_1, \tilde{g}_1^{-1} z_1, \dots, \tilde{g}_r, \tilde{g}_r^{-1} z_r) d\tilde{g}. \quad (16)$$

Now observe that, for each face  $F$ ,

$$\tilde{h}_{\partial F}(\tilde{g}_1, \tilde{g}_1^{-1} z_1, \dots, \tilde{g}_r, \tilde{g}_r^{-1} z_r) = h_{\partial F}^{\tilde{G}}(\tilde{g}_1, \dots, \tilde{g}_r) \prod_{i \in [1, r] : L(e_i^{-1}) = F} z_i.$$

Set  $z_F = \prod_{i \in [1, r] : L(e_i^{-1}) = F} z_i$ . For two distinct faces  $F$  and  $F'$ ,  $z_F$  and  $z_{F'}$  are products of disjoint collections of  $z_i$ 's. Actually, the sets  $\{i \in [1, r] : L(e_i^{-1}) = F\}$  form a partition of  $[1, r]$ . Since the image of the uniform measure on the product of a finite number of copies of  $\Pi$  by multiplication of the factors is the uniform measure on  $\Pi$ , the right hand side of (16) equals

$$\begin{aligned} & \frac{|\Pi^{\mathbb{E}^+}|}{Z_T^{\mathbb{G}} |\Pi^{\mathbb{F}}|} \int_{\tilde{G}^r} f(\pi(\tilde{g})) \sum_{\substack{(z_F) \in \Pi^{\mathbb{F}} \\ \prod z_F = z}} \prod_{F \in \mathbb{F}} \tilde{p}_{T\sigma(F)}(h_{\partial F}^{\tilde{G}}(\tilde{g}_1, \dots, \tilde{g}_r)z_F) d\tilde{g} \\ &= \frac{|\Pi|^{|\mathbb{E}| - |\mathbb{F}|}}{Z_T^{\mathbb{G}}} \int_{G^r} f(g) D_{T,z}^{\mathbb{G}}(g) dg. \end{aligned}$$

The second assertion is proved, as well as the third.  $\square$

**Remark 2.5** Let  $f$  be a continuous function on  $G^{\mathbb{E}^+}$ . The integral of  $f$  with respect to  $P_{T,z}^{\mathbb{G}}$  can be written as follows.

$$\begin{aligned} \int_{G^{\mathbb{E}^+}} f dP_{T,z}^{\mathbb{G}} &= \frac{1}{Z_{T,z}^{\mathbb{G}}} \int_{G^{\mathbb{E}^+}} f(g) \sum_{\substack{(z_F)_F \in \Pi^{\mathbb{F}} \\ \prod z_F = z}} \prod_{F \in \mathbb{F}} \tilde{p}_{T\sigma(F)}(h_{\partial F}^{\tilde{\mathbb{G}}}(\tilde{g})z_F) dg \\ &= \frac{1}{Z_{T,z}^{\mathbb{G}}} \frac{1}{|\Pi^{\mathbb{E}^+}|} \int_{\tilde{G}^{\mathbb{E}^+}} f(\pi(\tilde{g})) \sum_{\substack{(z_F)_F \in \Pi^{\mathbb{F}} \\ \prod z_F = z}} \prod_{F \in \mathbb{F}} \tilde{p}_{T\sigma(F)}(h_{\partial F}^{\tilde{\mathbb{G}}}(\tilde{g})z_F) d\tilde{g} \\ &= \frac{1}{Z_{T,z}^{\mathbb{G}}} \frac{1}{|\Pi^{\mathbb{E}^+}|} \sum_{\substack{(z_F)_F \in \Pi^{\mathbb{F}} \\ \prod z_F = z}} \int_{\tilde{G}^{\mathbb{E}^+}} f(\pi(\tilde{g})) \prod_{F \in \mathbb{F}} \tilde{p}_{T\sigma(F)}(h_{\partial F}^{\tilde{\mathbb{G}}}(\tilde{g})z_F) d\tilde{g}. \end{aligned}$$

Thanks to the invariance by translation of the Haar measure, all the terms in the last sum are equal. This statement is also the content of [15, Lemma 7.5]. Hence, in particular, if we choose a face  $F_*$  in  $\mathbb{F}$ , then

$$\int_{G^{\mathbb{E}^+}} f dP_{T,z}^{\mathbb{G}} = \frac{1}{Z_{T,z}^{\mathbb{G}}} \frac{|\Pi|^{\mathbb{F}} - 1}{|\Pi^{\mathbb{E}^+}|} \int_{\tilde{G}^{\mathbb{E}^+}} f(\pi(\tilde{g})) \tilde{p}_{T\sigma(F_*)}(h_{\partial F_*}^{\tilde{\mathbb{G}}}(\tilde{g})z) \prod_{F \in \mathbb{F} - \{F_*\}} \tilde{p}_{T\sigma(F)}(h_{\partial F}^{\tilde{\mathbb{G}}}(\tilde{g})z_F) d\tilde{g}.$$

Comparing this expression with [15, Theorem 8.4] shows that our definition of the discrete Yang-Mills measure associated to a specific isomorphism class of  $G$ -bundles is consistent with that previously given by A. Sengupta.

**Corollary 2.6** The covering map  $\pi : \tilde{G}^{\mathbb{E}} \longrightarrow G^{\mathbb{E}^+}$  satisfies  $\pi_* \tilde{P}_T^{\mathbb{G}} = P_T^{\mathbb{G}}$ . Moreover,  $\tilde{Z}_T^{\mathbb{G}} = |\Pi|^{|\mathbb{E}| - |\mathbb{F}|} Z_T^{\mathbb{G}}$  and  $\sum_{z \in \Pi} Z_{T,z}^{\mathbb{G}} P_{T,z}^{\mathbb{G}} = Z_T^{\mathbb{G}} P_T^{\mathbb{G}}$ .

**Proof** – By definition of  $D_{T,z}^{\mathbb{G}}$  and by Lemma 2.2,

$$\sum_{z \in \Pi} D_{T,z}^{\mathbb{G}}(g) = \prod_{F \in \mathbb{F}} p_{T\sigma(F)}(h_{\partial F}^{\mathbb{G}}(g)).$$

Hence, by Proposition 2.4,  $\sum_{z \in \Pi} Z_{T,z}^{\mathbb{G}} P_{T,z}^{\mathbb{G}} = Z_T^{\mathbb{G}} P_T^{\mathbb{G}}$ . On the other hand,

$$\sum_{z \in \Pi} Z_{T,z}^{\mathbb{G}} P_{T,z}^{\mathbb{G}} = \frac{1}{|\Pi|^{|\mathbb{E}| - |\mathbb{F}|}} \pi_* \left[ \sum_{z \in \Pi} \tilde{Z}_{T,z}^{\mathbb{G}} \tilde{P}_{T,z}^{\mathbb{G}} \right] = \frac{1}{|\Pi|^{|\mathbb{E}| - |\mathbb{F}|}} \pi_* (\tilde{Z}_T^{\mathbb{G}} \tilde{P}_T^{\mathbb{G}}).$$

The result follows.  $\square$

## 2.2 The general case

Let us drop the assumption that  $G$  is semi-simple. Then, except in trivial cases like  $\mathbb{E} = \emptyset$ , definitions (10) and (11) do not make sense anymore, because they involve infinite normalization constants. Fortunately, Proposition 2.4 is still meaningful, as the following result shows.

**Lemma 2.7** *Let  $T > 0$  and  $z \in \Pi$ . The definition of  $D_{T,z}^{\mathbb{G}}$  (see Proposition 2.4) makes sense on any compact connected Lie group. If  $G$  is such a group, the function thus defined is bounded on  $G$ .*

**Proof** – The trouble is that  $D_{T,z}^{\mathbb{G}}$  might take infinite values. However, for all  $g \in G^{\mathbb{E}^+}$ , and by Lemma 2.2, the sum of positive functions  $\sum_{z \in \Pi} D_{T,z}^{\mathbb{G}}$  is equal to  $\prod_{F \in \mathbb{F}} p_{T\sigma(F)} \circ h_{\partial F}^G$ , which is the density of the usual discrete Yang-Mills measure and is finite and even bounded on  $G$ . The result follows.  $\square$

**Definition 2.8** *Let  $\mathbb{G}$  be a graph on  $M$ . Let  $z$  be an element of  $\Pi$ . Let  $T$  be a positive real number. Then the discrete Yang-Mills measure on  $\mathbb{G}$  associated with the isomorphism class of bundles corresponding to  $z$  and at temperature  $T$  is the Borel probability measure  $P_{T,z}^{\mathbb{G}}$  on  $G^{\mathbb{E}^+}$  defined by*

$$dP_{T,z}^{\mathbb{G}} = \frac{1}{Z_{T,z}^{\mathbb{G}}} D_{T,z}^{\mathbb{G}} dg^{\otimes \mathbb{E}^+}.$$

The following lemma follows immediately from Lemma 2.2.

**Lemma 2.9** *The relation  $\sum_{z \in \Pi} Z_{T,z}^{\mathbb{G}} P_{T,z}^{\mathbb{G}} = Z_T^{\mathbb{G}} P_T^{\mathbb{G}}$  holds.*

At this point, we have written the usual discrete Yang-Mills measure as a convex combination of probability measures, one for each isomorphism class of principal  $G$ -bundle over  $M$ . In the case of a semi-simple structure group and a simple graph, this decomposition has been given a strong geometrical motivation. In the next subsection, we check that our construction is consistent with some observations made in [8, Chapter 3] in the case of an Abelian structure group.

## 2.3 The case of an Abelian structure group

Let us compare our definition of the discrete Yang-Mills measures with the study of the Abelian case presented in [8, Section 1.9]. As explained in [8, Lemma 1.34], when  $G$  is Abelian, the non-trivial information about any gauge-invariant measure on  $G^{\mathbb{E}^+}$  is contained in the law of the discrete holonomies along the boundaries of the faces. This is why, in what follows, we focus on this law.

Assume that  $G = SO(2)^m$ . Let  $e : \mathbb{R}^m \rightarrow SO(2)^m$  be defined by  $e(x_1, \dots, x_m) = (e^{2i\pi x_1}, \dots, e^{2i\pi x_m})$ . Assume that  $\mathbb{F} = \{F_1, \dots, F_n\}$ . For each  $i = 1 \dots n$ , set  $\sigma_i = \sigma(F_i)$ . Set also  $\sigma_M = \sigma(M)$ . Let  $Y_1, \dots, Y_n$  be independent  $\mathbb{R}^m$ -valued Gaussian random variables with  $Y_i \sim \mathcal{N}(0, T\sigma_i I_m)$  for each  $i$ , where  $I_m$  is the identity matrix. Let  $S = Y_1 + \dots + Y_n$  be their sum. For each  $i = 1 \dots n$ , set  $X_i = Y_i - \frac{\sigma_i}{\sigma_M} S$ .

Let  $Z$  be a  $\mathbb{Z}^m$ -valued random variable, independent of  $Y_1, \dots, Y_n$ , such that, for all  $z \in \mathbb{Z}^m$ ,

$$\mathbb{P}(Z = z) = C \exp -\frac{|z_1|^2 + \dots + |z_m|^2}{2T\sigma_M},$$

where  $C$  is the correct normalization constant.

**Theorem 2.10** *For all  $T > 0$ , all  $z \in \mathbb{Z}^m$ , all  $f$  continuous on  $G^{\mathbb{E}^+}$ , one has*

$$\int_{G^{\mathbb{E}^+}} f(h_{\partial F_1}, \dots, h_{\partial F_n}) dP_{T,z}^G = \mathbb{E} \left[ f \left( e(X_1 + \frac{\sigma_1}{\sigma_M} z), \dots, e(X_n + \frac{\sigma_1}{\sigma_M} z) \right) \right]. \quad (17)$$

Moreover,

$$\int_{G^{\mathbb{E}^+}} f(h_{\partial F_1}, \dots, h_{\partial F_n}) dP_T^G = \mathbb{E} \left[ f \left( e(X_1 + \frac{\sigma_1}{\sigma_M} Z), \dots, e(X_n + \frac{\sigma_1}{\sigma_M} Z) \right) \right]. \quad (18)$$

The second relation is proved in the case  $T = m = 1$  in [8, Proposition 1.38]. We observed there that  $Z$  plays the role of a total curvature, that is, of the obstruction class of the bundle, and that replacing  $Z$  by a deterministic element of  $\mathbb{Z}^m$  would be equivalent to selecting an isomorphism class of  $SO(2)^m$ -bundles. We prove now that our definition of  $P_{T,z}^G$  is consistent with this observation.

**Proof** – The proof is very similar to that of [8, Prop. 1.38]. For the convenience of the reader and because [8] deals with a more restrictive situation, we present a detailed sketch of proof.

By using the definition of  $P_{T,z}^G$  and the fact that, under the Haar measure on  $G^{\mathbb{E}^+}$ , the holonomies along the boundaries of all faces except one are independent and uniformly distributed on  $G$ , one finds that the left hand side of (17) is equal to

$$\left( (2\pi)^{n-1} \frac{\sigma_1 \dots \sigma_n}{\sigma_M} \right)^{-\frac{m}{2}} e^{\frac{|z|^2}{2T\sigma_M}} \int_{([0,1]^m)^{n-1}} f(e(x_1), \dots, e(x_n)) D_L(x_1, \dots, x_{n-1}) dx_1 \dots dx_{n-1}, \quad (19)$$

with  $x_n = -x_1 - \dots - x_{n-1}$  and

$$D_L(x_1, \dots, x_{n-1}) = \sum_{z_1, \dots, z_{n-1} \in \mathbb{Z}^m} \exp -\frac{1}{2T} \left[ \sum_{i=1}^n \frac{|x_i + z_i|^2}{\sigma_i} \right], \quad (20)$$

where we have set  $z_n = z - z_1 - \dots - z_{n-1}$ .

By computing the covariance of the Gaussian vector  $(X_1, \dots, X_n)$ , one finds that the right hand side of (17) is equal to

$$\left( (2\pi)^{n-1} \frac{\sigma_1 \dots \sigma_n}{\sigma_M} \right)^{-\frac{m}{2}} \int_{([0,1]^m)^{n-1}} f(e(x_1), \dots, e(x_n)) D_R(x_1, \dots, x_{n-1}) dx_1 \dots dx_{n-1}, \quad (21)$$

with again  $x_n = -x_1 - \dots - x_{n-1}$  and

$$D_R(x_1, \dots, x_{n-1}) = \sum_{w_1, \dots, w_{n-1} \in \mathbb{Z}^m} \exp -\frac{1}{2T} \left[ \sum_{i=1}^{n-1} \frac{1}{\sigma_i} \left| x_i + w_i - \frac{\sigma_i}{\sigma_M} z \right|^2 + \frac{1}{\sigma_n} \left| \sum_{i=1}^n x_i + w_i - \frac{\sigma_i}{\sigma_M} z \right|^2 \right], \quad (22)$$

where we have set  $w_n = z - w_1 - \dots - w_{n-1}$ .

An elementary computation shows that  $\exp(\frac{|z|^2}{2T\sigma_M})D_L = D_R$  and this is most easily seen by identifying  $w_i$  and  $z_i$  for  $i = 1 \dots n-1$ .

This proves (17) and (18) follows by summing both sides over  $z \in \mathbb{Z}^m$ , with a weight  $\mathbb{P}(Z = z)$ .  $\square$

### 3 The continuous measures

#### 3.1 Invariance under subdivision

Let  $\mathbb{G}_1 = (\mathbb{V}_1, \mathbb{E}_1, \mathbb{F}_1)$  and  $\mathbb{G}_2 = (\mathbb{V}_2, \mathbb{E}_2, \mathbb{F}_2)$  be two graphs on  $M$ , not necessarily simple ones. We say that  $\mathbb{G}_2$  is finer than  $\mathbb{G}_1$  if  $\mathbb{E}_1^* \subset \mathbb{E}_2^*$ , or equivalently  $\mathbb{E}_1 \subset \mathbb{E}_2^*$ . We denote this by  $\mathbb{G}_1 \leq \mathbb{G}_2$ .

Let  $\mathbb{E}_1^+$  and  $\mathbb{E}_2^+$  be orientations of  $\mathbb{G}_1$  and  $\mathbb{G}_2$  respectively. We have two probability spaces  $(G^{\mathbb{E}_1^+}, P_{T,z}^{\mathbb{G}_1})$  and  $(G^{\mathbb{E}_2^+}, P_{T,z}^{\mathbb{G}_2})$ , and there is a natural mapping  $f_{\mathbb{G}_1\mathbb{G}_2}$  from the second one to the first one, defined by  $f_{\mathbb{G}_1\mathbb{G}_2}(g) = (h_e(g))_{e \in \mathbb{E}_1}$ .

**Proposition 3.1** *The mapping  $f_{\mathbb{G}_1\mathbb{G}_2} : G^{\mathbb{E}_2^+} \rightarrow G^{\mathbb{E}_1^+}$  is onto. Moreover, for all  $T > 0$  and  $z \in \Pi$ , it satisfies*

$$(f_{\mathbb{G}_1\mathbb{G}_2})_* P_{T,z}^{\mathbb{G}_2} = P_{T,z}^{\mathbb{G}_1}. \quad (23)$$

**Proof** – That  $f_{\mathbb{G}_1\mathbb{G}_2}$  is onto has been proved in [8, Theorem 1.22] and is anyway easy to check.

The proof of (23) is also essentially the same than that of [8, Theorem 1.22]. We give a detailed sketch of proof and give full details for the only non-trivial and new step. First, one proves that there exists a finite sequence  $\mathbb{G}_1 = \mathbb{G}'_1 \leq \dots \leq \mathbb{G}'_n = \mathbb{G}_2$  of graphs such that, for each  $k = 1 \dots n-1$ , one can deduce  $\mathbb{G}'_{k+1}$  from  $\mathbb{G}'_k$  by one of the following elementary operations:

- $V$  : Adding a vertex in the middle of an edge.
- $E_1$  : Adding a ‘loose’ edge, that is, an edge such that exactly one of its two endpoints already belong to the graph.
- $E_2$  : Adding a ‘tight’ edge, that is, joining two vertices by a new edge.

There is a probability space associated to each of them and, for each  $k$ , a mapping  $f_{\mathbb{G}'_k\mathbb{G}'_{k+1}}$  from the space associated to  $\mathbb{G}'_{k+1}$  to the one associated to  $\mathbb{G}'_k$ . In fact,  $f_{\mathbb{G}_1\mathbb{G}_2} = f_{\mathbb{G}'_1\mathbb{G}'_2} \circ \dots \circ f_{\mathbb{G}'_{n-1}\mathbb{G}'_n}$  and it is sufficient to prove the result when  $\mathbb{G}_2$  can be deduced from  $\mathbb{G}_1$  by one of the elementary operations described above.

In the cases of operations  $V$  and  $E_1$ , the result follows immediately from the basic properties of the Haar measure. The proof in the case of operation  $E_2$  involves the properties of the heat kernel.

Assume that  $\mathbb{E}_1^+ = \{e_1, \dots, e_{r-1}\}$  and  $\mathbb{E}_2^+ = \{e_1, \dots, e_r\}$ , with  $\{\underline{e}_r, \overline{e}_r\} \subset \mathbb{V}_1$ . We claim that, for all  $g_1, \dots, g_{r-1} \in G$ ,

$$\int_G D_{T,z}^{\mathbb{G}_2}(g_1, \dots, g_r) dg_r = D_{T,z}^{\mathbb{G}_1}(g_1, \dots, g_{r-1}). \quad (24)$$

The edge  $e_r$  cuts a face of  $\mathbb{F}_1$  into two faces. Set  $\mathbb{F}_1 = \{F_1, \dots, F_n, F\}$  and  $\mathbb{F}_2 = \{F_1, \dots, F_n, F', F''\}$ , with  $F' \cap F'' = e_r$  and  $F' \cup F'' = F$ . Assume that  $L(e_r) = F'$  and  $L(e_r^{-1}) = F''$ . Then there



exists two paths  $c'$  and  $c''$  in  $\mathbb{E}_1^*$  such that  $e_r c'$  is the boundary of  $F'$ ,  $c'' e_r^{-1}$  that of  $F''$  and  $c' c''$  that of  $F$ . Let us fix  $\tilde{g} = (\tilde{g}_1, \dots, \tilde{g}_{r-1})$  some arbitrary lift of  $(g_1, \dots, g_{r-1})$ . Then the left hand side of (24) is equal to

$$\begin{aligned}
& \sum_{z_1, \dots, z_n \in \Pi} \prod_{i=1}^n \tilde{p}_{T\sigma(F_i)}(h_{\partial F_i}^{\tilde{G}}(\tilde{g})z_i) \int_G \sum_{\pi(\tilde{x})=x} \tilde{p}_{T\sigma(F')} (h_{c'}^{\tilde{G}}(\tilde{g})\tilde{x}) \tilde{p}_{T\sigma(F'')}(\tilde{x}^{-1} h_{c''}^{\tilde{G}}(\tilde{g})z) \prod_{i=1}^n z_i^{-1} dx \\
&= \sum_{z_1, \dots, z_n \in \Pi} \prod_{i=1}^n \tilde{p}_{T\sigma(F_i)}(h_{\partial F_i}^{\tilde{G}}(\tilde{g})z_i) \int_{\tilde{G}} \tilde{p}_{T\sigma(F')} (h_{c'}^{\tilde{G}}(\tilde{g})\tilde{x}) \tilde{p}_{T\sigma(F'')}(\tilde{x}^{-1} h_{c''}^{\tilde{G}}(\tilde{g})z) \prod_{i=1}^n z_i^{-1} d\tilde{x} \\
&= \sum_{z_1, \dots, z_n \in \Pi} \prod_{i=1}^n \tilde{p}_{T\sigma(F_i)}(h_{\partial F_i}^{\tilde{G}}(\tilde{g})z_i) \tilde{p}_{T\sigma(F)}(h_{c'c''}^{\tilde{G}}(\tilde{g})z) \prod_{i=1}^n z_i^{-1} \\
&= D_{T,z}^{\mathbb{G}_1}(g_1, \dots, g_{r-1}).
\end{aligned}$$

We have used the convolution property of the heat kernel between the second and the third line. This finishes the proof in the case of a transformation  $E_2$ .  $\square$

### 3.2 Random holonomy along piecewise geodesic paths

Recall that  $M$  is endowed with a Riemannian metric. Let  $\Pi M$  denote the set of piecewise geodesic paths on  $M$ , so that  $\Pi M \subset PM$ . Let  $\mathcal{G}$  denote the set of graphs with geodesic edges.

Recall [8, Section 2.3] that  $(\mathcal{G}, \leq)$  is a directed set, that is, a partially ordered set such that any two elements admit an upper bound. This is a consequence of the rigidity of geodesics and it is the property which makes  $\mathcal{G}$  an interesting set for us. Recall that, if  $J$  is a subset of  $PM$  stable by concatenation,  $f : J \rightarrow G$  is said to be multiplicative if, whenever  $c_1$  and  $c_2$  belong to  $J$  and satisfy  $\overline{c_1} = \underline{c_2}$ , one has  $f(c_1 c_2) = f(c_2) f(c_1)$ .

The projective limit of the system  $((G^{\mathbb{E}^+})_{\mathbb{G} \in \mathcal{G}}, (f_{\mathbb{G}\mathbb{G}'}))_{\mathbb{G} \leq \mathbb{G}'}$  is canonically isomorphic to the set  $\mathcal{M}(\Pi M, G)$  of multiplicative functions from  $\Pi M$  to  $G$ , with projection mappings  $f_{\mathbb{G}} : \mathcal{M}(\Pi M, G) \rightarrow G^{\mathbb{E}^+}$  given by restriction.

According to Proposition 3.1 and general results on projective limits of measure spaces (see [14]), it is possible to take the projective limit of the compact Borel probability spaces  $(G^{\mathbb{E}^+}, P_{T,z}^{\mathbb{G}})_{\mathbb{G} \in \mathcal{G}}$  with respect to the mappings  $(f_{\mathbb{G}\mathbb{G}'}))_{\mathbb{G} \leq \mathbb{G}'}$ .

**Proposition 3.2** *Let  $\mathcal{C}$  be the cylinder  $\sigma$ -field of  $\mathcal{M}(\Pi M, G)$ . There exists on the measurable space  $(\mathcal{M}(\Pi M, G), \mathcal{C})$  a unique probability measure  $P_{T,z}$  such that, for all graph  $\mathbb{G}$  with geodesic edges,  $(f_{\mathbb{G}})_* P_{T,z} = P_{T,z}^{\mathbb{G}}$ .*

Let  $(H_{\zeta})_{\zeta \in \Pi M}$  denote the canonical process on  $(\mathcal{M}(\Pi M, G), \mathcal{C})$ . Let us also denote, for each piecewise geodesic path  $\zeta$ , by  $\ell(\zeta)$  the length of  $\zeta$ . Finally, let  $d_G$  denote the Riemannian distance on  $G$ . The main property of  $P_{T,z}$  is the following.

**Proposition 3.3** *There exist two constants  $K, L > 0$  such that, for any loop  $\zeta \in \Pi M$ ,  $\ell(\zeta) \leq L$  implies  $\mathbb{E}_{P_{T,z}}[d_G(1, H_{\zeta})] \leq K\ell(\zeta)$ .*

**Proof** – Assume first that  $\zeta$  is a simple loop. Then, if  $L$  is small enough,  $\ell(\zeta) \leq L$  implies that  $\zeta$  is homotopic to a constant loop and, by a local isoperimetric inequality [8, Proposition 2.15],

bounds a domain  $D \subset M$  such that  $\sigma(D) \leq K_1 \ell(\zeta)^2$ , where  $K_1$  depends only on  $L$ . Since the Riemannian metric is smooth,  $K_1$  remains bounded when  $L$  gets smaller and we may assume, by taking  $L$  small enough, that  $K_1 L^2 \leq \frac{1}{2} \sigma(M)$ .

The law of  $H_\zeta$  can be computed in any graph  $\mathbb{G}$  such that  $\zeta$  belongs to  $\mathbb{E}^*$ . We choose a graph with only two faces, namely  $\overline{D}$  and  $M \setminus D$ . Then, if  $g$  denotes the genus of  $M$ , the expectation we want to estimate is equal to

$$\frac{1}{Z_{T,z}^{\mathbb{G}}} \int_{G^{2g+1}} d_G(1, x) \sum_{\pi(\tilde{x})=x} \tilde{p}_{T\sigma(D)}(\tilde{x}) \tilde{p}_{T\sigma(D^c)}(\tilde{x}^{-1}[a_1, b_1]^\sim \dots [a_g, b_g]^\sim z) dx da_1 db_1 \dots da_g db_g,$$

where  $[a, b]^\sim$  denotes the lift to  $\tilde{G}$  of  $[a, b]$ , that is, the value of  $[\tilde{a}, \tilde{b}]$  for any lift  $(\tilde{a}, \tilde{b})$  of  $(a, b)$ . By the convolution property of the heat kernel,

$$Z_{T,z}^{\mathbb{G}} = \int_{G^{2g}} \tilde{p}_{T\sigma(M)}([a_1, b_1]^\sim \dots [a_g, b_g]^\sim z) da_1 db_1 \dots da_g db_g,$$

which does not depend on  $\mathbb{G}$  nor on  $\zeta$ . Hence,

$$\begin{aligned} \mathbb{E}_{P_{T,z}}[d_G(1, H_\zeta)] &\leq K_2 \|\tilde{p}_{T\sigma(M)}\|_\infty \int_G d_G(1, x) \sum_{\pi(\tilde{x})=x} \tilde{p}_{T\sigma(D)}(\tilde{x}) dx \\ &\leq K_3 \int_G d_G(1, x) p_{T\sigma(D)}(x) dx \\ &\leq K_4 \sqrt{\sigma(D)} \\ &\leq K \ell(\zeta). \end{aligned}$$

We have used Lemma 2.2 to pass from  $\tilde{p}$  to  $p$  and then a classical estimation on the heat kernel, see for example [8, Proposition 1.31].

The case where  $\zeta$  is not assumed to be a simple loop anymore follows now easily along the lines of [8, Sections 2.4 and 2.5].  $\square$

### 3.3 The Yang-Mills measures

Once Proposition 3.3 is proved, we can go through the construction of the Yang-Mills measure as in [8, Sections 2.6, 2.7 and 2.10]. For the convenience of the reader, we recall the main steps and indicate where the proofs can be found.

The topology on  $PM$  is that of convergence in length, that is, uniform convergence plus convergence of the length. It is induced by the distance  $d_\ell(c, c') = \inf \sup_{t \in [0, 1]} d(c(t), c'(t)) + |\ell(c) - \ell(c')|$ , where the infimum is taken over all parametrizations of  $c$  and  $c'$ .

1. One defines for each  $c \in PM$  a random variable  $H_c$  characterized by the fact that, if  $(\zeta_n)_{n \geq 0}$  is a sequence of  $PM$  converging in length to  $c$  and such that each  $\zeta_n$  shares the same endpoints as  $c$ , then  $\mathbb{E}_{P_{T,z}}[d_G(H_{\zeta_n}, H_c)] \rightarrow 0$ . [8, Proposition 2.35, Section 2.6.4]

2. One checks that the same convergence property holds without the assumption that the approximating paths are piecewise geodesic. [8, Proposition 2.42]

3. One checks that the finite-dimensional marginals of the family  $(H_c)_{c \in PM}$  are consistent with the discrete theory. More precisely, for each graph  $\mathbb{G}$ , not necessarily piecewise geodesic, the law of  $(H_e)_{e \in \mathbb{E}^+}$  is  $P_{T,z}^{\mathbb{G}}$ . As a side result one gets the fact that the number  $Z_{T,z}^{\mathbb{G}}$  is independent of  $\mathbb{G}$ . [8, Proposition 2.46,2.50]

4. Then, one considers, for each finite subset  $I$  of  $PM$ , the probability spaces  $(\mathcal{M}(I, G), P_{T,z}^I)$ , where  $P_{T,z}^I$  is the law of  $(H_c)_{c \in I}$  together with the natural projections  $\mathcal{M}(I, G) \rightarrow \mathcal{M}(J, G)$  defined whenever  $J \subset I$ . The projective limit of these probability spaces is canonically isomorphic to  $(\mathcal{M}(PM, G), \mathcal{C})$ , where  $\mathcal{C}$  is the cylinder  $\sigma$ -field, endowed with a probability measure that we denote by  $P_{T,z}$ . [8, Theorem 2.62]

**Theorem 3.4** *Let  $M$  be a compact surface without boundary endowed with a volume 2-form  $\sigma$ . Let  $G$  be a compact connected Lie group and  $\pi : \tilde{G} \rightarrow G$  a universal cover of  $G$ . Let  $P \rightarrow M$  be a principal  $G$ -bundle over  $M$  with obstruction class  $z \in \ker \pi$ . Let  $T > 0$  be a positive real number. Let  $(\mathcal{M}(PM, G), \mathcal{C})$  denote the set of multiplicative functions from  $PM$  to  $G$  endowed with the cylinder  $\sigma$ -field. Let  $(H_c)_{c \in PM}$  denote the evaluation process on this space.*

*There exists on  $(\mathcal{M}(PM, G), \mathcal{C})$  a unique probability measure  $P_{T,z}$  such that the following two properties hold.*

1. *For all graph  $\mathbb{G} = (\mathbb{V}, \mathbb{E}, \mathbb{F})$  on  $M$ , with a choice of orientation  $\mathbb{E}^+ = \{e_1, \dots, e_r\}$ , the law of  $(H_{e_1}, \dots, H_{e_r})$  under  $P_{T,z}$  is equal to  $P_{T,z}^{\mathbb{G}}$ .*

2. *Whenever  $c$  belongs to  $PM$  and  $(c_n)_{n \geq 0}$  is a sequence of  $PM$  converging in length to  $c$  such that for each  $n \geq 0$ ,  $c_n$  and  $c$  share the same endpoints,  $\mathbb{E}_{P_{T,z}}[d_G(H_{c_n}, H_c)]$  tends to 0 as  $n$  tends to  $\infty$ .*

Recall from Lemma 2.9 that, for each graph  $\mathbb{G}$  and each  $T > 0$ , one has  $\sum_{z \in \Pi} Z_{T,z}^{\mathbb{G}} P_{T,z}^{\mathbb{G}} = Z_T^{\mathbb{G}} P_T^{\mathbb{G}}$ . As stated above, the numbers  $Z_{T,z}^{\mathbb{G}}, Z_T^{\mathbb{G}}$  do not depend on the graph  $\mathbb{G}$ . Hence, by performing simultaneously the construction of the measure  $P_{T,z}$  for each  $z \in \Pi$  and also of the measure  $P_T$  as defined in [8], one gets the following result.

**Proposition 3.5** *Let us keep the notation of the theorem above. Let  $P_T$  be the probability measure on  $(\mathcal{M}(PM, G), \mathcal{C})$  constructed in [8, Theorem 2.62]. The following equality holds:*

$$\sum_{z \in \Pi} Z_{T,z} P_{T,z} = Z_T P_T. \quad (25)$$

### 3.4 Mutual singularity of the measures $P_{T,z}$

The reader may wonder why we have not taken the projective limit of the covering probability spaces  $(\tilde{G}^{(\mathbb{E})}, \tilde{P}_{T,z}^{\mathbb{G}})$ . It seems indeed that something has been lost by projecting the mutually singular measures  $(\tilde{P}_{T,z}^{\mathbb{G}})_{z \in \Pi}$  to define the mutually absolutely continuous measures  $(P_{T,z}^{\mathbb{G}})_{z \in \Pi}$ .

There are at least two answers to this question. The first is, there is no natural projective structure of the probability spaces  $(\tilde{G}^{(\mathbb{E})}, \tilde{P}_{T,z}^{\mathbb{G}})$ , as the reader will convince himself easily by looking at simple examples. The fact that the covering  $\tilde{G}^{(\mathbb{E})}/\tilde{\mathcal{J}}_{\mathbb{G}} \rightarrow G^{\mathbb{E}^+}/\mathcal{J}_G$  has degree  $|\Pi^{\mathbb{F}}|$ , which even when  $G$  is semi-simple tends to infinity as  $\mathbb{G}$  gets finer, may be an indication of this lack of projective structure. This means that there is no covering of the measurable space  $(\mathcal{M}(PM, G), \mathcal{C})$  which plays a role similar to that of the spaces  $\tilde{G}^{(\mathbb{E})}$ .

The second answer is that such a covering would be useless, as the following result shows.

**Theorem 3.6** *Let  $T, T'$  be two positive real numbers. Let  $z, z'$  be two elements of  $\Pi$ . Then the probability measures  $P_{T,z}$  and  $P_{T',z'}$  are mutually singular on  $(\mathcal{M}(PM, G), \mathcal{C})$ , unless  $(T, z) = (T', z')$ .*

The dependence in  $T$  is accessory with respect to our previous discussion but we include it here because it does not make the proof significantly harder. A closely related question has been discussed by Fleischhack [5]. The main point is thus that there are disjoint sectors on the space  $(\mathcal{M}(PM, G), \mathcal{C})$  corresponding to different temperatures and different isomorphism classes of bundles. This result should be compared to the fact that, if  $(W_t)_{t \geq 0}$  is a standard real Brownian motion, then, given two real numbers  $T, T' > 0$ , the laws of  $(W_{Tt})_{t \geq 0}$  and  $(W_{T't})_{t \geq 0}$  are mutually singular unless  $T = T'$ .

To prove Theorem 3.6, we construct a pair  $(\tau, \mathbf{o})$  of random variables on  $(\mathcal{M}(PM, G), \mathcal{C})$  with values in  $[0, +\infty] \times \Pi$  and we show that,  $P_{T,z}$  almost-surely,  $(\tau, \mathbf{o}) = (T, z)$ .

For this, we focus on the random holonomy along a simple family of loops which we start by defining.

Let  $g$  denote the genus of  $M$ . Let  $D$  denote the closed unit disk in  $\mathbb{R}^2$ . Let  $q : D \rightarrow M$  denote a continuous onto mapping such that the restriction of  $q$  to the interior of  $D$  is an orientation-preserving diffeomorphism and  $q$  maps the boundary of  $D$  onto  $2g$  loops  $a_1, b_1, \dots, a_g, b_g$  which generate  $\pi_1(M)$  and such that  $q(\partial D)$  is the cycle  $[b_g^{-1}, a_g^{-1}] \dots [b_1^{-1}, a_1^{-1}]$ , where  $[a, b] = aba^{-1}b^{-1}$ .

For each  $s \in [0, 1]$ , let  $c_s$  denote the loop in  $\mathbb{R}^2$  based at  $(1, 0)$  going once counterclockwise along the circle of center  $(1-s, 0)$  and radius  $s$ . We project the loops  $c_s$  on  $M$  and re-index them. For this, let  $D_s$  denote the disk bounded by  $c_s$ . Set, for each  $s \in [0, 1]$ ,  $A(s) = \sigma(q(D_s))$ . This defines an increasing diffeomorphism  $A : [0, 1] \rightarrow [0, \sigma(M)]$  and we set, for each  $t \in [0, \sigma(M)]$ ,  $l_t = q(c_{A^{-1}(t)})$ . It is an element of  $PM$  which bounds with positive orientation the domain  $q(D_{A^{-1}(t)})$  whose area is  $t$ . Finally, for each  $t \in [0, \sigma(M)]$ , set  $X_t = H_{l_t}$ .

Recall that, if  $x, y \in G$ , then the commutator  $[\tilde{x}, \tilde{y}]$  does not depend on the choice of  $\tilde{x}$  and  $\tilde{y}$  such that  $\pi(\tilde{x}) = x$  and  $\pi(\tilde{y}) = y$ . We denote this commutator by  $[\widetilde{x, y}]$ .

**Lemma 3.7** *The conditional law of the family of random variables  $(X_t)_{t \in [0, \sigma(M)]}$  under  $P_{T,z}$  given  $(H_{a_1}, H_{b_1}, \dots, H_{a_g}, H_{b_g})$  is the same as that of  $(\pi(B_{Tt}))_{t \in [0, \sigma(M)]}$ , where  $B$  is a Brownian bridge on  $\widetilde{G}$  of length  $T\sigma(M)$  from the unit element to  $[H_{a_1}, H_{b_1}] \dots [H_{a_g}, H_{b_g}]z$ .*

**Proof** – Let  $0 \leq t_1 \leq \dots \leq t_n \leq \sigma(M)$  be  $n$  real numbers. Let  $\phi : G^n \rightarrow \mathbb{R}$  and  $\psi : G^{2g} \rightarrow \mathbb{R}$  be two continuous functions. Let us compute the quantity  $\mathbb{E}_{P_{T,z}}[\psi(H_{a_1}, H_{b_1}, \dots, H_{a_g}, H_{b_g})\phi(X_{t_1}, \dots, X_{t_n})]$ . It is equal to

$$\begin{aligned} & \frac{1}{Z_{T,z}} \int_{G^{2g+n}} \psi(a_1, b_1, \dots, a_g, b_g) \phi(x_1, \dots, x_n) \\ & \sum_{\pi(\tilde{x}_1)=x_1, \dots, \pi(\tilde{x}_n)=x_n} \tilde{p}_{Tt_1}(\tilde{x}_1) \tilde{p}_{T(t_2-t_1)}(\tilde{x}_2 \tilde{x}_1^{-1}) \dots \tilde{p}_{T(t_n-t_{n-1})}(\tilde{x}_n \tilde{x}_{n-1}^{-1}) \\ & \tilde{p}_{T(\sigma(M)-t_n)}([\widetilde{a_1, b_1}] \dots [\widetilde{a_g, b_g}] \tilde{x}_n^{-1} z) dx_1 \dots dx_n da_1 db_1 \dots da_g db_g \\ & = \frac{1}{Z_{T,z}} \int_{G^{2g}} \psi(a_1, b_1, \dots, a_g, b_g) \left[ \frac{1}{\tilde{p}_{T\sigma(M)}([\widetilde{a_1, b_1}] \dots [\widetilde{a_g, b_g}] z)} \int_{\widetilde{G}^n} \phi(\pi(\tilde{x}_1), \dots, \pi(\tilde{x}_n)) \right. \\ & \left. \tilde{p}_{Tt_1}(\tilde{x}_1) \tilde{p}_{T(t_2-t_1)}(\tilde{x}_2 \tilde{x}_1^{-1}) \dots \tilde{p}_{T(t_n-t_{n-1})}(\tilde{x}_n \tilde{x}_{n-1}^{-1}) \tilde{p}_{T\sigma(M)}([\widetilde{a_1, b_1}] \dots [\widetilde{a_g, b_g}] z \tilde{x}_n^{-1}) d\tilde{x}_1 \dots d\tilde{x}_n \right] \end{aligned}$$

$$\tilde{p}_{T\sigma(M)}([\widetilde{a_1, b_1}] \dots [\widetilde{a_g, b_g}]z) da_1 db_1 \dots da_g db_g.$$

For all  $\tilde{y} \in \tilde{G}$ , let  $(B_t^{\tilde{y}})_{t \in [0, T\sigma(M)]}$  denote a Brownian bridge on  $\tilde{G}$  of length  $T\sigma(M)$  starting at the unit element and finishing at  $\tilde{y}$ . Then the expression between the brackets is exactly  $\mathbb{E}[\phi(\pi(B_{Tt_1}^{\tilde{y}}), \dots, \pi(B_{Tt_n}^{\tilde{y}})))]$  with  $\tilde{y} = [\widetilde{a_1, b_1}] \dots [\widetilde{a_g, b_g}]z$ . The result follows.  $\square$

Let  $\mathcal{M}_c(PM, G)$ , or simply  $\mathcal{M}_c$ , denote the subset of  $\mathcal{M}(PM, G)$  consisting of those multiplicative functions  $f : PM \rightarrow G$  such that the mapping from  $[0, \sigma(M)] \cap \mathbb{Q}$  to  $G$  which sends  $t$  to  $f(l_t)$  is uniformly continuous. Observe that  $\mathcal{M}_c$  belongs to the cylinder  $\sigma$ -field  $\mathcal{C}$ . As a corollary of the Lemma above, we have

$$\forall T > 0, \forall z \in \Pi, P_{T,z}(\mathcal{M}_c) = 1. \quad (26)$$

We can now define two random variables on  $(\mathcal{M}(PM, G), \mathcal{C})$ .

**Definition 3.8** 1. Let  $(X'_t)_{t \in [0, \sigma(M)]}$  be the unique continuous extension of  $(X_t)_{t \in [0, \sigma(M)] \cap \mathbb{Q}}$ . Let  $(\tilde{X}'_t)_{t \in [0, \sigma(M)]}$  be the lift of  $(X'_t)_{t \in [0, \sigma(M)]}$  to  $\tilde{G}$ , starting at the unit element. Set

$$\mathfrak{o} = \tilde{X}'_1 \left( [\widetilde{H_{a_1}, H_{b_1}}] \dots [\widetilde{H_{a_g}, H_{b_g}}] \right)^{-1}. \quad (27)$$

2. Set

$$\tau = \limsup_{t \downarrow 0, t \in \mathbb{Q}} \frac{d_G(1, X_t)}{\sqrt{2t \log |\log t|}}. \quad (28)$$

The random variable  $\mathfrak{o}$  is well defined on  $\mathcal{M}_c$ , hence, by (26),  $P_{T,z}$ -almost surely for all  $T > 0$  and  $z \in \Pi$ . The variable  $\tau$  is well-defined everywhere, possibly equal to  $+\infty$ .

The next result implies Theorem 3.6.

**Proposition 3.9** *There exists a real positive constant  $C$ , which depends only on  $G$ , such that*

$$\forall T > 0, \forall z \in \Pi, P_{T,z}[\{\mathfrak{o} = z\} \cap \{\tau = C\sqrt{T}\}] = 1.$$

**Proof** – It follows immediately from Lemma 3.7 that  $\mathfrak{o} = z$   $P_{T,z}$ -almost surely.

Let us now fix  $T > 0$  and  $z \in \Pi$  and prove that  $\tau = C\sqrt{T}$   $P_{T,z}$ -almost surely for some constant  $C$ .

By Levy's iterated logarithm law (see [11]), there exists a constant  $C > 0$  such that the Brownian motion  $(W_t)_{t \geq 0}$  on  $\tilde{G}$  started from 1 satisfies almost surely

$$\limsup_{t \downarrow 0} \frac{d_{\tilde{G}}(1, W_t)}{\sqrt{2t \log |\log t|}} = C, \quad (29)$$

where  $\tilde{G}$  is endowed with the metric such that  $\pi : \tilde{G} \rightarrow G$  is a local isometry. We denote by  $d_{\tilde{G}}$  the corresponding distance.

For all  $\tilde{y} \in \tilde{G}$ , let  $(B_t^{\tilde{y}})_{t \in [0, T\sigma(M)]}$  be the Brownian bridge defined in Lemma 3.7. For all  $\tilde{y} \in \tilde{G}$  and up to any time  $s < T\sigma(M)$ , the law of  $(B_t^{\tilde{y}})_{t \in [0, s]}$  is absolutely continuous with respect to that of  $(W_t)_{t \in [0, s]}$ . Hence,  $B^{\tilde{y}}$  satisfies the iterated logarithm law (29).

Now, for all  $\tilde{x} \in \tilde{G}$ ,  $d_{\tilde{G}}(1, \tilde{x}) = d_G(1, \pi(\tilde{x}))$ . In particular, using the fact that  $T = (\deg \rho)^{-\frac{2}{d}} T$ ,

$$\limsup_{t \downarrow 0, t \in \mathbb{Q}} \frac{d_G(1, B_{Tt}^{\tilde{y}})}{\sqrt{2t \log |\log t|}} = C\sqrt{T}. \quad (30)$$

Finally, by Lemma 3.7, given  $H_{a_1}, H_{b_1}, \dots, H_{a_g}, H_{b_g}$ ,  $(X_t)_{t \in [0, \sigma(M)]}$  has the law of a Brownian bridge  $(B_{Tt}^{\tilde{y}})_{t \in [0, \sigma(M)]}$  for some  $\tilde{y}$ . The result follows.  $\square$

## 4 Combinatorial computation of the partition functions

Let  $T > 0$  and  $z \in \Pi$  be fixed. Let  $\mathbb{G}$  be a graph on  $M$ . The fact that the number  $Z_{T,z}^{\mathbb{G}}$  does not depend on  $\mathbb{G}$  is obtained in [8] as a consequence of a rather tedious approximation procedure. On the other hand, it is explained in a very convincing, if not very rigorous, way in [19]. In this section, we show that it is possible to compute  $Z_{T,z}^{\mathbb{G}}$  in a combinatorial way, by using among other tools the formalism of fat graphs.

**Theorem 4.1** *Let  $g \geq 0$  denote the genus of  $M$ . Let  $\mathbb{G}$  be a graph on  $M$ . Then, for each  $T > 0$  and each  $z \in \Pi$ , one has*

$$Z_{T,z}^{\mathbb{G}} = \int_{G^{2g}} \tilde{p}_{T\tilde{\sigma}(M)}([\widetilde{a_1, b_1}] \dots [\widetilde{a_g, b_g}]z) da_1 db_1 \dots da_g db_g.$$

### 4.1 Reduction to the case of a graph with a single face

Let  $\mathbb{G} = (\mathbb{V}, \mathbb{E}, \mathbb{F})$  be a graph on  $M$ . We do not assume that it is simple. Let us consider the dual combinatorial graph<sup>3</sup> of  $\mathbb{G}$ . It is a pair  $\mathbb{G}' = (V, E)$  of finite sets, namely  $V = \mathbb{F}$ ,  $E = \mathbb{E}$ , endowed with two mappings  $s, t : E \rightarrow V$ , respectively defined by  $s(e) = L(e)$  and  $t(e) = L(e^{-1})$  (see Section 1.1 for the definition of  $L$ ). This graph  $\mathbb{G}'$  is clearly connected.

A *subtree* of  $\mathbb{G}'$  is a connected subset  $\mathcal{T} \subset E$  stable by inversion and containing no cycle. Our main tool will be a *spanning tree* of  $\mathbb{G}'$ , that is, a subtree which is maximal for the inclusion. Such a subtree satisfies  $s(\mathcal{T}) = t(\mathcal{T}) = V$ .

The following properties are elementary and we leave their proof to the reader.

**Lemma 4.2** *Let  $\mathcal{T} \subset E$  be a subtree of  $\mathbb{G}'$ .*

1. *The set  $\bigcup_{e \in \mathbb{E} \setminus \mathcal{T}} e$  is connected.*
2. *The set of endpoints of  $\mathbb{E} \setminus \mathcal{T}$  is  $\mathbb{V}$ .*
3. *If  $\mathcal{T}$  is a spanning tree, the the set  $M \setminus \bigcup_{e \in \mathbb{E} \setminus \mathcal{T}} e$  is connected.*

**Proposition 4.3** *Let  $\mathbb{G} = (\mathbb{V}, \mathbb{E}, \mathbb{F})$  be a graph. Let  $\mathbb{G}' = (V, E)$  be the dual combinatorial graph of  $\mathbb{G}$ . Let  $\mathcal{T} \subset E$  be a spanning tree of  $\mathbb{G}'$ . Then  $\mathbb{G}_{\mathcal{T}} = (\mathbb{V}, \mathbb{E} \setminus \mathcal{T}, \{M\})$  is a graph with a single face and  $Z_{T,z}^{\mathbb{G}} = Z_{T,z}^{\mathbb{G}_{\mathcal{T}}}$ .*

**Proof** – We proceed by induction on the cardinal of  $\mathcal{T}$ . If  $\mathcal{T} = \emptyset$ , then  $\mathbb{G}'$  has only one vertex, so that  $\mathbb{G}$  has a single face and the result is true.

<sup>3</sup>We consider in fact the combinatorial graph underlying the dual fat graph to the fat graph induced by  $\mathbb{G}$ .

Assume that the result has been proved under the assumption  $|\mathcal{T}| \leq n - 1$  for some integer  $n \geq 1$ . Assume  $\mathbb{G}$  and  $\mathcal{T}$  are given with  $|\mathcal{T}| = n$ . Let  $F_1$  be a leaf of  $\mathcal{T}$ , that is, an element of  $V$  such that  $|\{e \in E : s(e) = F_1\}| = |\{e \in E : t(e) = F_1\}| = 1$ . Let  $e \in \mathcal{T}$  be the edge such that  $t(e) = F_1$ . Set  $s(e) = F_2$ . Since  $\mathcal{T}$  is a tree,  $F_1 \neq F_2$ .

Set  $\mathbb{E}_e = \mathbb{E} \setminus \{e, e^{-1}\}$  and  $\mathbb{F}_e = (\mathbb{F} \setminus \{F_1, F_2\}) \cup \{F_1 \cup F_2\}$ . Then  $\mathbb{G}_e = (\mathbb{V}, \mathbb{E}_e, \mathbb{F}_e)$  is still a graph. The fact that the set of vertices is  $\mathbb{V}$  comes from the second assertion of Lemma 4.2. Moreover,  $\mathcal{T}_e = \mathcal{T} \setminus \{e, e^{-1}\} \subset \mathbb{E}_e$  is a spanning tree of  $\mathbb{G}'_e$  and  $|\mathcal{T}_e| = n - 2 \leq n - 1$ .

Now,  $\mathbb{G}_e \leq \mathbb{G}$ , so that, by the proof of the invariance under subdivision of the discrete measures, more precisely by (24),  $Z_{T,z}^{\mathbb{G}_e} = Z_{T,z}^{\mathbb{G}}$ . The result follows by induction.  $\square$

## 4.2 Fat graphs with a single face

In order to reduce further the problem, we introduce more carefully the structure of fat graph. The reader may consult [7] for further details and also M. Imbert's paper [6] from which the strategy of our proof is inspired.

Let  $\mathbb{E}$  be a set of cardinal  $2a$ . A structure of fat graph on  $\mathbb{E}$  is the data of two permutations  $\sigma$  and  $\alpha$  of  $\mathbb{E}$  such that  $\alpha$  is a fixed-point free involution<sup>4</sup>.

If  $\mathbb{G} = (\mathbb{V}, \mathbb{E}, \mathbb{F})$  is a graph on  $M$ , the structure of fat graph on  $\mathbb{E}$  induced by  $\mathbb{G}$  is given by setting, for all  $e \in \mathbb{E}$ ,  $\alpha(e) = e^{-1}$  and defining, for each  $e \in \mathbb{E}$ ,  $\sigma(e)$  as the incoming edge at  $\bar{e}$  which follows immediately  $e$  in the cyclic order induced by the orientation of  $M$ .

The vertices, edges, faces of the fat graph  $\Gamma = (\mathbb{E}; \sigma, \alpha)$  are respectively the cycles of the permutations  $\sigma, \alpha, \alpha\sigma^{-1}$ . We denote the last permutation by  $\varphi$ , so that  $\sigma\alpha\varphi = 1$ . If  $e \in \mathbb{E}$ , the finishing (resp. starting) point  $\bar{e}$  (resp.  $\underline{e}$ ) of  $e$  is defined as the cycle of  $\sigma$  containing  $e$  (resp.  $\alpha(e)$ ). We use the notation  $e^{-1} = \alpha(e)$ .

The number of vertices, edges, faces, are denoted respectively by  $s, a, f$ . The genus of the graph is the number  $g$  defined<sup>5</sup> by Euler's relation  $s - a + f = 2 - 2g$ . If  $\mathbb{G}$  is a graph on  $M$ , it induces on  $\mathbb{E}$  a structure of fat graph with genus equal to that of  $M$ .

Let  $\Gamma = (\mathbb{E}; \sigma, \alpha)$  be a fat graph with a single face, that is, such that  $\varphi = \alpha\sigma^{-1}$  is a cyclic permutation of  $\mathbb{E}$ . Consider the adjoint action  $\text{Ad}$  of  $\tilde{G}$  on itself. We associate to  $\Gamma$  a probability measure on  $\tilde{G}/\text{Ad}$  as follows.

Assume  $\mathbb{E} = \{e_1, \dots, e_a, \alpha(e_1), \dots, \alpha(e_a)\}$ . Write  $\varphi = (\alpha^{i_1}(e_{n_1}), \dots, \alpha^{i_{2a}}(e_{n_{2a}}))$ , with  $i_k \in \{0, 1\}$  for  $k = 1 \dots 2a$ . Then one can define a mapping  $h_\Gamma : G^a \rightarrow \tilde{G}/\text{Ad}$  by setting, for each  $g = (g_1, \dots, g_a)$ ,  $h_\Gamma(g) = \tilde{g}_{n_{2a}}^{\varepsilon_{2a}} \dots \tilde{g}_{n_1}^{\varepsilon_1}$ , where  $\tilde{g} = (\tilde{g}_1, \dots, \tilde{g}_r)$  is an arbitrary lift of  $g$  and, for each  $k = 1 \dots 2a$ ,  $\varepsilon_k = 1 - 2i_k$ . Since each edge appears twice in  $\varphi$ , once with each orientation, this definition does not depend on the choice of  $\tilde{g}$ .

The mapping  $h_\Gamma$  depends on the choice of  $e_1, \dots, e_a \in \mathbb{E}$  but the image of the Haar measure on  $G^a$  by  $h_\Gamma$  does not. We associate to  $\Gamma$  the probability measure  $\nu_\Gamma = (h_\Gamma)_*(dg^{\otimes a})$ . We think of  $\nu_\Gamma$  as a measure on  $\tilde{G}/\text{Ad}$  or a measure on  $\tilde{G}$  invariant by adjunction. The following lemma is a straightforward consequence of the definition of  $\nu_\Gamma$ .

**Lemma 4.4** *Let  $\mathbb{G}$  be a graph on  $M$  with a single face. Let  $\Gamma$  be the fat graph induced by  $\mathbb{G}$ . Then*

$$Z_{T,z}^{\mathbb{G}} = \int_{\tilde{G}} \tilde{p}_{T\bar{\sigma}(M)}(\tilde{g}z) d\nu_\Gamma(\tilde{g}). \quad (31)$$

<sup>4</sup>The notation  $\sigma$  is standard and we keep it. There should be no confusion with the volume 2-form on  $M$ .

<sup>5</sup>Here again, the conflict of notation with the generic element of  $G$  should not lead to any ambiguity.

Let us recall two classical operations on fat graphs, namely the contraction of an edge, or Whitehead's move, and the cut-and-paste operation.

**Definition 4.5** Let  $\Gamma = (\mathbb{E}; \sigma, \alpha)$  be a fat graph.

1. *Whitehead's move* – Let  $e \in \mathbb{E}$  be given such that  $\underline{e} \neq \bar{e}$ . Set  $\mathbb{E}' = \mathbb{E} \setminus \{e, \alpha(e)\}$  and define  $\alpha' = \alpha|_{\mathbb{E}'}$ . Decompose  $\sigma$  in a product of commuting cycles and write  $\sigma = (e, e_1, \dots, e_k)(\alpha(e), e_{k+1}, \dots, e_l)\sigma_0$  where  $\sigma_0$  is the product of the cycles which contain neither  $e$  nor  $\alpha(e)$ . Set  $\sigma' = (e_1, \dots, e_k, e_{k+1}, \dots, e_l)\sigma_0$ . The fat graph  $(\mathbb{E}'; \sigma', \alpha')$  is by definition the result of the contraction of the edge  $e$  in  $\Gamma$ . It is denoted by  $W_e(\Gamma)$ .

2. *Cut and paste* – Assume that  $\Gamma$  has a single face. Let  $e \in \mathbb{E}$  be given. Write  $\varphi = \alpha\sigma^{-1}$  as  $\varphi = (e, e_1, \dots, e_r, d, e^{-1}, e_{r+1}, \dots, e_s)$ , where we have emphasized  $d = \varphi^{-1}(e^{-1})$ . Set  $\varphi' = (e, d, e_1, \dots, e_r, e^{-1}, e_{r+1}, \dots, e_s)$  and  $\sigma' = (\varphi')^{-1}\alpha$ . Then the fat graph  $(\mathbb{E}; \sigma', \alpha)$  is by definition the result of the cut-and-paste operation along  $e$ . It is denoted by  $K_e(\Gamma)$ .

The following properties are classical (see [6]).

**Lemma 4.6** Let  $\Gamma$  be a fat graph. Let  $e, e'$  be two edges of  $\Gamma$ .

1. Assume that  $\underline{e} \neq \bar{e}$ . Then  $W_e(\Gamma)$  is a fat graph with the same number of faces, same genus, and one less vertex as  $\Gamma$ .
2. Whitehead's moves commute : if  $W_e(W_{e'}(\Gamma))$  and  $W_{e'}(W_e(\Gamma))$  are both defined, they are equal.
3. Assume that  $\Gamma$  has one single face and one single vertex. Then  $K_e(\Gamma)$  still has one single face and one single vertex.

Let us describe how the measure  $\nu$  is transformed by these operations.

**Proposition 4.7** Let  $\Gamma = (\mathbb{E}; \sigma, \alpha)$  be a fat graph with a single face. Pick  $e \in \mathbb{E}$ .

1. One has  $\nu_\Gamma = \nu_{K_e(\Gamma)}$ .
2. If  $e$  has distinct endpoints, then  $\nu_\Gamma = \nu_{W_e(\Gamma)}$ .

**Proof** – 1. Write  $\mathbb{E} = \{e_1, \dots, e_a, \alpha(e_1), \dots, \alpha(e_a)\}$  with  $e_1 = e$  and  $e_2 = \sigma(e_1) = \varphi^{-1}(e^{-1})$ . We construct a change of variables  $k = (k_1, \dots, k_a) : G^a \rightarrow G^a$  as follows. Define  $k_1 = g_2^{-1}g_1$  and  $k_i = g_i$  for  $i = 2 \dots a$ . This change of variables satisfies the relation  $h_\Gamma(g) = h_{K_e(\Gamma)}(k)$  and preserves the Haar measure. The result follows.

2. Consider the other change of variables  $w = (w_1, \dots, w_a) : G^a \rightarrow G^a$  defined by setting  $w_1 = g_1$  and, for each  $i = 2 \dots a$ ,

$$w_i = \begin{cases} g_i g_1 & \text{if } \underline{e_i} = \bar{e} \text{ and } \bar{e_i} \neq \bar{e}, \\ g_1^{-1} g_i & \text{if } \underline{e_i} \neq \bar{e} \text{ and } \bar{e_i} = \bar{e}, \\ g_1^{-1} g_i g_1 & \text{if } \underline{e_i} = \bar{e} \text{ and } \bar{e_i} = \bar{e}. \end{cases}$$

This change of variables satisfies the relation  $h_\Gamma(g_1, \dots, g_a) = h_{W_e(\Gamma)}(w_1, \dots, w_a)$  and preserves the Haar measure. This finishes the proof.  $\square$

**Definition 4.8** Let  $\Gamma = (\mathbb{E}; \sigma, \alpha)$  be a fat graph of genus  $g$  with a single face and a single vertex. Let  $m$  be an integer such that  $0 \leq m \leq g$ . We say that  $\mathbb{G}$  is standard of order  $m$  if it is possible to label the elements of  $\mathbb{E}$  in such a way that  $\varphi = (a_1, b_1, a_1^{-1}, b_1^{-1}, \dots, a_m, b_m, a_m^{-1}, b_m^{-1}, e_{4m+1}, \dots, e_{4g})$ .



If  $\Gamma$  is standard of order  $g$ , it is very easy to write down the measure  $\nu_\Gamma$ . The next result allows us to extend this observation to the general case.

**Proposition 4.9** *Let  $\Gamma = (\mathbb{E}; \sigma, \alpha)$  be a fat graph of genus  $g$  with a single face and a single vertex. Assume that  $g > 0$  and  $\Gamma$  is standard of order  $m$  for some  $m < g$ . Then there exists a fat graph  $\Gamma'$  standard of order  $m + 1$  and such that  $\nu_\Gamma = \nu_{\Gamma'}$ .*

**Proof** – By Euler’s relation,  $\Gamma$  has  $2g$  edges, so that  $|\mathbb{E}| = 4g \geq 4$ . Write

$$\varphi = (a_1, b_1, a_1^{-1}, b_1^{-1}, \dots, a_m, b_m, a_m^{-1}, b_m^{-1}, e, e_1, \dots, e_k, e^{-1}, e_{k+1}, \dots, e_{k+l})$$

with  $k, l \geq 0$  and  $k + l = 4(g - m) - 2$ . From now on, we abbreviate  $a_1, \dots, b_m^{-1}$  by  $S_m$ .

We claim that  $k$  is positive. Otherwise,  $\sigma = \varphi^{-1}\alpha$  would fix  $e$ , in contradiction with the fact the  $\Gamma$  has a single vertex.

For the same reason, it is not possible that  $\{e_1, \dots, e_k\}$  be stable by the permutation  $\alpha$ . Otherwise,  $\sigma$  would stabilize  $\{e, e_1, \dots, e_k\}$  which is a proper subset of  $\mathbb{E}$  since it does not contain  $e^{-1}$ . Hence,  $\varphi$  can be written, with a new labeling of the edges,

$$\varphi = (S_m, e, e_1, \dots, e_r, f, e_{r+1}, \dots, e_s, e^{-1}, e_{s+1}, \dots, e_t, f^{-1}, e_{t+1}, \dots, e_u)$$

with  $0 \leq r \leq s \leq t \leq u$ . Consider now

$$\Gamma' = K_{f^{-1}}^t \circ K_e^{t-r} \circ K_f^{t-s}(\Gamma).$$

Then, if  $\Gamma' = (\mathbb{E}; \sigma', \alpha)$ , with the same labeling of  $\mathbb{E}$ , one has

$$\varphi' = \alpha(\sigma')^{-1} = (S_m, e, f, e^{-1}, f^{-1}, e_{s+1}, \dots, e_t, e_{r+1}, \dots, e_s, e_1, \dots, e_r, e_{t+1}, \dots, e_u).$$

Thus,  $\Gamma'$  is standard of order  $m + 1$  and, by Proposition 4.7, it satisfies  $\nu_{\Gamma'} = \nu_\Gamma$ . The result is proved.  $\square$

**Proof of Theorem 4.1** – Let  $\mathbb{G}$  be a graph on  $M$ . Pick  $T > 0$  and  $z \in \Pi$ . Let us compute  $Z_{T,z}^{\mathbb{G}}$ .

By Proposition 4.3, we may assume that  $\mathbb{G}$  has a single face. Let  $\Gamma = (\mathbb{E}; \sigma, \alpha)$  be the fat graph induced by  $\mathbb{G}$ . By Lemma 4.4, it is enough to compute  $\nu_\Gamma$ .

Assume that  $\Gamma$  has at least two vertices. By applying Whitehead’s moves along the edges of a spanning tree of  $\Gamma$ , we transform  $\Gamma$  into a fat graph with one single vertex. According to Proposition 4.7, this leaves the measure  $\nu_\Gamma$  unchanged. Thus, we may assume that  $\Gamma$  has one single face and one single vertex.

If the genus of  $\Gamma$  is 0, that is, if  $M$  is a sphere, then  $\Gamma$  has no edges and the measure  $\nu_\Gamma$  is the Dirac mass at the unit element of  $\tilde{G}$ .

If the genus of  $\Gamma$  is positive, let  $m \geq 0$  be the greatest integer such that  $\Gamma$  is standard of order  $m$ . If  $m = g$ , then  $\nu_\Gamma$  is the image on  $\tilde{G}$  of the Haar measure on  $G^{2g}$  by the mapping  $(a_1, b_1, \dots, a_g, b_g) \mapsto [\widetilde{a_1, b_1}] \dots [\widetilde{a_g, b_g}]$ . If  $m < g$ , then by induction on  $g - m$ , Proposition 4.9 implies that  $\nu_\Gamma$  is the same as if  $m = g$ .

In conclusion,  $\nu_\Gamma$  is always the image on  $\tilde{G}$  of the Haar measure on  $G^{2g}$  by the mapping  $(a_1, b_1, \dots, a_g, b_g) \mapsto [\widetilde{a_1, b_1}] \dots [\widetilde{a_g, b_g}]$ . By Lemma 4.4, this proves the result.  $\square$

## 5 Appendix: The new discrete theory as a singular covering of the old one

In this appendix, we study the structure of the vertical arrow of Diagram (6). We prove that it is a covering outside a closed singular set of codimension one. First, let us recall a nice description of the base space  $G^{\mathbb{E}^+}/\mathcal{J}_{\mathbb{G}}$  which is explained in [4] and [9]. In what follows,  $\mathbb{G} = (\mathbb{V}, \mathbb{E}, \mathbb{F})$  is a graph, that we do not assume to be simple. We assume an orientation  $\mathbb{E}^+ = \{e_1, \dots, e_r\}$  has been chosen.

Let  $T \subset \mathbb{E}$  be a spanning tree of  $\mathbb{G}$ . By this we mean, as at the beginning of Section 4.1, that  $T$  is a connected subset of  $\mathbb{E}$  stable by inversion, that no simple loop can be made by concatenating edges of  $T$  and that  $T$  is maximal for inclusion with these properties. In particular, the set of endpoints of the edges of  $T$  is  $\mathbb{V}$  itself. Hence, if  $v, w \in \mathbb{V}$ , there is a unique injective path from  $v$  to  $w$  within  $T$ . We denote this path by  $[v, w]$ . Finally, let us choose a vertex  $r$  that we call the root. For each edge  $e \in \mathbb{E}$ , define the loop  $\lambda_e$  based at  $r$  by  $\lambda_e = [r, \underline{e}]e[\bar{e}, r]$ .

**Proposition 5.1** *The mapping  $G^{\mathbb{E}^+} \rightarrow G^{\mathbb{E}^+ \setminus T^+}/G$  which sends  $g = (g_1, \dots, g_r)$  to the orbit of  $(h_{\lambda_e}(g))_{e \in \mathbb{E}^+ \setminus T^+}$  under the action of  $G$  by diagonal conjugation on  $G^{\mathbb{E}^+ \setminus T^+}$  induces a homeomorphism*

$$G^{\mathbb{E}^+}/\mathcal{J}_{\mathbb{G}} \rightarrow G^{\mathbb{E}^+ \setminus T^+}/G.$$

This proposition says that any configuration is gauge-equivalent to a configuration which is equal to 1 on the edges of  $T^6$  and also that two such configurations are equivalent if and only if they differ by simultaneous conjugation by some element of  $G$ .

Using this homeomorphism, it is possible to describe in a simple way most of the space  $\tilde{G}^{(\mathbb{E})}$ . In order to explain what *most* means, we use the following result.

**Proposition 5.2** *Let  $x$  be an element of  $G$ . Let  $\tilde{x} \in \tilde{G}$  be an element of  $\pi^{-1}(x)$ . The subgroup  $\{z \in \Pi \mid z\tilde{x} \in \text{Ad}(\tilde{G})\tilde{x}\}$  of  $\Pi$  does not depend on the choice of  $\tilde{x}$ . Let  $S \subset G$  be the set of those  $x$  for which this subgroup is not reduced to  $\{1\}$ . Then  $S$  is stable by conjugation, closed and of codimension 1 in  $G$ . More precisely, it is contained in the smooth image of a manifold of dimension at most  $\dim G - 1$ .*

**Proof** – Only the two last assertions are not elementary. According to a general structure theorem ([1, Theorem V.8.1]),  $\tilde{G}$  is isomorphic to  $\mathbb{R}^m \times K$ , where  $m \geq 0$  and  $K$  is simply connected and semi-simple. Define  $\Pi_K = \{c \in K \mid (1, c) \in \Pi\}$ . Let  $\tilde{x}, \tilde{y} \in \tilde{G}$  and  $z \in \Pi$  be such that  $\tilde{y}\tilde{x}\tilde{y}^{-1} = z\tilde{x}$ . Then, decomposing this identity according to  $\tilde{G} \simeq \mathbb{R}^m \times K$  shows that  $\tilde{x}$  belongs to  $\{1\} \times S_K$ , where  $S_K = \{k \in K \mid \exists c \in \Pi_K, ck \in \text{Ad}(K)k\}$ . In fact, the equality  $S = \pi(\{1\} \times S_K)$  holds.

Since  $\Pi_K \subset Z(K)$  is finite and  $K$  is compact,  $S_K$  and hence  $S$  are closed. Let  $T$  be a maximal torus of  $K$ . We claim that  $S_K \cap T$  is a finite union of cosets of proper subgroups of  $T$ .

Indeed, let  $t \in S_K \cap T$ . There exists  $c \in \Pi_K$ ,  $c \neq 1$ , such that  $t$  and  $ct$  are conjugate. Let  $W$  be the Weyl group of  $T$ . Since both  $t$  and  $zt$  belong to  $T$ , there exists  $w \in W$ ,  $w \neq \text{id}$ , such that  $w(t) = ct$ . So, we have proved that

$$S_K \cap T \subset \bigcup_{\substack{c \in \Pi_K - \{1\} \\ w \in W - \{\text{id}\}}} T_{c,w},$$

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<sup>6</sup>Considering configurations which vanish on the edges of  $T$  is the discrete analog to putting connections in axial gauge with respect to some coordinate system.

where  $T_{c,w} = \{t \in T \mid w(t) = ct\}$ .

Let  $c, w$  be given as above. If  $T_{c,w}$  is not empty, it is a coset of the closed subgroup  $\{t \in T \mid w(t)t^{-1} = 1\}$ . Since  $w \neq \text{id}$ , this subgroup is a proper subgroup and our claim is proved.

Finally,  $S_K$  is the union of the images of the mappings  $K/T \times T_{c,w} \longrightarrow K$ ,  $(kT, t) \mapsto \text{Ad}(k)t$  and  $\dim(K/T \times T_{c,w}) \leq \dim K - 1 \leq \dim G - 1$ .  $\square$

As usual, we denote by  $G/\text{Ad}$  and  $\tilde{G}/\text{Ad}$  the spaces of conjugacy classes of  $G$  and  $\tilde{G}$  respectively. We identify  $S$  with a subset of  $G/\text{Ad}$  and  $\tilde{S} = \pi^{-1}(S)$  with a subset of  $\tilde{G}/\text{Ad}$ .

**Example 5.3** Take  $\tilde{G} = SU(n)$  and  $\Pi = Z(SU(n))$ . An element of  $SU(n)$  is in the set  $\tilde{S}$  if and only if its spectrum is invariant by multiplication by some  $n$ -th root of unity distinct from 1. If the spectrum of a matrix is invariant by multiplication by a primitive  $d$ -th root  $\zeta$  of unity, with  $d \mid n$ , then its spectrum is of the form  $\{\alpha_1, \alpha_1\zeta, \dots, \alpha_1\zeta^{d-1}, \dots, \alpha_{\frac{n}{d}}, \dots, \alpha_{\frac{n}{d}}\zeta^{d-1}\}$ , with  $\alpha_1 \dots \alpha_{\frac{n}{d}} = 1$ . In a maximal torus of  $SU(n)$ , of dimension  $n - 1$ , such matrices form a union of submanifolds of codimension  $n - \frac{n}{d}$ . Hence,  $\tilde{S}$  is of codimension at least  $n - \frac{n}{d}$  in  $SU(n)$ , where  $d$  is the smallest divisor of  $n$  greater or equal to 2. In any case,  $\text{codim } \tilde{S} \geq \frac{n}{2}$ .

The main consequence of the definition of  $S$  is the following.

**Lemma 5.4** The group  $\Pi$  acts freely and properly by translations on  $\tilde{G}/\text{Ad} - \tilde{S}$ , and the quotient space of this action is canonically homeomorphic to  $G/\text{Ad} - S$ .

**Proof** – The translate of a conjugacy class of  $\tilde{G}$  by a central element is still a conjugacy class. Hence,  $\Pi$  acts by translations on  $\tilde{G}/\text{Ad}$ . It is elementary to check that the orbits of this action are in bijective correspondence with the conjugacy classes of  $G$ . Moreover, the natural projection  $\tilde{G} \longrightarrow G/\text{Ad}$  induces a continuous mapping  $\Pi \backslash \tilde{G} \longrightarrow G/\text{Ad}$ . Since the adjoint action commutes to that of  $\Pi$ , this induces in turn a continuous bijective mapping  $\Pi \backslash \tilde{G}/\text{Ad} \longrightarrow G/\text{Ad}$ . Since the source space of this mapping is compact, it is a homeomorphism.

The subset  $S$  of  $G$  has been defined precisely in such a way that the restriction of the action of  $\Pi$  to  $\tilde{G}/\text{Ad} - \tilde{S}$  is free. To show that it is proper, write as in the last proof  $\tilde{G} \simeq \mathbb{R}^m \times K$ , with  $K$  semi-simple. Observe then that  $\tilde{G}/\text{Ad} \simeq \mathbb{R}^m \times K/\text{Ad}$ . Then  $\Pi$  is a subgroup of  $L \times Z(K)$ , where  $L$  is some lattice in  $\mathbb{R}^m$ , for example the projection of  $\Pi$  on the first factor. Now since  $Z(K)$  is finite,  $L \times Z(K)$  acts properly on  $\mathbb{R}^m \times K/\text{Ad}$ . Hence, so does  $\Pi$  on  $\tilde{G}/\text{Ad}$ .  $\square$

**Remark 5.5** It is not always true that  $\tilde{G}/\text{Ad} - \tilde{S}$  is connected. In particular, one cannot say that the projection of  $\tilde{G}/\text{Ad} - \tilde{S}$  on  $G/\text{Ad} - S$  is a Galois covering with automorphism group  $\Pi$ . For example, take  $G = SO(3)$ ,  $\tilde{G} = SU(2)$  and  $\Pi = \{I_2, -I_2\}$ . Then  $S$  is the set of matrices of rotations with angle  $\pi$  in  $\mathbb{R}^3$  and  $\tilde{S}$  is the set of  $2 \times 2$  unitary matrices with spectrum  $\{i, -i\}$ . This is the equatorial 2-sphere of  $SU(2) \simeq S^3$  so that  $SU(2) - \tilde{S}$  is not connected. On the other hand, take  $\tilde{G} = SU(3)$  and  $\Pi = \{I_3, jI_3, j^2I_3\}$ . Then  $\tilde{S}$  is the set of  $3 \times 3$  unitary matrices with spectrum  $\{1, j, j^2\}$ , which is a smooth submanifold of codimension 2. In this case,  $SU(3) - \tilde{S}$  is connected and the projection is a Galois covering. In fact, according to Example 5.3,  $\tilde{G}/\text{Ad} - \tilde{S}$  is always connected when  $\tilde{G} = SU(n)$  with  $n \geq 3$ .

Let us define  $U \subset G^{\mathbb{R}^+}/\mathcal{J}_G$  as the open set of configurations such that the holonomy along the boundary of each face belongs to the complement of  $S$ . We shall now analyze the vertical arrow of (6) restricted to  $\pi^{-1}(U)$ .

**Proposition 5.6** *The restriction of the mapping  $\pi : \tilde{G}^{(\mathbb{E})}/\tilde{\mathcal{J}}_{\mathbb{G}} \longrightarrow G^{\mathbb{E}^+}/\mathcal{J}_G$  to  $\pi^{-1}(U)$  is a covering of  $U$  whose automorphism group acts transitively on the fibers and is isomorphic to  $\Pi^{\mathbb{F}}$ .*

**Proof** – In this proof, the generic element of  $\tilde{G}/\text{Ad}$  is denoted by  $\tilde{\mathfrak{x}}$ . We denote also by  $\pi : \tilde{G}/\text{Ad} \longrightarrow G/\text{Ad}$  the natural projection. Let  $X \subset G^{\mathbb{E}^+}/\mathcal{J}_G \times (\tilde{G}/\text{Ad})^{\mathbb{F}}$  be defined as

$$X = \{([g], (\tilde{\mathfrak{x}}_F)_{F \in \mathbb{F}}) \mid \forall F \in \mathbb{F}, h_{\partial F}([g]) = \pi(\tilde{\mathfrak{x}}_F)\}.$$

Set  $X_U = X \cap (\pi^{-1}(U) \times (\tilde{G}/\text{Ad})^{\mathbb{F}})$ . Define a mapping  $\kappa : \tilde{G}^{(\mathbb{E})}/\tilde{\mathcal{J}}_{\mathbb{G}} \longrightarrow X$  by setting  $\kappa(\tilde{g}) = (\pi(\tilde{g}), (\tilde{h}_{\partial F}(\tilde{g}))_{F \in \mathbb{F}})$ . By definition,  $\pi = pr_1 \circ \kappa$ . We claim two things which together imply our result.

1. The mapping  $\kappa$  induces a homeomorphism between  $\pi^{-1}(U)$  and  $X_U$ .
2. The projection  $pr_1 : X_U \longrightarrow U$  is a covering whose automorphism group acts transitively on the fibers and is isomorphic to  $\Pi^{\mathbb{F}}$ .

*Proof of claim 1* – To begin with,  $\kappa$  is a continuous mapping. Let us prove that it is onto. For this, choose  $([g], (\tilde{\mathfrak{x}}_F)_{F \in \mathbb{F}})$  in  $X_U$ . Pick  $\tilde{g} \in \tilde{G}^{(\mathbb{E})}$  such that  $\pi(\tilde{g}) = g$ . Let  $F$  be a face of  $\mathbb{G}$ . We have  $\pi(\tilde{h}_{\partial F}(\tilde{g})) = \pi(\tilde{\mathfrak{x}}_F)$ . Hence, there exists  $z \in \Pi$  such that  $\tilde{h}_{\partial F}(\tilde{g})z = \tilde{\mathfrak{x}}_F$ . Let  $e \in \mathbb{E}$  be such that  $L(e) = F$ . Replacing  $\tilde{g}(e)$  by  $\tilde{g}(e)z$ , we transform  $\tilde{g}$  without changing  $\pi(\tilde{g})$  into a configuration such that the class of the holonomy along the boundary of  $F$  is exactly  $\tilde{\mathfrak{x}}_F$ . We can do this for each face successively and we get a configuration  $\tilde{g}$  such that  $\kappa(\tilde{g}) = ([g], (\tilde{\mathfrak{x}}_F)_{F \in \mathbb{F}})$ .

Let us prove that it is one-to-one. For this, consider again the spanning tree  $T \subset \mathbb{E}$  and the root  $r \in \mathbb{V}$ . Let  $\tilde{g}$  and  $\tilde{g}'$  be two configurations such that  $\kappa(\tilde{g}) = \kappa(\tilde{g}')$ . Let us prove that they are equivalent. First, each of them is equivalent to a configuration which takes its values in  $\Pi$  on the edges of  $T$ . Indeed, take  $\tilde{g}$  for example. The relations  $j_r = 1$  and  $j_e^{-1}\tilde{g}(e)j_e = 1$  for each  $e \in T^+$  determine uniquely  $j_v$  for each vertex  $v$ . By letting  $j = ((j_v)_{v \in \mathbb{V}}, 1) \in \tilde{\mathcal{J}}_{\mathbb{G}}$  act on  $\tilde{g}$ , we get a configuration of the desired form. Assume now that, for each  $e \in T$ , both  $\tilde{g}(e)$  and  $\tilde{g}'(e)$  belong to  $\Pi$ . Since  $\kappa(\tilde{g}) = \kappa(\tilde{g}')$ , it is in particular true that the configurations  $(\pi(\tilde{g}(e)))_{e \in \mathbb{E}^+}$  and  $(\pi(\tilde{g}'(e)))_{e \in \mathbb{E}^+}$  of  $G^{\mathbb{E}^+}$  are equivalent. Since both take the value 1 on the edges of  $T$  and according to Proposition 5.1, they differ by simultaneous conjugation by some element of  $G$ , say  $(\pi(\tilde{g}'(e))) = \text{Ad}(x)(\pi(\tilde{g}(e)))$ . Choose  $\tilde{x}$  in  $\pi^{-1}(x)$  and set  $j_v = \tilde{x}$  for all  $v \in \mathbb{V}$ . Then, replacing  $\tilde{g}$  by  $((j_v), 1) \cdot \tilde{g}$ , we may assume that, for each  $e \in \mathbb{E}$ ,  $\pi(\tilde{g}(e)) = \pi(\tilde{g}'(e))$ . In other words, there exists a function  $z : \mathbb{E} \longrightarrow \Pi$  such that, for each  $e \in \mathbb{E}$ ,  $\tilde{g}'(e) = \tilde{g}(e)z(e)$ . In particular, this implies that, for each face  $F$  of  $\mathbb{G}$ , the relation  $\tilde{h}_{\partial F}(\tilde{g}') = \tilde{h}_{\partial F}(\tilde{g}) \prod_{L(e)=F} z(e)$  holds. On the other hand, since  $\kappa(\tilde{g}) = \kappa(\tilde{g}')$ , both sides of this equality are conjugate. But, by the assumption that both  $\tilde{g}$  and  $\tilde{g}'$  belong to  $\pi^{-1}(U)$ , this imposes the relation  $\prod_{L(e)=F} z(e) = 1$ . Hence,  $(1, (z(e))_{e \in \mathbb{E}})$  belongs to the gauge group and transforms  $\tilde{g}$  into  $\tilde{g}'$ , which are henceforth equivalent. Finally,  $\kappa$  is a continuous bijection.

We prove now that its inverse is continuous. Pick  $([g], (\tilde{\mathfrak{x}}_F)_{F \in \mathbb{F}})$  in  $X_U$ . Choose a small neighbourhood  $V$  of  $\bigcup_{e \in \mathbb{E}^+} \{g(e)\}$  in  $G$  and a continuous section  $\tau : V \longrightarrow \tilde{G}$  of  $\pi$ . This section induces another continuous section  $v : W \subset G^{\mathbb{E}^+} \longrightarrow \tilde{G}^{(\mathbb{E})}$  of the natural projection, defined on a neighbourhood of  $g$  by  $v(g')(e) = \tau(g'(e))$  if  $e \in \mathbb{E}^+$  and  $\tau(g'(e^{-1}))^{-1}$  if  $e^{-1} \in \mathbb{E}^+$ . Finally, let  $\lambda : \mathbb{F} \longrightarrow \mathbb{E}$  be a section of  $L$ , that is, the choice of an edge on the boundary of each face. For  $g'$  in a neighbourhood of  $g$  and  $(\tilde{\mathfrak{x}}'_F)_{F \in \mathbb{F}}$  in a neighbourhood of  $(\tilde{\mathfrak{x}}_F)_{F \in \mathbb{F}}$ , define  $\psi(g', (\tilde{\mathfrak{x}}'_F)) \in \tilde{G}^{(\mathbb{E})}$  by setting  $\psi(g', (\tilde{\mathfrak{x}}'_F))(e) = v(g')(e)$  if  $e \notin \lambda(\mathbb{F})$  and  $v(g')(e)z_F$  if  $e = \lambda(F)$ , where  $z_F$  is the unique element of  $\Pi$  such that  $\tilde{h}_{\partial F}(g)z_F = \tilde{\mathfrak{x}}_F$ . Then, by Lemma 5.4  $\tilde{h}_{\partial F}(\psi(g', (\tilde{\mathfrak{x}}'_F))) = \tilde{\mathfrak{x}}'_F$  for  $g'$  close enough to  $g$  and  $\tilde{\mathfrak{x}}'_F$  close enough to  $\tilde{\mathfrak{x}}_F$ . This construction provides us locally with a continuous inverse to  $\kappa$ , which is thus a homeomorphism.

*Proof of claim 2* – To prove this assertion, observe that  $U$  is homeomorphic to the subset  $\{([g], (\mathfrak{r}_F)_{F \in \mathbb{F}}) \mid \forall F \in \mathbb{F}, h_{\partial F}([g]) = \mathfrak{r}_F\}$  of  $G^{\mathbb{E}^+} \times (G/\text{Ad})^{\mathbb{F}}$ . Now the result is a consequence of Lemma 5.4.  $\square$

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