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Easiness in graph models

Chantal Berline and Antonino Salibra

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Abstract

We generalize Baeten and Boerboom's method of forcing, and apply it to show that there is a fixed sequence $(u_k)_{k \in \omega}$ of closed (untyped) λ -terms satisfying the following properties:

a) For any countable sequence $(g_k)_{k \in \omega}$ of continuous functions (of arbitrary arity) on the power set $\mathcal{P}(D)$ of an arbitrary countable set D , there is a graph model (D, p) such that $(\lambda x.xx)(\lambda x.xx)u_k$ represents g_k in the model.

b) For any countable sequence $(t_k)_{k \in \omega}$ of closed λ -terms there is a graph model that satisfies $(\lambda x.xx)(\lambda x.xx)u_k = t_k$ for all k .

These two results bring information on the landscape of graph theories (= λ -theories that can be realized as theories of graph models), and more generally on the lattice of λ -theories (ordered by inclusion).

Keywords. *untyped λ -calculus, graph models, easy terms, Scott's continuous semantics, lattices of λ -theories, webbed models.*

1 Introduction.

Lambda theories are equational extensions of the untyped λ -calculus that are closed under derivation. They arise by syntactical and semantic considerations: a lambda theory may correspond to an operational semantics of the lambda calculus, as well as it may be induced by a model of lambda calculus through the kernel congruence relation of the interpretation function (see e.g. [6] [14]). The set of lambda-theories ordered by inclusion is naturally equipped with a structure of complete lattice (see Chapter 4 in [6]), where the meet of a family of lambda theories is their intersection, and the join is the least equivalence relation containing their union. The bottom element of this lattice is the minimal λ -theory λ_β , while the top element is the inconsistent λ -theory. The lattice of lambda theories, hereafter denoted by λT , has a continuum of elements and a complex structure. Since researchers have mainly focused their interest on a limited number of λ -theories, very little is known about the structure and equational theory of λT (see [38, 45]).

Since syntactical techniques are usually difficult to use in the study of λ -theories, then semantic methods have been extensively investigated. Topology

is at the center of the known approaches to giving models of the untyped lambda calculus. The first model, found by Scott in 1969 in the category of complete lattices and Scott continuous functions, was successfully used to show that all the unsolvable λ -terms can be consistently equated. After Scott, a large number of mathematical models for lambda calculus, arising from syntax-free constructions, have been introduced in various categories of domains and were classified into semantics according to the nature of their representable functions, see e.g. [1, 6, 14, 43]. Scott's continuous semantics [48] is given in the category whose objects are complete partial orders and morphisms are Scott continuous functions. The stable semantics introduced by Berry [15] and the strongly stable semantics introduced by Bucciarelli-Ehrhard [16] are a strengthening of the continuous semantics, introduced to capture the sequential features of lambda calculus. All these semantics are structurally and equationally rich in the sense that, in each of them, it is possible to build up 2^{\aleph_0} models inducing distinct and pairwise inconsistent λ -theories [31, 32, 35].

Although a rich host of different λ -theories have a syntax-free model (i.e., a model that induces precisely those identities which hold in the given theory), the above denotational semantics are *equationally incomplete*: they do not even match all the possible operational semantics of lambda calculus. The problem of the equational incompleteness was positively solved by Honsell-Ronchi della Rocca [25] for the continuous semantics, by Bastonero-Gouy [24, 10, 11] for the stable semantics, and by Salibra [46, 47] for the strongly stable semantics. As for λT , the results on the structure of the set of λ -theories induced by a semantics are still rare, and there exist several longstanding very basic open questions (see [14] for a survey). In particular it is still open to know whether λ_β , the least λ -theory, could be the theory of a non-syntactic model in Scott's continuous semantics.

In this paper we concentrate on the semantics \mathcal{G} of lambda calculus given in terms of graph models, *graph semantics* for short. These models, isolated in the seventies by Plotkin, Scott and Engeler [37] within the continuous semantics, have been proved useful for giving proofs of consistency of extensions of lambda calculus and for studying operational features of lambda calculus (see [14]). For example, the simplest graph model, namely Engeler-Plotkin's model, has been used by Berline [14] to give concise proofs of the head-normalization theorem and of the left-normalization theorem of lambda calculus. Bucciarelli and Salibra [17, 18] have very recently proved that the set \mathcal{GT} , consisting of all the graph theories (= λ -theories that can be realized as theories of graph models), admits a least element, which is strictly greater than λ_β ; in particular λ_β cannot be the theory of a graph model. These authors have also proved in [18] results about the "smaller" class \mathcal{G}_sT of all sensible graph theories (a theory is *sensible* if all the unsolvable terms are congruent). Smaller means here only that \mathcal{G}_sT is strictly included in \mathcal{GT} , since from Kerth [33] [36] plus David [21] it follows that \mathcal{G}_sT also contains 2^ω λ -theories (the result is however much harder to prove than for \mathcal{GT}).

The reasons to concentrate on \mathcal{G} are the following. First \mathcal{G} is, from far, the simplest class of models, in the sense that the webs of graph models are

the simplest existing webs. Second \mathcal{GT} contains nevertheless a continuum of elements [31], so it is a rich class, in the sense that its cardinality is the maximal possible one, but it is a proper subclass of λT , since no graph model can be extensional. Third it is quite clear that the techniques and results for \mathcal{G} and \mathcal{GT} can often be transferred to other classes of webbed models, whether more general or belonging to other semantics.

It is a well known result of Jacopini [27] that Ω can be consistently equated to any closed term t of the (untyped) λ -calculus, where Ω is the paradigmatic unsolvable term $(\lambda x.xx) \lambda x.xx$ (this is called *the easiness of Ω*). Baeten-Boerboom gave in [5] the first semantic proof of this result by showing that for all closed terms t one can build a graph model satisfying the equation $\Omega = t$. This semantic result extends to other classes of models and to some other terms which share with Ω enough of its good will (cf. [14] for a survey of such results). We recall that a graph model is, by definition, a reflexive Scott domain, which is generated by a web of the form (D, p) , where D is an infinite set and $p : D^* \times D \rightarrow D$ is a total injection, D^* being the set of finite subsets of D (see Section 2.2). For short we will confuse graph models and their webs, but one should have present in mind that the underlying domain of the model (D, p) is the full powerset $\mathcal{P}(D)$ ordered by inclusion, which is therefore independent of p . Starting from the set $D = \mathbb{N}$ of natural numbers, Baeten and Boerboom build p by a method of “forcing”, which, although much simpler than the forcing techniques used in set theory, is somewhat in the same spirit. In the Baeten-Boerboom setting a *forcing condition* is a partial injection $q : D^* \times D \rightarrow D$ and “ q forces $\alpha \in t$ ”, abbreviated by $q \Vdash \alpha \in t$, means that for all total injection $p \supseteq q$ we have that $(D, p) \models \alpha \in t$. The game is to build p as an increasing union of forcing conditions which successively put in Ω all the elements which are forced to be in t and exclude all the other ones.

In this paper we address the question of the “easiness” of sequences of λ -terms and of the λ -representability of sequences of continuous functions on $\mathcal{P}(D)$, where D is any countable infinite set. Given two sequences \bar{t} and \bar{v} of the same length, we denote by $\bar{t} = \bar{v}$ the set consisting of all the equations $t_k = v_k$. We say that a (possibly infinite) sequence \bar{t} of closed λ -terms is

1. *easy* if, for every other sequence \bar{v} (of same length) of closed λ -terms, the set $\bar{t} = \bar{v}$ is consistent.
2. *graph easy* if, for every other sequence \bar{v} (of same length) of closed λ -terms, there is a graph model satisfying $\bar{t} = \bar{v}$. (Of course, “graph easy” implies “easy”).
3. *graph easy for functionals* if, for every sequence \bar{f} (of same length) of Scott continuous functions on $\mathcal{P}(D)$, there exists a graph model (D, p) such that t_k represents f_k in the model for every k .

We generalize Baeten and Boerboom’s method of forcing, and apply it to show that there is a sequence $(u_k)_{k \in \omega}$ of closed λ -terms satisfying the conditions expressed in the following two theorems.

Theorem 1. The sequence $(\Omega u_k : k < \omega)$ is graph easy.

Theorem 2. The sequence $(\Omega u_k : k < \omega)$ is graph easy for functionals.

The above theorems have clear incidence on our knowledge of λT and on all the subsets $\mathcal{C}T$ of λT , where \mathcal{C} is any interesting class of models of λ -calculus in the continuous semantics which contains the class \mathcal{G} of all graph-models, and $\mathcal{C}T$ is the set consisting of the λ -theories of the models in \mathcal{C} .

The question of the λ -representability of (sequences of) continuous functions has not yet been addressed, as far as we know. Related works are only the very recent papers by Alessi et al. [3] and Dezani-Lusin [22], where the authors use intersection type systems (see [3, 7, 20]) for synthesizing filter models of lambda calculus in which the interpretation of a simple easy term can be any filter described by a continuous predicate. The notion of simple easiness was introduced by Alessi-Lusin [4] as a semantical tool to prove easiness. In fact, simple easiness implies easiness, while it is an open question whether easiness implies simple easiness. We would like to point out here that the main result in [3] (that the interpretation of a simple easy term can be any filter described by a continuous predicate) can be also interpreted as a generalization of Baeten and Boerboom's method of forcing via the use of intersection type systems. We believe that the framework we have developed in this paper is however more direct than the one used in [3]. We illustrate this with two examples, concerning the λ -representability of the minimal fixed point operator and of the pair union/intersection.

One application of Theorem 2 that we develop here concerns the lattice λT of all λ -theories ordered by inclusion. In particular, instantiating Theorem 2 we get the existence of a λ -theory T such that the upwards closure of T in λT is a distributive lattice. The existence of a distributive sub-lattice of λT was an open question, which arises naturally since Salibra [45] proved that the lattice λT *does not satisfy the modularity law* (which is a weak form of distributivity), and since Lusin and Salibra [38] have shown, among other results on λT , the existence of a sub-lattice satisfying a restricted form of distributivity (called meet semi-distributivity) expressed in the form of a quasi-identity.

Another application that we develop here concerns the variety (i.e., equational class) of lambda abstraction algebras (**LAA**'s) introduced by Pigozzi and Salibra in [40, 41] as a purely algebraic theory of the untyped lambda calculus which nevertheless, and in contrast to Combinatory Logic, keeps all the functional intuitions. There is a close relationship between the lattice λT of lambda theories and the variety **LAA**. In [44] Salibra has shown that, for every variety of **LAA**'s, there exists exactly one lambda theory whose term algebra generates the variety. Thus, the properties of an arbitrary lambda theory can be studied by means of the variety of **LAA**'s generated by its term algebra. Many longstanding open problems of lambda calculus can be restated in terms of algebraic properties of varieties of **LAA**'s. For example, the open problem of the order-incompleteness of lambda calculus [49, 47] asks for the existence of a lambda theory not arising as the equational theory of a non-trivially partially ordered model of lambda calculus. The order-incompleteness of lambda calculus is equivalent to the existence of an n -permutable variety of **LAA**'s for

some natural number $n \geq 2$ (see the remark after Theorem 3.4 in [49]; the definition of n -permutability can be found in [39]). As a consequence of Theorem 2, we show that there exists a congruence distributive variety of **LAA**'s. The existence of varieties of **LAA**'s satisfying strong algebraic properties, such as n -permutability or congruence distributivity, was an open problem since Salibra [45] proved that the variety **LAA** is not congruence modular. The existence of a congruence distributive variety of **LAA**'s shows, against the common belief, that the lambda calculus satisfies strong algebraic properties. We express hope to positively solve in the future the order-incompleteness problem by showing the existence of an n -permutable variety of **LAA**'s.

The paper is organized as follows. Section 2 is a preliminary section containing the definition of a graph model and recalling the two possible ways of building graph models out of partial webs, namely “*canonical completion*” and “*completion by forcing*”. This section also surveys the most recent results about the lambda theories induced by graph models.

In Section 3 we introduce the generalized terms, which allow continuous functions of arbitrary arity as first-order function symbols, and we extend the classic notion of easiness of Ω to sequences of generalized terms.

In Section 4 we show that Baeten-Boerboom’s method works not only for forcing but more generally for weakly continuous operators, and also for generalized terms. This allows for the (optional) use of the (continuous) notion of partial interpretation as an alternative to forcing. We provide sequences of lambda terms of arbitrary finite length that are functionally graph easy.

In Section 5 we introduce the technical notions of flattening and of an orthogonal system of representatives (osr); then we give examples of infinite sequences of terms that admit an osr. These technicalities are applied to get infinite sequences of terms that are functionally graph easy.

In Section 6 it is shown that there exist a distributive sub-lattice of the lattice of lambda theories and a congruence distributive variety of lambda abstraction algebras.

Section 7 is devoted to conclusions and future work.

2 Preliminaries.

2.1 Basic notations and conventions.

λ -calculus. In this paper λ -calculus will always mean *untyped* λ -calculus, and we adopt the notations of [6]. In particular Λ and Λ° are, respectively, the set of λ -terms and of closed λ -terms. A λ -theory is a congruence on Λ (with respect to the operators of abstraction and application), which contains (α) - and (β) -conversion. There is a smallest λ -theory, denoted here by λ_β , which is nothing else than (α) - and (β) -conversion itself. λ -theories can of course also be seen as (specific) sets of equations between λ -terms. A λ -theory is *sensible* if all the unsolvable terms are congruent, and *semi-sensible* if no solvable term

can equate an unsolvable term. The smallest sensible λ -theory is traditionally denoted by \mathcal{H} .

The lattice of λ -theories. The set of lambda-theories ordered by inclusion is naturally equipped with a structure of complete lattice (see Chapter 4 in [6]), where the meet of a family of λ -theories is their intersection, and the join is the least equivalence relation containing their union. The bottom element of this lattice is the minimal λ -theory λ_β , while the top element is the inconsistent λ -theory. The lattice of λ -theories will be denoted by λT . The sets of semi-sensible λ -theories and of sensible λ -theories constitute sub-lattices of λT .

Lattice identities. In the context of lattices an identity in the binary symbols $\{+, \cdot\}$ is called a *lattice identity*. (“+” is intended for *sup* and “ \cdot ” for *inf*). A lattice identity is trivial if it holds in every lattice and nontrivial otherwise.

Given the lattice λT of λ -theories, we interpret the variables of a lattice identity as λ -theories, and for arbitrary binary relations T and S between λ -terms we interpret $T + S$ as the lambda theory generated by the union of the two relations, and $T \cdot S$ as the intersection (as usual, we will write TS for $T \cdot S$).

A *quasi-identity* is an implication with an equational conclusion and a finite number of equational premises. A quasi-identity in the language of lattices is satisfied by the lattice of lambda-theories if, the conclusion of the quasi-identity is satisfied by all the lambda theories that satisfy the premises.

Scott’s semantics. *Cpos* (complete partial orders) and (*Scott-*) *continuous functions* between cpos are defined in [6, Chapter I.2]. Given a set S , and an element \perp not in S , the *flat cpo* S_\perp is the order $(S \cup \{\perp\}, \leq)$ where $x \leq y$ if and only if $x = \perp$ or $x = y$. If C, C' are cpos then $[C \rightarrow C']$ denotes the cpo of all the continuous functions from C into C' . A reflexive cpo is a triple (C, A, λ) such that $\lambda \in [[C \rightarrow C] \rightarrow C]$ and $A \in [C \rightarrow [C \rightarrow C]]$ and $A \circ \lambda = id$. Reflexive cpos model λ -calculus in a way which is recalled in Section 2.2 (for more details see [6, Chapter V.5]). We will mainly (but not always) be interested in cpos of the form $(\mathcal{P}(D), \subseteq)$, for some infinite countable set D . In this case \subseteq will be understood as set inclusion. By “a continuous function g of arity n on $\mathcal{P}(D)$ ” we mean: $g \in [\mathcal{P}(D)^n \rightarrow \mathcal{P}(D)]$.

Sets. For every set S , S^* is the set of all finite subsets of S , while $\mathcal{P}(S)$ is the powerset of S and $S^{<\omega}$ (resp. S^ω , $S^{\leq\omega}$) is the set of all finite (resp. infinite, resp. finite or infinite) sequences of elements of S ; $l(\bar{s})$ denotes the length of the sequence \bar{s} . When writing $g(\bar{x})$, where g is a function and \bar{x} a sequence of elements of the domain of g , we will of course always understand that $l(\bar{x})$ is the arity of g .

Further conventions. Greek letters α, β, \dots will always understand elements of a set D specified by the context (from Section 3 on, D will be any

fixed countable infinite set). Small Latin letters a, b, c will understand elements of D^* , and $\bar{a}, \bar{b}, \bar{c} \dots$ elements of $(D^*)^{<\omega}$. Also, (a, α) is the usual set-theoretical pair, and (\bar{a}, α) is defined by induction as follows: $(b\bar{c}, \alpha) =_{def} (b, (\bar{c}, \alpha))$.

Traces of continuous functions. A continuous function g on $\mathcal{P}(D)$, of any arity, is completely determined by its *trace*, which is defined by:

$$tr(g) =_{def} \{ (\bar{a}, \alpha) : \alpha \in g(\bar{a}) \} \quad (1)$$

The trace is, essentially, the relevant part of the graph $graph(g)$ of g ; “essentially” refers to the fact that, if g is unary, say, then $tr(g) \subseteq D^* \times D \subseteq \mathcal{P}(D) \times D$, while $graph(g) \subseteq \mathcal{P}(D) \times \mathcal{P}(D)$

2.2 Graph models.

The class of graph models belongs to Scott’s continuous semantics. Graph models owe their name to the fact that continuous functions are encoded in them via (a sufficient fragment of) their graphs, namely their traces.

A *graph model* is a model of untyped λ -calculus, which is generated from a web in a way that will be recalled below. Historically, the first graph model was Plotkin and Scott’s P_ω (see e.g. [6]), which is also known in the literature as “the graph model”. The simplest graph model, \mathcal{E} , was introduced soon afterwards, and independently, by Engeler [23] and Plotkin [42]. More examples can be found in [14].

For short we will confuse the model and its web and so we define:

Definition 1 A graph model is a pair (D, p) , where D is an infinite set and $p : D^* \times D \rightarrow D$ is an injective total function.

Such a pair will also be called a *total pair*. A total pair (D, p) generates a reflexive cpo $(\mathcal{P}(D), \lambda_p, A_p)$, and hence a model of λ -calculus. The continuous function $\lambda_p \in [[\mathcal{P}(D) \rightarrow \mathcal{P}(D)] \rightarrow \mathcal{P}(D)]$ is defined by $\lambda_p = p^+ \circ tr$. This definition extends to continuous functions of arbitrary arity on $\mathcal{P}(D)$; in other words, for any such function g , we have:

$$\lambda_p(g) = \{ p(\bar{a}, \alpha) : \alpha \in g(\bar{a}) \} \quad (2)$$

The left inverse $A_p \in [\mathcal{P}(D) \rightarrow [\mathcal{P}(D) \rightarrow \mathcal{P}(D)]]$ of λ_p (which allows to interpret application in the model) is defined by:

$$A_p(X)(Y) = \{ \alpha \in D : (\exists a \subseteq Y) p(a, \alpha) \in X \}.$$

where X, Y are arbitrary subsets of D . When no ambiguity will occur we will write XY instead of $A_p(X)(Y)$. More generally, $X\bar{Y}$ is defined, for $\bar{Y} = (Y_1, \dots, Y_n)$, as $(..((XY_1)...)Y_n)$.

Let Env_D be the set of D -environments ρ mapping the set of the variables of λ -calculus into $\mathcal{P}(D)$. For $\rho \in Env_D$ and $X \in \mathcal{P}(D)$ let $\rho[x : X]$ be the environment which takes value X on x and coincides with ρ on all other variables. The interpretation $t^p : Env_D \rightarrow \mathcal{P}(D)$ of a λ -term t which is relative to (D, p) is defined by induction by:

- $x_\rho^p = \rho(x)$
- $(tu)_\rho^p = A_p(t_\rho^p)(u_\rho^p) = \{\alpha : (\exists a \subseteq u_\rho^p) p(a, \alpha) \in t_\rho^p\}$
- $(\lambda x.t)_\rho^p = \lambda_p(X \in \mathcal{P}(D) \mapsto t_{\rho[x:=X]}^p) = \{p(a, \alpha) : \alpha \in t_{\rho[x:=a]}^p\}$

Since t_ρ^p only depends on the value of ρ on the free variables of t , we only write t^p if t is closed. The following trivial example will be used in the Appendix.

Example 2 $(\lambda x.x)^p = \{p(a, \alpha) : \alpha \in a\}$

We turn now to the interpretation of $\Omega = \delta\delta$ in graph models, where $\delta = \lambda x.xx$. It is easy to check that the interpretation of Ω in P_ω and \mathcal{E} is \emptyset ; but, fortunately, this is not always the case. The following lemma gives one necessary condition and one sufficient condition for $\alpha \in D$ to be in the interpretation of Ω in (D, p) , but, first, two remarks on the interpretation of δ are in order.

Remark 3 (i) $p(a, \alpha) \in \delta^p \iff \alpha \in a$.

(ii) $(\alpha \in XX \text{ and } X \subseteq \delta^p) \implies \exists a \subseteq X (p(a, \alpha) \in X \text{ and } \alpha \in aa)$

Lemma 4 [5] Let (D, p) be a graph model and $\alpha \in D$, then:

(i) If $\alpha \in \Omega^p$, then there exists a such that $p(a, \alpha) \in a$.

(ii) If there exists $\beta \in D$ such that $p(\{\beta\}, \alpha) = \beta$, then $\alpha \in \Omega^p$.

Proof. (ii) By definition of application, $p(\{\beta\}, \alpha) = \beta$ implies $\alpha \in \{\beta\}\{\beta\}$, hence $p(\{\beta\}, \alpha) \in \delta^p$ (Remark 3 (i)); hence $\beta \in \delta^p$ and $\alpha \in \delta^p\delta^p = \Omega^p$, since application is monotone wrt inclusion.

(i) If $\alpha \in \Omega^p = \delta^p\delta^p$ then:

$\exists a_1 \subseteq \delta^p (p(a_1, \alpha) \in \delta \text{ and } \alpha \in a_1a_1)$ (Remark 3 (ii) with $X = \delta^p$)

$\exists a_2 \subseteq a_1 (p(a_2, \alpha) \in a_1 \text{ and } \alpha \in a_2a_2)$ (Remark 3 (ii) with $X = a_1$)

.....
 $\exists a_{n+1} \subseteq a_n (p(a_{n+1}, \alpha) \in a_n \text{ and } \alpha \in a_{n+1}a_{n+1})$ (Rem. 3 (ii) with $X = a_n$)

Now, since a_1 is a finite set and the sequence a_n is decreasing, there is an n such that $a_n = a_{n+1}$; hence $p(a_n, \alpha) \in a_n$. ■

A graph model (D, p) satisfies $t = u$, written $(D, p) \models t = u$, if $t^p = u^p$, or, equivalently, if $t_\rho^p = u_\rho^p$ for all environments ρ . The λ -theory $Th(D, p)$ induced by (D, p) is defined as

$$Th(D, p) = \{t = u : t, u \in \Lambda \text{ and } t^p = u^p\}.$$

A λ -theory induced by a graph model will be called a *graph theory*. A graph model is called *sensible* (rep. *semi-sensible*) if its theory is.

Notation 5 \mathcal{G} and \mathcal{G}_s are the classes of graph models and sensible graph models respectively, while λT , $\mathcal{G}T$, \mathcal{G}_sT are respectively the classes of λ -theories, graph theories, and sensible graph-theories.

2.3 Building graph models from partial pairs.

There are other classes of models that can be generated from webs, but graph models are the models with the simplest (=less structured) webs, and the most feasible to deal with the interpretation of terms. Some of these classes belong to the continuous semantics and include \mathcal{G} , others belong to other semantics (for example the Berry/Girard stable semantics). These classes of webbed models, as well as the techniques for studying these models and their λ -theories are surveyed in [14].

For proving the consistency of extensions of λ -calculus, or more generally for studying the lattice λT of λ -theories one is interested in building models submitted to specified equational or /and inequational constraints. The class of graph models offers a great wealth of models that are furthermore feasible. For this reason this is the first class of models to experiment with.

There are two known methods for building graph models, namely: by forcing or by canonical completion. Both methods can be extended to the other classes of webbed models (with more or less facility!), both methods consist in completing a partial pair into a total one, i.e. into a graph model.

In the setting of graph models, the *general definition of a partial pair* (see [14]), which allows to cover both methods, is the following: A *partial pair* is a pair (A, q) where A is any set and q is a partial (possibly total) injection from $A^* \times A$ to A , written $q : A^* \times A \rightarrow A$, examples of partial pairs are: all the graph models, and the empty pair (\emptyset, \emptyset) . For dealing only with the forcing method a more restricted definition is sufficient, which we will introduce later on.

The canonical completion method was, de facto, introduced by Plotkin and Engeler, since their model \mathcal{E} is nothing else than the canonical completion of (\emptyset, \emptyset) . It was systematized by Longo for graph models [37], who proved in particular that the graph model P_ω is the canonical completion of $(\{0\}, \{(\emptyset, 0), 0\})$, up to isomorphism. It was then used on a larger scale by Kerth, who used it, for example, to prove the existence of 2^ω pairwise inconsistent graph theories, and also transferred it to other semantics [33, 32, 35], and by Bucciarelli-Salibra in [17, 18]. *Canonical* refers here to the fact that the graph model (D, p) is built in an inductive (and “canonical”) way from the partial pair (A, q) we start with, and is completely determined by it. Furthermore, if the partial web is *positive* (in the sense of [14]) then (D, p) is sensible. Finally if one can apply the strong approximation theorem in the spirit of Hyland [26] and Wadsworth [51], which is the case for P_ω and \mathcal{E} , then $Th(D, p)$ is completely known: (D, p) equates two terms if and only if they have the same Böhm tree. For more details, and for the extension of the method to the main other classes of webbed models see [14].

The forcing method that we will present below, originates in Baeten-Boerboom [5]. In the simpler presentation proposed by Zylberajch [52], it starts

from a partial pair (D, p_0) ,¹ where D is an infinite countable set, and builds by induction a total $p : D^* \times D \rightarrow D$, hence a graph model (D, p) . The inductive construction depends here not only on p_0 but also on the consistency problem we are interested in, and it exploits heavily the fact that the interpretation of Ω can be quite freely constrained. The method was generalized to other classes of webbed models in Jiang [29, 30], Kerth [33, 34], and such a generalization was used by Bastonero to build an extensional model of the continuous semantic, whose theory could be realized neither by a model in the stable semantics nor by an hypercoherence model (such models belong to the strongly stable semantics) [8, 9]. It was also generalized to families of terms having a similar behavior as Ω by Zylberajch [52]. Note that, although (D, \emptyset) is a positive web, no model built by completing (D, \emptyset) by forcing will be sensible, and most of them will furthermore be clearly non semi-sensible.

A last difference between both methods is that if we start with a recursive partial web, the canonical completion will build a recursive total web (hence a graph model that can be viewed as a reasonable intersection type system), while forcing always create a non recursive web.

The partial interpretation method. In this paper we highlight the fact that the key reason why constructions by forcing are possible is that forcing induces a family of “weakly continuous functions” (see Definition 10). We also introduce the notion of a partial interpretation of a term and note that it induces a family of continuous functions. Hence partial interpretations can be used as an alternative to forcing to build models by using a similar method; in particular all the results proved in this paper can be obtained in both ways. The two notions are distinct (forcing is not continuous, as proved in the Appendix), but their use is essentially equivalent: in most cases it is a question of taste, even if sometimes one or the other can appear as more direct.

Convention. *From now on we will only deal with the forcing-like methods, and hence we will work with some fixed countable infinite set D .*

3 Generalized terms and easy sequences: basic definitions.

Generalized terms. In the next section we will extend the classic notion of easiness of Ω to a more general class of terms, which allows continuous functions of arbitrary arity as first-order function symbols. All the results proved in the remaining sections, could be proved working only with pure λ -terms (we first did it that way), but with more sophisticated tools. The interest of adding continuous functions in the language is that it allows for cleaner statements,

¹As a matter of fact $p_0 = \emptyset$ in [5] and in all the other authors quoted, but we will need here this more general setting.

simpler and more straightforward proofs, and that finally all the applications are evident corollaries. One may also wonder why adding genuine functions and not only elements of $\mathcal{P}(D)$ is necessary, since after all every function f is coded in each (D, p) by $\lambda_p(f) \in \mathcal{P}(D)$; once more the answer is that it is much simpler to do it that way.

Definition 6 *The set Λ_D of the generalized λ -terms (relatively to D), or gen-terms is defined as the smallest set such that:*

- (i) $V \subseteq \Lambda_D$, where V is the set of variables of Λ
 - (ii) $\mathcal{P}(D) \subseteq \Lambda_D$
 - (iii) if $t, u \in \Lambda_D$, then tu is in Λ_D
 - (iv) if $t \in \Lambda_D$ and $x \in V$ then $\lambda x.t \in \Lambda_D$
 - (v) if $f \in [\mathcal{P}(D)^n \rightarrow \mathcal{P}(D)]$, $1 \leq n$, and $\bar{t} \in \Lambda_D^n$, then $f(\bar{t}) \in \Lambda_D$.
- Λ_D° is defined as the set of closed gen-terms.

Recall that Λ was the set of terms obtained when removing (ii) and (v) from the above definition. Hereafter the element of Λ will be called *pure terms*.

Thus, f is not a gen-term, while $\lambda \bar{x}.f(\bar{x})$ is. To be more formal we should have introduced one new symbol for each element of $\mathcal{P}(D) \cup \cup_{n \in \omega} [\mathcal{P}(D)^n \rightarrow \mathcal{P}(D)]$. A *redex* is a gen-term of the form $(\lambda x.t)u$, where t, u are gen-terms, and its reduct is defined as usual. We extend β -equivalence to gen-terms in a straightforward way: we just add to the usual rules the fact that it should be a congruence also with respect to the first-order functions, in other words $t_1 =_\beta t'_1, \dots, t_n =_\beta t'_n$ should imply $f(t_1, \dots, t_n) =_\beta f(t'_1, \dots, t'_n)$; in particular no rule taking account of the evaluation of functions is given at the syntactic level. *The interpretation t^p* of the gen-term t in the graph model (D, p) is once more defined by induction on t . Cases (i), (iii) and (iv) are as in Section 2.2, while the interpretations of $X \subseteq D$ and $f(t_1, \dots, t_n)$ are the obvious ones:

$$X_\rho^p = X; \quad f(t_1, \dots, t_n)_\rho^p =_{def} f((t_1)_\rho^p, \dots, (t_n)_\rho^p).$$

It is clear that this interpretation coincides with that of Section 2.2 for pure λ -terms. Satisfaction in (D, p) of an equation $t = t'$, for $t, t' \in \Lambda_D$ is defined as usual by $t_\rho^p = t'_\rho^p$ for all ρ . It is then clear that any graph model equates β -equivalent gen-terms and respects the behavior of the added functions: if f is an n -ary continuous function which takes value $Y \in \mathcal{P}(D)$ on $X_1, \dots, X_n \in \mathcal{P}(D)$ then all graph models on D will satisfy $f(X_1, \dots, X_n) = Y$. Furthermore it is easy to check that

$$(D, p) \models \lambda \bar{x}.f(\bar{x}) = \lambda_p(f),$$

where $\lambda_p(f)$ is the code of f in (D, p) .

Partial interpretations. We extend the notion of interpretation of a gen-term from total pairs to partial pairs. In the sequel we will always have the choice between using total interpretations plus forcing, or partial interpretations (and no forcing).

Definition 7 Let (D, q) be a partial pair. Given $t \in \Lambda_D$ we define t^q by induction on t :

- (i) $x_\rho^q = \rho(x)$
- (ii) $X_\rho^q = X$
- (iii) $(tu)_\rho^q =_{def} \{ \alpha \in D : (\exists a \subseteq u_\rho^q) [(a, \alpha) \in \text{dom}(q) \wedge q(a, \alpha) \in t_\rho^q] \}$
- (iv) $(\lambda x.t)_\rho^q = \{ q(c, \gamma) \in D : (c, \gamma) \in \text{dom}(q) \wedge \gamma \in t_{\rho[x:c]}^q \}$
- (v) $(f(t_1, \dots, t_n))_\rho^q = f((t_1)_\rho^q, \dots, (t_n)_\rho^q)$

We write t^q for t_ρ^q if $t \in \Lambda_D^\circ$ is a closed gen-term.

Easy sequences of terms. We now define easy sequences of terms.

Given two sequences \bar{t} and \bar{t}' of the same length, we denote by $\bar{t} = \bar{t}'$ the set consisting of all the equations $t_k = u_k$.

Definition 8 Let \bar{s} be a (possibly infinite) sequence of closed pure λ -terms, then:

- (i) \bar{s} is easy if for all sequences $\bar{t} \in (\Lambda^\circ)^{l(\bar{s})}$ the set $\bar{t} = \bar{s}$ is consistent.
- (ii) \bar{s} is graph easy if for all sequences $\bar{t} \in (\Lambda^\circ)^{l(\bar{s})}$ there is a graph model satisfying $\bar{t} = \bar{s}$.
- (iii) \bar{s} is functionally graph easy if for all countable sets D and all sequences $\bar{t} \in (\Lambda_D^\circ)^{l(\bar{s})}$ there is a graph model of web D satisfying $\bar{t} = \bar{s}$.

Of course (iii) \implies (ii) \implies (i).

4 Baeten-Boerboom's proof revisited.

4.1 Weakly continuous operators are the point.

We observe here that Baeten-Boerboom's proof, in Zylberajch's style, works for any weakly continuous operator (instead of forcing) and that easyness with respect to all closed gen-terms hold.

Notation 9 Q is the cpo of partial (including total) injections $q : D^* \times D \rightarrow D$, partially ordered by inclusion of their graphs.

By "a total p " we will always mean "an element of Q which is total" (equivalently: which is a maximal element of Q). The domain and range of $q \in Q$ are denoted by $\text{dom}(q)$ and $\text{range}(q)$, we will also confuse the partial injections and their graphs.

Given any function $H : Q \rightarrow \mathcal{P}(S)$, S any set, we will use H_q for $H(q)$ when more convenient.

Definition 10 A function $H : Q \rightarrow \mathcal{P}(S)$, where S is any countable infinite set, is weakly continuous if it is monotone with respect to inclusion and if furthermore, for all total $p \in Q$ and $\alpha \in H(p)$, there is a finite $q \subseteq p$ such that $\alpha \in H(q)$.

Since we are working with a countable D , the difference with continuity comes of course from the fact that there exist infinite elements of Q which are not total.

Theorem 11 *Given any weakly continuous function $H : Q \rightarrow \mathcal{P}(D)$, there is a total p such that $(D, p) \models \Omega = H_p$.*

Proof. We are going to build an increasing sequence of partial injective maps p_n , starting from p_0 , and a sequence of elements $\alpha_n \in D \cup \{v\}$, where v is some new element, such that: $p =_{def} \cup p_n$ is a total injection (in fact a bijection), and $(D, p) \models \Omega = A = H_p$, where $A =_{def} \{\alpha_n / n \in \omega\} \cap D$.

We fix an enumeration of D , and an enumeration of $D^* \times D$.

We start from $p_0 = \emptyset$.

Assume that p_n and $\alpha_0, \dots, \alpha_{n-1}$ have been built.

Let α_n be the first element of $H_{p_n} - \{\alpha_0, \dots, \alpha_{n-1}\}$ if this set is non-empty, and v otherwise.

Let (b_n, δ_n) be the first element in $D^* \times D - \text{dom}(p_n)$ and γ_n be the first element in $D - (\text{range}(p_n) \cup b_n)$.

Case 1. $\alpha_n = v$ we let

$$p_{n+1} = p_n \cup \{(b_n, \delta_n), \gamma_n\}$$

Case 2. $\alpha_n \in D$ we let :

$$p_{n+1} = p_n \cup \{(b_n, \delta_n), \gamma_n, (\{\beta_n\}, \alpha_n), \beta_n\}$$

where β_n is the first element of D such that :

$$\begin{aligned} (\{\beta_n\}, \alpha_n) &\in D^* \times D - (\text{dom}(p_n) \cup \{(b_n, \delta_n)\}) \text{ and} \\ \beta_n &\in D - (\text{range}(p_n) \cup \{\gamma_n\}) \end{aligned}$$

It is clear that p_n is a strictly increasing sequence of well-defined partial injective maps and that $p = \cup p_n$ is total. It is also surjective since there are infinitely many elements of $D^* \times D$ of the form (\emptyset, δ) , $\delta \in D$: these elements are successively introduced at steps, say, n_k (where the n_k form a strictly increasing sequence of integers), and are then given as image the first element in $D - \text{range}(p_{n_k})$, hence the k -th element of D will necessarily belong to $\text{range}(p_{n_{k+1}})$.

There remains to see that $(D, p) \models \Omega = A = H_p$.

$A \subseteq H_p$ follows from the definition of α_n and from the fact that $H_{p_n} \subseteq H_p$.

$H_p \subseteq A$: suppose $\gamma \in H_p$; then, since H is weakly continuous, $\gamma \in H_{p_m}$ for some m (and for all the larger ones). If $\gamma \notin A$ then, for all $n \geq m$, $\alpha_n \in D$ has smaller rank than γ in the enumeration of D , contradicting the fact that there is only a finite number of such elements.

$A \subseteq \Omega^p$: $\alpha_n \in \Omega^p$ follows immediately from the fact that $(\{\beta_n\}, \alpha_n), \beta_n \in p_{n+1} \subseteq p$ and from Lemma 4 (ii).

$\Omega^p \subseteq A$: if $\varepsilon \in \Omega^p$ then there is an $a \in D^*$ such that $p(a, \varepsilon) \in a$ (by Lemma 4 (i)). Since $p = \cup p_n$ and because of the choices of the γ_n this can only occur if ε is one of the α_n . ■

For showing the existence of infinite graph easy sequences we will need to have available the following slight extension of Theorem 11.

Definition 12 $p_0 \in Q$ is free for Ω if:

- (i) $D^* \times D - \text{dom}(p_0)$ and $D - \text{range}(p_0)$ are infinite, and
- (ii) $(a, \alpha) \in \text{dom}(p_0)$ implies $p_0(a, \alpha) \notin a$.

Theorem 13 If $H : Q \rightarrow \mathcal{P}(D)$ is weakly continuous and $p_0 \in Q$ is free for Ω , then there is a total $p \supseteq p_0$ such that $(D, p) \models \Omega = H_p$.

Proof. Indeed, the proof of Theorem 11 only used that \emptyset was free for Ω . ■

We now show that Theorem 11 and Theorem 13 can be applied to two different classes of functions $H : Q \rightarrow \mathcal{P}(D)$ arising respectively from forcing (as defined below) and partial interpretation (as defined in Definition 7).

Definition 14 (Forcing) For $t \in \Lambda_D^\circ$, $q \in Q$ and $\alpha \in D$, the abbreviation $q \Vdash \alpha \in t$ means that for all total injections $p \supseteq q$ we have that $(D, p) \models \alpha \in t^p$. Furthermore $q \Vdash X \subseteq t$ means that $q \Vdash \alpha \in t$ for all $\alpha \in X$.

Thus, for p is total, $p \Vdash \alpha \in t$ if and only if $\alpha \in t^p$. Moreover if $q_i \Vdash \alpha_i \in t$ for all $i \in I$ then $\cup q_i \Vdash \{\alpha_i : i \in I\} \subseteq t$.

Lemma 15 For all $t \in \Lambda_D^\circ$ the function $F_t : Q \rightarrow \mathcal{P}(D)$ defined by $F_t(q) = \{\alpha \in D / q \Vdash \alpha \in t\}$ is weakly continuous, and we have $F_t(p) = t^p$ for each total p .

Proof. The proof of the weak continuity of F_t is a straightforward induction on the complexity of the closed gen-term t ; we detail it anyway.

If t is an element X of $\mathcal{P}(D)$ then F_t is the constant function with value X .

Let now $p \in Q$ be total.

If $t = uv$ and $\alpha \in t^p$, then there exists $a \subseteq v^p$ such that $p(a, \alpha) \in u^p$. Choose such an a and let $\gamma = p(a, \alpha)$. By induction hypothesis there is a finite $q \subseteq p$ such that $q \Vdash a \subseteq v$ and a finite $r \subseteq p$ such that $r \Vdash \gamma \in u$; then it is clear that $q \cup r \cup \{(a, \alpha), \gamma\} \Vdash \alpha \in t$.

If $t = \lambda x.u$ and $\alpha \in t^p$ then there is a unique pair (b, β) such that $\alpha = p(b, \beta)$ and $\beta \in u[x : b]^p$. By induction hypothesis there is a finite $q \subseteq p$ such that $q \Vdash \beta \in u[x : b]$; then it is clear that $q \cup \{(b, \beta), \alpha\} \Vdash \alpha \in t$.

If $t = f(t_1, \dots, t_n)$ and $\alpha \in f(t_1, \dots, t_n)^p = f(t_1^p, \dots, t_n^p)$, then from the continuity of f it follows the existence of finite $b_1 \subseteq t_1^p \dots b_n \subseteq t_n^p$ such that $\alpha \in f(b_1, \dots, b_n)$. Since the b_i 's are finite and the t_i 's are of lower complexity than t , there are finite $q_1, \dots, q_n \subseteq p$ such that $q_i \Vdash b_i \subseteq t_i$ for all i ; then we clearly have $q \Vdash b_i \subseteq t_i$ for all i , where $q =_{def} \cup \{q_i : i \leq n\}$. The conclusion

$q \Vdash \alpha \in f(t_1, \dots, t_n)$ follows from $\alpha \in f(b_1, \dots, b_n)$, $q \Vdash b_i \subseteq t_i$ for all i , and the monotonicity of f . ■

We note that the function F_t defined in the above lemma is not continuous as shown in Appendix 7.

Lemma 16 *For all $t \in \Lambda_D^\circ$, the function $I_t : Q \rightarrow \mathcal{P}(D)$ defined by $I_t(q) = t^q$ is continuous (where t^q is the interpretation of the gen-term t in the partial pair (D, q) as defined in Definition 7).*

Proof. The proof of the continuity of I_t is a straightforward induction on the complexity of the closed gen-term t . ■

4.2 Easy terms.

In this section we show that the λ -term Ω is functionally easy. Then every continuous function on $\mathcal{P}(D)$ is λ -represented by Ω .

Theorem 17 *Ω is functionally easy, that is, for all closed gen-terms $t \in \Lambda_D^\circ$ there is a p such that $(D, p) \models \Omega = t$.*

Proof. It is enough to apply Theorem 11 either to the weakly continuous function F_t defined in Lemma 15 or to the continuous function I_t defined in Lemma 16. ■

Let us give now a few applications of this result.

The following is the classic result by Baeten and Boerboom.

Corollary 18 [5] *Ω is easy, that is, for all closed pure terms $t \in \Lambda^\circ$ there is a p such that $(D, p) \models \Omega = t$.*

Definition 19 (i) *A continuous function f on $\mathcal{P}(D)$ is λ -represented by a pure term $t \in \Lambda^\circ$ in a graph model (D, p) if $(D, p) \models t = \lambda\bar{x}.f(\bar{x})$.*

(ii) *A (possibly infinite) sequence \bar{f} of continuous function is λ -represented by $\bar{t} \in (\Lambda^\circ)^{l(\bar{f})}$ in (D, p) if (D, p) satisfies $t_k = \lambda\bar{x}.f_k(\bar{x})$ for all k .*

The preceding definition would of course trivialize if the term t in (i) and the sequence \bar{t} of terms in (ii) were asked to be gen-terms.

Corollary 20 *Each continuous function f on $\mathcal{P}(D)$ is λ -represented by Ω in some graph model.*

Proof. From Theorem 17 there is a graph model satisfying $\Omega = \lambda\bar{x}.f(\bar{x})$, which implies clearly that Ω represents f in this graph model. ■

The least fixed point operator on a cpo C is the continuous function $L \in [[\mathcal{C} \rightarrow \mathcal{C}] \rightarrow \mathcal{C}]$ defined by $L(f) = \cup_{n \in \omega} f^n(\perp)$, where \perp is the least element of

the C . Using the formalism of intersection type systems and filter models, Alessi et al. [3] proved that there exists a reflexive cpo where L is represented by Ω , in the sense that the least fixed point operator of the underlying cpo is the interpretation of Ω in the model. It is hence interesting to note that we can get this result in a more economical way, and with a simpler model.

Corollary 21 *There is a graph-model where Ω represents L .*

Proof. The smallest element of the cpo $\mathcal{P}(D)$ is $\perp = \emptyset$. By Corollary 20 there is a graph model (D, p) where Ω represents the unary continuous function defined on $\mathcal{P}(D)$ by: $h(X) = \cup_{n \in \omega} X^n \emptyset$ (where e.g. $X^2 \emptyset$ means $X(X\emptyset)$). But, then, for all unary continuous function f , we automatically have: $[\Omega(\lambda x.f(x))]^p = \cup_{n \in \omega} f^n(\emptyset)$. Thus Ω represents L in (D, p) . ■

We now define easy sequences of terms.

4.3 Finite easy sequences of terms.

The existence of finite easy sequences of pure terms could be proved without using generalized terms (using variations of the tools built in Section 5), and it will also show up as a particular case of a result proved in Section 5.3. But in the present setting, which allows for the use of the standard tricks in λ -calculus, it appears as a direct corollary of Theorem 17.

Theorem 22 *For each $n \in \omega$ there is a sequence $\bar{u} \in (\Lambda^\circ)^n$ of pure terms such that $(\Omega u_k)_{k \leq n}$ is a functionally graph easy sequence.*

Proof. We only treat the case $n = 2$, and claim that the two projections $T = \lambda x.\lambda y.x$ and $F = \lambda x.\lambda y.y$ work. Using Theorem 17, for all gen-terms t_1 and t_2 , we get a graph model satisfying $\Omega = \lambda z.zt_1t_2$. Then it is clear that in the same graph model we have that $\Omega T = t_1$ and $\Omega F = t_2$. ■

Proposition 23 *For each $n \in \omega$ there is a sequence $\bar{u} \in (\Lambda^\circ)^n$ of pure terms such that each sequence \bar{f} of continuous functions on $\mathcal{P}(D)$ is λ -represented by $(\Omega u_k)_{k \leq n}$ in some graph model over D .*

Proof. By the above theorem. ■

Corollary 24 *The pair (\cup, \cap) consisting of union and intersection on $\mathcal{P}(D)$, is λ -represented by ΩT and ΩF .*

Proof. Immediate consequence of the preceding corollary since \cup, \cap are continuous functions (continuity of \cap follows from the fact that $\mathcal{P}(D)$ is a distributive lattice). ■

Interesting applications of this result to the structure of the lattice of lambda theories are shown in Section 6.1.

In [22] Dezani-Lusin have shown the existence of a filter model of lambda calculus representing union, and the existence of a filter model representing intersection could be derived along the same way; but the existence of a unique model for both, as we have shown in Corollary 24, was left open.

5 Infinite easy sequences of terms.

In Section 4.3 we have proved the existence of functionally graph easy sequences of each finite length (Theorem 22). In this section we introduce the two technical notions of *flattening* and *osr* which give us another way to obtain easy sequences, and, in particular, to prove the existence of *infinite* easy sequences.

5.1 Flattenings.

Notation 25 Let $E =_{def} \bigcup_{n \geq 0} E_n$ where the E_n are defined by: $E_0 =_{def} D$ and $E_{n+1} =_{def} (E_n^* \times E_n) \cup E_n$.

It is easy to check that $E_{n+1} =_{def} (E_n^* \times E_n) \cup D$, and also that $tr(g) \subseteq E_{n+1}$ for any continuous function g of arity n .

It is also easy to check that each $q \in Q$ extends to a partial function $f_q : E \rightarrow D$, satisfying the following properties:

- (i) $f_q(x) =_{def} x$ if $x \in D$;
- (ii) $f_q(e, \varepsilon) =_{def} q(f_q^+(e), f_q(\varepsilon))$ if $e \cup \{\varepsilon\} \subseteq dom(f_q)$ and $(f_q^+(e), f_q(\varepsilon)) \in dom(q)$, undefined otherwise, where: $f_q^+(e) =_{def} \{f_q(x) / x \in e\}$.

Thus $f_q(x) \neq \perp$ if and only if q is hereditarily defined on all the internal components of x . We define now a total function $f_q^\bullet : \mathcal{P}(E) \rightarrow \mathcal{P}(D)$.

Notation 26 $f_q^\bullet(G) =_{def} \{f_q(x) / x \in G \cap dom(f_q)\}$, for any $G \subseteq E$.

Definition 27 For $x \in E$ and $G \subseteq E$ we will call respectively $f_q(x)$ and $f_q^\bullet(G)$ the q -flattening of x and G .

Example 28 For all $q \in Q$ and $G \subseteq D$ we have $f_q^\bullet(G) = G$.

In particular, for all $t \in \Lambda_D^\circ$ we have $f_q^\bullet(t^q) = t^q$.

We will see more sophisticated examples below (but these ones are relevant for our purpose).

Lemma 29 The function $f : E \times Q \rightarrow D_\perp$, defined by $f(x, q) = f_q(x)$ if $x \in dom(f_q)$ and \perp otherwise, is continuous with respect to q .

Proof. Since D_\perp is flat, continuity is here equivalent to saying that:

- (i) If $q \subseteq q'$ and $f_q(x)$ is defined then $f_{q'}(x)$ is defined and $f_{q'}(x) = f_q(x)$.

(ii) If q is the increasing union of a sequence $(q_n)_{n \in \omega}$ then there exists n such that $f_q(x) = f_{q_n}(x)$.

The first point is trivial, and the second follows easily from the fact that the computation of $f_q(x)$ requires only a finite part of the graph of q . ■

Lemma 30 *The function $f^\bullet : \mathcal{P}(E) \times Q \rightarrow \mathcal{P}(D)$, defined by $f^\bullet(G, q) = f_q^\bullet(G)$, is continuous. It is even additive wrt the first component G (i.e. commutes with all unions).*

Proof. A binary function is continuous iff it is continuous componentwise. Continuity with respect to q follows easily from the preceding lemma, and additivity with respect to G is trivial. ■

Lemma 31 *Let $G : Q \rightarrow \mathcal{P}(E)$ and let $H : Q \rightarrow \mathcal{P}(D)$ be defined by $H_q = f_q^\bullet(G_q)$ for all $q \in Q$, then:*

- (i) *If G is continuous then H is also continuous,*
- (ii) *If G is weakly continuous then H is weakly continuous.*

Proof. Since $H =_{def} f^\bullet \circ (id \times G)$ the monotonicity of H follows from the monotonicity of G , and similarly for continuity, using the continuity of f^\bullet (previous Lemma). Suppose now that G is only weakly continuous and suppose $\alpha \in H_p$, where p is a total injection. By definition of H there is an $x \in G(p) \subseteq E$ such that $\alpha = f_p(x)$. By Lemma 29 there is a finite $s \subseteq p$ such that $\alpha = f_s(x)$; furthermore, since G is weakly continuous there is a finite $r \subseteq p$ such that $x \in G(r)$. Then, if $q = r \cup s$ we have $\alpha \in H_q = f_q^\bullet(G_q)$, by monotonicity of f^\bullet and G . ■

5.2 Orthogonal system of representatives.

Definition 32 *Let $\bar{u} \in (\Lambda^\circ)^{\leq \omega}$ be a sequence of closed pure terms and $p_0 \in Q$ be free for Ω . The sequence $\bar{\varepsilon} \in E^{\leq \omega}$ is an orthogonal system of representatives (an osr, for short) for \bar{u} modulo p_0 if: $l(\bar{\varepsilon}) = l(\bar{u})$ and, for all $j, k \leq l(\bar{u})$ and all total $p \supseteq p_0$, we have $f_p(\varepsilon_k) \in u_j^p$ iff $k = j$.*

Definition 33 *The sequence \bar{u} admits an osr if there exist $\bar{\varepsilon}, p_0$ such that $\bar{\varepsilon}$ is an osr for \bar{u} modulo p_0 .*

Of course not all sequences of pure terms admit an *osr*. Examples of finite and infinite sequences of (simple) terms admitting an osr will be given in Lemma 36 below. It is clear that any subsequence or permutation of a sequence admitting an osr also admits an osr. The interest of the notion of osr comes from the fact that, for all $\bar{u} \in \Lambda^{\leq \omega}$ admitting an osr, the sequence $(\Omega u_k)_{k \leq l(\bar{u})}$ is functionally graph easy (Theorem 37 in the next section).

Notation 34 $\pi_{n,k} =_{def} \lambda x_1 \dots \lambda x_n . x_k \in \Lambda$, for $1 \leq k \leq n$.
 $\pi'_k =_{def} \pi_{k+1, k+1}$, for $k \in \omega$.

Notation 35 Let α be some fixed element of D .

$$\varepsilon_{n,k} =_{def} (\emptyset^{k-1}\{\alpha\}\emptyset^{n-k}, \alpha) \in E, \text{ for } 1 \leq k \leq n.$$

$$\varepsilon'_k =_{def} \varepsilon_{k+1,k+1} = (\emptyset^k\{\alpha\}, \alpha) \in E, \text{ for } k \in \omega.$$

(where $(\bar{a}_1\bar{a}_2\dots\bar{a}_n, \alpha)$ is defined as (\bar{a}, α) where \bar{a} is the concatenation of $\bar{a}_1, \bar{a}_2, \dots, \bar{a}_n$).

Lemma 36 (i) $(\varepsilon_{n,k})_{k \leq n}$ is an osr for $(\pi_{n,k})_{k \leq n}$ modulo \emptyset .

(ii) $(\varepsilon'_k)_{k \in \omega}$ is an osr for $(\pi'_k)_{k \in \omega}$ modulo $p_0 = \{((\emptyset, \alpha), \alpha)\}$.

Proof. (i) is clear, by definition of $\pi_{n,k}^p$.

(ii) Suppose that p is total and $p(\emptyset, \alpha) = \alpha$. Then it is easy to check successively that (D, p) satisfies:

1. $\alpha \notin (\lambda x.x)^p$.
2. $\{\alpha\}\emptyset^n = \{\alpha\}$ for all $n \geq 0$.
3. $\{\alpha\} = \{p(\{\alpha\}, \alpha)\}\{\alpha\}$.
4. $\forall n \geq 0 (\alpha \notin \pi'_n)$
(this follows from 1,2, and the monotonicity of application).
5. $\forall n \geq 1 (p(\{\alpha\}, \alpha) \notin \pi'_n)$
(this follows from 1,3, and the monotonicity of application).
6. $f_p(\emptyset^m, \{\alpha\}, \alpha) \in \pi'_n$ iff $m = n$.
(the case $n > m$ is excluded by 5 and the monotonicity of application, and $m > n$ contradicts 4).

■

5.3 Infinite easy sequences of terms.

Theorem 37 For all $\bar{u} \in (\Lambda^\circ)^{\leq \omega}$ admitting an osr, the sequence $(\Omega u_k)_{k \leq l(\bar{u})}$ is functionally graph easy (and then easy).

Proof. Let $\bar{\varepsilon}, p_0$ be such that $\bar{\varepsilon}$ is an osr for \bar{u} modulo p_0 , and let $\bar{t} \in \Lambda_D^{l(\bar{u})}$. For all $q \in Q$, let $G_q = \{(\{\varepsilon_k\}, \alpha) / 1 \leq k \leq l(\bar{t}), \alpha \in t_k^q\} \subseteq E$. Since G_q is essentially the disjoint union of the subsets t_k^q of D , which are continuous wrt q , the function $G : Q \rightarrow \mathcal{P}(E)$ is continuous. From Lemma 31 the function F defined by $F(q) = f_q^\bullet(G_q)$ is also continuous. From Theorem 13 there is a total $p \supseteq p_0$ such that $\Omega^p = f_p^\bullet(G_p)$. Now, since $\bar{\varepsilon}$ is an osr for \bar{u} relatively to p_0 we have that $f_p^\bullet(G_p).u_k = f_p^\bullet(G_p).\{\varepsilon_k\} = f_p^\bullet(t_k^p) = t_k^p$ (by definition of application in (D, p)), thus $(\Omega u_k)^p = t_k^p$, and $(D, p) \models \Omega u_k = t_k$ for all k .

The alternative proof using forcing works in a similar way, using case (ii) of Lemma 31. ■

Recall that the pure λ -terms π'_k are defined in Notation 34.

Corollary 38 *The infinite sequence $(\Omega\pi'_k)_{k \geq 0}$ is functionally graph easy.*

Corollary 39 *For all infinite sequences \bar{g} of continuous functions on $\mathcal{P}(D)$, there is a graph model (D, p) such that for all k we have: $(D, p) \models \Omega\pi'_k = \lambda\bar{x}.g_k(\bar{x})$, where $l(\bar{x})$ is the arity of g_k .*

As an application we recover the following Kerth's result for free, which asserts that \mathcal{GT} is as "wide" as possible. Note that the proof in [31] was via canonical completion of partial pairs, and, even if not difficult, required some non trivial observations, and some computations.

Corollary 40 [31] *There exist 2^ω pairwise inconsistent graph theories.*

Proof. Let \bar{s} be an infinite graph easy sequence and let \bar{t} be the sequence of Church integers. For any permutation σ on usual integers let p_σ be such that $(D, p_\sigma) \models s_k = t_{\sigma(k)}$ for all k . It is clear that the graph models (D, p_σ) are non equationally equivalent, and that their theories are pairwise inconsistent. ■

But Kerth-David's result which asserts the existence of 2^ω sensible graph theories, mentioned in the introduction, is out of the scope of our techniques.

6 Applications.

In this section we show that there exist

1. a distributive sublattice of the lattice of λ -theories, and
2. a congruence distributive variety of lambda abstraction algebras.

6.1 The Lattice of λ -Theories.

The set of the λ -theories ordered by inclusion is naturally equipped with a structure of complete lattice (see Section 2). The lattice λT of λ -theories has a very rich and complex structure. For example, Visser [50] has shown in first eighties that every countable partially ordered set embeds into λT by an order-preserving map, and that every interval of λT , whose bounds are recursively enumerable lambda theories, has a continuum of elements.

Lusin-Salibra [38] and Salibra [45] have employed results and techniques from universal algebra, in particular commutator theory and the theory of Mal'cev conditions, to obtain some results characterizing the structure and the equational theory of the lattice of lambda theories. Very little had previously been known about the equational theory of this lattice.

We briefly outline the approach developed in [38]. Consider the absolutely free algebra of pure terms:

$$\mathbf{\Lambda} := (\Lambda, \cdot^\Lambda, \lambda x^\Lambda, x^\Lambda)_{x \in Va}, \quad (3)$$

where Λ is the set of pure terms over an infinite set Va of variables and, for all $M, N \in \Lambda$,

$$M \cdot^{\Lambda} N = (MN); \quad \lambda x^{\Lambda}(M) = (\lambda x.M); \quad x^{\Lambda} = x.$$

An equivalence relation T over the set Λ of pure terms is a lambda theory if, and only if, it is a congruence over Λ including (α) and (β) -conversion. For every lambda theory T , the congruence lattice of the term algebra $\mathbf{\Lambda}^T$, the quotient of Λ by T , is isomorphic to the interval sub-lattice $[T] = \{S : T \subseteq S\}$ of the lattice of the lambda theories. In particular, the isomorphism between the lattice λT and the congruence lattice of $\mathbf{\Lambda}^{\lambda\beta}$ is the starting point for studying the structure of λT by universal algebraic methods.

It was shown by Salibra [45] that the lattice λT is not modular, i.e., it does not satisfy the following modular law (that is a weakened form of distributivity):

$$T(S + TR) = TS + TR,$$

while Lusin and Salibra [38] have shown that λT satisfies interesting quasi-identities in the language of bounded lattices. For example, the following quasi-identity holds in the lattice λT :

$$S + T = 1, \quad SG = TG \rightarrow G = GS = GT,$$

where 1 is the inconsistent lambda theory.

The same authors have shown in [38] that there exists a sub-lattice of the lattice λT which satisfies the following restricted form of distributivity (called meet semidistributivity)

$$TR = TS \rightarrow TR = T(R + S),$$

and a nontrivial identity in the language of lattices enriched by the composition of binary relations.

In [38] it was conjectured that the lattice λT does not satisfy any nontrivial lattice identity. To support this conjecture the authors have shown in [38] that, for every nontrivial lattice identity e , there exists a natural number n such that e fails in the lattice of lambda theories in a language of λ -terms with n constants. We can relax the above conjecture by asking whether there exists an infinite sub-lattice of λT satisfying interesting lattice identities. In this section we show that there exists an infinite distributive sub-lattice of λT .

First we introduce a lambda theory \mathcal{L} , whose consistency is obtained by using the methods introduced in the previous sections. Then the equations defining \mathcal{L} are used to define lattice term operations on the term algebra $\mathbf{\Lambda}^{\lambda\beta}$, the quotient of Λ by the congruence \mathcal{L} . It follows from this result that the lattice of all lambda theories including \mathcal{L} is distributive.

Recall that $T =_{def} \lambda xy.x$ and $F =_{def} \lambda xy.y$.

Lemma 41 *The lambda theory \mathcal{L} , axiomatized by*

1. $\Omega Txx = x; \quad \Omega Fxx = x.$
2. $\Omega Txy = \Omega Tyx; \quad \Omega Fxy = \Omega Fyx.$
3. $\Omega Tx(\Omega Tyz) = \Omega T(\Omega Txy)z; \quad \Omega Fx(\Omega Fyz) = \Omega F(\Omega Fxy)z.$
4. $\Omega Tx(\Omega Fxy) = x; \quad \Omega Fx(\Omega Txy) = x.$
5. $\Omega Tx(\Omega Fyz) = \Omega F(\Omega Txy)(\Omega Txz); \quad \Omega Fx(\Omega Tyz) = \Omega T(\Omega Fxy)(\Omega Fxz).$

is consistent.

Proof. From Corollary 24 it follows that there exists a graph model (D, p) , where the set-theoretical union and intersection are λ -represented by ΩT and ΩF . Since $(\mathcal{P}(D), \cup, \cap)$ is a distributive lattice, then the λ -theory \mathcal{L} is contained in the theory $Th(D, p)$ of (D, p) . ■

Lemma 42 *The congruence lattice of the term algebra $\mathbf{\Lambda}^{\mathcal{L}}$ is isomorphic to the interval sub-lattice $[\mathcal{L}] = \{T : \mathcal{L} \subseteq T\}$ of the lattice of lambda theories.*

Proof. A λ -theory T satisfying the condition $\mathcal{L} \subseteq T$ can be interpreted as a congruence \equiv_T on the term algebra $\mathbf{\Lambda}^{\mathcal{L}}$ (see [38]): for every $\tau, \sigma \in \mathbf{\Lambda}^{\mathcal{L}}$, $\tau \equiv_T \sigma$ if, and only if, there exist pure terms $t \in \tau$ and $u \in \sigma$ such that $T \vdash t = u$ (recall that τ, σ are equivalence classes of pure terms). ■

Lemma 43 *Let \mathbf{A} be any algebra. If \mathbf{A} admits two binary term operations satisfying the axioms of a distributive lattice, then the congruence lattice of \mathbf{A} is distributive.*

Proof. Let A be the universe of the algebra \mathbf{A} , and $+, \cdot$ be the binary term operations of \mathbf{A} satisfying the axioms of a distributive lattice. It is well known that the congruence lattice of every distributive lattice is distributive (see [39]), so that the congruence lattice of the algebra $(A, +, \cdot)$ is distributive. We get the conclusion if we show that the congruence lattice of \mathbf{A} is a sub-lattice of the distributive congruence lattice of the algebra $(A, +, \cdot)$. First every congruence over \mathbf{A} is a congruence over $(A, +, \cdot)$, because “+” and “.” are term operations. This implies that the set of congruences over \mathbf{A} is a subset of the set of congruences over $(A, +, \cdot)$. The conclusion is now immediate because the meet and the join in both congruence lattices are the same: they are defined set-theoretically as intersection and least equivalence relation. ■

As a matter of notation, for every lambda theory T , we denote by $[t]_T$ the equivalence class of the pure terms u such that $T \vdash t = u$.

Theorem 44 *The interval sub-lattice $[\mathcal{L}] = \{T : \mathcal{L} \subseteq T\}$ of the lattice of lambda theories is distributive.*

Proof. By Lemma 42 it is sufficient to prove that the congruence lattice of the term algebra $\mathbf{\Lambda}^{\mathcal{L}}$ is distributive. Define the following term operations over $\mathbf{\Lambda}^{\mathcal{L}}$, for every $\tau, \sigma \in \mathbf{\Lambda}^{\mathcal{L}}$:

$$\tau + \sigma = [\Omega Tts]_{\mathcal{L}}; \quad \tau \cdot \sigma = [\Omega Fts]_{\mathcal{L}}, \quad \text{for some } t \in \tau \text{ and } s \in \sigma. \quad (4)$$

Then it is easy to verify by using the axioms defining \mathcal{L} that the term operations “+” and “ \cdot ” satisfy the axioms of a distributive lattice. For example, the identity $\Omega T x(\Omega F x y) = x$, specified in Lemma 41(4), corresponds to the absorption law $x + (x \cdot y) = x$, while the identity $\Omega T x(\Omega F y z) = \Omega F(\Omega T x y)(\Omega T x z)$, specified in Lemma 41(5), corresponds to the distributive law $x + (y \cdot z) = (x + y) \cdot (x + z)$. Then the term algebra $\mathbf{\Lambda}^{\mathcal{L}}$ satisfies the hypothesis of Lemma 43, so that it admits a distributive congruence lattice. ■

6.2 Lambda abstraction algebras.

Another application of the main results of the paper that we develop here concerns the variety (i.e., equational class) of lambda abstraction algebras (**LAA**’s) introduced by Pigozzi and Salibra in [40, 41] as a purely algebraic theory of the untyped lambda calculus. The equational theory of **LAA**’s is intended as an alternative to combinatory logic in this regard since it is a first-order algebraic description of lambda calculus, which keeps the lambda notation and hence all the functional intuitions. Lambda abstraction algebras are axiomatized by the equations that hold between contexts of the lambda calculus (i.e., λ -terms with ‘holes’ [6, Def. 14.4.1]), as opposed to lambda terms with free variables. The essential feature of a context is that a free variable in a λ -term may become bound when we substitute it for a ‘hole’ within the context. Thus, ‘holes’ play the role of algebraic variables, and the contexts are the algebraic terms in the similarity type of lambda abstraction algebras. There is a rather peculiar relation between the lattice λT of lambda theories and the variety **LAA**. In [44] Salibra has shown that the lattice λT is isomorphic to the lattice of the equational theories of **LAA**’s. In fact, the correspondence, which maps an arbitrary λ -theory T into the equational theory of the variety generated by the term algebra of T , is an isomorphism of complete lattices. Thus, the properties of an arbitrary lambda theory can be studied by means of the variety of **LAA**’s generated by its term algebra. As we have specified in the introduction, many longstanding open problems of lambda calculus can be restated in terms of algebraic properties of varieties of **LAA**’s. In this section we show that there exists a congruence distributive variety of **LAA**’s, that is, a variety \mathcal{V} of **LAA**’s such that every algebra in \mathcal{V} has a distributive congruence lattice. The existence of varieties of **LAA**’s satisfying strong algebraic properties, such congruence distributivity, was an open problem since Salibra [45] proved that the variety **LAA** is not congruence modular and Lusin-Salibra [38] proved that every variety \mathcal{V} of **LAA**’s generated by the term algebra of a semi-sensible λ -theory does not satisfy any lattice identity.

Theorem 45 *There exists a congruence distributive variety of lambda abstraction algebras.*

Proof. Let \mathcal{V} be the variety of **LAA**'s generated by the term algebra $\mathbf{A}^{\mathcal{L}}$ of the lambda theory \mathcal{L} defined in Lemma 41. We claim that \mathcal{V} is congruence distributive, that is, every algebra $\mathbf{A} \in \mathcal{V}$ has a distributive congruence lattice. We have shown in the proof of Theorem 44 that the term algebra $\mathbf{A}^{\mathcal{L}}$ has two term operations $+$ and \cdot (defined in (4)), which satisfy the axioms of a distributive lattice. Since $\mathbf{A}^{\mathcal{L}}$ generates the variety \mathcal{V} and $+, \cdot$ are term operations, then every algebra $\mathbf{A} \in \mathcal{V}$ has also two term operations satisfying the axioms of a distributive lattice. The conclusion is obtained from Lemma 43. ■

7 Conclusion and future work.

We have generalized Baeten-Boerboom's method of forcing first to generalized terms involving all the continuous functions on a given power set $\mathcal{P}(D)$, and, second, to all weakly continuous operators. This approach allows us to prove very directly results about the lambda-representability of continuous functions on power sets, and also to generalize these results to countable sequences of continuous functions.

Related works are only the very recent papers by Alessi et al. [3] and Dezani-Lusin [22], where the authors use intersection type systems (see [3, 7, 20]) for synthesizing filter models of lambda calculus in which the interpretation of a simple easy term can be any filter described by a continuous predicate. This result can be interpreted as a generalization of Baeten and Boerboom's method of forcing via the use of intersection type systems. We believe that the framework we have developed in this paper is however more direct than the one used in [3]. We illustrate this with two examples, concerning the λ -representability of the minimal fixed point operator (Corollary 21) and of the pair union/intersection (Corollary 24).

As an application of the existence of (finite/infinite) sequences of terms that are functionally graph easy, we get strong results concerning the structure of the lattice of lambda theories and the existence of varieties of lambda abstraction algebras with very strong algebraic properties. More precisely, we show the existence of a distributive sub-lattice of the lattice of lambda theories and of a congruence distributive variety of lambda abstraction algebras.

In the present paper we only consider domains $\mathcal{P}(D)$, and, correlatively, graph models of lambda-calculus, and we concentrate on a limited number of applications. A further application, which is not treated here, is the question of the lambda-representability of first-order, say, structures in graph models. This will be the subject of another paper.

We would like to extend the results of the present paper to more sophisticated Scott-domains and webs. Various interesting classes of webbed models of lambda-calculus, concerning the main semantics of lambda-calculus, were surveyed in [14]. For the continuous semantics they range from graph models to

filter models, with a clear preference for the models whose underlying domain is prime-algebraic (which excludes some filter models), since they can be represented via feasible webs. All are accessible to Baeten-Boerboom's technique (see Section 2.3), but with less facility than for graph models. Compatibility conditions have to be met, depending on the class we consider, which do not occur when dealing with graph models. However no systematic study has been made so far if one excepts filter models [3, 22]. Our intention is hence to extend the methods and results presented in this paper to more general Scott-domains and webs.

Appendix.

Proposition 46 *Forcing is not continuous.*

Proof. Let α be a fixed element of D and q be a bijection between $D^* \times D - \{(\{\alpha\}, \alpha)\}$ and $D - \{\alpha\}$. It is clear that $q \Vdash \alpha \in \lambda x.x$, since the only total injection p which extends q satisfies $p(\{\alpha\}, \alpha) = \alpha$. Let $r \subseteq q$ be the partial sub-injection of q such that $dom(r) = \{(b, \beta) / \beta \in b\} \cap dom(q)$. Since $q - r$ is infinite and countable there is a countable strictly increasing sequence q_n starting from r and whose union is q .

We claim now that no q' such that $r \subseteq q' \subsetneq q$ can force $\alpha \in \lambda x.x$. Let indeed $(c, \gamma) \in dom(q) - dom(q')$ and let p be a total injection which extends q' and satisfies $\alpha = p(c, \gamma)$. From the hypothesis on q, q', p we have that $\alpha \notin (\lambda x.x)^p$. Hence $q' \not\Vdash \alpha \in \lambda x.x$. ■

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