



**HAL**  
open science

## Branching rules, Kostka-Foulkes polynomials and $q$ -multiplicities in tensor product for the root systems

$B_n, C_n$  and  $D_n$

Cédric Lecouvey

► **To cite this version:**

Cédric Lecouvey. Branching rules, Kostka-Foulkes polynomials and  $q$ -multiplicities in tensor product for the root systems  $B_n, C_n$  and  $D_n$ . *Algebras and Representation Theory*, 2006, 9 (4), pp.377-402. 10.1007/s10468-006-9020-7. hal-00003733v2

**HAL Id: hal-00003733**

**<https://hal.science/hal-00003733v2>**

Submitted on 17 Jan 2005

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Branching rules, Kostka-Foulkes polynomials and $q$ -multiplicities in tensor product for the root systems $B_n, C_n$ and $D_n$

Cédric Lecouvey  
lecouvey@math.unicaen.fr

## Abstract

The Kostka-Foulkes polynomials  $K_{\lambda,\mu}^\phi(q)$  related to a root system  $\phi$  can be defined as alternated sums running over the Weyl group associated to  $\phi$ . By restricting these sums over the elements of the symmetric group when  $\phi$  is of type  $B_n, C_n$  or  $D_n$ , we obtain again a class  $\tilde{K}_{\lambda,\mu}^\phi(q)$  of Kostka-Foulkes polynomials. When  $\phi$  is of type  $C_n$  or  $D_n$  there exists a duality between these polynomials and some natural  $q$ -multiplicities  $u_{\lambda,\mu}(q)$  and  $U_{\lambda,\mu}(q)$  in tensor product [14]. In this paper we first establish identities for the  $\tilde{K}_{\lambda,\mu}^\phi(q)$  which implies in particular that they can be decomposed as sums of Kostka-Foulkes polynomials  $K_{\lambda,\mu}^{A_{n-1}}(q)$  with nonnegative integer coefficients. Moreover these coefficients are branching rule coefficients. This allows us to clarify the connection between the  $q$ -multiplicities  $u_{\lambda,\mu}(q), U_{\lambda,\mu}(q)$  and the polynomials  $K_{\lambda,\mu}^\diamond(q)$  defined in [27]. Finally we show that  $u_{\lambda,\mu}(q)$  and  $U_{\lambda,\mu}(q)$  coincide up to a power of  $q$  with the one dimension sum introduced in [4] when all the parts of  $\mu$  are equal to 1 which partially proves some conjectures of [14] and [27].

## 1 Introduction

Consider  $\lambda$  and  $\mu$  two partitions of the set  $\mathcal{P}_n$  of partitions with  $n$  parts. The Schur-Weyl duality establishes that the dimension  $K_{\lambda,\mu}^{A_{n-1}}$  of the weight space  $\mu$  in the finite dimensional irreducible  $sl_n$ -module  $V^{A_{n-1}}(\lambda)$  of highest weight  $\lambda$  is equal to the multiplicity of  $V^{A_{n-1}}(\lambda)$  in the tensor product

$$V_\mu^{A_{n-1}} = V^{A_{n-1}}(\mu_1 \Lambda_1) \otimes \dots \otimes V^{A_{n-1}}(\mu_n \Lambda_1).$$

It follows from the Weyl character formula that  $K_{\lambda,\mu}^{A_{n-1}} = \sum_{\sigma \in \mathcal{S}_n} (-1)^{l(\sigma)} \mathcal{P}^{A_{n-1}}(\sigma(\lambda + \rho) - (\mu + \rho))$  where  $\mathcal{P}^{A_{n-1}}$  is the Kostant partition function which counts, in the root system of type  $A_{n-1}$ , the number of decomposition of  $\beta \in \mathbb{Z}^n$  as a sum of positive roots. The Kostka-Foulkes polynomials can be defined by setting  $K_{\lambda,\mu}^{A_{n-1}}(q) = \sum_{\sigma \in \mathcal{S}_n} (-1)^{l(\sigma)} \mathcal{P}_q^{A_n}(\sigma(\lambda + \rho) - (\mu + \rho))$  where  $\mathcal{P}_q^{A_{n-1}}$  is the  $q$ -Kostant partition function characterized by

$$\prod_{\alpha \text{ positive root}} \frac{1}{(1 - qe^\alpha)} = \sum_{\beta \in \mathbb{Z}^n} \mathcal{P}_q^{A_n}(\beta) e^\beta$$

with  $\rho = (n-1, \dots, 0)$  the half sum of the positive roots. One can prove that they are the coefficients of the expansion of the Schur function  $s_\mu(x)$  on the basis of Hall polynomials  $\{P_\lambda(x, q), \lambda \in \mathcal{P}_n\}$  (see [19]). Then it follows from the theory of affine Hecke algebras that the Kostka-Foulkes polynomials are Kazhdan-Lusztig polynomials [18]. In particular they have nonnegative integer coefficients. As proved by Lascoux and Schützenberger this positivity result can also be obtained by using the

charge statistic  $\text{ch}$  on semistandard tableaux. More precisely we have  $K_{\lambda,\mu}^{A_{n-1}}(q) = \sum_{T \in ST(\lambda)_\mu} q^{\text{ch}(T)}$  where  $ST(\lambda)_\mu$  is the set of semistandard tableaux of shape  $\lambda$  and weight  $\mu$ . In [20], Nakayashiki and Yamada have shown that the charge can be computed from the combinatorial  $R$ -matrix corresponding to Kashiwara's crystals associated to some  $U_q(\widehat{sl}_n)$ -modules.

Now consider  $\phi$  a root system of type  $B_n, C_n$  or  $D_n$ . Write  $\mathfrak{g}_\phi$  for the corresponding simple Lie algebra. The Kostka-Foulkes polynomials  $K_{\lambda,\mu}^\phi(q)$  associated to  $\phi$  are defined by setting

$$K_{\lambda,\mu}^\phi(q) = \sum_{w \in W_\phi} (-1)^{l(w)} \mathcal{P}_q^\phi(w(\lambda + \rho_\phi) - (\mu + \rho_\phi))$$

where  $W_\phi, \rho_\phi$  and  $\mathcal{P}_q^\phi$  are respectively the Weyl group, the half sum of the positive roots and the  $q$ -partition function corresponding to  $\phi$ . The polynomial  $K_{\lambda,\mu}^\phi(q)$  can be considered as a  $q$ -analogue of the dimension of the weight space  $\mu$  in  $V^\phi(\lambda)$ . As Kazhdan-Lusztig polynomials, they have also nonnegative coefficients. In [16], we have obtained for the root systems  $B_n, C_n$  and  $D_n$  a statistic on Kashiwara-Nakashima's tableaux from which it is possible to deduce this positivity for particular pairs of partitions  $(\lambda, \mu)$ . Nevertheless as far as the author is aware, no combinatorial proof of this positivity result is known in general.

The Kostka-Foulkes polynomials  $K_{\lambda,\mu}^\phi(q)$  cannot be directly interpreted as  $q$ -multiplicities in tensor products. So there does not exist an equivalent result to the Schur Weyl duality for the root system  $\phi$ . Denote by  $V^\phi(\lambda)$  the finite dimensional irreducible  $\mathfrak{g}_\phi$ -module of highest weight  $\lambda$ . In [14], we have introduced from determinantal expressions of the Schur functions associated to  $\phi$ , two polynomials  $u_{\lambda,\mu}(q)$  and  $U_{\lambda,\mu}(q)$  which can be respectively regarded as quantizations of the multiplicities of  $V^\phi(\lambda)$  in the tensor products

$$V_\mu^\phi = V^\phi(\mu_1 \Lambda_1) \otimes \cdots \otimes V^\phi(\mu_n \Lambda_1) \text{ and } W_\mu^\phi = W^\phi(\mu_1 \Lambda_1) \otimes \cdots \otimes W^\phi(\mu_n \Lambda_1)$$

where for any  $i = 1, \dots, n$ ,  $W^\phi(\mu_i \Lambda_1) = V^\phi(\mu_i \Lambda_1) \oplus V^\phi((\mu_i - 2)\Lambda_1) \oplus \cdots \oplus V^\phi((\mu_i \bmod 2)\Lambda_1)$ . When  $n$  is sufficiently large they do not depend on the root system  $\phi$  considered and we have established a duality result between the  $q$ -multiplicities  $u_{\lambda,\mu}(q)$ ,  $U_{\lambda,\mu}(q)$  and the polynomials

$$\tilde{K}_{\lambda,\mu}^\phi(q) = \sum_{\sigma \in \mathcal{S}_n} (-1)^{l(\sigma)} \mathcal{P}_q^\phi(w(\lambda + \rho_n) - (\mu + \rho_n))$$

where  $\rho_n = (n, \dots, 1)$ . These polynomials  $\tilde{K}_{\lambda,\mu}^\phi(q)$  are also Kostka-Foulkes polynomials. So this result can be interpreted as a duality between  $q$ -analogues of weight multiplicities and  $q$ -analogues of tensor product multiplicities for the root systems  $B_n, C_n$  and  $D_n$ .

At the same time Shimozono and Zabrocki [27] have independently defined by using creating operators some polynomials  $K_{\lambda,R}^\diamond(q)$  where  $R$  is a sequence of rectangular partitions and  $\diamond$  a partition of the set  $\{\emptyset, (1), (11), (2)\}$ . These polynomials can also be regarded as  $q$ -multiplicities in tensor products. In [4], Hatayama, Kuniba, Okado and Takagi have introduced for type  $C_n$  a quantization  $X_{\lambda,\mu}(q)$  of the multiplicity of  $V^{C_n}(\lambda)$  in  $W^{C_n}$ . This quantization is based on the determination of the combinatorial  $R$ -matrix of some  $U'_q(\widehat{sp}_{2n})$ -crystals in the spirit of [20]. It can be regarded as a one dimension sum for the affine root system  $C_n^{(1)}$ . In [14] and [27], the authors conjecture that the polynomials  $X_{\lambda,\mu}(q)$ ,  $U_{\lambda,\mu}(q)$  and  $K_{\lambda,\mu}^{(2)}(q)$  coincide up to simple renormalizations. As observed in [27], this conjecture can be related to the  $X = M$  conjecture which gives fermionic formulas for the one dimension sum  $X$ . Note that the  $X = M$  conjecture have been proved in various cases for all nonexceptional affine types [22], [23] and [24].

In this article we first obtained identities for the polynomials  $\tilde{K}_{\lambda,\mu}^\phi(q)$  which imply that they can be decomposed as sums of polynomials  $K_{\lambda,\mu}^{A_{n-1}}(q)$ . Moreover the coefficients of these decompositions can be simply expressed in terms of branching rules coefficients. This gives in particular an elementary proof of the positivity of the Kostka-Foulkes polynomials  $\tilde{K}_{\lambda,\mu}^\phi(q)$ . Next we obtain similar decompositions for the polynomials  $u_{\lambda,\mu}(q)$  and  $U_{\lambda,\mu}(q)$ . By comparing these identities with those obtained for the polynomials  $K_{\lambda,R}^\diamond(q)$  in [27], we derive the equalities  $K_{\lambda,\mu}^{(1,1)}(q) = u_{\lambda,\mu}(q^2)$  and  $K_{\lambda,\mu}^{(2)}(q) = U_{\lambda,\mu}(q^2)$ . Finally we establish some conjectures of [14] and [27] when all the parts of  $\mu$  are equal to 1 (i.e. for the  $q$ -multiplicities defined in the tensor powers of the vector representation), namely we have

$$K_{\lambda,(1^n)}^{(1,1)}(q) = u_{\lambda,(1^n)}(q^2) = q^{n-|\lambda|} X_{\lambda,(1^n)}(q^2) \text{ and } K_{\lambda,(1^n)}^{(2)}(q) = U_{\lambda,(1^n)}(q^2) = q^{2(n-|\lambda|)} X_{\lambda,(1^n)}(q^2). \quad (1)$$

In Section 2 we review some material on root systems, branching rules coefficients, Kostka-Foulkes polynomials and  $q$ -multiplicities  $u_{\lambda,\mu}(q), U_{\lambda,\mu}(q)$  we need in the sequel. In section 3 we obtain identities for the polynomials  $\tilde{K}_{\lambda,\mu}^\phi(q), u_{\lambda,\mu}(q)$  and  $U_{\lambda,\mu}(q)$  from which we clarify the relations between  $u_{\lambda,\mu}(q), U_{\lambda,\mu}(q)$  and  $K_{\lambda,\mu}^{(1,1)}(q), K_{\lambda,\mu}^{(2)}(q)$ . Section 4 is devoted to the proof of (1). Note that the  $X = M$  conjecture is in particular true when all the parts of  $\mu$  are equal to 1 [22]. Thus in this case the one dimension sums  $X$  and their corresponding fermionic formulas  $M$  are, up to simple renormalizations, Kazhdan-Lusztig polynomials.

**Notation:** In the sequel we frequently define similar objects for the root systems  $B_n, C_n$  and  $D_n$ . When they are related to type  $B_n$  (resp.  $C_n, D_n$ ), we implicitly attach to them the label  $B$  (resp. the labels  $C, D$ ). To avoid cumbersome repetitions, we sometimes omit the labels  $B, C$  and  $D$  when our definitions or statements are identical for the three root systems.

## 2 Background

### 2.1 Convention for the root systems of types $B_n, C_n$ and $D_n$

Consider an integer  $n \geq 1$ . The weight lattice for the root system  $C_n$  (resp.  $B_n$  and  $D_n$ ) can be identified with  $P_{C_n} = \mathbb{Z}^n$  (resp.  $P_{B_n} = P_{D_n} = \left(\frac{\mathbb{Z}}{2}\right)^n$ ) equipped with the orthonormal basis  $\varepsilon_i, i = 1, \dots, n$ . We take for the simple roots

$$\begin{cases} \alpha_n^{B_n} = \varepsilon_n \text{ and } \alpha_i^{B_n} = \varepsilon_i - \varepsilon_{i+1}, i = 1, \dots, n-1 \text{ for the root system } B_n \\ \alpha_n^{C_n} = 2\varepsilon_n \text{ and } \alpha_i^{C_n} = \varepsilon_i - \varepsilon_{i+1}, i = 1, \dots, n-1 \text{ for the root system } C_n \\ \alpha_n^{D_n} = \varepsilon_n + \varepsilon_{n-1} \text{ and } \alpha_i^{D_n} = \varepsilon_i - \varepsilon_{i+1}, i = 1, \dots, n-1 \text{ for the root system } D_n \end{cases} \quad (2)$$

Then the set of positive roots are

$$\begin{cases} R_{B_n}^+ = \{\varepsilon_i - \varepsilon_j, \varepsilon_i + \varepsilon_j \text{ with } 1 \leq i < j \leq n\} \cup \{\varepsilon_i \text{ with } 1 \leq i \leq n\} \text{ for the root system } B_n \\ R_{C_n}^+ = \{\varepsilon_i - \varepsilon_j, \varepsilon_i + \varepsilon_j \text{ with } 1 \leq i < j \leq n\} \cup \{2\varepsilon_i \text{ with } 1 \leq i \leq n\} \text{ for the root system } C_n \\ R_{D_n}^+ = \{\varepsilon_i - \varepsilon_j, \varepsilon_i + \varepsilon_j \text{ with } 1 \leq i < j \leq n\} \text{ for the root system } D_n \end{cases} \quad .$$

Denote respectively by  $P_{B_n}^+, P_{C_n}^+$  and  $P_{D_n}^+$  the sets of dominant weights of  $so_{2n+1}, sp_{2n}$  and  $so_{2n}$ .

Let  $\lambda = (\lambda_1, \dots, \lambda_n)$  be a partition with  $n$  parts. We will classically identify  $\lambda$  with the dominant weight  $\sum_{i=1}^n \lambda_i \varepsilon_i$ . Note that there exists dominant weights associated to the orthogonal root systems whose coordinates on the basis  $\varepsilon_i$ ,  $i = 1, \dots, n$  are not positive integers (hence which cannot be regarded as partitions). For each root system of type  $B_n, C_n$  or  $D_n$ , the set of weights having nonnegative integer coordinates on the basis  $\varepsilon_1, \dots, \varepsilon_n$  can be identify with the set  $\mathcal{P}_n$  of partitions of length  $n$ . For any partition  $\lambda$ , the weights of the finite dimensional  $so_{2n+1}$ ,  $sp_{2n}$  or  $so_{2n}$ -module of highest weight  $\lambda$  are all in  $\mathbb{Z}^n$ . For any  $\alpha \in \mathbb{Z}^n$  we write  $|\alpha| = \alpha_1 + \dots + \alpha_n$  and  $\|\alpha\| = \sum_{i=1}^{n-1} (n-i)\alpha_i$ . The conjugate partition of the partition  $\lambda$  is denoted  $\lambda'$  as usual. Consider  $\lambda, \mu$  two partitions of length  $n$  and set  $m = \max(\lambda_1, \mu_1)$ . Then by adding to  $\lambda'$  and  $\mu'$  the required numbers of parts 0 we will consider them as partitions of length  $m$ .

The Weyl group  $W_{B_n} = W_{C_n}$  of  $so_{2n+1}$  and  $sp_{2n}$  is identified to the subgroup of the permutation group of the set  $\{\bar{n}, \dots, \bar{2}, \bar{1}, 1, 2, \dots, n\}$  generated by  $s_i = (i, i+1)(\bar{i}, \bar{i}+1)$ ,  $i = 1, \dots, n-1$  and  $s_n = (n, \bar{n})$  where for  $a \neq b$   $(a, b)$  is the simple transposition which switches  $a$  and  $b$ . We denote by  $l_B$  the length function corresponding to the set of generators  $s_i$ ,  $i = 1, \dots, n$ .

The Weyl group  $W_{D_n}$  of  $so_{2n}$  is identified to the subgroup of  $W_{B_n}$  generated by the transpositions  $s_i = (i, i+1)(\bar{i}, \bar{i}+1)$ ,  $i = 1, \dots, n-1$  and  $s'_n = (n, \bar{n}-1)(n-1, \bar{n})$ . We denote by  $l_D$  the length function corresponding to the set of generators  $s'_n$  and  $s_i$ ,  $i = 1, \dots, n-1$ .

Note that  $W_{D_n} \subset W_{B_n}$  and any  $w \in W_{B_n}$  verifies  $w(\bar{i}) = \overline{w(i)}$  for  $i \in \{1, \dots, n\}$ . The action of  $w$  on  $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{Z}^n$  is given by

$$w \cdot (\beta_1, \dots, \beta_n) = (\beta_1^w, \dots, \beta_n^w)$$

where  $\beta_i^w = \beta_{w(i)}$  if  $\sigma(i) \in \{1, \dots, n\}$  and  $\beta_i^w = -\beta_{w(\bar{i})}$  otherwise.

The half sums  $\rho_{B_n}, \rho_{C_n}$  and  $\rho_{D_n}$  of the positive roots associated to each root system  $B_n, C_n$  and  $D_n$  verify:

$$\rho_{B_n} = (n - \frac{1}{2}, n - \frac{3}{2}, \dots, \frac{1}{2}), \rho_{C_n} = (n, n-1, \dots, 1) \text{ and } \rho_{D_n} = (n-1, n-2, \dots, 0).$$

In the sequel we identify the symmetric group  $\mathcal{S}_n$  (which is the Weyl group of the root system  $A_{n-1}$ ) with the subgroup of  $W_{B_n}$  or  $W_{D_n}$  generated by the  $s_i$ 's,  $i = 1, \dots, n-1$ .

## 2.2 Branching rules coefficients

For any partition  $\lambda$ , we denote by  $V_n^B(\lambda), V_n^C(\lambda)$ , and  $V_n^D(\lambda)$  the finite dimensional irreducible modules of highest weight  $\lambda$  respectively for  $sp_{2n}, so_{2n+1}$  and  $so_{2n}$ . Then  $V_n^B(\lambda), V_n^C(\lambda)$ , and  $V_n^D(\lambda)$  can also be regarded as irreducible representations respectively of the groups  $Sp_{2n}, So_{2n+1}$  and  $So_{2n}$ . By restriction to  $GL_n$ , they decompose in a direct sum of irreducible rational representations. Recall that the irreducible rational representations of  $GL_n$  are indexed by the  $n$ -tuples

$$(\gamma^+, \gamma^-) = (\gamma_1^+, \gamma_2^+, \dots, \gamma_p^+, 0, \dots, 0, -\gamma_q^-, \dots, -\gamma_1^-) \quad (3)$$

where  $\gamma^+$  and  $\gamma^-$  are partitions of length  $p$  and  $q$  such that  $p+q \leq n$ . Write  $V_n^A(\gamma^+, \gamma^-)$  for the irreducible rational representations of  $GL_n$  of highest weight  $(\gamma^+, \gamma^-)$ . When  $\gamma^- = \emptyset$ , we write simply  $V_n^A(\gamma)$  instead of  $V_n^A(\gamma^+, \gamma^-)$ .

As customary, we use for a basis of the group algebra  $\mathbb{Z}[\mathbb{Z}^n]$ , the formal exponentials  $(e^\beta)_{\beta \in \mathbb{Z}^n}$  satisfying the relations  $e^{\beta_1} e^{\beta_2} = e^{\beta_1 + \beta_2}$ . We furthermore introduce  $n$  independent indeterminates

$x_1, \dots, x_n$  in order to identify  $\mathbb{Z}[\mathbb{Z}^n]$  with the ring of polynomials  $\mathbb{Z}[x_1, \dots, x_n, x_1^{-1}, \dots, x_n^{-1}]$  by writing  $e^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n} = x^\beta$  for any  $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{Z}^n$ .

Set

$$\begin{aligned} \prod_{1 \leq r < s \leq n} \left(1 - \frac{1}{x_r x_s}\right)^{-1} \prod_{1 \leq i \leq n} \left(1 - \frac{1}{x_i}\right)^{-1} &= \sum_{\beta \in L_B} b(\beta) x^{-\beta}, \\ \prod_{1 \leq r \leq s \leq n} \left(1 - \frac{1}{x_r x_s}\right)^{-1} &= \sum_{\beta \in L_C} c(\beta) x^{-\beta}, \\ \prod_{1 \leq r < s \leq n} \left(1 - \frac{1}{x_r x_s}\right)^{-1} &= \sum_{\beta \in L_D} d(\beta) x^{-\beta} \end{aligned} \quad (4)$$

where

$$\begin{aligned} L_B &= \{\beta \in \mathbb{Z}^n, \beta = \sum_{1 \leq r < s \leq n} e_{r,s}(\varepsilon_r + \varepsilon_s) + \sum_{1 \leq i \leq n} e_i \varepsilon_i \text{ with } e_{r,s} \geq 0 \text{ and } e_i \geq 0\} \\ L_C &= \{\beta \in \mathbb{Z}^n, \beta = \sum_{1 \leq r \leq s \leq n} e_{r,s}(\varepsilon_r + \varepsilon_s) \text{ with } e_{r,s} \geq 0\} \text{ and} \\ L_D &= \{\beta \in \mathbb{Z}^n, \beta = \sum_{1 \leq r < s \leq n} e_{r,s}(\varepsilon_r + \varepsilon_s) \text{ with } e_{r,s} \geq 0\}. \end{aligned}$$

Denote respectively by  $[V_n^A(\gamma^+, \gamma^-) : V_n^B(\lambda)]$ ,  $[V_n^A(\gamma^+, \gamma^-) : V_n^C(\lambda)]$  and  $[V_n^A(\gamma^+, \gamma^-) : V_n^D(\lambda)]$ , the multiplicities of  $V_n^A(\gamma^+, \gamma^-)$  in the restrictions of  $V_n^B(\lambda)$ ,  $V_n^C(\lambda)$  and  $V_n^D(\lambda)$  to  $GL_n$ .

**Proposition 2.2.1** *With the above notation, we have:*

1.  $[V_n^A(\gamma^+, \gamma^-) : V_n^B(\lambda)] = \sum_{w \in W_{B_n}} (-1)^{l(w)} b(w \circ \lambda - (\gamma^+, \gamma^-))$ ,
2.  $[V_n^A(\gamma^+, \gamma^-) : V_n^C(\lambda)] = \sum_{w \in W_{C_n}} (-1)^{l(w)} c(w \circ \lambda - (\gamma^+, \gamma^-))$ ,
3.  $[V_n^A(\gamma^+, \gamma^-) : V_n^D(\lambda)] = \sum_{w \in W_{D_n}} (-1)^{l(w)} d(w \circ \lambda - (\gamma^+, \gamma^-))$ .

**Proof.** The proposition can be considered as a corollary of Theorem 8.2.1 of [3] with  $G$  one of the Lie groups  $So_{2n+1}$ ,  $Sp_{2n}$ ,  $So_{2n}$  and  $H = GL_n$ . ■

For any partitions  $\lambda$  and  $\nu$  of length  $n$ , write  $[V_n^D(\nu) : V_n^B(\lambda)]$  for the multiplicity of  $V_n^D(\nu)$  in the restriction of  $V_n^B(\lambda)$  to  $So_{2n}$ .

**Lemma 2.2.2** *With the notation above we have*

$$[V_n^D(\nu) : V_n^B(\lambda)] = \sum_{w \in W_{B_n}} (-1)^{l(w)} \mathbf{1}_{\mathbb{N}}(w \circ \lambda - \nu)$$

where for any  $\beta \in \mathbb{Z}^n$ ,  $\mathbf{1}_{\mathbb{N}}(\beta) = 1$  if all the coordinates of  $\beta$  are nonnegative integers and  $\mathbf{1}_{\mathbb{N}}(\beta) = 0$  otherwise.

**Proof.** The lemma also follows from Theorem 8.2.1 of [3]. ■

For any partitions  $\lambda$  and  $\nu$  of length  $n$ , write  $[V_n^B(\lambda) : V_{2n}^A(\nu)]$  for the multiplicity of  $V_n^B(\lambda)$  in the restriction of  $V_{2n}^A(\nu)$  from  $GL_{2n}$  to  $SO_{2n}$ . Similarly write  $[V_n^C(\lambda) : V_{2n}^A(\nu)]$  for the multiplicity of  $V_n^C(\lambda)$  in the restriction of  $V_{2n}^A(\nu)$  from  $GL_{2n}$  to  $Sp_{2n}$  and  $[V_n^D(\lambda) : V_{2n}^A(\nu)]$  for the multiplicity of  $V_n^D(\lambda)$  in the restriction of  $V_{2n}^A(\nu)$  from  $GL_{2n}$  to  $SO_{2n+1}$ . Denote respectively by  $\mathcal{P}_n^{(2)}$  and  $\mathcal{P}_n^{(1,1)}$  the sub-sets of  $\mathcal{P}_n$  containing the partitions with even rows and the partitions with even columns. The Littlewood-Richardson coefficients are denoted  $c_{\gamma,\lambda}^\nu$  as usual. Let us recall a classical result by Littelwood (see [17] appendix p 295)

**Proposition 2.2.3** *Consider  $\lambda$  and  $\mu$  in  $\mathcal{P}_n$ . Then:*

1.  $[V_n^B(\lambda) : V_{2n}^A(\nu)] = \sum_{\gamma \in \mathcal{P}_n} c_{\gamma,\lambda}^\nu$ ,
2.  $[V_n^C(\lambda) : V_{2n}^A(\nu)] = \sum_{\gamma \in \mathcal{P}_n^{(1,1)}} c_{\gamma,\lambda}^\nu$ ,
3.  $[V_n^D(\lambda) : V_{2n}^A(\nu)] = \sum_{\gamma \in \mathcal{P}_n^{(2)}} c_{\gamma,\lambda}^\nu$ .

The proposition below follows immediately from Theorem  $A_1$  of [10].

**Proposition 2.2.4** *Consider  $\nu \in \mathcal{P}_n$  and  $\lambda^+, \lambda^-$  two partitions such that  $(\lambda^+, \lambda^-)$  has length  $n$ . Then:*

1.  $[V_n^A(\lambda^+, \lambda^-) : V_n^B(\nu)] = \sum_{\gamma, \delta \in \mathcal{P}_n} c_{\gamma,\delta}^\nu c_{\lambda^+, \lambda^-}^\delta$ ,
2.  $[V_n^A(\lambda^+, \lambda^-) : V_n^C(\nu)] = \sum_{\gamma, \delta \in \mathcal{P}_n^{(2)}} c_{\gamma,\delta}^\nu c_{\lambda^+, \lambda^-}^\delta$ ,
3.  $[V_n^A(\lambda^+, \lambda^-) : V_n^D(\nu)] = \sum_{\gamma, \delta \in \mathcal{P}_n^{(1,1)}} c_{\gamma,\delta}^\nu c_{\lambda^+, \lambda^-}^\delta$ .

When  $(\lambda^+, \lambda^-) = \lambda$  is a partition (that is  $\lambda^- = \emptyset$ ), we obtain the following dualities:

**Corollary 2.2.5** *Consider  $\lambda, \nu$  two partitions of length  $n$ , then*

1.  $[V_n^A(\lambda) : V_n^B(\nu)] = [V_n^B(\lambda) : V_{2n}^A(\nu)] = \sum_{\gamma \in \mathcal{P}_n} c_{\gamma,\lambda}^\nu$ ,
2.  $[V_n^A(\lambda) : V_n^C(\nu)] = [V_n^D(\lambda) : V_{2n}^A(\nu)] = \sum_{\gamma \in \mathcal{P}_n^{(2)}} c_{\gamma,\lambda}^\nu$ ,
3.  $[V_n^A(\lambda) : V_n^D(\nu)] = [V_n^C(\lambda) : V_{2n}^A(\nu)] = \sum_{\gamma \in \mathcal{P}_n^{(1,1)}} c_{\gamma,\lambda}^\nu$ .

## 2.3 Kostka-Foulkes polynomials

For any  $w \in W_{B_n}$ , the dot action of  $w$  on  $\beta \in \mathbb{Z}^n$  is defined by

$$w \circ \beta = w(\beta + \rho_{B_n}) - \rho_{B_n}.$$

The  $q$ -analogue  $\mathcal{P}_q^{B_n}$  of the Kostant partition function corresponding to the root system  $B_n$  is defined by the equality

$$\prod_{\alpha \in R_{B_n}^+} \frac{1}{1 - qx^\alpha} = \sum_{\beta \in \mathbb{Z}^n} \mathcal{P}_q^{B_n}(\beta) x^\beta.$$

Note that  $\mathcal{P}_q^{B_n}(\beta) = 0$  if  $\beta$  is not a linear combination of positive roots of  $R_{B_n}^+$  with nonnegative coefficients. We write similarly  $\mathcal{P}_q^{C_n}$  and  $\mathcal{P}_q^{D_n}$  for the  $q$ -partition functions associated respectively to the root systems  $C_n$  and  $D_n$ . Given  $\lambda$  and  $\mu$  two partitions of length  $n$ , the Kostka-Foulkes polynomials of types  $B_n, C_n$  and  $D_n$  are then respectively defined by

$$\begin{aligned} K_{\lambda,\mu}^{B_n}(q) &= \sum_{\sigma \in W_{B_n}} (-1)^{l(\sigma)} \mathcal{P}_q^{B_n}(\sigma(\lambda + \rho_{B_n}) - (\mu + \rho_{B_n})), \\ K_{\lambda,\mu}^{C_n}(q) &= \sum_{\sigma \in W_{C_n}} (-1)^{l(\sigma)} \mathcal{P}_q^{C_n}(\sigma(\lambda + \rho_{C_n}) - (\mu + \rho_{C_n})), \\ K_{\lambda,\mu}^{D_n}(q) &= \sum_{\sigma \in W_{D_n}} (-1)^{l(\sigma)} \mathcal{P}_q^{D_n}(\sigma(\lambda + \rho_{D_n}) - (\mu + \rho_{D_n})). \end{aligned}$$

Set

$$\begin{aligned} \tilde{K}_{\lambda,\mu}^{B_n}(q) &= \sum_{\sigma \in \mathcal{S}_n} (-1)^{l(\sigma)} \mathcal{P}_q^{B_n}(\sigma(\lambda + \rho_{B_n}) - (\mu + \rho_n)), \\ \tilde{K}_{\lambda,\mu}^{C_n}(q) &= \sum_{\sigma \in \mathcal{S}_n} (-1)^{l(\sigma)} \mathcal{P}_q^{C_n}(\sigma(\lambda + \rho_{C_n}) - (\mu + \rho_n)), \\ \tilde{K}_{\lambda,\mu}^{D_n}(q) &= \sum_{\sigma \in \mathcal{S}_n} (-1)^{l(\sigma)} \mathcal{P}_q^{D_n}(\sigma(\lambda + \rho_{D_n}) - (\mu + \rho_n)) \end{aligned}$$

where  $\rho_n = (n, \dots, 1)$ . In [14], we have proved that the polynomials  $\tilde{K}_{\lambda,\mu}(q)$  are also Kostka-Foulkes polynomials. More precisely we have:

**Lemma 2.3.1** *Consider  $\lambda, \mu$  two partitions of length  $n$  such that  $|\lambda| \geq |\mu|$ . Let  $k$  be any integer such that  $k \geq \frac{|\lambda| - |\mu|}{2}$ . Then we have*

$$\tilde{K}_{\lambda,\mu}(q) = K_{\lambda+k\kappa_n, \mu+k\kappa_n}(q)$$

where  $\kappa_n = (1, \dots, 1) \in \mathbb{Z}^n$ .

**Remark:** Since  $\sigma(\kappa_n) = \kappa_n$  for any  $\sigma \in \mathcal{S}_n$ , we have  $\tilde{K}_{\lambda+k\kappa_n, \mu+k\kappa_n}(q) = \tilde{K}_{\lambda,\mu}(q)$  for any integer  $k \geq 0$ . So we can extend the above definition of  $\tilde{K}_{\lambda,\mu}(q)$  for  $\lambda$  and  $\mu$  decreasing sequences of integers (positive or not).

## 2.4 The $q$ -multiplicities $u_{\lambda,\mu}(q)$ and $U_{\lambda,\mu}(q)$

Set

$$\begin{aligned} \prod_{1 \leq i < j \leq n} \frac{1}{1 - q \frac{x_i}{x_j}} \prod_{1 \leq r < s \leq n} \frac{1}{1 - \frac{q}{x_i x_j}} &= \sum_{\beta \in \mathbb{Z}^n} f_q(\beta) x^\beta \text{ and} \\ \prod_{1 \leq i < j \leq n} \frac{1}{1 - q \frac{x_i}{x_j}} \prod_{1 \leq r \leq s \leq n} \frac{1}{1 - \frac{q}{x_i x_j}} &= \sum_{\beta \in \mathbb{Z}^n} F_q(\beta) x^\beta. \end{aligned}$$

Given  $\lambda$  and  $\mu$  two partitions of length  $n$ , let  $u_{\lambda,\mu}(q)$  and  $U_{\lambda,\mu}(q)$  be the two polynomials defined by



$$u_{\lambda,\mu}(q) = \sum_{\sigma \in \mathcal{S}_n} (-1)^{l(\sigma)} f_q(\sigma(\lambda + \rho_n) - \mu - \rho_n) \text{ and } U_{\lambda,\mu}(q) = \sum_{\sigma \in \mathcal{S}_n} (-1)^{l(\sigma)} F_q(\sigma(\lambda + \rho_n) - \mu - \rho_n)$$

where  $\rho_n = (n, \dots, 1)$ . Then  $u_{\lambda,\mu}(q)$  and  $U_{\lambda,\mu}(q)$  can be regarded as quantizations of tensor product multiplicities [14]. Consider the tensor products

$$\begin{aligned} V_\mu^B &= V^B(\mu_1 \Lambda_1) \otimes \cdots \otimes V^B(\mu_n \Lambda_1), \quad V_\mu^C = V^C(\mu_1 \Lambda_1) \otimes \cdots \otimes V^C(\mu_n \Lambda_1), \\ V_\mu^D &= V^D(\mu_1 \Lambda_1) \otimes \cdots \otimes V^D(\mu_n \Lambda_1) \end{aligned}$$

and

$$\begin{aligned} W_\mu^B &= W^B(\mu_1 \Lambda_1) \otimes \cdots \otimes W^B(\mu_n \Lambda_1), \quad W_\mu^C = W^C(\mu_1 \Lambda_1) \otimes \cdots \otimes W^C(\mu_n \Lambda_1), \\ W_\mu^D &= W^D(\mu_1 \Lambda_1) \otimes \cdots \otimes W^D(\mu_n \Lambda_1) \end{aligned}$$

where for any  $k \in \mathbb{N}$ ,  $W(k_1) = V(k\Lambda_1) \oplus V((k-2)\Lambda_1) \oplus \cdots \oplus V((k \bmod 2)\Lambda_1)$ . Then we have the following proposition:

**Proposition 2.4.1** [14] *Let  $\lambda$  and  $\mu$  be two partitions of length  $n$ . Then*

1.  $u_{\lambda,\mu}(q)$  is a  $q$ -analogue of the multiplicity of the representation  $V(\lambda)$  in  $V_\mu$ ,
2.  $U_{\lambda,\mu}(q)$  is a  $q$ -analogue of the multiplicity of the representation  $V(\lambda)$  in  $W_\mu$ .

**Remarks:**

- (i) : It follows from the definition of  $f_q$  and  $F_q$  that  $u_{\lambda,\mu}(q) = U_{\lambda,\mu}(q) = 0$  if  $|\lambda| > |\mu|$ .
- (ii) : When  $q = 1$ , we recover that the multiplicities of  $V^B(\lambda), V^C(\lambda)$  and  $V^D(\lambda)$  respectively in  $V_\mu^B, V_\mu^C$ , and  $V_\mu^D$  are equal [10].
- (iii) : In [14], we have also obtained that  $u_{\lambda,\mu}(q)$  and  $U_{\lambda,\mu}(q)$  can be regarded as  $q$ -multiplicities in tensor product of column shaped representations.
- (iv) : Like the definition of  $\tilde{K}_{\lambda,\mu}(q)$ , the definitions of  $u_{\lambda,\mu}(q)$  and  $U_{\lambda,\mu}(q)$  can also be extended for  $\lambda$  and  $\mu$  decreasing sequences of integers.
- (v) : Consider  $\lambda, \mu \in \mathcal{P}_n$  and set  $\lambda^\# = (\lambda_1, \dots, \lambda_n, 0)$ ,  $\mu^\# = (\mu_1, \dots, \mu_n, 0)$ . Then  $u_{\lambda^\#, \mu^\#}(q) = u_{\lambda,\mu}(q)$  and  $U_{\lambda^\#, \mu^\#}(q) = U_{\lambda,\mu}(q)$ .

**Theorem 2.4.2** [14] *Consider  $\lambda, \mu$  two partitions of length  $n$  and set  $m = \max(\lambda_1, \mu_1)$ . Then  $\hat{\lambda} = (m - \lambda_n, \dots, m - \lambda_1)$  and  $\hat{\mu} = (m - \mu_n, \dots, m - \mu_1)$  are partitions of length  $n$  and*

$$u_{\lambda,\mu}(q) = \tilde{K}_{\hat{\lambda}, \hat{\mu}}^{D_n}(q), \quad U_{\lambda,\mu}(q) = \tilde{K}_{\hat{\lambda}, \hat{\mu}}^{C_n}(q).$$

Write  $I$  for the involution defined on  $\mathbb{Z}^n$  by  $I(\beta_1, \dots, \beta_n) = (-\beta_n, \dots, -\beta_1)$ . For any decreasing sequence of integers  $\gamma$ ,  $I(\gamma)$  is also a decreasing sequence of integers. By Theorem 2.4.2, this means that the correspondences

$$u_{\lambda,\mu}(q) \longleftrightarrow \tilde{K}_{I(\lambda), I(\mu)}^{D_n}(q) \text{ and } U_{\lambda,\mu}(q) \longleftrightarrow \tilde{K}_{I(\lambda), I(\mu)}^{C_n}(q) \quad (5)$$

where  $\lambda, \mu$  are decreasing sequence of integers can be interpreted as a duality result for the  $q$ -multiplicities associated to the classical root systems.

## 2.5 Crystals of type $C_n$ and the one dimension sum $X_{\lambda,\mu}(q)$

We have seen that  $U_{\lambda,\mu}(q)$  can be regarded as a  $q$ -analogue of the multiplicity of  $V(\lambda)$  in  $W_\mu^C$ . In [4], Hatayama, Kuniba, Okado and Takagi have introduced another quantification  $X_{\lambda,\mu}(q)$  of this multiplicity based on the determination of the combinatorial  $R$ -matrix of certain  $U'_q(C_n^{(1)})$ -crystals  $B_k^C$ . Considered as the crystal graph of a  $U_q(C_n)$ -module,  $B_k^C$  is isomorphic to

$$B^C(k\Lambda_1) \oplus B^C((k-2)\Lambda_1) \oplus \cdots \oplus B^C(k \bmod 2\Lambda_1) \quad (6)$$

where for any  $i \in \{k, k-2, \dots, k \bmod 2\}$ ,  $B^C(k\Lambda_1)$  is the crystal graph of the irreducible finite dimensional  $U_q(C_n)$ -module of highest weight  $k\Lambda_1$ . In [9], Kashiwara and Nakashima have obtained a natural labelling of the vertices of  $B^C(k\Lambda_1)$  by one-row tableaux of length  $k$  filled by letters of the alphabet

$$\mathcal{C}_n = \{1 < \cdots < n-1 < n < \bar{n} < \overline{n-1} < \cdots < \bar{1}\}$$

such that the letters increase from left to right (that is by semistandard one-row tableaux on  $\mathcal{C}_n$ ). Then the vertices of  $B_k^C$  can be depicted by one-row tableaux of length  $k$  by adding  $p$  pairs  $(0, \bar{0})$  to the tableaux appearing in the crystals  $B^C((k-2p)\Lambda_1)$  of the decomposition (6). Then by setting

$$\mathcal{C}_{n+1} = \{0 < 1 < \cdots < n-1 < n < \bar{n} < \overline{n-1} < \cdots < \bar{1} < \bar{0}\}$$

the crystal  $B_k^C$  can be regarded as a subcrystal of the  $C_{n+1}$ -crystal associated to the dominant weight  $k\Lambda_1$  (labelled by the one-row semistandard tableaux on  $\mathcal{C}_{n+1}$  with length  $k$ ).

Recall that the combinatorial  $R$ -matrix associated to crystals  $B_k^C$  is equivalent to the description of the crystal graph isomorphisms

$$\begin{cases} B_l^C \otimes B_k^C \xrightarrow{\cong} B_k^C \otimes B_l^C \\ b_1 \otimes b_2 \mapsto b'_2 \otimes b'_1 \end{cases}$$

together with the energy function  $H^C$  on  $B_l^C \otimes B_k^C$ . As proved in [4], this can be done by using the insertion algorithm for  $C_{n+1}$ -tableaux of [1] or [15]. In the sequel we only need the description of the energy function  $H^C$ . Consider  $b_1 \in B_l^C$  and  $b_2 \in B_k^C$  and denote by  $z = \min(\#0 \text{ in } b_1, \#0 \text{ in } b_2) = \min(\#\bar{0} \text{ in } b_1, \#\bar{0} \text{ in } b_2)$ . Let  $b_1^*$  and  $b_2^*$  be the row tableaux obtained by erasing  $z$  pairs of letters  $(0, \bar{0})$  in  $b_1$  and  $b_2$ . Write  $l^*$  and  $k^*$  for the lengths of  $b_1^*$  and  $b_2^*$ . Denote by  $P_{n+1}^C(b_1^* \otimes b_2^*)$  the tableau obtained by inserting the row  $b_2^*$  into the row  $b_1^*$  following the  $C_{n+1}$ -insertion algorithm. Then  $P_{n+1}^C(b_1^* \otimes b_2^*)$  is a two-row  $C_{n+1}$ -tableau which contains  $k^* + l^*$  letters. Since the plactic relations for the root system  $C_{n+1}$  are not homogeneous, pairs of letters  $(0, \bar{0})$  can appear in  $P_{n+1}^C(b_1^* \otimes b_2^*)$ . Write  $m$  for the length of the shortest row of  $P_{n+1}^C(b_1^* \otimes b_2^*)$ .

**Proposition 2.5.1** [4] *For any  $b_1 \otimes b_2$  in  $B_l^C \otimes B_k^C$  we have*

$$H^C(b_1 \otimes b_2) = \min(l^*, k^*) - m.$$

The multiplicity of  $V(\lambda)$  in  $W_\mu^C$  is then equal to the number of highest weight vertices of weight  $\lambda$  in the crystal  $B_\mu^C = B_{\mu_1}^C \otimes \cdots \otimes B_{\mu_n}^C$ . Then the one dimension sum  $X_{\lambda,\mu}(q)$  for  $C_n^{(1)}$ -crystals is defined in [4] by

$$X_{\lambda,\mu}(q) = \sum_{b \in E_\lambda} q^{H^C(b)} \text{ with } H^C(b) = \sum_{0 \leq i < j \leq n} H^C(b_i \otimes b_j^{(i+1)})$$

where  $E_\lambda$  is the set of highest weight vertices  $b = b_1 \otimes \cdots \otimes b_n$  in  $B_\mu^C$  of highest weight  $\lambda$ ,  $b_j^{(i)}$  is determined by the crystal isomorphism

$$\begin{aligned} B_{\mu_i}^C \otimes B_{\mu_{i+1}}^C \otimes B_{\mu_{i+2}}^C \otimes \cdots \otimes B_{\mu_j}^C &\rightarrow B_{\mu_i}^C \otimes B_{\mu_j}^C \otimes B_{\mu_{i+1}}^C \cdots \otimes B_{\mu_{j-1}}^C \\ b_i \otimes b_{i+1} \otimes \cdots \otimes b_j &\rightarrow b_j^{(i)} \otimes b'_i \otimes \cdots \otimes b'_{j-1} \end{aligned} \quad (7)$$

and for any  $j = 1, \dots, n$ ,  $H(b_0 \otimes b_j^{(1)})$  is equal to the number of letters 0 in  $b_j^{(1)}$ .

When  $|\lambda| = |\mu|$  the vertices of  $E_\lambda$  contain only unbarred letters. In this case  $H^A(b_i \otimes b_j^{(i+1)}) = H^C(b_i \otimes b_j^{(i+1)})$  that is the energy function of type  $A$  defined on the vertices  $b$  which do not contain any barred letter is the restriction of that of type  $C$ . In [20] Nakayashiki and Yamada have proved the equality  $H^A(b) = \text{ch}(Q(b))$  where  $Q(b)$  is the semistandard tableau associated to  $b$  by generalizing the Robinson-Schensted correspondence and  $\text{ch}$  is the charge defined by Lascoux and Schützenberger in [12] and [13]. Recall that the charge statistic verifies

$$K_{\lambda, \mu}^{A_{n-1}}(q) = \sum_{T \in SST_\mu(\lambda)} q^{\text{ch}(T)} \quad (8)$$

where  $SST_\mu(\lambda)$  is the set of semistandard tableaux of shape  $\lambda$  and weight  $\mu$ . This implies that  $X_{\lambda, \mu}(q) = K_{\lambda, \mu}^{A_{n-1}}(q)$  when  $|\lambda| = |\mu|$ .

Many computations suggest the following identities:

**Conjecture 2.5.2** *Consider  $\lambda \in \mathcal{P}_n$ . Then we have*

$$\begin{aligned} \text{(i)} : u_{\lambda, (1^n)}(q) &= q^{\frac{|\mu| - |\lambda|}{2}} X_{\lambda, (1^n)}(q), \\ \text{(ii)} : U_{\lambda, \mu}(q) &= q^{|\mu| - |\lambda|} X_{\lambda, \mu}(q) \text{ for any } \mu \in \mathcal{P}_n. \end{aligned}$$

**Remarks:**

(i): A  $q$ -analogue for the multiplicity of  $V_n^C(\lambda)$  in  $W^C(\mu)$  can also be defined from rigged configurations. The  $X = M$  conjecture gives a simple relation called fermionic formula between  $X$  and  $M$ . In [22], Okado, Schilling and Shimozono have proved this conjecture when all the parts of  $\mu$  are equal to 1. In 4, we will prove (i) of Conjecture 2.5.2 and (ii) when  $\mu = (1, \dots, 1)$ . Thus by combining our results with those of [22] we obtain very simple relations between the three different  $q$ -analogues for the multiplicity of  $V_n^C(\lambda)$  in  $W^C(1^n) = V(1^n)$ .

(ii): By Theorem 2.4.2, the polynomials  $U_{\lambda, \mu}(q)$  are Kostka-Foulkes polynomials. Thus (ii) of Conjecture 2.5.2 implies that, up to a simple renormalization, the one dimension sums  $X_{\lambda, \mu}(q)$  are affine Kazhdan-Lusztig polynomials.

### 3 Identities for the $q$ -multiplicities

#### 3.1 Decomposition of the $\tilde{K}_{\lambda, \mu}(q)$ in terms of the $K_{\lambda, \mu}^{A_{n-1}}(q)$

Denote by  $\mathcal{P}_q^A$  the  $q$ -Kostant partition function defined by

$$\prod_{1 \leq i < j \leq n} \left(1 - q \frac{x_i}{x_j}\right)^{-1} = \sum_{\eta \in \mathbb{Z}^n} \mathcal{P}_q^A(\eta) x^\eta$$

Then for any  $\lambda, \mu \in \mathcal{P}_n$ , the Kostka-Foulkes polynomial  $K_{\lambda, \mu}^{A_{n-1}}(q)$  is such that

$$K_{\lambda, \mu}^{A_{n-1}}(q) = \sum_{\sigma \in \mathcal{S}_n} (-1)^{l(\sigma)} \mathcal{P}_q^A(\sigma \circ \lambda - \mu).$$

Since  $K_{\lambda+k\kappa_n, \mu+k\kappa_n}^{A_{n-1}}(q) = K_{\lambda, \mu}^{A_{n-1}}(q)$  for any positive integer  $k$ , the definition of  $K_{\lambda, \mu}^{A_{n-1}}(q)$  can be extended for  $\lambda$  and  $\mu$  decreasing sequence of integers. By definition of the  $q$ -partition function  $\mathcal{P}_q^C$  we must have

$$\sum_{\beta \in \mathbb{Z}^n} \mathcal{P}_q^C(\beta) x^\beta = \prod_{1 \leq r \leq s \leq n} (1 - qx_r x_s)^{-1} \prod_{1 \leq i < j \leq n} \left(1 - q \frac{x_i}{x_j}\right)^{-1}.$$

Now we can write by (4)

$$\prod_{1 \leq r \leq s \leq n} (1 - qx_r x_s)^{-1} = \sum_{\delta \in L_C} q^{|\delta|/2} c(\delta) x^\delta \quad (9)$$

since the number of roots appearing in a decomposition of  $\delta \in L_C$  as a sum of positive roots  $\varepsilon_r + \varepsilon_s$  with  $1 \leq r \leq s \leq n$  is always equal to  $|\delta|/2$ . Thus we obtain

$$\sum_{\beta \in \mathbb{Z}^n} \mathcal{P}_q^C(\beta) x^\beta = \sum_{\eta \in \mathbb{Z}^n} \sum_{\delta \in L_C} q^{|\delta|/2} \mathcal{P}_q^A(\eta) c(\delta) x^{\delta+\eta}. \quad (10)$$

By using similar arguments for  $\mathcal{P}_q^D$  we derive the following lemma:

**Lemma 3.1.1** *For any  $\beta \in \mathbb{Z}^n$  we have*

$$\mathcal{P}_q^C(\beta) = \sum_{\delta \in L_C^{|\beta|}} q^{|\delta|/2} c(\delta) \mathcal{P}_q^A(\beta - \delta) \text{ and } \mathcal{P}_q^D(\beta) = \sum_{\delta \in L_D^{|\beta|}} q^{|\delta|/2} d(\delta) \mathcal{P}_q^A(\beta - \delta)$$

where  $L_C^{|\beta|} = \{\delta \in L_C, |\delta| = |\beta|\}$  and  $L_D^{|\beta|} = \{\delta \in L_D, |\delta| = |\beta|\}$ .

**Proof.** >From (10) we derive the equality  $\mathcal{P}_q^C(\beta) = \sum_{\eta+\delta=\beta} c(\delta) q^{|\delta|/2} \mathcal{P}_q^{A_n}(\eta)$ . Since  $\mathcal{P}_q^{A_n}(\eta) = 0$  when  $|\eta| \neq 0$ , we can suppose  $|\eta| = 0$  and  $|\delta| = |\beta|$  in the previous sum. Then  $\delta \in L_C^{|\beta|}$  and the result follows immediately. The proof for  $\mathcal{P}_q^D(\beta)$  is similar. ■

**Remark:** A similar result for the  $q$ -partition function  $\mathcal{P}_q^B$  does not exit. Indeed the number of roots appearing in a decomposition of  $\delta \in L_B$  as a sum of positive roots  $\varepsilon_r + \varepsilon_s$  with  $1 \leq r < s \leq n$  and  $\varepsilon_i$  with  $1 \leq i \leq n$  does not depend only of  $|\delta|$  since  $|\varepsilon_r + \varepsilon_s| \neq |\varepsilon_i|$ .

**Proposition 3.1.2** *Consider  $\lambda, \mu \in \mathcal{P}_n$  such that  $|\lambda| \geq |\mu|$ . Then we have:*

1.  $\tilde{K}_{\lambda, \mu}^{C_n}(q) = q^{\frac{|\lambda| - |\mu|}{2}} \sum_{\gamma \in \tilde{\mathcal{P}}_n} \sum_{\sigma \in \mathcal{S}_n} (-1)^{l(\sigma)} c(\sigma \circ \lambda - \gamma) K_{\gamma, \mu}^{A_{n-1}}(q)$
2.  $\tilde{K}_{\lambda, \mu}^{D_n}(q) = q^{\frac{|\lambda| - |\mu|}{2}} \sum_{\gamma \in \tilde{\mathcal{P}}_n} \sum_{\sigma \in \mathcal{S}_n} (-1)^{l(\sigma)} d(\sigma \circ \lambda - \gamma) K_{\gamma, \mu}^{A_{n-1}}(q)$

where  $\tilde{\mathcal{P}}_n = \{\gamma = (\gamma_1, \dots, \gamma_n) \in \mathbb{Z}^n, \gamma_1 \geq \gamma_2 \geq \dots \geq \gamma_n\}$ .

**Proof.** We have

$$\tilde{K}_{\lambda,\mu}^{C_n}(q) = \sum_{\sigma \in \mathcal{S}_n} (-1)^{l(\sigma)} \mathcal{P}_q^{C_n}(\sigma(\lambda + \rho_n) - (\mu + \rho_n)).$$

Hence from the previous lemma we derive

$$\tilde{K}_{\lambda,\mu}^{C_n}(q) = \sum_{\delta \in L_C^{|\beta|}} c(\delta) q^{|\delta|/2} \sum_{\sigma \in \mathcal{S}_n} (-1)^{l(\sigma)} \mathcal{P}_q^{A_{n-1}}(\sigma(\lambda + \rho_n) - (\mu + \delta + \rho_n))$$

and

$$\tilde{K}_{\lambda,\mu}^{C_n}(q) = \sum_{\sigma \in \mathcal{S}_n} (-1)^{l(\sigma)} \sum_{\delta \in L_C^{|\beta|}} c(\delta) q^{|\delta|/2} \mathcal{P}_q^{A_{n-1}}(\sigma(\lambda + \rho_n - \sigma^{-1}(\delta)) - (\mu + \rho_n))$$

For any  $\sigma \in \mathcal{S}_n$ , we have  $\sigma^{-1}(L_C^{|\beta|}) = L_C^{|\beta|}$  and  $c(\delta) = c(\sigma(\delta))$ . Thus we obtain

$$\tilde{K}_{\lambda,\mu}^{C_n}(q) = \sum_{\sigma \in \mathcal{S}_n} (-1)^{l(\sigma)} \sum_{\delta \in L_C^{|\beta|}} c(\delta) q^{|\delta|/2} \mathcal{P}_q^{A_{n-1}}(\sigma(\lambda + \rho_n - \delta) - (\mu + \rho_n)) = \sum_{\delta \in L_C^{|\beta|}} c(\delta) q^{\frac{|\mu| - |\lambda|}{2}} K_{\lambda - \delta, \mu}^{A_{n-1}}(q). \quad (11)$$

Now  $K_{\lambda - \delta, \mu}^{A_{n-1}}(q) = 0$  or there exists  $\sigma \in \mathcal{S}_n$  and  $\gamma \in \tilde{\mathcal{P}}_n$  such that  $\gamma = \sigma^{-1} \circ (\lambda - \delta)$ . It follows that

$$\tilde{K}_{\lambda,\mu}^{C_n}(q) = \sum_{\sigma \in \mathcal{S}_n} (-1)^{l(\sigma)} \sum_{\gamma \in \tilde{\mathcal{P}}_n} c((\lambda + \rho) - \sigma(\gamma + \rho)) K_{\gamma, \mu}^{A_{n-1}}(q).$$

Since  $c(\delta) = c(\sigma(\delta))$  for any  $\sigma \in \mathcal{S}_n$ , and  $\delta \in L_C$  we obtain the desired equality

$$\tilde{K}_{\lambda,\mu}^{C_n}(q) = \sum_{\gamma \in \tilde{\mathcal{P}}_n} \sum_{\sigma \in \mathcal{S}_n} (-1)^{l(\sigma)} c(\sigma(\lambda + \rho) - (\gamma + \rho)) K_{\gamma, \mu}^{A_{n-1}}(q)$$

The proof is similar for  $\tilde{K}_{\lambda,\mu}^{D_n}(q)$ . ■

By Lemma 2.3.1,  $\tilde{K}_{\lambda,\mu}^{C_n}(q)$  and  $\tilde{K}_{\lambda,\mu}^{D_n}(q)$  are Kazhdan-Lusztig polynomials. Thus they have nonnegative integer coefficients. This property can also be obtained from the proposition below:

**Corollary 3.1.3** *Consider  $\lambda, \mu \in \mathcal{P}_n$ . For  $k$  a sufficiently large integer we have:*

1.  $\tilde{K}_{\lambda,\mu}^{C_n}(q) = q^{\frac{|\lambda| - |\mu|}{2}} \sum_{\gamma \in \tilde{\mathcal{P}}_n} [V_n^A(\gamma + k\kappa_n), V_n^C(\lambda + k\kappa_n)] K_{\gamma, \mu}^{A_{n-1}}(q),$
2.  $\tilde{K}_{\lambda,\mu}^{D_n}(q) = q^{\frac{|\lambda| - |\mu|}{2}} \sum_{\gamma \in \tilde{\mathcal{P}}_n} [V_n^A(\gamma + k\kappa_n), V_n^D(\lambda + k\kappa_n)] K_{\gamma, \mu}^{A_{n-1}}(q).$

**Proof.** Consider  $\gamma \in \tilde{\mathcal{P}}_n$  and write  $\gamma = (\gamma^+, \gamma^-)$  as in (3). We have by Proposition 2.2.1  $[V_n^A(\gamma) : V_n^C(\lambda)] = \sum_{w \in W_{C_n}} (-1)^{l(w)} c(w \circ \lambda - \gamma)$ . Now if  $k$  is sufficiently large  $\gamma + k\kappa_n$  is a partition and  $[V_n^A(\gamma) : V_n^C(\lambda)] = \sum_{w \in W_{C_n}} (-1)^{l(w)} c(w \circ \lambda - \gamma + w(k\kappa_n) - k\kappa_n)$ . Suppose that  $w \notin \mathcal{S}_n$ , then  $|w(k\kappa_n) - k\kappa_n| \leq -2k$ . Thus we can choose  $k$  such that  $\gamma + k\kappa_n$  is a partition and  $w \circ \lambda - \gamma + w(k\kappa_n) - k\kappa_n \notin L_C$  for any  $w \in W_{C_n}$  such that  $w \notin \mathcal{S}_n$ . For such an integer  $k$  we have  $c(w \circ \lambda - \gamma + w(k\kappa_n) - k\kappa_n) = 0$ . Then we derive 1 from 1 of Proposition 3.1.2. We prove 2 by using similar arguments. ■

To obtain a decomposition of the polynomial  $\tilde{K}_{\lambda,\mu}^{B_n}(q)$  as a sum of polynomials  $K_{\gamma,\mu}^{A_{n-1}}(q)$ , we need first to obtain its decomposition in terms of the polynomials  $\tilde{K}_{\nu,\mu}^{D_n}(q)$ .

**Proposition 3.1.4** Consider  $\lambda, \mu \in \mathcal{P}_n$  such that  $|\lambda| \geq |\mu|$ . Then for  $k$  a sufficiently large integer we have:

$$\tilde{K}_{\lambda, \mu}^{B_n}(q) = \sum_{\nu \in \tilde{\mathcal{P}}_n} q^{|\lambda| - |\nu|} [V_n^D(\nu + k\kappa_n), V_n^B(\lambda + k\kappa_n)] \tilde{K}_{\nu, \mu}^{D_n}(q).$$

**Proof.** By definition of the partition functions  $\mathcal{P}_q^{B_n}$  and  $\mathcal{P}_q^{D_n}$  we can write

$$\sum_{\beta \in \mathbb{Z}^n} \mathcal{P}_q^{B_n}(\beta) x^\beta = \sum_{\gamma \in \mathbb{Z}^n} \mathcal{P}_q^{D_n}(\gamma) x^\gamma \prod_{k=1}^n \frac{1}{1 - qx_k}.$$

We have  $\prod_{k=1}^n \frac{1}{1 - qx_k} = \sum_{\delta \in \mathbb{N}^n} q^{|\delta|} x^\delta$ . Thus we derive

$$\sum_{\beta \in \mathbb{Z}^n} \mathcal{P}_q^{B_n}(\beta) x^\beta = \sum_{\gamma \in \mathbb{Z}^n} \sum_{\delta \in \mathbb{N}^n} q^{|\delta|} \mathcal{P}_q^{D_n}(\gamma) x^{\gamma + \delta}.$$

This implies that  $\mathcal{P}_q^{B_n}(\beta) = \sum_{\delta \in \mathbb{N}^n} q^{|\delta|} \mathcal{P}_q^{D_n}(\beta - \delta)$  and

$$\tilde{K}_{\lambda, \mu}^{B_n}(q) = \sum_{\sigma \in \mathcal{S}_n} (-1)^{l(\sigma)} \sum_{\delta \in \mathbb{N}^n} q^{|\delta|} \mathcal{P}_q^{D_n}(\sigma(\lambda + \rho_n) - (\mu + \delta + \rho_n)).$$

By using similar arguments to that of the proof of Proposition 3.1.2, we obtain

$$\tilde{K}_{\lambda, \mu}^{B_n}(q) = q^{|\lambda| - |\mu|} \sum_{\nu \in \tilde{\mathcal{P}}_n} \sum_{\sigma \in \mathcal{S}_n} (-1)^{l(\sigma)} \mathbf{1}_{\mathbb{N}}(\sigma(\lambda + \rho_n) - (\nu + \rho_n)) K_{\nu, \mu}^{D_n}(q).$$

There exists a sufficiently large integer  $k$  for which  $w(\lambda + \rho_n + k\kappa_n) - (\nu + \rho_n + k\kappa_n) \notin \mathbb{N}^n$  for any  $w \in W_{B_n}$  such that  $w \notin \mathcal{S}_n$ . Then it follows from Lemma 2.2.2 that  $[V_n^D(\nu + k\kappa_n) : V_n^B(\lambda + k\kappa_n)] = \sum_{\sigma \in \mathcal{S}_n} (-1)^{l(\sigma)} \mathbf{1}_{\mathbb{N}}(\sigma \circ \lambda - \nu)$  and the proposition is proved. ■

**Remark:** From 2 of Corollary 3.1.3 and the above proposition we obtain

$$\tilde{K}_{\lambda, \mu}^{B_n}(q) = \sum_{\nu \in \tilde{\mathcal{P}}_n} \sum_{\gamma \in \tilde{\mathcal{P}}_n} q^{\frac{2|\lambda| - |\nu| - |\mu|}{2}} [V_n^D(\nu + k\kappa_n), V_n^B(\lambda + k\kappa_n)] [V_n^A(\gamma + k\kappa_n), V_n^D(\nu + k\kappa_n)] K_{\gamma, \mu}^{A_{n-1}}(q)$$

which implies in particular that the polynomials  $\tilde{K}_{\lambda, \mu}^{B_n}(q)$  have non negative coefficients.

### 3.2 Decomposition of $u_{\lambda, \mu}(q)$ and $U_{\lambda, \mu}(q)$ in terms of the $K_{\lambda, \mu}^{A_{n-1}}(q)$

By using similar arguments to the proof of Lemma 3.1.1, we can establish that for any  $\beta \in \mathbb{Z}^n$  we have

$$f_q(\beta) = \sum_{\delta \in L_C^{|\beta|}} q^{|\delta|/2} d(\delta) \mathcal{P}_q^A(\beta + \delta) \text{ and } F_q(\beta) = \sum_{\delta \in L_D^{|\beta|}} q^{|\delta|/2} c(\delta) \mathcal{P}_q^A(\beta + \delta)$$

which implies the following proposition:

**Proposition 3.2.1** Consider  $\lambda, \mu \in \mathcal{P}_n$  such that  $|\mu| \leq |\lambda|$ . Then we have

1.  $U_{\lambda,\mu}(q) = q^{\frac{|\mu| - |\lambda|}{2}} \sum_{\nu \in \mathcal{P}_n} [V_n^A(\lambda), V_n^C(\nu)] K_{\nu,\mu}^{A_{n-1}}(q),$
2.  $u_{\lambda,\mu}(q) = q^{\frac{|\mu| - |\lambda|}{2}} \sum_{\nu \in \mathcal{P}_n} [V_n^A(\lambda), V_n^D(\nu)] K_{\nu,\mu}^{A_{n-1}}(q).$

**Proof.** We proceed as in the proof of Proposition 3.1.2 and (11) is replaced by

$$U_{\lambda,\mu}(q) = \sum_{\delta \in L_C^{|\beta|}} c(\delta) q^{\frac{|\mu| - |\lambda|}{2}} K_{\lambda+\delta,\mu}^{A_{n-1}}(q).$$

This time  $K_{\lambda+\delta,\mu}^{A_{n-1}}(q) = 0$  or there exists  $\sigma \in \mathcal{S}_n$  and  $\nu \in \mathcal{P}_n$  such that  $\nu = \sigma^{-1} \circ (\lambda + \delta)$ . Note that  $\nu$  can not have negative coordinates for  $\lambda + \delta$  have non negative coordinates. We deduce:

$$U_{\lambda,\mu}(q) = q^{\frac{|\mu| - |\lambda|}{2}} \sum_{\sigma \in \mathcal{S}_n} (-1)^{l(\sigma)} \sum_{\nu \in \mathcal{P}_n} c(\sigma(\nu + \rho_n) - (\lambda + \rho_n)) K_{\gamma,\mu}^{A_{n-1}}(q).$$

We have seen in the proof of Corollary 3.1.3 that for  $k$  a sufficiently large integer we have

$$[V_n^A(\lambda + k\kappa_n), V_n^C(\nu + k\kappa_n)] = \sum_{\sigma \in \mathcal{S}_n} (-1)^{l(\sigma)} c(\sigma \circ \nu - \lambda).$$

Since  $\nu$  is a partition and  $c_{\gamma,\lambda+k\kappa_n}^{\nu+k\kappa_n} = c_{\gamma,\lambda}^{\nu}$  for any  $k \in \mathbb{N}$ , we deduce from Corollary 2.2.5 that  $[V_n^A(\lambda), V_n^C(\nu)] = [V_n^A(\lambda + k\kappa_n), V_n^C(\nu + k\kappa_n)]$  which proves 1. Assertion 2 is obtained similarly. ■

### 3.3 Link with the polynomials $K_{\lambda,\mu}^{\diamond}(q)$ of Shimozono and Zabrocki

We deduce from Corollary 2.2.5 and Proposition 3.2.1:

**Theorem 3.3.1** *Consider  $\lambda, \mu \in \mathcal{P}_n$  such that  $|\mu| \geq |\lambda|$  Then we have the following equalities*

$$\begin{aligned} \text{(i)} : u_{\lambda,\mu}(q) &= q^{\frac{|\mu| - |\lambda|}{2}} \sum_{\nu \in \mathcal{P}_n} [V_n^C(\lambda), V_n^A(\nu)] K_{\nu,\mu}^{A_{n-1}}(q) = q^{\frac{|\mu| - |\lambda|}{2}} \sum_{\nu \in \mathcal{P}_n} [V_n^A(\lambda), V_n^D(\nu)] K_{\nu,\mu}^{A_{n-1}}(q) \\ &= q^{\frac{|\mu| - |\lambda|}{2}} \sum_{\nu \in \mathcal{P}_n} \sum_{\gamma \in \mathcal{P}_n^{(1,1)}} c_{\gamma,\lambda}^{\nu} K_{\nu,\mu}^{A_{n-1}}(q), \end{aligned}$$

$$\begin{aligned} \text{(ii)} : U_{\lambda,\mu}(q) &= q^{\frac{|\mu| - |\lambda|}{2}} \sum_{\nu \in \mathcal{P}_n} [V_n^D(\lambda), V_n^A(\nu)] K_{\nu,\mu}^{A_{n-1}}(q) = q^{\frac{|\mu| - |\lambda|}{2}} \sum_{\nu \in \mathcal{P}_n} [V_n^A(\lambda), V_n^C(\nu)] K_{\nu,\mu}^{A_{n-1}}(q) \\ &= q^{\frac{|\mu| - |\lambda|}{2}} \sum_{\nu \in \mathcal{P}_n} \sum_{\gamma \in \mathcal{P}_n^{(2)}} c_{\gamma,\lambda}^{\nu} K_{\nu,\mu}^{A_{n-1}}(q). \end{aligned}$$

By comparing the leftmost equalities of the above theorem with equality (7.6) of [27] we obtain

$$K_{\lambda,\mu}^{(1,1)}(q) = u_{\lambda,\mu}(q^2) \text{ and } K_{\lambda,\mu}^{(2)}(q) = U_{\lambda,\mu}(q^2) \tag{12}$$

where  $K_{\lambda,\mu}^{(1,1)}(q)$  and  $K_{\lambda,\mu}^{(2)}(q)$  are polynomials defined by Shimozono and Zabrocki by using creating operators on formal series. In particular the polynomials  $K_{\lambda,\mu}^{(1,1)}(q)$  and  $K_{\lambda,\mu}^{(2)}(q)$  are Kazhdan-Lusztig polynomials specialized at  $q^2$ .

**Remark:** In [27] the authors have also defined another polynomial denoted  $K_{\lambda,\mu}^{(1)}(q)$  verifying

$$K_{\lambda,\mu}^{(1)}(q) = q^{|\mu|-|\lambda|} \sum_{\nu \in \mathcal{P}_n} \sum_{\gamma \in \mathcal{P}_n} c_{\gamma,\lambda}^\nu K_{\nu,\mu}^{A_{n-1}}(q^2). \quad (13)$$

From the duality (5), it is tempting to introduce the polynomial  $V_{\lambda,\mu}(q) = \tilde{K}_{I(\lambda),I(\mu)}(q)$ . A similar result than Proposition 3.2.1 can not exist for  $V_{\lambda,\mu}(q)$  (see the remark following Lemma 3.1.1). Nevertheless, by using Corollary 2.2.5, one can establish that

$$V_{\lambda,\mu}(1) = \sum_{\nu \in \mathcal{P}_n} [V_n^B(\lambda), V_{2n}^A(\nu)] K_{\nu,\mu}^{A_{n-1}}(1) = \sum_{\nu \in \mathcal{P}_n} \sum_{\gamma \in \mathcal{P}_n} c_{\gamma,\lambda}^\nu K_{\nu,\mu}^{A_{n-1}}(1).$$

Thus  $K_{\lambda,\mu}^{(1)}(1) = V_{\lambda,\mu}(1)$ . However we have  $K_{\lambda,\mu}^{(1)}(q) \neq V_{\lambda,\mu}(q^2)$  in general. For example if we take  $\lambda = (1, 0, 0)$  and  $\mu = (1, 1, 1)$  we obtain  $K_{\lambda,\mu}^{(1)}(q) = q^8 + 2q^6 + 2q^4 + q^2$  and  $V_{\lambda,\mu}(q^2) = q^{10} + q^8 + 2q^6 + q^4 + q^2$ .

Consider  $\nu$  in  $\mathcal{P}_n$ . For any standard tableau  $\tau$  of shape  $\nu$  and weight  $(1^n)$ , let  $\nu'$  be the standard tableau of shape  $\nu'$  and weight  $(1^n)$  obtained by reflecting  $\tau$  among the diagonal. Then one can verify that  $\text{ch}(\tau') = \frac{n(n-1)}{2} - \text{ch}(\tau)$  which by (8) implies the identity:

$$K_{\nu',(1^n)}^{A_{n-1}}(q) = q^{\frac{n(n-1)}{2}} K_{\nu,(1^n)}^{A_{n-1}}(q^{-1}). \quad (14)$$

The following proposition will be useful in Section 4.

**Proposition 3.3.2** *Consider  $\lambda \in \mathcal{P}_n$  such that  $n \geq |\lambda|$ . Then we have*

$$(i) : U_{\lambda',(1^n)}(q) = q^{\frac{n(n-1)}{2} + n - |\lambda|} u_{\lambda,(1^n)}(q^{-1}) \quad \text{and} \quad (ii) : u_{\lambda',(1^n)}(q) = q^{\frac{n(n-1)}{2} + n - |\lambda|} U_{\lambda,(1^n)}(q^{-1}).$$

**Proof.** By (ii) of Theorem 3.3.1 we have

$$U_{\lambda',(1^n)}(q) = q^{\frac{n-|\lambda|}{2}} \sum_{\nu \in \mathcal{P}_n} \sum_{\gamma \in \mathcal{P}_n^{(2)}} c_{\gamma,\lambda'}^\nu K_{\nu,(1^n)}^{A_{n-1}}(q). \quad (15)$$

Since  $K_{\nu,(1^n)}^{A_{n-1}}(q) = 0$  when  $|\nu| \neq n$ , we can suppose that  $\nu$  belongs to  $\widehat{\mathcal{P}}_n = \{\nu \in \mathcal{P}_n, |\nu| = n\}$  in the above sum. For any  $\nu$  in  $\widehat{\mathcal{P}}_n$ , we have  $c_{\gamma,\lambda}^\nu = 0$  unless  $|\gamma| = n - |\lambda|$ . So we can suppose that  $\gamma$  belongs to  $\widehat{\mathcal{P}}_n^{(2)} = \{\gamma \in \mathcal{P}_n^{(2)}, |\gamma| \leq n\}$  in (15). The map  $\Gamma : \nu \mapsto \nu'$  is an involution of  $\widehat{\mathcal{P}}_n$ . Moreover  $\Gamma(\widehat{\mathcal{P}}_n^{(2)}) = \widehat{\mathcal{P}}_n^{(1,1)} = \{\gamma \in \mathcal{P}_n^{(1,1)}, |\gamma| \leq n\}$ . Thus we can write

$$U_{\lambda',(1^n)}(q) = q^{\frac{n-|\lambda|}{2}} \sum_{\nu \in \widehat{\mathcal{P}}_n} \sum_{\gamma \in \widehat{\mathcal{P}}_n^{(1,1)}} c_{\gamma',\lambda'}^\nu K_{\nu',(1^n)}^{A_{n-1}}(q).$$

Now since  $c_{\gamma',\lambda'}^\nu = c_{\gamma,\lambda}^\nu$ , we derive by (14)

$$U_{\lambda',(1^n)}(q) = q^{\frac{n(n-1)}{2} + n - |\lambda|} \sum_{\nu \in \widehat{\mathcal{P}}_n} \sum_{\gamma \in \widehat{\mathcal{P}}_n^{(1,1)}} c_{\gamma,\lambda}^\nu K_{\nu,(1^n)}^{A_{n-1}}(q^{-1}).$$



Finally the equality  $U_{\lambda',(1^n)}(q) = q^{\frac{n(n-1)}{2}+n-|\lambda|}u_{\lambda,(1^n)}(q^{-1})$  is deduced from (i) of Theorem 3.3.1. We obtain (ii) similarly. ■

**Remark:** By introducing for the root system  $A_{n-1}$  generalized Kostka-Foulkes polynomials  $K_{\nu,R}(q)$  where  $R = (R_1, \dots, R_n)$  is a sequence of rectangular partitions, Schilling and Warnarr [25] have proved the equality  $K_{\nu',R'}(q) = q^{\|R\|}K_{\nu,R'}(q^{-1})$  where  $R' = (R'_1, \dots, R'_l)$  and  $\|R\| = \sum_{i < j} |R_i \cap R_j|$  which can be considered as a generalization of (14). In [27], Shimozono and Zabrocki have also defined their polynomials  $K_{\lambda,R}^{\diamond}(q)$  when  $R$  is a sequence of rectangular partitions. By (12), the above proposition can also be regarded as a Corollary of Proposition 28 of [27].

## 4 Proof of Conjecture 2.5.2 when $\mu = (1, \dots, 1)$

### 4.1 The crystals $B_{\Xi}$

For any integer  $l \geq 0$ , let  $B^A(l)$  be the crystal graph of the irreducible finite dimensional  $U_q(\mathfrak{sl}_{2n})$ -module of highest weight  $l\Lambda_1$ . In the sequel we choose to label the vertices of  $B^A(1)$  by the letters of  $\mathcal{C}_n$ , that is we identify  $B^A(1)$  with the crystal

$$1 \xrightarrow{1} 2 \cdots \rightarrow n-1 \xrightarrow{n-1} n \xrightarrow{n} \bar{n} \xrightarrow{n+1} \overline{n-1} \xrightarrow{n+2} \cdots \rightarrow \bar{2} \xrightarrow{2n-1} \bar{1}.$$

Recall that the crystal graph  $B^C(1)$  has been identified in 2.5 with

$$1 \xrightarrow{1} 2 \cdots \rightarrow n-1 \xrightarrow{n-1} n \xrightarrow{n} \bar{n} \xrightarrow{n-1} \overline{n-1} \xrightarrow{n-2} \cdots \rightarrow \bar{2} \xrightarrow{1} \bar{1}.$$

Thus the crystals  $B^A(1)$  and  $B^C(1)$  have the same vertices. For any partition  $\mu \in \mathcal{P}_n$ , set  $B_{(\mu)}^A = B^A(\mu_1) \otimes \cdots \otimes B^A(\mu_n)$  and  $B_{(\mu)}^C = B^C(\mu_1\Lambda_1) \otimes \cdots \otimes B^C(\mu_n\Lambda_1)$  (note that  $B_{(\mu)}^C \neq B_{\mu}^C$  defined in 2.5). Then  $B_{(\mu)}^A$  and  $B_{(\mu)}^C$  have also the same vertices. Nevertheless their crystal structure are distinct and their decompositions in connected components do not coincide.

Denote by  $H^A$  the energy function associated to  $B^A(l) \otimes B^A(k)$ . Then for any  $b_1 \otimes b_2$  belonging to  $B^A(l) \otimes B^A(k)$ , we have  $H^A(b_1 \otimes b_2) = \min(l, k) - m$  where  $m$  is the length of the shortest row of the semistandard tableau  $P^A(b_1 \otimes b_2)$  obtained by inserting the row  $b_2$  in the row  $b_1$  following the column bumping algorithm for semistandard tableaux. Given  $b = b_1 \otimes \cdots \otimes b_n \in B_{(\mu)}^A$ , set

$$H^A(b) = \sum_{1 \leq i < j \leq n} H^A(b_i \otimes b_j^{(i+1)})$$

where the vertices  $b_j^{(i+1)}$  are defined as in (7). In the sequel we need the following result due to Nakayashiki and Yamada:

**Theorem 4.1.1** [20] *Consider  $\nu$  and  $\mu$  two partitions of  $\mathcal{P}_n$ . Then*

$$K_{\nu,\mu}^{A_{n-1}}(q) = \sum_{b \in G_{\nu}} q^{H^A(b)}$$

where  $G_{\nu}$  is the set of highest weight vertices of weight  $\nu$  in  $B_{(\mu)}^A$ .

**Remark:** It is possible to show that  $P^A(b_1 \otimes b_2) = P_{n+1}^C(b_1 \otimes b_2)$  (see 2.5) for any  $b_1 \otimes b_2$  belonging to  $B^A(l) \otimes B^A(k)$  if and only if  $l = k = 1$ . Moreover if we choose  $l \geq 2$  and  $k \geq 2$ , these two tableaux can have distinct shapes. For example, by taking  $n = 2$ ,  $l = k = 2$ ,  $b_1 = b_2 = \bar{2}2$ , we obtain

$$P^A(b_1 \otimes b_2) = \begin{array}{|c|c|c|} \hline 2 & 2 & \bar{2} \\ \hline \hline 2 & & \\ \hline \end{array} \text{ and } P_{n+1}^C(b_1 \otimes b_2) = \begin{array}{|c|c|} \hline 1 & \bar{2} \\ \hline 2 & \bar{1} \\ \hline \end{array}.$$

Hence the two statistics  $H^A$  and  $H^C$  do not coincide in general on the vertices of  $B_{(\mu)}^A$ .

Suppose now that  $\mu = (1, \dots, 1)$ . Then any vertex  $b$  of  $B_{(1^n)}^A$  or  $B_{(1^n)}^C$  can be written

$$b = x_1 \otimes \dots \otimes x_n$$

where  $x_1, \dots, x_n$  are letters of  $\mathcal{C}_n$ . We have

$$H^A(b) = H^C(b) = \sum_{i=1}^{n-1} (n-i) H(x_i \otimes x_{i+1})$$

where  $H(x_i \otimes x_{i+1}) = 1$  if  $x_i \geq x_{i+1}$  and  $H(x_i \otimes x_{i+1}) = 0$  otherwise.

To each vertex  $b$ , we associate the  $(n-1)$ -tuple  $\Xi(b) = (\xi_1, \dots, \xi_{n-1})$  such that for any  $i = 1, \dots, n-1$ ,  $\xi_i = 0$  if  $x_i < x_{i+1}$  and  $\xi_i = 1$  otherwise. For any  $(n-1)$ -tuple  $\Xi$  with coordinates equal to 0 or 1, set

$$B_{\Xi} = \{b \in B_{(1^n)}^A, \Xi(b) = \Xi\}.$$

Then the statistics  $H^A$  and  $H^C$  are invariant on the vertices of  $B_{\Xi}$  and we have

$$H^A(b) = H^C(b) = \theta(b) = \sum_{i=1}^{n-1} (n-i) \xi_i \text{ for any } b \in B_{\Xi} \text{ with } \Xi = (\xi_1, \dots, \xi_{n-1}).$$

**Lemma 4.1.2** *Let  $\Xi = (\xi_1, \dots, \xi_{n-1})$  be a  $(n-1)$ -tuple with coordinates equal to 0 or 1. Then the set  $B_{\Xi}$  has a structure of  $A_{2n-1}$ -crystal and a structure of  $C_n$ -crystal.*

**Proof.** Consider  $\tilde{K}$  a Kashiwara crystal operator for  $U_q(\mathfrak{sl}_{2n-1})$  or  $U_q(\mathfrak{sp}_{2n})$  and  $x, y$  two letters of  $\mathcal{C}_n$ . Set  $\tilde{K}(x \otimes y) = x' \otimes y'$ . From the description of the crystals  $B^A(1)^{\otimes 2}$  and  $B^C(1)^{\otimes 2}$  given by Kashiwara and Nakashima in [9] we derive the equivalence:

$$x \leq y \iff x' \leq y'.$$

This implies that  $B_{\Xi}$  is stable under the action of any Kashiwara crystal operator. Thus  $B_{\Xi}$  has a structure of  $A_{2n-1}$ -crystal and a structure of  $C_n$ -crystal. ■

## 4.2 The $X = u$ conjecture when $\mu = (1, \dots, 1)$

**Theorem 4.2.1** *For any partition  $\lambda$  of length  $n$  we have*

$$u_{\lambda, (1^n)}(q) = q^{\frac{n-|\lambda|}{2}} X_{\lambda, (1^n)}(q).$$

**Proof.** Denote by  $\mathfrak{E}$  the set of the  $(n-1)$ -tuples  $\Xi$  with coordinates equal to 0 or 1. By Lemma 4.1.2, for any  $\Xi \in \mathfrak{E}$ ,  $B_\Xi$  has a structure of  $A_{2n-1}$ -crystal and a structure of  $C_n$ -crystal. Let us denote respectively by  $B_\Xi^A$  and  $B_\Xi^C$  these two crystals. There exists a  $U_q(\mathfrak{gl}_{2n})$ -module  $M_\Xi^A$  whose crystal is isomorphic to  $B_\Xi^A$ . Similarly there exists a  $U_q(\mathfrak{sp}_{2n})$ -module  $M_\Xi^C$  whose crystal is isomorphic to  $B_\Xi^A$ . Recall that the weight  $\text{wt}^A(b)$  of a vertex  $b \in B_\Xi^A$  is equal to  $(d_1, \dots, d_n, d_{n+1}, \dots, d_{2n})$  where for any  $i \in \{1, \dots, 2n\}$ ,  $d_i$  is the number of letters  $i$  in  $b$ . Similarly the weight  $\text{wt}^C(b)$  of a vertex  $b \in B_\Xi^C$  is equal to  $(\delta_1, \dots, \delta_n)$  where for any  $i \in \{1, \dots, 2n\}$ ,  $\delta_i$  is the number of letters  $i$  in  $b$  minus the number of letters  $\bar{i}$ . Then the characters of the modules  $M_\Xi^A$  and  $M_\Xi^C$  verify

$$\text{char}(M_\Xi^A)(x_1, \dots, x_n) = \sum_{b \in B_\Xi^A} x^{\text{wt}^A(b)} \quad \text{and} \quad \text{char}(M_\Xi^C)(x_1, \dots, x_n) = \sum_{b \in B_\Xi^C} x^{\text{wt}^C(b)}.$$

Let  $N_\Xi^A$  and  $N_\Xi^C$  be two representations respectively of  $GL_{2n}$  and  $Sp_{2n}$  such that  $\text{char}(N_\Xi^A) = \text{char}(M_\Xi^A)$  and  $\text{char}(N_\Xi^C) = \text{char}(M_\Xi^C)$ . Then, by definition of  $\text{wt}^A(b)$  and  $\text{wt}^C(b)$ , we have

$$\text{char}(N_\Xi^C)(x_1, \dots, x_n) = \text{char}(N_\Xi^A)(x_1, \dots, x_n, \frac{1}{x_n}, \dots, \frac{1}{x_1})$$

that is  $\text{ch}(N_\Xi^C)$  is obtained by changing  $x_{n+i}$  in  $\frac{1}{x_i}$  in  $\text{char}(N_\Xi^A)$ . Thus  $\text{char}(N_\Xi^C) = \text{char}(N_\Xi^A \downarrow_{Sp_{2n}}^{GL_{2n}})$  where  $N_\Xi^A \downarrow_{Sp_{2n}}^{GL_{2n}}$  is the restriction of  $N_\Xi^A$  to the action of  $Sp_{2n}$ . This implies that  $N_\Xi^C$  and  $N_\Xi^A \downarrow_{Sp_{2n}}^{GL_{2n}}$  are isomorphic representations of  $Sp_{2n}$ .

Write  $E_\lambda^\Xi = \{b \in B_\Xi^C, b \in E_\lambda\}$  for the set of highest weight vertices of weight  $\lambda$  in  $B_\Xi^C$ . Then we obtain:

$$\text{card}(E_\lambda^\Xi) = [V_n^C(\lambda), N_\Xi^C] = [V_n^C(\lambda), N_\Xi^A]$$

where  $[V_n^C(\lambda), N_\Xi^A]$  is the multiplicity of  $V_n^C(\lambda)$  in  $N_\Xi^A \downarrow_{Sp_{2n}}^{GL_{2n}}$ . Recall that  $X_{\lambda, (1^n)}(q) = \sum_{b \in E_\lambda} q^{H^C(b)}$  where  $E_\lambda$  is the set of highest weight vertices of weight  $\lambda$  in  $B_{(1^n)}^C$ . By Lemma 4.1.2 and the equality above we can write

$$X_{\lambda, (1^n)}(q) = \sum_{\Xi \in \mathfrak{E}} \sum_{b \in E_\lambda^\Xi} q^{H^C(b)} = \sum_{\Xi \in \mathfrak{E}} \text{card}(E_\lambda^\Xi) q^{\theta_\Xi} = \sum_{\Xi \in \mathfrak{E}} [V_n^C(\lambda), N_\Xi^A] q^{\theta_\Xi} \quad (16)$$

where for any  $\Xi = (\xi_1, \dots, \xi_{n-1})$  in  $\mathfrak{E}$ ,  $\theta_\Xi = \sum_{i=1}^{n-1} (n-i)\xi_i$ .

By definition of the representations  $N_\Xi^A$  we have

$$\bigoplus_{\Xi \in \mathfrak{E}} N_\Xi^A \simeq V_{2n}^A(\Lambda_1)^{\otimes n} \quad (17)$$

as  $GL_{2n}$ -representations. The irreducible components of  $B_{(1^n)}^A$  are indexed by standard tableaux with letters in  $\{1, \dots, n\}$ . For any  $b \in B_{(1^n)}^A$  denote by  $Q(b)$  the recording tableau (which is a standard tableau with letters in  $\{1, \dots, n\}$ ) associated to  $b$  by the Robinson-Schensted correspondence. The  $Q$ -symbol yields a one to one correspondence between the highest weight vertices of  $B_{(1^n)}^A$  (thus the irreducible components of  $V_{2n}^A(\Lambda_1)^{\otimes n}$ ) and the set  $ST$  of standard tableaux with letters in  $\{1, \dots, n\}$ . For any  $\tau \in ST$ , denote by  $V_n^A(\tau)$  the irreducible component of  $V_{2n}^A(\Lambda_1)^{\otimes n}$  associated to  $\tau$ . Then  $V_{2n}^A(\tau) \simeq V_{2n}^A(\nu)$  where  $\nu$  is the partition corresponding to the shape of  $\tau$ . Write  $ST_\Xi$  for the set of standards tableaux  $\tau \in ST$  such that there exists a highest weight vertex  $b \in B_\Xi^A$  (thus for the

$A_{2n-1}$ -structure of graph) with  $Q(b) = \tau$ . Then from (17) the sets  $ST_{\Xi}$ ,  $\Xi \in \mathfrak{E}$  are disjoint and  $\cup_{\mathfrak{E}} ST_{\Xi} = ST$ . For any  $\Xi \in \mathfrak{E}$ , we have

$$N_{\Xi}^A = \bigoplus_{\tau \in ST_{\Xi}} V_{2n}^A(\tau).$$

Thus we derive from (16) that

$$X_{\lambda, (1^n)}(q) = \sum_{\Xi \in \mathfrak{E}} \sum_{\tau \in ST_{\Xi}} [V_n^C(\lambda), V_{2n}^A(\tau)] q^{\theta_{\Xi}}.$$

Now the multiplicity  $[V_n^C(\lambda), V_{2n}^A(\tau)]$  depends only on the shape  $\nu$  of the standard tableau  $\tau$ . This means that we have the equality  $[V_n^C(\lambda), V_{2n}^A(\tau)] = [V_n^C(\lambda), V_{2n}^A(\nu)]$  for any  $\tau$  of shape  $\nu \in \mathcal{P}_n$ . We deduce that

$$X_{\lambda, (1^n)}(q) = \sum_{\Xi \in \mathfrak{E}} \sum_{\nu \in \mathcal{P}_n} \sum_{\tau \in ST_{\Xi}^{\nu}} [V_n^C(\lambda), V_{2n}^A(\nu)] q^{\theta_{\Xi}}$$

where  $ST_{\Xi}^{\nu} = \{\tau \in ST_{\Xi} \text{ of shape } \nu\}$ . So we obtain

$$X_{\lambda, (1^n)}(q) = \sum_{\nu \in \mathcal{P}_n} [V_n^C(\lambda), V_{2n}^A(\nu)] \sum_{\Xi \in \mathfrak{E}} \text{card}(ST_{\Xi}^{\nu}) q^{\theta_{\Xi}}.$$

But we have  $\sum_{\Xi \in \mathfrak{E}} \text{card}(ST_{\Xi}^{\nu}) q^{\theta_{\Xi}} = K_{\nu, (1^n)}^{A_{n-1}}(q)$  since  $\theta_{\Xi} = H^A(b)$  for any highest weight vertex  $b$  of weight  $\nu$  in  $B_{\Xi}^A$ . Thus

$$X_{\lambda, (1^n)}(q) = \sum_{\nu \in \mathcal{P}_n} [V_n^C(\lambda), V_{2n}^A(\nu)] K_{\nu, (1^n)}^{A_{n-1}}(q).$$

Finally by Theorem 3.3.1, we obtain

$$q^{\frac{n-|\lambda|}{2}} X_{\lambda, (1^n)}(q) = u_{\lambda, (1^n)}(q)$$

and the Theorem is proved. ■

**Remark:** One can define, from the crystal  $B_{(\mu)}^C$ , the sum

$$Y_{\lambda, \mu}(q) = \sum_{b \in F_{\lambda}} q^{H^C(b)}$$

where  $F_{\lambda}$  is the set of highest weight vertices  $b = b_1 \otimes \cdots \otimes b_n$  in  $B_{(\mu)}^C$  of weight  $\lambda$ . However, the polynomial  $Y_{\lambda, \mu}(q)$  is not a one dimension sum related to an affine root system and the identity

$$u_{\lambda, \mu}(q) = q^{\frac{|\mu|-|\lambda|}{2}} Y_{\lambda, \mu}(q)$$

is false in general. This is in particular the case for  $\mu = (2, 2, 2)$  and  $\lambda = (2, 0, 0)$ . Nevertheless, from (12) and Conjecture 32 of [27], one can conjecture that the  $q$ -multiplicities  $u_{\lambda, \mu}(q)$  are equal, up to the multiplication by a power of  $q$ , to the one dimension sums related to the affine root system  $A_{2n-1}^{(2)}$ .

### 4.3 The $X = U$ conjecture when $\mu = (1, \dots, 1)$

We want to establish the equality  $U_{\lambda, (1^n)}(q) = q^{n-|\lambda|} X_{\lambda, (1^n)}(q)$ . By Proposition 3.3.2 we know that

$$U_{\lambda, (1^n)}(q) = q^{\frac{n(n-1)}{2} + n - |\lambda|} u_{\lambda', (1^n)}(q^{-1}).$$

Thus by Theorem 4.2.1 we obtain

$$U_{\lambda, (1^n)}(q) = q^{\frac{n(n-1)}{2} + \frac{n-|\lambda|}{2}} X_{\lambda', (1^n)}(q^{-1})$$

and it suffices to prove the equality

$$X_{\lambda', (1^n)}(q^{-1}) = q^{-\frac{n(n-1)}{2} + \frac{n-|\lambda|}{2}} X_{\lambda, (1^n)}(q)$$

which is equivalent to

$$X_{\lambda', (1^n)}(q) = q^{\frac{n(n-1)}{2} - \frac{n-|\lambda|}{2}} X_{\lambda, (1^n)}(q^{-1}). \quad (18)$$

In [15] we have introduced a Robinson-Schensted type correspondence for the vertices of  $B_{(1^n)}^C$ . In particular we have obtained a one to one correspondence between the highest weight vertices of  $b \in B_{(1^n)}^C$  and the oscillating tableaux of length  $n$ . Recall that an oscillating tableau of length  $n$  is a sequence  $Q = (Q_1, \dots, Q_n)$  of Young diagrams such that  $Q_i$  and  $Q_{i+1}$  differs by exactly one box (that is  $Q_{k+1}/Q_k = \square$  or  $Q_k/Q_{k+1} = \square$ ). We denote by  $Q(b)$  the oscillating tableau associated to the highest weight vertex  $b \in B_{(1^n)}^C$  under this correspondence. More precisely set  $b = x_1 \otimes \dots \otimes x_n$ . Then  $Q(b)$  is defined recursively as follows:

$$Q_1 = \square \text{ and } Q_{i+1} = x_{i+1} \rightarrow Q_i \quad (19)$$

where  $x_{i+1} \rightarrow Q_i$  is the Young diagram obtained from  $Q_i$  by adding a box on the  $k$ -th row of  $Q_i$  if  $x_{i+1} = k \in \{1, \dots, n\}$  and by deleting a box on the  $k$ -th row of  $Q_i$  if  $x_{i+1} = \bar{k} \in \{\bar{1}, \dots, \bar{n}\}$ . Given any highest weight vertex  $b$ , it is easy to verify that  $Q(b)$  is an oscillating tableau of length  $n$ .

Suppose  $Q(b) = (Q_1, \dots, Q_n)$ . Then  $Q' = (Q'_1, \dots, Q'_n)$ , where for any  $i \in \{1, \dots, n\}$   $Q'_i$  is the conjugate diagram of  $Q_i$ , is also an oscillating tableau. There exists a highest weight vertex  $b' \in B_{(1^n)}^C$  such that  $Q(b') = Q'$ . Moreover if the weight of  $b$  is equal to  $\lambda$ , then the weight of  $b'$  is equal to  $\lambda'$ . To prove our conjecture we need the two following technical lemmas:

**Lemma 4.3.1** *Suppose that  $b = x_1 \otimes \dots \otimes x_n$  is a highest weight vertex of  $B_{(1^n)}^C$  and write  $b' = x'_1 \otimes \dots \otimes x'_n$ . Then:*

1. *for any  $i = 1, \dots, n$ , the two letters  $x_i$  and  $x'_i$  are simultaneously barred or unbarred,*

$$2. H(x'_i \otimes x'_{i+1}) = \begin{cases} 1 - H(x_i \otimes x_{i+1}) & \text{if } x_i \text{ and } x_{i+1} \text{ are simultaneously barred or unbarred} \\ H(x_i \otimes x_{i+1}) & \text{otherwise} \end{cases} .$$

**Proof.** We obtain 1 immediately from the definition (19) of  $Q$  and  $Q'$ .

As usual we enumerate the rows (resp. the columns) of the Young diagrams from top to bottom (resp. from left to right).

Suppose that  $x_i = p$  and  $x_{i+1} = q$  with  $p, q \in \{1, \dots, n\}$ . Then  $Q_{i+1}$  is obtained by adding first a box in the  $p$ -th row of  $Q_i$  to give  $Q_i$ , next a box in the  $q$ -th row of  $Q_i$ . Thus  $Q'_{i+1}$  is obtained

by adding first a box in the  $p$ -th column of  $Q'_{i-1}$ , next a box in the  $q$ -th column of  $Q'_i$  if  $p \neq q$ , by adding two boxes in the  $p$ -th column of  $Q'_{i-1}$  otherwise. This implies that we have  $x'_i \geq x'_{i+1}$  when  $p < q$  (i.e.  $x_i < x_{i+1}$ ) and  $x'_i < x'_{i+1}$  when  $p \geq q$  (i.e.  $x_i \geq x_{i+1}$ ).

Now suppose that  $x_i = \bar{p}$  and  $x_{i+1} = \bar{q}$  with  $p, q \in \{1, \dots, n\}$ . Then  $Q_{i+1}$  is obtained by deleting first a box in the  $p$ -th row of  $Q_{i-1}$  to give  $Q_i$ , next a box in the  $q$ -th row of  $Q_i$ . Thus  $Q'_{i+1}$  is obtained by deleting first a box in the  $p$ -th column of  $Q'_{i-1}$ , next a box in the  $q$ -th column of  $Q'_i$  if  $p \neq q$ , by deleting two boxes in the  $p$ -th column of  $Q'_{i-1}$  otherwise. This implies that we have  $x'_i \geq x'_{i+1}$  when  $\bar{p} < \bar{q}$  (i.e.  $x_i < x_{i+1}$ ) and  $x'_i < x'_{i+1}$  when  $\bar{p} \geq \bar{q}$  (i.e.  $x_i \geq x_{i+1}$ ).

So in all cases we obtain  $H(x'_i \otimes x'_{i+1}) = 1 - H(x_i \otimes x_{i+1})$  when  $x_i$  and  $x_{i+1}$  are simultaneously barred or unbarred.

The equality  $H(x'_i \otimes x'_{i+1}) = H(x_i \otimes x_{i+1})$  when  $x_i$  and  $x_{i+1}$  are not simultaneously barred or unbarred follows from 1 since a barred letter is always greater than an unbarred one. ■

For any vertex  $b = x_1 \otimes \dots \otimes x_n \in B_{(1^n)}^C$ , set  $Z_b = \{i \in \{1, \dots, n-1\}, x_i \text{ and } x_{i+1} \text{ are not simultaneously barred or unbarred}\}$

**Lemma 4.3.2** *With the above notation we have*

$$\sum_{i \in Z_b} (n-i)(1 - 2H(x_i \otimes x_{i+1})) = \frac{n - |\lambda|}{2}$$

for any highest weight vertex  $b = x_1 \otimes \dots \otimes x_n \in B_{(1^n)}^C$  of weight  $\lambda$ .

**Proof.** Observe first that we have

$$1 - 2H(x_i \otimes x_{i+1}) = \begin{cases} 1 & \text{if } x_i \leq x_{i+1} \\ -1 & \text{otherwise} \end{cases}.$$

Since  $b$  is a highest weight vertex, it follows from the description of the action of the Kashiwara operators [8] on a tensor product of crystals that for any  $i \in \{1, \dots, n\}$ ,  $b_i = x_1 \otimes \dots \otimes x_i$  is a highest weight vertex. In particular we must have  $b_1 = x_1 = 1$ . Thus we obtain

$$\sum_{i \in Z_b} (1 - 2H(x_i \otimes x_{i+1})) = \begin{cases} 0 & \text{if } x_n \text{ is unbarred} \\ 1 & \text{otherwise} \end{cases}. \quad (20)$$

To prove the lemma we proceed by induction on  $n$ . When  $n = 1$ , we have  $b = 1$ ,  $Z_b = \emptyset$  and  $|\lambda| = 1$ . Hence the lemma is true. Now suppose the lemma proved for any highest weight vertex of  $B_{(1^{n-1})}^C$  and consider  $b = x_1 \otimes \dots \otimes x_n \in B_{(1^n)}^C$  a highest weight vertex of weight  $\lambda$ . Set  $s_b = \sum_{i \in Z_b} (n-i)(1 - 2H(x_i \otimes x_{i+1}))$ . We have  $Z_b = Z_{b_{n-1}}$  if  $x_{n-1}$  and  $x_n$  are simultaneously barred or unbarred and  $Z_b = Z_{b_{n-1}} \cup \{n-1\}$  otherwise. Write  $\tilde{\lambda}$  for the weight of the highest weight vertex  $b_{n-1} = x_1 \otimes \dots \otimes x_{n-1}$ . In the two cases  $Z_b = Z_{b_{n-1}}$  and  $Z_b = Z_{b_{n-1}} \cup \{n-1\}$  we derive by using the induction hypothesis

$$s_b = \frac{n-1 - |\tilde{\lambda}|}{2} + \sum_{i \in Z_b} (1 - 2H(x_i \otimes x_{i+1})).$$

When  $x_n$  is unbarred, we have  $|\tilde{\lambda}| = |\lambda| - 1$ . Hence  $\frac{n-1-|\tilde{\lambda}|}{2} = \frac{n-|\lambda|}{2}$  and we obtain  $s_b = \frac{n-|\lambda|}{2}$  by (20). When  $x_n$  is barred, we have  $|\tilde{\lambda}| = |\lambda| + 1$  thus  $\frac{n-1-|\tilde{\lambda}|}{2} = \frac{n-|\lambda|}{2} - 1$  and by (20) we also derive  $s_b = \frac{n-|\lambda|}{2}$  which proves the lemma. ■

**Theorem 4.3.3** For any partition  $\lambda$  of length  $n$  we have

$$U_{\lambda, (1^n)}(q) = q^{n-|\lambda|} X_{\lambda, (1^n)}(q).$$

**Proof.** Consider  $b = x_1 \otimes \cdots \otimes x_n$  a highest weight vertex of  $B_{(1^n)}^C$  of weight  $\lambda$  and set  $b' = x'_1 \otimes \cdots \otimes x'_n$ . Then

$$H(b') = \sum_{i=1}^{n-1} (n-i)H(x'_i \otimes x'_{i+1}) = \sum_{i \in Z_b} (n-i)H(x_i \otimes x_{i+1}) + \sum_{i \notin Z_b} (n-i)(1 - H(x_i \otimes x_{i+1}))$$

by Lemma 4.3.1. Thus we have

$$H(b') = \sum_{i=1}^{n-1} (n-i)(1 - H(x_i \otimes x_{i+1})) + \sum_{i \in Z_b} (n-i)(2H(x_i \otimes x_{i+1}) - 1) = \frac{n(n-1)}{2} - \frac{n-|\lambda|}{2} - H(b)$$

where the last equality follows from Lemma 4.3.2. Finally we derive

$$X_{\lambda', (1^n)}(q) = \sum_{b \in F_\lambda} q^{H(b')} = q^{\frac{n(n-1)}{2} - \frac{n-|\lambda|}{2}} \sum_{b \in F_\lambda} q^{-H(b)} = q^{\frac{n(n-1)}{2} - \frac{n-|\lambda|}{2}} X_{\lambda, (1^n)}(q^{-1})$$

which by (18) proves the Theorem. ■

**Remark:** Theorem 4.3.3 can be regarded as an analogue for the affine root system  $C_n^{(1)}$  of Theorem 4.1.1 when  $\mu = (1, \dots, 1)$ .

#### 4.4 Appendix on the one dimension sums

In [7], Kang, Kashiwara and Misra have defined crystals  $B_l^\phi$  for quantum affine algebras associated to the affine root systems  $\phi$  of types  $A_n^{(1)}$ ,  $C_n^{(1)}$ ,  $A_{2n-1}^{(2)}$ ,  $A_{2n}^{(2)}$ ,  $D_{n+1}^{(2)}$ ,  $B_n^{(1)}$  and  $D_n^{(1)}$ . We give below the decomposition of the crystals  $B_l^\phi$  as classical crystals for each affine root system  $\phi$ :

$C_n^{(1)}$	$A_{2n-1}^{(2)}$	$A_{2n}^{(2)}$	$D_{n+1}^{(2)}$	$B_n^{(1)}$	$D_n^{(1)}$	(21)
$\bigoplus_{k=0, k \equiv l \pmod{2}}^l B^C(k\Lambda_1)$	$B^C(k\Lambda_1)$	$\bigoplus_{k=0}^l B^C(k\Lambda_1)$	$\bigoplus_{k=0}^l B^B(k\Lambda_1)$	$B^B(k\Lambda_1)$	$B^D(k\Lambda_1)$	

where  $B^C(k\Lambda_1)$ ,  $B^D(k\Lambda_1)$  and  $B^B(k\Lambda_1)$  are the crystal graphs of the finite dimensional modules of highest weight  $k\Lambda_1$  respectively for  $U_q(\mathfrak{sp}_{2n})$ ,  $U_q(\mathfrak{so}_{2n+1})$  and  $U_q(\mathfrak{so}_{2n})$ . Given any partition  $\mu \in \mathcal{P}_n$ , set  $B_\mu^\phi = B_{\mu_1}^\phi \otimes \cdots \otimes B_{\mu_n}^\phi$ . Then, by using the energy functions explicited in [4] and [5], one can define a statistic  $H^\phi$  on the vertices of  $B_\mu^\phi$  from which it is possible to calculate the one dimension sum  $X_{\lambda, \mu}^\phi(q)$ . In particular, with our notation, we have

$$X_{\lambda, \mu}^{C_n^{(1)}}(q) = X_{\lambda, \mu}(q)$$

since the polynomial  $X_{\lambda, \mu}(q)$  is a one dimension sum for the affine root system  $C_n^{(1)}$ . Denote by  $m$  the number of nonzero parts of  $\mu$ . One proves that the highest weight vertices of weight  $\lambda$  in  $B_\mu^{B_n^{(1)}}$

(considered as a  $B_n$ -crystal) are in one to one correspondence with those of  $B_\mu^{A_{2n-1}^{(2)}}$  (considered as a  $C_n$ -crystal). Similarly when  $m < n$  the highest weight vertices of weight  $\lambda$  in  $B_\mu^{D_n^{(1)}}$  (considered as a  $D_n$ -crystal) are also in one to one correspondence with those of  $B_\mu^{A_{2n-1}^{(2)}}$ . Moreover the statistics  $H^{A_{2n-1}^{(2)}}$ ,  $H^{B_n^{(1)}}$  and  $H^{D_n^{(1)}}$  are conserved via these correspondences. Thus we obtain by (21) the equality

$$X_{\lambda,\mu}^{A_{2n}^{(2)}}(q) = X_{\lambda,\mu}^{D_{n+1}^{(2)}}(q)$$

for any  $\mu \in \mathcal{P}_n$ . Similarly, when  $m < n$ , we have

$$X_{\lambda,\mu}^{A_{2n-1}^{(2)}}(q) = X_{\lambda,\mu}^{B_n^{(1)}}(q) = X_{\lambda,\mu}^{D_n^{(1)}}(q). \quad (22)$$

From Theorems 2.4.2 and 4.3.3 we obtain:

**Corollary 4.4.1** *For any  $\lambda \in \mathcal{P}_n$  we have*

$$X_{\lambda,\mu}^{C_n^{(1)}}(q) = q^{-\frac{|\mu|-|\lambda|}{2}} U_{\lambda,\mu}(q) = q^{-\frac{|\mu|-|\lambda|}{2}} \tilde{K}_{\lambda,\hat{\mu}}^{C_n}(q)$$

when  $\mu = (1, \dots, 1) \in \mathcal{P}_n$ .

When  $\phi$  is one of the two affine root systems  $A_{2n}^{(2)}$  or  $D_{n+1}^{(2)}$ , the structure of classical crystal of  $B_\mu^\phi$  implies that the one dimension sum  $X_{\lambda,\mu}^\phi(q)$  cannot be naturally related to the  $q$ -multiplicities  $u_{\lambda,\mu}(q)$  or  $U_{\lambda,\mu}(q)$  (see 2.4). However they can be expressed in terms of the polynomials  $K_{\lambda,\mu}^{(1)}(q)$  [26].

**Note:** *While revising this work, the author have been informed that, in a paper in preparation [26], Shimozono obtains a proof of the  $X = K$  conjecture for tensor product of the “symmetric power” Kirilov-Reshetikin modules for nonexceptional affine algebras of type  $D_{n+1}^{(2)}$ ,  $A_{2n}^{(2)}$ ,  $C_n^{(1)}$ ,  $A_{2n}^{(1)}$  or  $A_{2n}^{(2)\dagger}$ . With our convention for the definition of the one dimension sums (which is that of [4]) this result can be reformulated by writing*

$$K_{\lambda,\mu}^{(2)}(q) = q^{|\mu|-|\lambda|} X_{\lambda,\mu}^{C_n^{(1)}}(q^2) = q^{|\mu|-|\lambda|} X_{\lambda,\mu}^{A_{2n}^{(2)\dagger}}(q^2) \text{ and } K_{\lambda,\mu}^{(1)}(q) = q^{|\mu|-|\lambda|} X_{\lambda,\mu}^{A_{2n}^{(2)}}(q^2) = q^{|\mu|-|\lambda|} X_{\lambda,\mu}^{D_n^{(2)}}(q^2)$$

for any  $\mu \in \mathcal{P}_n$ . By (12) the first equality above establishes (ii) of Conjecture 2.5.2 since  $X_{\lambda,\mu}(q)$  is a one dimension sum for  $C_n^{(1)}$ -crystals. Thus assertion 2 of Corollary 4.4.1 holds for any  $\mu \in \mathcal{P}_n$ .

## References

- [1] T. H.BAKER *An insertion scheme for  $C_n$  crystals*, in M. Kashiwara and T. Miwa, eds., Physical Combinatorics, Birkhäuser, Boston, 2000, **191**: 1-48.
- [2] W. FULTON, J. HARRIS, *Representation theory*, Graduate Texts in Mathematics, Springer-Verlag.
- [3] G. GOODMAN, N. R WALLACH, *Representation theory and invariants of the classical groups*, Cambridge University Press.



- [4] G. HATAYAMA, A. KUNIBA, M. OKADO, T. TAKAGI, *Combinatorial R matrices for a family of crystals:  $C_n^{(1)}$  and  $A_{2n-1}^{(2)}$  cases*, Physical Combinatorics edited by M Kashiwara and T Miwa, Birkhauser, 105-139 (2000).
- [5] H. HATAYAMA, A. KUNIBA, M. OKADO, T. TAKAGI, *Combinatorial R matrices for a family of crystals:  $B_n^{(1)}$ ,  $D_n^{(1)}$ ,  $A_{2n}^{(2)}$  and  $D_{n+1}^{(2)}$  cases*, Journal of Algebra, **247**, 577-615 (2002).
- [6] G. HATAYAMA, A. KUNIBA, M. OKADO, T. TAKAGI, Y. YAMADA, *Remarks on fermionic formula*, in N. Jing and K. C. Misra, eds. Recent Developments in Quantum Affine Algebras and Related Topics, Contemporary Mathematics **248**, AMS, Providence, 243-291, (1999).
- [7] S-J. KANG, M. KASHIWARA, K-C. MISRA, *Crystal bases of Verma modules for quantum affine Lie algebras*, Compositio. Math. **92**, 299-325 (1994).
- [8] M. KASHIWARA, *On crystal bases of the  $q$ -analogue of universal enveloping algebras*, Duke Math. J, **63** (1991), 465-516.
- [9] M. KASHIWARA, T. NAKASHIMA, *Crystal graphs for representations of the  $q$ -analogue of classical Lie algebras*, Journal of Algebra, **165**, 295-345 (1994).
- [10] K. KOIKE, I. TERADA, *Young diagrammatic methods for the representations theory of the classical groups of type  $B_n, C_n$  and  $D_n$* , Journal of Algebra, **107**, 466-511 (1987).
- [11] K. KOIKE, I. TERADA, *Restriction Rules for  $GL, SO, Sp$* , Adv. in Math., **79**, 104-135 (1990).
- [12] A. LASCoux, M-P. SCHÜTZENBERGER, *Le monoïde plaxique*, in non commutative structures in algebra and geometric combinatorics A. de Luca Ed., Quaderni della Ricerca Scientifica del C.N.R., Roma, (1981).
- [13] A. LASCoux, M-P. SCHÜTZENBERGER, *Sur une conjecture de H.O Foulkes*, CR Acad Sci Paris, **288**, 95-98 (1979).
- [14] C. LECOUCVEY, *A duality between  $q$ -multiplicities in tensor products and  $q$ -multiplicities of weights for the root systems  $B, C$  or  $D$*  (submitted), ArXiv: CO/04007522.
- [15] C. LECOUCVEY, *Schensted-type correspondence, Plactic Monoid and Jeu de Taquin for type  $C_n$* , Journal of Algebra, **247**, 295-331 (2002).
- [16] C. LECOUCVEY, *Combinatorics of crystal graphs and Kostka-Foulkes polynomials for the root systems  $B_n, C_n$  and  $D_n$* , to appear in Journal of European Combinatorics.
- [17] D-E. LITTLEWOOD, *The theory of group characters and matrix representations of groups*, Oxford University Press, second edition (1958).
- [18] G. LUSZTIG, *Singularities, character formulas, and a  $q$ -analog of weight multiplicities*, Analyse et topologie sur les espaces singuliers (II-III), Asterisque **101-102**, 208-227 (1983).
- [19] I-G. MACDONALD, *Symmetric functions and Hall polynomials*, Second edition, Oxford Mathematical Monograph, Oxford University Press, New York, (1995).
- [20] A. NAKAYASHIKI, Y. YAMADA, *Kostka-Foulkes polynomials and energy function in solvable lattice models*, Selecta Mathematica New Series, Vol 3 N°4, 547-599, (1997).

- [21] K. NELSEN, A. RAM, *Kostka-Foulkes polynomials and Macdonald spherical functions*, preprint (2004), ArXiv: RT/0401298.
- [22] M. OKADO, A. SCHILLING, M. SHIMOZONO, *A crystal to rigged configuration bijection for nonexceptional affine algebras*, Preprint (2002), Arxiv QA/0203163.
- [23] M. OKADO, A. SCHILLING, M. SHIMOZONO, *Virtual Crystals and Fermionic Formulas of Type  $D_{n+1}^{(2)}$ ,  $A_{2n}^{(2)}$  and  $C_n^{(1)}$* , Representation Theory, **7**, 101-163 (2003).
- [24] A. SCHILLING, M. SHIMOZONO,  *$X = M$  for symmetric powers*, Preprint (2004), Arxiv QA/0412376.
- [25] A. SCHILLING, S. O. WARNARR, *Inhomogeneous lattice paths, generalized Kostka-Foulkes polynomials and  $A_{n-1}$  supernomials*, Comm. Math. Phys. **202**, 359-401 (1999).
- [26] M. SHIMOZONO, *On the  $X = M = K$  conjecture*, personal communication.
- [27] M. SHIMOZONO, M. ZABROCKI, *Deformed universal characters for classical and affine algebras*, Preprint (2004), ArXiv: CO/0404288.