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Branching rules, Kostka-Foulkes polynomials and q -multiplicities in tensor product for the root systems B_n, C_n and D_n

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Abstract

The Kostka-Foulkes polynomials $K_{\lambda,\mu}^\phi(q)$ related to a root system ϕ can be defined as alternated sums running over the Weyl group associated to ϕ . By restricting these sums over the elements of the symmetric group when ϕ is of type B_n, C_n or D_n , we obtain again a class $\tilde{K}_{\lambda,\mu}^\phi(q)$ of Kostka-Foulkes polynomials. When ϕ is of type C_n or D_n there exists a duality between these polynomials and some natural q -multiplicities $u_{\lambda,\mu}(q)$ and $U_{\lambda,\mu}(q)$ in tensor product [14]. In this paper we first establish identities for the $\tilde{K}_{\lambda,\mu}^\phi(q)$ which implies in particular that they can be decomposed as sums of Kostka-Foulkes polynomials $K_{\lambda,\mu}^{A_{n-1}}(q)$ with nonnegative integer coefficients. Moreover these coefficients are branching rule coefficients. This allows us to clarify the connection between the q -multiplicities $u_{\lambda,\mu}(q), U_{\lambda,\mu}(q)$ and the polynomials $K_{\lambda,\mu}^\diamond(q)$ defined in [25]. Finally we show that $u_{\lambda,\mu}(q)$ and $U_{\lambda,\mu}(q)$ coincide up to a power of q with the one dimension sum introduced in [4] when all the parts of μ are equal to 1 which partially proves some conjectures of [14] and [25].

1 Introduction

Consider λ and μ two partitions of the set \mathcal{P}_n of partitions with n parts. The Schur-Weyl duality establishes that the dimension $K_{\lambda,\mu}^{A_{n-1}}$ of the weight space μ in the finite dimensional irreducible sl_n -module $V^{A_{n-1}}(\lambda)$ of highest weight λ is equal to the multiplicity of $V^{A_{n-1}}(\lambda)$ in the tensor product

$$V_\mu^{A_{n-1}} = V^{A_{n-1}}(\mu_1\Lambda_1) \otimes \cdots \otimes V^{A_{n-1}}(\mu_n\Lambda_1).$$

It follows from the Weyl character formula that $K_{\lambda,\mu}^{A_{n-1}} = \sum_{\sigma \in \mathcal{S}_n} (-1)^{l(\sigma)} \mathcal{P}^{A_{n-1}}(\sigma(\lambda + \rho) - (\mu + \rho))$ where $\mathcal{P}^{A_{n-1}}$ is the Kostant partition function which counts, in the root system of type A_{n-1} , the number of decomposition of $\beta \in \mathbb{Z}^n$ as a sum of positive roots. The Kostka-Foulkes polynomials can be defined by setting $K_{\lambda,\mu}^{A_{n-1}}(q) = \sum_{\sigma \in \mathcal{S}_n} (-1)^{l(\sigma)} \mathcal{P}_q^{A_n}(\sigma(\lambda + \rho) - (\mu + \rho))$ where $\mathcal{P}_q^{A_{n-1}}$ is the q -Kostant partition function characterized by

$$\prod_{\alpha \text{ positive root}} \frac{1}{(1 - qe^\alpha)} = \sum_{\beta \in \mathbb{Z}^n} \mathcal{P}_q^{A_n}(\beta) e^\beta$$

with $\rho = (n-1, \dots, 0)$ the half sum of the positive roots. One can prove that they are the coefficients of the expansion of the Schur function $s_\mu(x)$ on the basis of Hall polynomials $\{P_\lambda(x, q), \lambda \in \mathcal{P}_n\}$ (see [19]). Then it follows from the theory of affine Hecke algebras that the Kostka-Foulkes polynomials are Kazhdan-Lusztig polynomials [18]. In particular they have nonnegative integer coefficients. As proved by Lascoux and Schützenberger this positivity result can also be obtained by using the

charge statistic ch on semistandard tableaux. More precisely we have $K_{\lambda,\mu}^{A_{n-1}}(q) = \sum_{T \in ST(\lambda)_\mu} q^{\text{ch}(T)}$ where $ST(\lambda)_\mu$ is the set of semistandard tableaux of shape λ and weight μ . In [20], Nakayashiki and Yamada have shown that the charge can be computed from the combinatorial R -matrix corresponding to Kashiwara's crystals associated to some $U_q(\widehat{sl}_n)$ -modules.

Now consider ϕ a root system of type B_n, C_n or D_n . Write \mathfrak{g}_ϕ for the corresponding simple Lie algebra. The Kostka-Foulkes polynomials $K_{\lambda,\mu}^\phi(q)$ associated to ϕ are defined by setting

$$K_{\lambda,\mu}^\phi(q) = \sum_{w \in W_\phi} (-1)^{l(w)} \mathcal{P}_q^\phi(w(\lambda + \rho_\phi) - (\mu + \rho_\phi))$$

where W_ϕ, ρ_ϕ and \mathcal{P}_q^ϕ are respectively the Weyl group, the half sum of the positive roots and the q -partition function corresponding to ϕ . The polynomial $K_{\lambda,\mu}^\phi(q)$ can be considered as a q -analogue of the dimension of the weight space μ in $V^\phi(\lambda)$. As Kazhdan-Lusztig polynomials, they have also nonnegative coefficients. In [16], we have obtained for the root systems B_n, C_n and D_n a statistic on Kashiwara-Nakashima's tableaux from which it is possible to deduce this positivity for particular pairs of partitions (λ, μ) . Nevertheless as far as the author is aware, no combinatorial proof of this positivity result is known in general.

In general the Kostka-Foulkes polynomials $K_{\lambda,\mu}^\phi(q)$ cannot be interpreted as q -multiplicities in tensor products. So there does not exist an equivalent result to the Schur Weyl duality for the root system ϕ . Denote by $V^\phi(\lambda)$ the finite dimensional irreducible \mathfrak{g}_ϕ -module of highest weight λ . In [14], we have introduced from determinantal expressions of the Schur functions associated to ϕ , two polynomials $u_{\lambda,\mu}(q)$ and $U_{\lambda,\mu}(q)$ which can be respectively regarded as quantizations of the multiplicities of $V^\phi(\lambda)$ in the tensor products

$$V_\mu^\phi = V^\phi(\mu_1 \Lambda_1) \otimes \cdots \otimes V^\phi(\mu_n \Lambda_1) \text{ and } W_\mu^\phi = W^\phi(\mu_1 \Lambda_1) \otimes \cdots \otimes W^\phi(\mu_n \Lambda_1)$$

where for any $i = 1, \dots, n$, $W^\phi(\mu_i \Lambda_1) = V^\phi(\mu_i \Lambda_1) \oplus V^\phi((\mu_i - 2)\Lambda_1) \oplus \cdots \oplus V^\phi((\mu_i \bmod 2)\Lambda_1)$. When n is sufficiently large they do not depend on the root system ϕ considered and we have established a duality result between the q -multiplicities $u_{\lambda,\mu}(q)$, $U_{\lambda,\mu}(q)$ and the polynomials

$$\tilde{K}_{\lambda,\mu}^\phi(q) = \sum_{\sigma \in \mathcal{S}_n} (-1)^{l(\sigma)} \mathcal{P}_q^\phi(w(\lambda + \rho_n) - (\mu + \rho_n))$$

where $\rho_n = (n, \dots, 1)$. These polynomials $\tilde{K}_{\lambda,\mu}^\phi(q)$ are also Kostka-Foulkes polynomials. So this result can be interpreted as a duality between q -analogues of weight multiplicities and q -analogues of tensor product multiplicities for the root systems B_n, C_n and D_n .

At the same time Shimozono and Zabrocki [25] have independently defined by using creating operators some polynomials $K_{\lambda,R}^\diamond(q)$ where R is a sequence of rectangular partitions and \diamond a partition of the set $\{\emptyset, (1), (11), (2)\}$. These polynomials can also be regarded as q -multiplicities in tensor products. In [4], Hatayama, Kuniba, Okado and Takagi have introduced for type C_n a quantization $X_{\lambda,\mu}(q)$ of the multiplicity of $V^{C_n}(\lambda)$ in $W_\mu^{C_n}$. This quantization is based on the determination of the combinatorial R -matrix of some $U'_q(\widehat{sp}_{2n})$ -crystals in the spirit of [20]. It can be regarded as a one dimension sum for the affine root system $C_n^{(1)}$. In [14] and [25], the authors conjecture that the polynomials $X_{\lambda,\mu}^{C_n}(q)$, $U_{\lambda,\mu}(q)$ and $K_{\lambda,\mu}^\diamond(q)$ coincide up to simple renormalizations. As observed in [25], this conjecture can be related to the $X = M$ conjecture which gives fermionic formulas for the one dimension sum X . Note that the $X = M$ conjecture have been very recently proved in various cases for the root systems B_n, C_n and D_n [22].

In this article we first obtained identities for the polynomials $\tilde{K}_{\lambda,\mu}^\phi(q)$ which imply that they can be decomposed as sums of polynomials $K_{\lambda,\mu}^{A_{n-1}}(q)$. Moreover the coefficients of these decompositions can be simply expressed in terms of branching rules coefficients. This gives in particular an elementary proof of the positivity of the Kostka-Foulkes polynomials $\tilde{K}_{\lambda,\mu}^\phi(q)$. Next we obtain similar decompositions for the polynomials $u_{\lambda,\mu}(q)$ and $U_{\lambda,\mu}(q)$. By comparing these identities with those obtained for the polynomials $K_{\lambda,R}^\diamond(q)$ in [25], we derive the equalities $K_{\lambda,\mu}^{(1,1)}(q) = u_{\lambda,\mu}(q^2)$ and $K_{\lambda,\mu}^{(2)}(q) = U_{\lambda,\mu}(q^2)$. Finally we establish that the conjectures of [14] and [25] are true when all the parts of μ are equal to 1 (i.e. for the q -multiplicities defined in the tensor powers of the vector representation), namely we have

$$K_{\lambda,(1^n)}^{(1,1)}(q) = u_{\lambda,(1^n)}(q^2) = q^{n-|\lambda|} X_{\lambda,(1^n)}(q^2) \text{ and } K_{\lambda,(1^n)}^{(2)}(q) = U_{\lambda,(1^n)}(q^2) = q^{2(n-|\lambda|)} X_{\lambda,(1^n)}(q^2). \quad (1)$$

In Section 2 we review some material on root systems, branching rules coefficients, Kostka-Foulkes polynomials and q -multiplicities $u_{\lambda,\mu}(q), U_{\lambda,\mu}(q)$ we need in the sequel. In section 3 we obtain identities for the polynomials $\tilde{K}_{\lambda,\mu}^\phi(q), u_{\lambda,\mu}(q)$ and $U_{\lambda,\mu}(q)$ from which we clarify the relations between $u_{\lambda,\mu}(q), U_{\lambda,\mu}(q)$ and $K_{\lambda,\mu}^{(1,1)}(q), K_{\lambda,\mu}^{(2)}(q)$. Section 4 is devoted to the proof of (1). Note that the $X = M$ conjecture is in particular true when all the parts of μ are equal to 1 [22]. Thus in this case the one dimension sums X are, up to simple renormalizations, Kazhdan-Lusztig polynomials.

Notation: In the sequel we frequently define similar objects for the root systems B_n, C_n and D_n . When they are related to type B_n (resp. C_n, D_n), we implicitly attach to them the label B (resp. the labels C, D). To avoid cumbersome repetitions, we sometimes omit the labels B, C and D when our definitions or statements are identical for the three root systems.

2 Background

2.1 Convention for the root systems of types B_n, C_n and D_n

Consider an integer $n \geq 1$. The weight lattice for the root system C_n (resp. B_n and D_n) can be identified with $P_{C_n} = \mathbb{Z}^n$ (resp. $P_{B_n} = P_{D_n} = \left(\frac{\mathbb{Z}}{2}\right)^n$) equipped with the orthonormal basis $\varepsilon_i, i = 1, \dots, n$. We take for the simple roots

$$\begin{cases} \alpha_n^{B_n} = \varepsilon_n \text{ and } \alpha_i^{B_n} = \varepsilon_i - \varepsilon_{i+1}, i = 1, \dots, n-1 \text{ for the root system } B_n \\ \alpha_n^{C_n} = 2\varepsilon_n \text{ and } \alpha_i^{C_n} = \varepsilon_i - \varepsilon_{i+1}, i = 1, \dots, n-1 \text{ for the root system } C_n \\ \alpha_n^{D_n} = \varepsilon_n + \varepsilon_{n-1} \text{ and } \alpha_i^{D_n} = \varepsilon_i - \varepsilon_{i+1}, i = 1, \dots, n-1 \text{ for the root system } D_n \end{cases} \quad (2)$$

Then the set of positive roots are

$$\begin{cases} R_{B_n}^+ = \{\varepsilon_i - \varepsilon_j, \varepsilon_i + \varepsilon_j \text{ with } 1 \leq i < j \leq n\} \cup \{\varepsilon_i \text{ with } 1 \leq i \leq n\} \text{ for the root system } B_n \\ R_{C_n}^+ = \{\varepsilon_i - \varepsilon_j, \varepsilon_i + \varepsilon_j \text{ with } 1 \leq i < j \leq n\} \cup \{2\varepsilon_i \text{ with } 1 \leq i \leq n\} \text{ for the root system } C_n \\ R_{D_n}^+ = \{\varepsilon_i - \varepsilon_j, \varepsilon_i + \varepsilon_j \text{ with } 1 \leq i < j \leq n\} \text{ for the root system } D_n \end{cases} \quad .$$

Denote respectively by $P_{B_n}^+, P_{C_n}^+$ and $P_{D_n}^+$ the sets of dominant weights of so_{2n+1}, sp_{2n} and so_{2n} . Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be a partition with n parts. We will classically identify λ with the dominant weight $\sum_{i=1}^n \lambda_i \varepsilon_i$. Note that there exists dominant weights associated to the orthogonal root systems

whose coordinates on the basis ε_i , $i = 1, \dots, n$ are not positive integers (hence which cannot be regarded as partitions). For each root system of type B_n, C_n or D_n , the set of weights having nonnegative integer coordinates on the basis $\varepsilon_1, \dots, \varepsilon_n$ can be identify with the set \mathcal{P}_n of partitions of length n . For any partition λ , the weights of the finite dimensional so_{2n+1}, sp_{2n} or so_{2n} -module of highest weight λ are all in \mathbb{Z}^n . For any $\alpha \in \mathbb{Z}^n$ we write $|\alpha| = \alpha_1 + \dots + \alpha_n$ and $\|\alpha\| = \sum_{i=1}^{n-1} (n-i)\alpha_i$. The conjugate partition of the partition λ is denoted λ' as usual. Consider λ, μ two partitions of length n and set $m = \max(\lambda_1, \mu_1)$. Then by adding to λ' and μ' the required numbers of parts 0 we will consider them as partitions of length m .

The Weyl group $W_{B_n} = W_{C_n}$ of so_{2n+1} and sp_{2n} is identified to the subgroup of the permutation group of the set $\{\bar{n}, \dots, \bar{2}, \bar{1}, 1, 2, \dots, n\}$ generated by $s_i = (i, i+1)(\bar{i}, \bar{i}+\bar{1})$, $i = 1, \dots, n-1$ and $s_n = (n, \bar{n})$ where for $a \neq b$ (a, b) is the simple transposition which switches a and b . We denote by l_B the length function corresponding to the set of generators s_i , $i = 1, \dots, n$.

The Weyl group W_{D_n} of so_{2n} is identified to the subgroup of W_{B_n} generated by the transpositions $s_i = (i, i+1)(\bar{i}, \bar{i}+\bar{1})$, $i = 1, \dots, n-1$ and $s'_n = (n, \bar{n}-\bar{1})(n-1, \bar{n})$. We denote by l_D the length function corresponding to the set of generators s'_n and s_i , $i = 1, \dots, n-1$.

Note that $W_{D_n} \subset W_{B_n}$ and any $w \in W_{B_n}$ verifies $w(\bar{i}) = \overline{w(i)}$ for $i \in \{1, \dots, n\}$. The action of w on $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{Z}^n$ is given by

$$w \cdot (\beta_1, \dots, \beta_n) = (\beta_1^w, \dots, \beta_n^w)$$

where $\beta_i^w = \beta_{w(i)}$ if $\sigma(i) \in \{1, \dots, n\}$ and $\beta_i^w = -\beta_{w(\bar{i})}$ otherwise.

The half sums ρ_{B_n}, ρ_{C_n} and ρ_{D_n} of the positive roots associated to each root system B_n, C_n and D_n verify:

$$\rho_{B_n} = (n - \frac{1}{2}, n - \frac{3}{2}, \dots, \frac{1}{2}), \rho_{C_n} = (n, n-1, \dots, 1) \text{ and } \rho_{D_n} = (n-1, n-2, \dots, 0).$$

In the sequel we identify the symmetric group \mathcal{S}_n (which is the Weyl group of the root system A_{n-1}) with the subgroup of W_{B_n} or W_{D_n} generated by the s_i 's, $i = 1, \dots, n-1$.

2.2 Branching rules coefficients

For any partition λ , we denote by $V_n^B(\lambda), V_n^C(\lambda)$, and $V_n^D(\lambda)$ the finite dimensional irreducible modules of highest weight λ respectively for sp_{2n}, so_{2n+1} and so_{2n} . Then $V_n^B(\lambda), V_n^C(\lambda)$, and $V_n^D(\lambda)$ can also be regarded as irreducible representations respectively of the groups Sp_{2n}, SO_{2n+1} and SO_{2n} . By restriction to GL_n , they decompose in a direct sum of irreducible rational representations. Recall that the irreducible rational representations of GL_n are indexed by the n -tuples

$$(\gamma^+, \gamma^-) = (\gamma_1^+, \gamma_2^+, \dots, \gamma_p^+, 0, \dots, 0, -\gamma_q^-, \dots, -\gamma_1^-) \quad (3)$$

where γ^+ and γ^- are partitions of length p and q such that $p+q \leq n$. Write $V_n^A(\gamma^+, \gamma^-)$ for the irreducible rational representations of GL_n of highest weight (γ^+, γ^-) . When $\gamma^- = \emptyset$, we write simply $V_n^A(\gamma)$ instead of $V_n^A(\gamma^+, \gamma^-)$.

As customary, we use for a basis of the group algebra $\mathbb{Z}[\mathbb{Z}^n]$, the formal exponentials $(e^\beta)_{\beta \in \mathbb{Z}^n}$ satisfying the relations $e^{\beta_1} e^{\beta_2} = e^{\beta_1 + \beta_2}$. We furthermore introduce n independent indeterminates x_1, \dots, x_n in order to identify $\mathbb{Z}[\mathbb{Z}^n]$ with the ring of polynomials $\mathbb{Z}[x_1, \dots, x_n, x_1^{-1}, \dots, x_n^{-1}]$ by writing $e^\beta = x_1^{\beta_1} \dots x_n^{\beta_n} = x^\beta$ for any $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{Z}^n$.

Set

$$\begin{aligned}
\prod_{1 \leq r < s \leq n} \left(1 - \frac{1}{x_r x_s}\right)^{-1} \prod_{1 \leq i \leq n} \left(1 - \frac{1}{x_i}\right)^{-1} &= \sum_{\beta \in L_B} b(\beta) x^{-\beta}, \\
\prod_{1 \leq r \leq s \leq n} \left(1 - \frac{1}{x_r x_s}\right)^{-1} &= \sum_{\beta \in L_C} c(\beta) x^{-\beta}, \\
\prod_{1 \leq r < s \leq n} \left(1 - \frac{1}{x_r x_s}\right)^{-1} &= \sum_{\beta \in L_D} d(\beta) x^{-\beta}
\end{aligned} \tag{4}$$

where

$$\begin{aligned}
L_B &= \left\{ \beta \in \mathbb{Z}^n, \beta = \sum_{1 \leq r < s \leq n} e_{r,s}(\varepsilon_r + \varepsilon_s) + \sum_{1 \leq i \leq n} e_i \varepsilon_i \text{ with } e_{r,s} \geq 0 \text{ and } e_i \geq 0 \right\} \\
L_C &= \left\{ \beta \in \mathbb{Z}^n, \beta = \sum_{1 \leq r \leq s \leq n} e_{r,s}(\varepsilon_r + \varepsilon_s) \text{ with } e_{r,s} \geq 0 \right\} \text{ and} \\
L_D &= \left\{ \beta \in \mathbb{Z}^n, \beta = \sum_{1 \leq r < s \leq n} e_{r,s}(\varepsilon_r + \varepsilon_s) \text{ with } e_{r,s} \geq 0 \right\}.
\end{aligned}$$

Denote respectively by $[V_n^A(\gamma^+, \gamma^-) : V_n^B(\lambda)]$, $[V_n^A(\gamma^+, \gamma^-) : V_n^C(\lambda)]$ and $[V_n^A(\gamma^+, \gamma^-) : V_n^D(\lambda)]$, the multiplicities of $V_n^A(\gamma^+, \gamma^-)$ in the restrictions of $V_n^B(\lambda)$, $V_n^C(\lambda)$ and $V_n^D(\lambda)$ to GL_n .

Proposition 2.2.1 *With the above notation, we have:*

1. $[V_n^A(\gamma^+, \gamma^-) : V_n^B(\lambda)] = \sum_{w \in W_{B_n}} (-1)^{l(w)} b(w \circ \lambda - (\gamma^+, \gamma^-))$,
2. $[V_n^A(\gamma^+, \gamma^-) : V_n^C(\lambda)] = \sum_{w \in W_{C_n}} (-1)^{l(w)} c(w \circ \lambda - (\gamma^+, \gamma^-))$,
3. $[V_n^A(\gamma^+, \gamma^-) : V_n^D(\lambda)] = \sum_{w \in W_{D_n}} (-1)^{l(w)} d(w \circ \lambda - (\gamma^+, \gamma^-))$.

Proof. The proposition can be considered as a corollary of Theorem 8.2.1 of [3] with G one of the Lie groups So_{2n+1} , Sp_{2n} , So_{2n} and $H = GL_n$. ■

For any partitions λ and ν of length n , write $[V_n^D(\nu) : V_n^B(\lambda)]$ for the multiplicity of $V_n^D(\nu)$ in the restriction of $V_n^B(\lambda)$ to So_{2n} .

Lemma 2.2.2 *With the notation above we have*

$$[V_n^D(\nu) : V_n^B(\lambda)] = \sum_{w \in W_{B_n}} (-1)^{l(w)} \mathbf{1}_{\mathbb{N}}(w \circ \lambda - \nu)$$

where for any $\beta \in \mathbb{Z}^n$, $\mathbf{1}_{\mathbb{N}}(\beta) = 1$ if all the coordinates of β are nonnegative integers and $\mathbf{1}_{\mathbb{N}}(\beta) = 0$ otherwise.

Proof. The lemma also follows from Theorem 8.2.1 of [3]. ■

For any partitions λ and ν of length n , write $[V_n^B(\lambda) : V_{2n}^A(\nu)]$ for the multiplicity of $V_n^B(\lambda)$ in the restriction of $V_{2n}^A(\nu)$ from GL_{2n} to So_{2n} . Similarly write $[V_n^C(\lambda) : V_{2n}^A(\nu)]$ for the multiplicity of

$V_n^C(\lambda)$ in the restriction of $V_{2n}^A(\nu)$ from GL_{2n} to Sp_{2n} and $[V_n^D(\lambda) : V_{2n}^A(\nu)]$ for the multiplicity of $V_n^D(\lambda)$ in the restriction of $V_{2n}^A(\nu)$ from GL_{2n} to So_{2n+1} . Denote respectively by $\mathcal{P}_n^{(2)}$ and $\mathcal{P}_n^{(1,1)}$ the sub-sets of \mathcal{P}_n containing the partitions with even rows and the partitions with even columns. The Littlewood-Richardson coefficients are denoted $c_{\gamma,\lambda}^\nu$ as usual. Let us recall a classical result by Littelwood (see [17] appendix p 295)

Proposition 2.2.3 *Consider λ and μ in \mathcal{P}_n . Then:*

1. $[V_n^B(\lambda) : V_{2n}^A(\nu)] = \sum_{\gamma \in \mathcal{P}_n} c_{\gamma,\lambda}^\nu$,
2. $[V_n^C(\lambda) : V_{2n}^A(\nu)] = \sum_{\gamma \in \mathcal{P}_n^{(1,1)}} c_{\gamma,\lambda}^\nu$,
3. $[V_n^D(\lambda) : V_{2n}^A(\nu)] = \sum_{\gamma \in \mathcal{P}_n^{(2)}} c_{\gamma,\lambda}^\nu$.

The proposition below follows immediately from Theorem A_1 of [10].

Proposition 2.2.4 *Consider $\nu \in \mathcal{P}_n$ and λ^+, λ^- two partitions such that (λ^+, λ^-) has length n . Then:*

1. $[V_n^A(\lambda^+, \lambda^-) : V_n^B(\nu)] = \sum_{\gamma, \delta \in \mathcal{P}_n} c_{\gamma,\delta}^\nu c_{\lambda^+, \lambda^-}^\delta$,
2. $[V_n^A(\lambda^+, \lambda^-) : V_n^C(\nu)] = \sum_{\gamma, \delta \in \mathcal{P}_n^{(2)}} c_{\gamma,\delta}^\nu c_{\lambda^+, \lambda^-}^\delta$,
3. $[V_n^A(\lambda^+, \lambda^-) : V_n^D(\nu)] = \sum_{\gamma, \delta \in \mathcal{P}_n^{(1,1)}} c_{\gamma,\delta}^\nu c_{\lambda^+, \lambda^-}^\delta$.

When $(\lambda^+, \lambda^-) = \lambda$ is a partition (that is $\lambda^- = \emptyset$), we obtain the following dualities:

Corollary 2.2.5 *Consider λ, ν two partitions of length n , then*

1. $[V_n^A(\lambda) : V_n^B(\nu)] = [V_n^B(\lambda) : V_{2n}^A(\nu)] = \sum_{\gamma \in \mathcal{P}_n} c_{\gamma,\lambda}^\nu$,
2. $[V_n^A(\lambda) : V_n^C(\nu)] = [V_n^D(\lambda) : V_{2n}^A(\nu)] = \sum_{\gamma \in \mathcal{P}_n^{(2)}} c_{\gamma,\lambda}^\nu$,
3. $[V_n^A(\lambda) : V_n^D(\nu)] = [V_n^C(\lambda) : V_{2n}^A(\nu)] = \sum_{\gamma \in \mathcal{P}_n^{(1,1)}} c_{\gamma,\lambda}^\nu$.

2.3 Kostka-Foulkes polynomials

For any $w \in W_{B_n}$, the dot action of w on $\beta \in \mathbb{Z}^n$ is defined by

$$w \circ \beta = w(\beta + \rho_{B_n}) - \rho_{B_n}.$$

The q -analogue $\mathcal{P}_q^{B_n}$ of the Kostant partition function corresponding to the root system B_n is defined by the equality

$$\prod_{\alpha \in R_{B_n}^+} \frac{1}{1 - qx^\alpha} = \sum_{\beta \in \mathbb{Z}^n} \mathcal{P}_q^{B_n}(\beta) x^\beta.$$

Note that $\mathcal{P}_q^{B_n}(\beta) = 0$ if β is not a linear combination of positive roots of $R_{B_n}^+$ with nonnegative coefficients. We write similarly $\mathcal{P}_q^{C_n}$ and $\mathcal{P}_q^{D_n}$ for the q -partition functions associated respectively

to the root systems C_n and D_n . Given λ and μ two partitions of length n , the Kostka-Foulkes polynomials of types B_n, C_n and D_n are then respectively defined by

$$\begin{aligned} K_{\lambda, \mu}^{B_n}(q) &= \sum_{\sigma \in W_{B_n}} (-1)^{l(\sigma)} \mathcal{P}_q^{B_n}(\sigma(\lambda + \rho_{B_n}) - (\mu + \rho_{B_n})), \\ K_{\lambda, \mu}^{C_n}(q) &= \sum_{\sigma \in W_{C_n}} (-1)^{l(\sigma)} \mathcal{P}_q^{C_n}(\sigma(\lambda + \rho_{C_n}) - (\mu + \rho_{C_n})), \\ K_{\lambda, \mu}^{D_n}(q) &= \sum_{\sigma \in W_{D_n}} (-1)^{l(\sigma)} \mathcal{P}_q^{D_n}(\sigma(\lambda + \rho_{D_n}) - (\mu + \rho_{D_n})). \end{aligned}$$

Set

$$\begin{aligned} \tilde{K}_{\lambda, \mu}^{B_n}(q) &= \sum_{\sigma \in \mathcal{S}_n} (-1)^{l(\sigma)} \mathcal{P}_q^{B_n}(\sigma(\lambda + \rho_{B_n}) - (\mu + \rho_n)), \\ \tilde{K}_{\lambda, \mu}^{C_n}(q) &= \sum_{\sigma \in \mathcal{S}_n} (-1)^{l(\sigma)} \mathcal{P}_q^{C_n}(\sigma(\lambda + \rho_{C_n}) - (\mu + \rho_n)), \\ \tilde{K}_{\lambda, \mu}^{D_n}(q) &= \sum_{\sigma \in \mathcal{S}_n} (-1)^{l(\sigma)} \mathcal{P}_q^{D_n}(\sigma(\lambda + \rho_{D_n}) - (\mu + \rho_n)) \end{aligned}$$

where $\rho_n = (n, \dots, 1)$. In [14], we have proved that the polynomials $\tilde{K}_{\lambda, \mu}(q)$ are also Kostka-Foulkes polynomials. More precisely we have:

Lemma 2.3.1 *Consider λ, μ two partitions of length n such that $|\lambda| \geq |\mu|$. Let k be any integer such that $k \geq \frac{|\lambda| - |\mu|}{2}$. Then we have*

$$\tilde{K}_{\lambda, \mu}(q) = K_{\lambda + k\kappa_n, \mu + k\kappa_n}(q)$$

where $\kappa_n = (1, \dots, 1) \in \mathbb{Z}^n$.

Remark: Since $\sigma(\kappa_n) = \kappa_n$ for any $\sigma \in \mathcal{S}_n$, we have $\tilde{K}_{\lambda + k\kappa_n, \mu + k\kappa_n}(q) = \tilde{K}_{\lambda, \mu}(q)$ for any integer $k \geq 0$. So we can extend the above definition of $\tilde{K}_{\lambda, \mu}(q)$ for λ and μ decreasing sequences of integers (positive or not).

2.4 The q -multiplicities $u_{\lambda, \mu}(q)$ and $U_{\lambda, \mu}(q)$

Set

$$\begin{aligned} \prod_{1 \leq i < j \leq n} \frac{1}{1 - q \frac{x_i}{x_j}} \prod_{1 \leq r < s \leq n} \frac{1}{1 - \frac{q}{x_i x_j}} &= \sum_{\beta \in \mathbb{Z}^n} f_q(\beta) x^\beta \text{ and} \\ \prod_{1 \leq i < j \leq n} \frac{1}{1 - q \frac{x_i}{x_j}} \prod_{1 \leq r \leq s \leq n} \frac{1}{1 - \frac{q}{x_i x_j}} &= \sum_{\beta \in \mathbb{Z}^n} F_q(\beta) x^\beta. \end{aligned}$$

Given λ and μ two partitions of length n , let $u_{\lambda, \mu}(q)$ and $U_{\lambda, \mu}(q)$ be the two polynomials defined by

$$u_{\lambda, \mu}(q) = \sum_{\sigma \in \mathcal{S}_n} (-1)^{l(\sigma)} f_q(\sigma(\lambda + \rho_n) - \mu - \rho_n) \text{ and } U_{\lambda, \mu}(q) = \sum_{\sigma \in \mathcal{S}_n} (-1)^{l(\sigma)} F_q(\sigma(\lambda + \rho_n) - \mu - \rho_n)$$

where $\rho_n = (n, \dots, 1)$. Then $u_{\lambda, \mu}(q)$ and $U_{\lambda, \mu}(q)$ can be regarded as quantizations of tensor product multiplicities [14]. Consider the tensor products

$$\begin{aligned} V_\mu^B &= V^B(\mu_1 \Lambda_1) \otimes \cdots \otimes V^B(\mu_n \Lambda_1), \quad V_\mu^C = V^C(\mu_1 \Lambda_1) \otimes \cdots \otimes V^C(\mu_n \Lambda_1), \\ V_\mu^D &= V^D(\mu_1 \Lambda_1) \otimes \cdots \otimes V^D(\mu_n \Lambda_1) \end{aligned}$$

and

$$\begin{aligned} W_\mu^B &= W^B(\mu_1 \Lambda_1) \otimes \cdots \otimes W^B(\mu_n \Lambda_1), \quad W_\mu^C = W^C(\mu_1 \Lambda_1) \otimes \cdots \otimes W^C(\mu_n \Lambda_1), \\ W_\mu^D &= W^D(\mu_1 \Lambda_1) \otimes \cdots \otimes W^D(\mu_n \Lambda_1) \end{aligned}$$

where for any $k \in \mathbb{N}$, $W(k\Lambda_1) = V(k\Lambda_1) \oplus V((k-2)\Lambda_1) \oplus \cdots \oplus V((k \bmod 2)\Lambda_1)$. Then we have the following proposition:

Proposition 2.4.1 [14] *Let λ and μ be two partitions of length n . Then*

1. $u_{\lambda, \mu}(q)$ is a q -analogue of the multiplicity of the representation $V(\lambda)$ in V_μ ,
2. $U_{\lambda, \mu}(q)$ is a q -analogue of the multiplicity of the representation $V(\lambda)$ in W_μ .

Remarks:

- (i) : It follows from the definition of f_q and F_q that $u_{\lambda, \mu}(q) = U_{\lambda, \mu}(q) = 0$ if $|\lambda| > |\mu|$.
- (ii) : When $q = 1$, we recover that the multiplicities of $V^B(\lambda)$, $V^C(\lambda)$ and $V^D(\lambda)$ respectively in V_μ^B , V_μ^C , and V_μ^D are equal [10].
- (iii) : In [14], we have also obtained that $u_{\lambda, \mu}(q)$ and $U_{\lambda, \mu}(q)$ can be regarded as q -multiplicities in tensor product of column shaped representations.
- (iv) : Like the definition of $\tilde{K}_{\lambda, \mu}(q)$, the definitions of $u_{\lambda, \mu}(q)$ and $U_{\lambda, \mu}(q)$ can also be extended for λ and μ decreasing sequences of integers.
- (v) : Consider $\lambda, \mu \in \mathcal{P}_n$ and set $\lambda^\# = (\lambda_1, \dots, \lambda_n, 0)$, $\mu^\# = (\mu_1, \dots, \mu_n, 0)$. Then $u_{\lambda^\#, \mu^\#}(q) = u_{\lambda, \mu}(q)$ and $U_{\lambda^\#, \mu^\#}(q) = U_{\lambda, \mu}(q)$.

Theorem 2.4.2 [14] *Consider λ, μ two partitions of length n and set $m = \max(\lambda_1, \mu_1)$. Then $\hat{\lambda} = (m - \lambda_n, \dots, m - \lambda_1)$ and $\hat{\mu} = (m - \mu_n, \dots, m - \mu_1)$ are partitions of length n and*

$$u_{\lambda, \mu}(q) = \tilde{K}_{\hat{\lambda}, \hat{\mu}}^{D_n}(q), \quad U_{\lambda, \mu}(q) = \tilde{K}_{\hat{\lambda}, \hat{\mu}}^{C_n}(q).$$

Write I for the involution defined on \mathbb{Z}^n by $I(\beta_1, \dots, \beta_n) = (-\beta_n, \dots, -\beta_1)$. For any decreasing sequence of integers γ , $I(\gamma)$ is also a decreasing sequence of integers. By Theorem 2.4.2, this means that the correspondences

$$u_{\lambda, \mu}(q) \longleftrightarrow \tilde{K}_{I(\lambda), I(\mu)}^{D_n}(q) \quad \text{and} \quad U_{\lambda, \mu}(q) \longleftrightarrow \tilde{K}_{I(\lambda), I(\mu)}^{C_n}(q) \quad (5)$$

where λ, μ are decreasing sequence of integers can be interpreted as a duality result for the q -multiplicities associated to the classical root systems.

2.5 Crystals of type C_n and the one dimension sums $X_{\lambda,\mu}(q)$ and $Y_{\lambda,\mu}(q)$

We have seen that $U_{\lambda,\mu}(q)$ can be regarded as a q -analogue of the multiplicity of $V(\lambda)$ in W_μ^C . In [4], Hatayama, Kuniba, Okado and Takagi have introduced another quantification $X_{\lambda,\mu}(q)$ of this multiplicity based on the determination of the combinatorial R -matrix of certain $U'_q(C_n^{(1)})$ -crystals B_k^C . Considered as the crystal graph of a $U_q(C_n)$ -module, B_k^C is isomorphic to

$$B^C(k\Lambda_1) \oplus B^C((k-2)\Lambda_1) \oplus \cdots \oplus B^C(k \bmod 2\Lambda_1) \quad (6)$$

where for any $i \in \{k, k-2, \dots, k \bmod 2\}$, $B^C(k\Lambda_1)$ is the crystal graph of the irreducible finite dimensional $U_q(C_n)$ -module of highest weight $k\Lambda_1$. In [9], Kashiwara and Nakashima have obtained a natural labelling of the vertices of $B^C(k\Lambda_1)$ by one-row tableaux of length k filled by letters of the alphabet

$$\mathcal{C}_n = \{1 < \cdots < n-1 < n < \bar{n} < \overline{n-1} < \cdots < \bar{1}\}$$

such that the letters increase from left to right (that is by semistandard one-row tableaux on \mathcal{C}_n). Then the vertices of B_k^C can be depicted by one-row tableaux of length k by adding p pairs $(0, \bar{0})$ to the tableaux appearing in the crystals $B^C((k-2p)\Lambda_1)$ of the decomposition (6). Then by setting

$$\mathcal{C}_{n+1} = \{0 < 1 < \cdots < n-1 < n < \bar{n} < \overline{n-1} < \cdots < \bar{1} < \bar{0}\}$$

the crystal B_k^C can be regarded as a subcrystal of the \mathcal{C}_{n+1} -crystal associated to the dominant weight $k\Lambda_1$ (labelled by the one-row semistandard tableaux on \mathcal{C}_{n+1} with length k).

Recall that the combinatorial R -matrix associated to crystals B_k^C is equivalent to the description of the crystal graph isomorphisms

$$\left\{ \begin{array}{l} B_l^C \otimes B_k^C \xrightarrow{\cong} B_k^C \otimes B_l^C \\ b_1 \otimes b_2 \mapsto b'_2 \otimes b'_1 \end{array} \right.$$

together with the energy function H^C on $B_l^C \otimes B_k^C$. As proved in [4], this can be done by using the insertion algorithm for \mathcal{C}_{n+1} -tableaux of [1] or [15]. In the sequel we only need the description of the energy function H^C . Let b_1^* and b_2^* be the row tableaux obtained by erasing the letters 0 and $\bar{0}$ in b_1 and b_2 . Let l^* and k^* be the lengths of b_1^* and b_2^* . Denote by $P_{n+1}(b_1^* \otimes b_2^*)$ the tableau obtained by inserting the row b_2^* into the row b_1^* following the \mathcal{C}_{n+1} -insertion algorithm. Then $P_{n+1}(b_1^* \otimes b_2^*)$ is a two-row \mathcal{C}_{n+1} -tableau which contains $k^* + l^*$ letters. Write m for the length of the shortest row of $P_{n+1}(b_1^* \otimes b_2^*)$.

Proposition 2.5.1 [4] *For any $b_1 \otimes b_2$ in $B_l^C \otimes B_k^C$ we have*

$$H^C(b_1 \otimes b_2) = \min(l^*, k^*) - m.$$

The multiplicity of $V(\lambda)$ in W_μ^C is then equal to the number of highest weight vertices of weight λ in the crystal $B_\mu^C = B_{\mu_1}^C \otimes \cdots \otimes B_{\mu_n}^C$. Then the one dimension sum $X_{\lambda,\mu}(q)$ for $C_n^{(1)}$ -crystals is defined in [4] by

$$X_{\lambda,\mu}(q) = \sum_{b \in E_\lambda} q^{H^C(b)} \quad \text{with} \quad H^C(b) = \sum_{0 \leq i < j \leq n} H^C(b_i \otimes b_j^{(i+1)})$$

where E_λ is the set of highest weight vertices $b = b_1 \otimes \cdots \otimes b_n$ in B_μ^C of highest weight λ , $b_j^{(i)}$ is determined by the crystal isomorphism

$$\begin{aligned} B_{\mu_i}^C \otimes B_{\mu_{i+1}}^C \otimes B_{\mu_{i+2}}^C \otimes \cdots \otimes B_{\mu_j}^C &\rightarrow B_{\mu_i}^C \otimes B_{\mu_j}^C \otimes B_{\mu_{i+1}}^C \cdots \otimes B_{\mu_{j-1}}^C \\ b_i \otimes b_{i+1} \otimes \cdots \otimes b_j &\rightarrow b_j^{(i)} \otimes b'_i \otimes \cdots \otimes b'_{j-1} \end{aligned} \quad (7)$$

and for any $j = 1, \dots, n$, $H(b_0 \otimes b_j^{(1)})$ is equal to the number of letters 0 in $b_j^{(1)}$.

It is also possible to define a q -analogue for the multiplicity of $V_n^C(\lambda)$ in V_μ^C from the crystal $B_{(\mu)}^C = B^C(\mu_1 \Lambda_1) \otimes B^C(\mu_2 \Lambda_1) \otimes \cdots \otimes B^C(\mu_n \Lambda_n)$ by setting

$$Y_{\lambda, \mu}(q) = \sum_{b \in F_\lambda} q^{H^C(b)}$$

where F_λ is the set of highest weight vertices $b = b_1 \otimes \cdots \otimes b_n$ in $B_{(\mu)}^C$ of weight λ . Since the crystals $B^C(k\Lambda_1)$ can be regarded as $U'_q(A_{2n-1}^{(2)})$ -crystals [4], $Y_{\lambda, \mu}(q)$ corresponds to a one dimension sum for $A_{2n-1}^{(2)}$ -crystals. Note that it is obtained by restricting the sum defining $X_{\lambda, \mu}(q)$ on the highest weight vertices of E_λ which do not contain any letter 0 or $\bar{0}$. For such a vertex we have $H(b_0 \otimes b_j^{(1)}) = 0$. Thus for any $b \in F_\lambda$, we can write $H^C(b) = \sum_{1 \leq i < j \leq n} H^C(b_i \otimes b_j^{(i+1)})$.

When $|\lambda| = |\mu|$ the vertices of F_λ contain only unbarred letters. In this case $H^A(b_i \otimes b_j^{(i+1)}) = H^C(b_i \otimes b_j^{(i+1)})$ that is the energy function of type A_{n-1} defined on the vertices b which do not contain any barred letter is the restriction of that of type C_n . Moreover we have $F_\lambda = E_\lambda$, thus $Y_{\lambda, \mu}(q) = X_{\lambda, \mu}(q)$. In [20] Nakayashiki and Yamada have proved the equality $H^A(b) = \text{ch}(Q(b))$ where $Q(b)$ is the semistandard tableau associated to b by generalizing the Robinson-Schensted correspondence and ch is the charge defined by Lascoux and Schützenberger in [12] and [13]. Recall that the charge statistic verifies

$$K_{\lambda, \mu}^{A_{n-1}}(q) = \sum_{T \in SST_\mu(\lambda)} q^{\text{ch}(T)} \quad (8)$$

where $SST_\mu(\lambda)$ is the set of semistandard tableaux of shape λ and weight μ . This implies that $Y_{\lambda, \mu}(q) = K_{\lambda, \mu}^{A_{n-1}}(q)$ when $|\lambda| = |\mu|$.

Many computations suggest the following identities:

Conjecture 2.5.2 *For any partitions λ and μ of length n with $|\mu| \geq |\lambda|$*

$$\begin{aligned} \text{(i)} : u_{\lambda, \mu}(q) &= q^{\frac{|\mu| - |\lambda|}{2}} Y_{\lambda, \mu}(q) \\ \text{(ii)} : U_{\lambda, \mu}(q) &= q^{|\mu| - |\lambda|} X_{\lambda, \mu}(q). \end{aligned}$$

Remarks:

- (i): The above conjecture is true when $|\lambda| = |\mu|$ since in this case $K_{\lambda, \mu}^{A_{n-1}}(q) = u_{\lambda, \mu}(q) = U_{\lambda, \mu}(q)$ and $Y_{\lambda, \mu}(q) = X_{\lambda, \mu}(q)$.
- (ii): When $\mu = (1, \dots, 1)$, we have $X_{\lambda, (1^n)}(q) = Y_{\lambda, (1^n)}(q)$.
- (iii): A q -analogue for the multiplicity of $V_n^C(\lambda)$ in $W^C(\mu)$ can also be defined from rigged configurations. The $X = M$ conjecture gives a simple relation called fermionic formula between X and M .

In [22], Okado, Schilling and Shimozono have proved this conjecture in specific cases. In particular the conjecture is true when all the parts of μ are equal to 1. In 4, we will prove the Conjecture 2.5.2 when $\mu = (1, \dots, 1)$. Thus by combining our result with that of [22] we obtain very simple relations between the three different q -analogues for the multiplicity of $V_n^C(\lambda)$ in $W^C(1^n)$.

(ii): By Theorem 2.4.2, the polynomials $u_{\lambda,\mu}(q)$ and $U_{\lambda,\mu}(q)$ are Kostka-Foulkes polynomials. Thus Conjecture 2.5.2 implies that, up to a simple renormalization, the one dimension sums $X_{\lambda,\mu}(q)$ and $Y_{\lambda,\mu}(q)$ are affine Kazhdan-Lusztig polynomials.

3 Identities for the q -multiplicities

3.1 Decomposition of the $\tilde{K}_{\lambda,\mu}(q)$ in terms of the $K_{\lambda,\mu}^{A_{n-1}}(q)$

Denote by \mathcal{P}_q^A the q -Kostant partition function defined by

$$\prod_{1 \leq i < j \leq n} \left(1 - q \frac{x_i}{x_j}\right)^{-1} = \sum_{\eta \in \mathbb{Z}^n} \mathcal{P}_q^A(\eta) x^\eta$$

Then for any $\lambda, \mu \in \mathcal{P}_n$, the Kostka-Foulkes polynomial $K_{\lambda,\mu}^{A_{n-1}}(q)$ is such that

$$K_{\lambda,\mu}^{A_{n-1}}(q) = \sum_{\sigma \in \mathcal{S}_n} (-1)^{l(\sigma)} \mathcal{P}_q^A(\sigma \circ \lambda - \mu).$$

Since $K_{\lambda+k\kappa_n, \mu+k\kappa_n}^{A_{n-1}}(q) = K_{\lambda,\mu}^{A_{n-1}}(q)$ for any positive integer k , the definition of $K_{\lambda,\mu}^{A_{n-1}}(q)$ can be extended for λ and μ decreasing sequence of integers. By definition of the q -partition function \mathcal{P}_q^C we must have

$$\sum_{\beta \in \mathbb{Z}^n} \mathcal{P}_q^C(\beta) x^\beta = \prod_{1 \leq r \leq s \leq n} (1 - qx_r x_s)^{-1} \prod_{1 \leq i < j \leq n} \left(1 - q \frac{x_i}{x_j}\right)^{-1}.$$

Now we can write by (4)

$$\prod_{1 \leq r \leq s \leq n} (1 - qx_r x_s)^{-1} = \sum_{\delta \in L_C} q^{|\delta|/2} c(\delta) x^\delta \quad (9)$$

since the number of roots appearing in a decomposition of $\delta \in L_C$ as a sum of positive roots $\varepsilon_r + \varepsilon_s$ with $1 \leq r \leq s \leq n$ is always equal to $|\delta|/2$. Thus we obtain

$$\sum_{\beta \in \mathbb{Z}^n} \mathcal{P}_q^C(\beta) x^\beta = \sum_{\eta \in \mathbb{Z}^n} \sum_{\delta \in L_C} q^{|\delta|/2} \mathcal{P}_q^A(\eta) c(\delta) x^{\delta+\eta}. \quad (10)$$

By using similar arguments for \mathcal{P}_q^D we derive the following lemma:

Lemma 3.1.1 *For any $\beta \in \mathbb{Z}^n$ we have*

$$\mathcal{P}_q^C(\beta) = \sum_{\delta \in L_C^{|\beta|}} q^{|\delta|/2} c(\delta) \mathcal{P}_q^A(\beta - \delta) \text{ and } \mathcal{P}_q^D(\beta) = \sum_{\delta \in L_D^{|\beta|}} q^{|\delta|/2} d(\delta) \mathcal{P}_q^A(\beta - \delta)$$

where $L_C^{|\beta|} = \{\delta \in L_C, |\delta| = |\beta|\}$ and $L_D^{|\beta|} = \{\delta \in L_D, |\delta| = |\beta|\}$.

Proof. From (10) we derive the equality $\mathcal{P}_q^C(\beta) = \sum_{\eta+\delta=\beta} c(\delta)q^{|\delta|/2}\mathcal{P}_q^{A_n}(\eta)$. Since $\mathcal{P}_q^{A_n}(\eta) = 0$ when $|\eta| \neq 0$, we can suppose $|\eta| = 0$ and $|\delta| = |\beta|$ in the previous sum. Then $\delta \in L_C^{|\beta|}$ and the result follows immediately. The proof for $\mathcal{P}_q^D(\beta)$ is similar. ■

Remark: A similar result for the q -partition function \mathcal{P}_q^B does not exist. Indeed the number of roots appearing in a decomposition of $\delta \in L_B$ as a sum of positive roots $\varepsilon_r + \varepsilon_s$ with $1 \leq r < s \leq n$ and ε_i with $1 \leq i \leq n$ does not depend only of $|\delta|$ since $|\varepsilon_r + \varepsilon_s| \neq |\varepsilon_i|$.

Proposition 3.1.2 Consider $\lambda, \mu \in \mathcal{P}_n$ such that $|\lambda| \geq |\mu|$. Then we have:

1. $\tilde{K}_{\lambda, \mu}^{C_n}(q) = q^{\frac{|\lambda| - |\mu|}{2}} \sum_{\gamma \in \tilde{\mathcal{P}}_n} \sum_{\sigma \in \mathcal{S}_n} (-1)^{l(\sigma)} c(\sigma \circ \lambda - \gamma) K_{\gamma, \mu}^{A_{n-1}}(q)$
2. $\tilde{K}_{\lambda, \mu}^{D_n}(q) = q^{\frac{|\lambda| - |\mu|}{2}} \sum_{\gamma \in \tilde{\mathcal{P}}_n} \sum_{\sigma \in \mathcal{S}_n} (-1)^{l(\sigma)} d(\sigma \circ \lambda - \gamma) K_{\gamma, \mu}^{A_{n-1}}(q)$

where $\tilde{\mathcal{P}}_n = \{\gamma = (\gamma_1, \dots, \gamma_n) \in \mathbb{Z}^n, \gamma_1 \geq \gamma_2 \geq \dots \geq \gamma_n\}$.

Proof. We have

$$\tilde{K}_{\lambda, \mu}^{C_n}(q) = \sum_{\sigma \in \mathcal{S}_n} (-1)^{l(\sigma)} \mathcal{P}_q^{C_n}(\sigma(\lambda + \rho_n) - (\mu + \rho_n)).$$

Hence from the previous lemma we derive

$$\tilde{K}_{\lambda, \mu}^{C_n}(q) = \sum_{\delta \in L_C^{|\beta|}} c(\delta)q^{|\delta|/2} \sum_{\sigma \in \mathcal{S}_n} (-1)^{l(\sigma)} \mathcal{P}_q^{A_{n-1}}(\sigma(\lambda + \rho_n) - (\mu + \delta + \rho_n))$$

and

$$\tilde{K}_{\lambda, \mu}^{C_n}(q) = \sum_{\sigma \in \mathcal{S}_n} (-1)^{l(\sigma)} \sum_{\delta \in L_C^{|\beta|}} c(\delta)q^{|\delta|/2} \mathcal{P}_q^{A_{n-1}}(\sigma(\lambda + \rho_n - \sigma^{-1}(\delta)) - (\mu + \rho_n))$$

For any $\sigma \in \mathcal{S}_n$, we have $\sigma^{-1}(L_C^{|\beta|}) = L_C^{|\beta|}$ and $c(\delta) = c(\sigma(\delta))$. Thus we obtain

$$\tilde{K}_{\lambda, \mu}^{C_n}(q) = \sum_{\sigma \in \mathcal{S}_n} (-1)^{l(\sigma)} \sum_{\delta \in L_C^{|\beta|}} c(\delta)q^{|\delta|/2} \mathcal{P}_q^{A_{n-1}}(\sigma(\lambda + \rho_n - \delta) - (\mu + \rho_n)) = \sum_{\delta \in L_C^{|\beta|}} c(\delta)q^{\frac{|\mu| - |\lambda|}{2}} K_{\lambda - \delta, \mu}^{A_{n-1}}(q). \quad (11)$$

Now $K_{\lambda - \delta, \mu}^{A_{n-1}}(q) = 0$ or there exists $\sigma \in \mathcal{S}_n$ and $\gamma \in \tilde{\mathcal{P}}_n$ such that $\gamma = \sigma^{-1} \circ (\lambda - \delta)$. It follows that

$$\tilde{K}_{\lambda, \mu}^{C_n}(q) = \sum_{\sigma \in \mathcal{S}_n} (-1)^{l(\sigma)} \sum_{\gamma \in \tilde{\mathcal{P}}_n} c((\lambda + \rho) - \sigma(\gamma + \rho)) K_{\gamma, \mu}^{A_{n-1}}(q).$$

Since $c(\delta) = c(\sigma(\delta))$ for any $\sigma \in \mathcal{S}_n$, and $\delta \in L_C$ we obtain the desired equality

$$\tilde{K}_{\lambda, \mu}^{C_n}(q) = \sum_{\gamma \in \tilde{\mathcal{P}}_n} \sum_{\sigma \in \mathcal{S}_n} (-1)^{l(\sigma)} c(\sigma(\lambda + \rho) - (\gamma + \rho)) K_{\gamma, \mu}^{A_{n-1}}(q)$$

The proof is similar for $\tilde{K}_{\lambda, \mu}^{D_n}(q)$. ■

By Lemma 2.3.1, $\tilde{K}_{\lambda, \mu}^{C_n}(q)$ and $\tilde{K}_{\lambda, \mu}^{D_n}(q)$ are Kazhdan-Lusztig polynomials. Thus they have nonnegative integer coefficients. This property can also be obtained from the proposition below:

Corollary 3.1.3 Consider $\lambda, \mu \in \mathcal{P}_n$. For k a sufficiently large integer we have:

1. $\tilde{K}_{\lambda, \mu}^{C_n}(q) = q^{\frac{|\lambda| - |\mu|}{2}} \sum_{\gamma \in \tilde{\mathcal{P}}_n} [V_n^A(\gamma + k\kappa_n), V_n^C(\lambda + k\kappa_n)] K_{\gamma, \mu}^{A_{n-1}}(q),$
2. $\tilde{K}_{\lambda, \mu}^{D_n}(q) = q^{\frac{|\lambda| - |\mu|}{2}} \sum_{\gamma \in \tilde{\mathcal{P}}_n} [V_n^A(\gamma + k\kappa_n), V_n^D(\lambda + k\kappa_n)] K_{\gamma, \mu}^{A_{n-1}}(q).$

Proof. Consider $\gamma \in \tilde{\mathcal{P}}_n$ and write $\gamma = (\gamma^+, \gamma^-)$ as in (3). We have by Proposition 2.2.1 $[V_n^A(\gamma) : V_n^C(\lambda)] = \sum_{w \in W_{C_n}} (-1)^{l(w)} c(w \circ \lambda - \gamma)$. Now if k is sufficiently large $\gamma + k\kappa_n$ is a partition and $[V_n^A(\gamma) : V_n^C(\lambda)] = \sum_{w \in W_{C_n}} (-1)^{l(w)} c(w \circ \lambda - \gamma + w(k\kappa_n) - k\kappa_n)$. Suppose that $w \notin \mathcal{S}_n$, then $|w(k\kappa_n) - k\kappa_n| \leq -2k$. Thus we can choose k such that $\gamma + k\kappa_n$ is a partition and $w \circ \lambda - \gamma + w(k\kappa_n) - k\kappa_n \notin L_C$ for any $w \in W_{C_n}$ such that $w \notin \mathcal{S}_n$. For such an integer k we have $c(w \circ \lambda - \gamma + w(k\kappa_n) - k\kappa_n) = 0$. Then we derive 1 from 1 of Proposition 3.1.2. We prove 2 by using similar arguments. ■

To obtain a decomposition of the polynomial $\tilde{K}_{\lambda, \mu}^{B_n}(q)$ as a sum of polynomials $K_{\gamma, \mu}^{A_{n-1}}(q)$, we need first to obtain its decomposition in terms of the polynomials $\tilde{K}_{\nu, \mu}^{D_n}(q)$.

Proposition 3.1.4 Consider $\lambda, \mu \in \mathcal{P}_n$ such that $|\lambda| \geq |\mu|$. Then for k a sufficiently large integer we have:

$$\tilde{K}_{\lambda, \mu}^{B_n}(q) = \sum_{\nu \in \tilde{\mathcal{P}}_n} q^{|\lambda| - |\nu|} [V_n^D(\nu + k\kappa_n), V_n^B(\lambda + k\kappa_n)] \tilde{K}_{\nu, \mu}^{D_n}(q).$$

Proof. By definition of the partitions functions $\mathcal{P}_q^{B_n}$ and $\mathcal{P}_q^{D_n}$ we can write

$$\sum_{\beta \in \mathbb{Z}^n} \mathcal{P}_q^{B_n}(\beta) x^\beta = \sum_{\gamma \in \mathbb{Z}^n} \mathcal{P}_q^{D_n}(\gamma) x^\gamma \prod_{k=1}^n \frac{1}{1 - qx_k}.$$

We have $\prod_{k=1}^n \frac{1}{1 - qx_k} = \sum_{\delta \in \mathbb{N}^n} q^{|\delta|} x^\delta$. Thus we derive

$$\sum_{\beta \in \mathbb{Z}^n} \mathcal{P}_q^{B_n}(\beta) x^\beta = \sum_{\gamma \in \mathbb{Z}^n} \sum_{\delta \in \mathbb{N}^n} q^{|\delta|} \mathcal{P}_q^{D_n}(\gamma) x^{\gamma + \delta}.$$

This implies that $\mathcal{P}_q^{B_n}(\beta) = \sum_{\delta \in \mathbb{N}^n} q^{|\delta|} \mathcal{P}_q^{D_n}(\beta - \delta)$ and

$$\tilde{K}_{\lambda, \mu}^{B_n}(q) = \sum_{\sigma \in \mathcal{S}_n} (-1)^{l(\sigma)} \sum_{\delta \in \mathbb{N}^n} q^{|\delta|} \mathcal{P}_q^{D_n}(\sigma(\lambda + \rho_n) - (\mu + \delta + \rho_n)).$$

By using similar arguments to that of the proof of Proposition 3.1.2, we obtain

$$\tilde{K}_{\lambda, \mu}^{B_n}(q) = q^{|\lambda| - |\mu|} \sum_{\nu \in \tilde{\mathcal{P}}_n} \sum_{\sigma \in \mathcal{S}_n} (-1)^{l(\sigma)} \mathbf{1}_{\mathbb{N}}(\sigma(\lambda + \rho_n) - (\nu + \rho_n)) K_{\nu, \mu}^{D_n}(q).$$

There exists a sufficiently large integer k for which $w(\lambda + \rho_n + k\kappa_n) - (\nu + \rho_n + k\kappa_n) \notin \mathbb{N}^n$ for any $w \in W_{B_n}$ such that $w \notin \mathcal{S}_n$. Then it follows from Lemma 2.2.2 that $[V_n^D(\nu + k\kappa_n) : V_n^B(\lambda + k\kappa_n)] = \sum_{\sigma \in \mathcal{S}_n} (-1)^{l(\sigma)} \mathbf{1}_{\mathbb{N}}(\sigma \circ \lambda - \nu)$ and the proposition is proved. ■

Remark: From 2 of Corollary 3.1.3 and the above proposition we obtain

$$\tilde{K}_{\lambda,\mu}^{B_n}(q) = \sum_{\nu \in \tilde{\mathcal{P}}_n} \sum_{\gamma \in \tilde{\mathcal{P}}_n} q^{\frac{2|\lambda| - |\nu| - |\mu|}{2}} [V_n^D(\nu + k\kappa_n), V_n^B(\lambda + k\kappa_n)] [V_n^A(\gamma + k\kappa_n), V_n^D(\nu + k\kappa_n)] K_{\gamma,\mu}^{A_{n-1}}(q)$$

which implies in particular that the polynomials $\tilde{K}_{\lambda,\mu}^{B_n}(q)$ have non negative coefficients.

3.2 Decomposition of $u_{\lambda,\mu}(q)$ and $U_{\lambda,\mu}(q)$ in terms of the $K_{\lambda,\mu}^{A_{n-1}}(q)$

By using similar arguments to the proof of Lemma 3.1.1, we can establish that for any $\beta \in \mathbb{Z}^n$ we have

$$f_q(\beta) = \sum_{\delta \in L_C^{|\beta|}} q^{|\delta|/2} d(\delta) \mathcal{P}_q^A(\beta + \delta) \text{ and } F_q(\beta) = \sum_{\delta \in L_D^{|\beta|}} q^{|\delta|/2} c(\delta) \mathcal{P}_q^A(\beta + \delta)$$

which implies the following proposition:

Proposition 3.2.1 *Consider $\lambda, \mu \in \mathcal{P}_n$ such that $|\mu| \leq |\lambda|$. Then for k a sufficiently large integer we have*

1. $U_{\lambda,\mu}(q) = q^{\frac{|\mu| - |\lambda|}{2}} \sum_{\nu \in \mathcal{P}_n} [V_n^A(\lambda + k\kappa_n), V_n^C(\nu + k\kappa_n)] K_{\nu,\mu}^{A_{n-1}}(q),$
2. $u_{\lambda,\mu}(q) = q^{\frac{|\mu| - |\lambda|}{2}} \sum_{\nu \in \mathcal{P}_n} [V_n^A(\lambda + k\kappa_n), V_n^D(\nu + k\kappa_n)] K_{\nu,\mu}^{A_{n-1}}(q).$

Proof. We proceed as in the proof of Proposition 3.1.2 and (11) is replaced by

$$U_{\lambda,\mu}(q) = \sum_{\delta \in L_C^{|\beta|}} c(\delta) q^{\frac{|\mu| - |\lambda|}{2}} K_{\lambda + \delta, \mu}^{A_{n-1}}(q).$$

This time $K_{\lambda + \delta, \mu}^{A_{n-1}}(q) = 0$ or there exists $\sigma \in \mathcal{S}_n$ and $\nu \in \mathcal{P}_n$ such that $\nu = \sigma^{-1} \circ (\lambda + \delta)$. Note that ν can not have negative coordinates for $\lambda + \delta$ have non negative coordinates. We deduce:

$$U_{\lambda,\mu}(q) = q^{\frac{|\mu| - |\lambda|}{2}} \sum_{\sigma \in \mathcal{S}_n} (-1)^{l(\sigma)} \sum_{\nu \in \mathcal{P}_n} c(\sigma(\nu + \rho_n) - (\lambda + \rho_n)) K_{\gamma,\mu}^{A_{n-1}}(q).$$

Since we have $[V_n^A(\lambda + k\kappa_n), V_n^C(\nu + k\kappa_n)] = \sum_{\sigma \in \mathcal{S}_n} (-1)^{l(\sigma)} c(\sigma \circ \nu - \lambda)$ for k sufficiently large, this proves 1. Assertion 2 is obtained similarly. ■

3.3 Link with the polynomials $K_{\lambda,\mu}^{\diamond}(q)$ of Shimozono and Zabrocki

Since $c_{\gamma, \lambda + k\kappa_n}^{\nu + k\kappa_n} = c_{\gamma, \lambda}^{\nu}$ for any $k \in \mathbb{N}$, we deduce from Corollary 2.2.5 and Proposition 3.2.1:

Theorem 3.3.1 *Consider $\lambda, \mu \in \mathcal{P}_n$ such that $|\mu| \geq |\lambda|$ Then we have the following equalities*

- (i) : $u_{\lambda,\mu}(q) = q^{\frac{|\mu| - |\lambda|}{2}} \sum_{\nu \in \mathcal{P}_n} [V_n^C(\lambda), V_{2n}^A(\nu)] K_{\nu,\mu}^{A_{n-1}}(q) = q^{\frac{|\mu| - |\lambda|}{2}} \sum_{\nu \in \mathcal{P}_n} \sum_{\gamma \in \mathcal{P}_n^{(1,1)}} c_{\gamma,\lambda}^{\nu} K_{\nu,\mu}^{A_{n-1}}(q),$
- (ii) : $u_{\lambda,\mu}(q) = q^{\frac{|\mu| - |\lambda|}{2}} \sum_{\nu \in \mathcal{P}_n} [V_n^D(\lambda), V_{2n}^A(\nu)] K_{\nu,\mu}^{A_{n-1}}(q) = q^{\frac{|\mu| - |\lambda|}{2}} \sum_{\nu \in \mathcal{P}_n} \sum_{\gamma \in \mathcal{P}_n^{(2)}} c_{\gamma,\lambda}^{\nu} K_{\nu,\mu}^{A_{n-1}}(q)$

By comparing the leftmost equalities of the above theorem with equality (7.6) of [25] we obtain

$$K_{\lambda,\mu}^{(1,1)}(q) = u_{\lambda,\mu}(q^2) \text{ and } K_{\lambda,\mu}^{(2)}(q) = U_{\lambda,\mu}(q^2) \quad (12)$$

where $K_{\lambda,\mu}^{(1,1)}(q)$ and $K_{\lambda,\mu}^{(2)}(q)$ are polynomials defined by Shimozono and Zabrocki by using creating operators on formal series. In particular the polynomials $K_{\lambda,\mu}^{(1,1)}(q)$ and $K_{\lambda,\mu}^{(2)}(q)$ are Kazhdan-Lusztig polynomials specialized at q^2 .

Remark: In [25] the authors have also defined another polynomial denoted $K_{\lambda,\mu}^{(1)}(q)$ verifying

$$K_{\lambda,\mu}^{(1)}(q) = q^{|\mu|-|\lambda|} \sum_{\nu \in \mathcal{P}_n} \sum_{\gamma \in \mathcal{P}_n} c_{\gamma,\lambda}^{\nu} K_{\nu,\mu}^{A_{n-1}}(q^2). \quad (13)$$

From the duality (5), it is tempting to introduce the polynomial $V_{\lambda,\mu}(q) = \widetilde{K}_{I(\lambda),I(\mu)}(q)$. A similar result than Proposition 3.2.1 can not exist for $V_{\lambda,\mu}(q)$ (see the remark following Lemma 3.1.1). Nevertheless, by using Corollary 2.2.5, one can establish that

$$V_{\lambda,\mu}(1) = \sum_{\nu \in \mathcal{P}_n} [V_n^B(\lambda), V_{2n}^A(\nu)] K_{\nu,\mu}^{A_{n-1}}(1) = \sum_{\nu \in \mathcal{P}_n} \sum_{\gamma \in \mathcal{P}_n} c_{\gamma,\lambda}^{\nu} K_{\nu,\mu}^{A_{n-1}}(1).$$

Thus $K_{\lambda,\mu}^{(1)}(1) = V_{\lambda,\mu}(1)$. However we have $K_{\lambda,\mu}^{(1)}(q) \neq V_{\lambda,\mu}(q^2)$ in general. For example if we take $\lambda = (1, 0, 0)$ and $\mu = (1, 1, 1)$ we obtain $K_{\lambda,\mu}^{(1)}(q) = q^8 + 2q^6 + 2q^4 + q^2$ and $V_{\lambda,\mu}(q^2) = q^{10} + q^8 + 2q^6 + q^4 + q^2$.

Consider ν in \mathcal{P}_n . For any standard tableau τ of shape ν and weight (1^n) , let ν' be the standard tableau of shape ν' and weight (1^n) obtained by reflecting τ among the diagonal. Then one can verify that $\text{ch}(\tau') = \frac{n(n-1)}{2} - \text{ch}(\tau)$ which by (8) implies the identity:

$$K_{\nu',(1^n)}^{A_{n-1}}(q) = q^{\frac{n(n-1)}{2}} K_{\nu,(1^n)}^{A_{n-1}}(q^{-1}). \quad (14)$$

The following proposition will be useful in Section 4.

Proposition 3.3.2 *Consider $\lambda \in \mathcal{P}_n$ such that $n \geq |\lambda|$. Then we have*

$$(i) : U_{\lambda',(1^n)}(q) = q^{\frac{n(n-1)}{2} + n - |\lambda|} u_{\lambda,(1^n)}(q^{-1}) \text{ and } (ii) : u_{\lambda',(1^n)}(q) = q^{\frac{n(n-1)}{2} + n - |\lambda|} U_{\lambda,(1^n)}(q^{-1}).$$

Proof. By (ii) of Theorem 3.3.1 we have

$$U_{\lambda',(1^n)}(q) = q^{\frac{n-|\lambda|}{2}} \sum_{\nu \in \mathcal{P}_n} \sum_{\gamma \in \mathcal{P}_n^{(2)}} c_{\gamma,\lambda'}^{\nu} K_{\nu,(1^n)}^{A_{n-1}}(q). \quad (15)$$

Since $K_{\nu,(1^n)}^{A_{n-1}}(q) = 0$ when $|\nu| \neq n$, we can suppose that ν belongs to $\widehat{\mathcal{P}}_n = \{\nu \in \mathcal{P}_n, |\nu| = n\}$ in the above sum. For any ν in $\widehat{\mathcal{P}}_n$, we have $c_{\gamma,\lambda}^{\nu} = 0$ unless $|\gamma| = n - |\lambda|$. So we can suppose that γ belongs to $\widehat{\mathcal{P}}_n^{(2)} = \{\gamma \in \mathcal{P}_n^{(2)}, |\gamma| \leq n\}$ in (15). The map $\Gamma : \nu \mapsto \nu'$ is an involution of $\widehat{\mathcal{P}}_n$. Moreover $\Gamma(\widehat{\mathcal{P}}_n^{(2)}) = \widehat{\mathcal{P}}_n^{(1,1)} = \{\gamma \in \mathcal{P}_n^{(1,1)}, |\gamma| \leq n\}$. Thus we can write

$$U_{\lambda',(1^n)}(q) = q^{\frac{n-|\lambda|}{2}} \sum_{\nu \in \widehat{\mathcal{P}}_n} \sum_{\gamma \in \widehat{\mathcal{P}}_n^{(1,1)}} c_{\gamma',\lambda'}^{\nu} K_{\nu',(1^n)}^{A_{n-1}}(q).$$

Now since $c_{\gamma', \lambda'}^{\nu'} = c_{\gamma, \lambda}^{\nu}$, we derive by (14)

$$U_{\lambda', (1^n)}(q) = q^{\frac{n(n-1)}{2} + \frac{n-|\lambda|}{2}} \sum_{\nu \in \widehat{\mathcal{P}}_n} \sum_{\gamma \in \widehat{\mathcal{P}}_n^{(1,1)}} c_{\gamma, \lambda}^{\nu} K_{\nu, (1^n)}^{A_{n-1}}(q^{-1}).$$

Finally the equality $U_{\lambda', (1^n)}(q) = q^{\frac{n(n-1)}{2} + n - |\lambda|} u_{\lambda, (1^n)}(q^{-1})$ is deduced from (i) of Theorem 3.3.1. We obtain (ii) similarly. ■

Remark: By introducing for the root system A_{n-1} generalized Kostka-Foulkes polynomials $K_{\nu, R}(q)$ where $R = (R_1, \dots, R_n)$ is a sequence of rectangular partitions, Schilling and Warnarr [23] have proved the equality $K_{\nu', R'}(q) = q^{\|R\|} K_{\nu, R'}(q^{-1})$ where $R' = (R'_1, \dots, R'_l)$ and $\|R\| = \sum_{i < j} |R_i \cap R_j|$ which can be considered as a generalization of (14). In [25], Shimozono and Zabrocki have also defined their polynomials $K_{\lambda, R}^{\diamond}(q)$ when R is a sequence of rectangular partitions. By (12), the above proposition can also be regarded as a Corollary of Proposition 28 of [25].

4 Proof of Conjecture 2.5.2 when $\mu = (1, \dots, 1)$

4.1 The crystals B_{Ξ}

For any integer $l \geq 0$, let $B^A(l)$ be the crystal graph of the irreducible finite dimensional $U_q(\mathfrak{sl}_{2n})$ -module of highest weight $l\Lambda_1$. In the sequel we choose to label the vertices of $B^A(1)$ by the letters of \mathcal{C}_n , that is we identify $B^A(1)$ with the crystal

$$1 \xrightarrow{1} 2 \cdots \rightarrow n-1 \xrightarrow{n-1} n \xrightarrow{n} \bar{n} \xrightarrow{n+1} \overline{n-1} \xrightarrow{n+2} \cdots \rightarrow \bar{2} \xrightarrow{2n-1} \bar{1}.$$

Recall that the crystal graph $B^C(1)$ has been identified in 2.5 with

$$1 \xrightarrow{1} 2 \cdots \rightarrow n-1 \xrightarrow{n-1} n \xrightarrow{n} \bar{n} \xrightarrow{n-1} \overline{n-1} \xrightarrow{n-2} \cdots \rightarrow \bar{2} \xrightarrow{1} \bar{1}.$$

Thus the crystals $B^A(1)$ and $B^C(1)$ have the same vertices. For any partition $\mu \in \mathcal{P}_n$, set $B_{(\mu)}^A = B^A(\mu_1) \otimes \cdots \otimes B^A(\mu_n)$ and $B_{(\mu)}^C = B^C(\mu_1) \otimes \cdots \otimes B^C(\mu_n)$. Then $B_{(\mu)}^A$ and $B_{(\mu)}^C$ have also the same vertices. Nevertheless their crystal structure are distinct and their decompositions in connected components do not coincide.

Denote by H^A the energy function associated to $B^A(l) \otimes B^A(k)$. Then for any $b_1 \otimes b_2$ belonging to $B^A(l) \otimes B^A(k)$, we have $H^A(b_1 \otimes b_2) = \min(l, k) - m$ where m is the length of the shortest row of the semistandard tableau $P^A(b_1 \otimes b_2)$ obtained by inserting the row b_2 in the row b_1 following the column bumping algorithm for semistandard tableaux. Given $b = b_1 \otimes \cdots \otimes b_n \in B_{(\mu)}^A$, set

$$H^A(b) = \sum_{1 \leq i < j \leq n} H^A(b_i \otimes b_j^{(i+1)})$$

where the vertices $b_j^{(i+1)}$ are defined as in (7). In the sequel we need the following result due to Nakayashiki and Yamada:

Theorem 4.1.1 [20] *Consider ν and μ two partitions of \mathcal{P}_n . Then*

$$K_{\nu, \mu}^{A_{n-1}}(q) = \sum_{b \in G_{\nu}} q^{H^A(b)}$$

where G_{ν} is the set of highest weight vertices of weight ν in $B_{(\mu)}^A$.

Remark: It is possible to show that $P^A(b_1 \otimes b_2) = P^C(b_1 \otimes b_2)$ (see 2.5) for any $b_1 \otimes b_2$ belonging to $B^A(l) \otimes B^A(k)$ if and only if $l = k = 1$. Moreover if we choose $l \geq 2$ and $k \geq 2$, these two tableaux can have distinct shapes. For example, by taking $l = k = 2$, $b_1 = b_2 = \bar{2}2$, we obtain

$$P^A(b_1 \otimes b_2) = \begin{array}{|c|c|c|} \hline 2 & 2 & \bar{2} \\ \hline \bar{2} & & \\ \hline \end{array} \text{ and } P^C(b_1 \otimes b_2) = \begin{array}{|c|c|} \hline 1 & \bar{2} \\ \hline 2 & \bar{1} \\ \hline \end{array}.$$

Hence the two statistics H^A and H^C do not coincide in general on the vertices of $B_{(\mu)}^A$.

Suppose now that $\mu = (1, \dots, 1)$. Then any vertex b of $B_{(1^n)}^A$ or $B_{(1^n)}^C$ can be written

$$b = x_1 \otimes \cdots \otimes x_n$$

where x_1, \dots, x_n are letters of \mathcal{C}_n . We have

$$H^A(b) = H^C(b) = \sum_{i=1}^{n-1} (n-i)H(x_i \otimes x_{i+1})$$

where $H(x_i \otimes x_{i+1}) = 1$ if $x_i \geq x_{i+1}$ and $H(x_i \otimes x_{i+1}) = 0$ otherwise.

To each vertex b , we associate the $(n-1)$ -tuple $\Xi(b) = (\xi_1, \dots, \xi_{n-1})$ such that for any $i = 1, \dots, n-1$, $\xi_i = 0$ if $x_i < x_{i+1}$ and $\xi_{i+1} = 1$ otherwise. For any $(n-1)$ -tuple Ξ with coordinates equal to 0 or 1, set

$$B_{\Xi} = \{b \in B_{(1^n)}^A, \Xi(b) = \Xi\}.$$

Then the statistics H^A and H^C are invariant on the vertices of B_{Ξ} and we have

$$H^A(b) = H^C(b) = \theta(b) = \sum_{i=1}^{n-1} (n-i)\xi_i \text{ for any } b \in B_{\Xi} \text{ with } \Xi = (\xi_1, \dots, \xi_{n-1}).$$

Lemma 4.1.2 *Let $\Xi = (\xi_1, \dots, \xi_{n-1})$ be a $(n-1)$ -tuple with coordinates equal to 0 or 1. Then the set B_{Ξ} has a structure of A_{2n-1} -crystal and a structure of C_n -crystal.*

Proof. Consider \tilde{K} a Kashiwara crystal operator for $U_q(sl_{2n-1})$ or $U_q(sp_{2n})$ and x, y two letters of \mathcal{C}_n . Set $\tilde{K}(x \otimes y) = x' \otimes y'$. From the description of the crystals $B^A(1)^{\otimes 2}$ and $B^C(1)^{\otimes 2}$ given by Kashiwara and Nakashima in [9] we derive the equivalence:

$$x \leq y \iff x' \leq y'.$$

This implies that B_{Ξ} is stable under the action of any Kashiwara crystal operator. Thus B_{Ξ} has a structure of A_{2n-1} -crystal and a structure of C_n -crystal. ■

4.2 The $Y = M = u$ conjecture when $\mu = (1, \dots, 1)$

Theorem 4.2.1 *For any partition λ of length n we have*

$$u_{\lambda, (1^n)}(q) = q^{\frac{n-|\lambda|}{2}} Y_{\lambda, (1^n)}(q) = q^{\frac{n-|\lambda|}{2}} X_{\lambda, (1^n)}(q).$$

Proof. Denote by \mathfrak{E} the set of the $(n-1)$ -tuples Ξ with coordinates equal to 0 or 1. By Lemma 4.1.2, for any $\Xi \in \mathfrak{E}$, B_Ξ has a structure of A_{2n-1} -crystal and a structure of C_n -crystal. Let us denote respectively by B_Ξ^A and B_Ξ^C these two crystals. There exists a $U_q(\mathfrak{gl}_{2n})$ -module M_Ξ^A whose crystal is isomorphic to B_Ξ^A . Similarly there exists a $U_q(\mathfrak{sp}_{2n})$ -module M_Ξ^C whose crystal is isomorphic to B_Ξ^C . Recall that the weight $\text{wt}^A(b)$ of a vertex $b \in B_\Xi^A$ is equal to $(d_1, \dots, d_n, d_{n+1}, \dots, d_{2n})$ where for any $i \in \{1, \dots, 2n\}$, d_i is the number of letters i in b . Similarly the weight $\text{wt}^C(b)$ of a vertex $b \in B_\Xi^C$ is equal to $(\delta_1, \dots, \delta_n)$ where for any $i \in \{1, \dots, 2n\}$, δ_i is the number of letters i in b minus the number of letters \bar{i} . Then the characters of the modules M_Ξ^A and M_Ξ^C verify

$$\text{char}(M_\Xi^A)(x_1, \dots, x_n) = \sum_{b \in B_\Xi^A} x^{\text{wt}^A(b)} \text{ and } \text{char}(M_\Xi^C)(x_1, \dots, x_n) = \sum_{b \in B_\Xi^C} x^{\text{wt}^C(b)}.$$

Let N_Ξ^A and N_Ξ^C be two representations respectively of GL_{2n} and Sp_{2n} such that $\text{char}(N_\Xi^A) = \text{char}(M_\Xi^A)$ and $\text{char}(N_\Xi^C) = \text{char}(M_\Xi^C)$. Then, by definition of $\text{wt}^A(b)$ and $\text{wt}^C(b)$, we have

$$\text{char}(N_\Xi^C)(x_1, \dots, x_n) = \text{char}(N_\Xi^A)(x_1, \dots, x_n, \frac{1}{x_n}, \dots, \frac{1}{x_1})$$

that is $\text{ch}(N_\Xi^C)$ is obtained by changing x_{n+i} in $\frac{1}{x_i}$ in $\text{char}(N_\Xi^A)$. Thus $\text{char}(N_\Xi^C) = \text{char}(N_\Xi^A \downarrow_{Sp_{2n}}^{GL_{2n}})$ where $N_\Xi^A \downarrow_{Sp_{2n}}^{GL_{2n}}$ is the restriction of N_Ξ^A to the action of Sp_{2n} . This implies that N_Ξ^C and $N_\Xi^A \downarrow_{Sp_{2n}}^{GL_{2n}}$ are isomorphic representations of Sp_{2n} .

Write $F_\lambda^\Xi = \{b \in B_\Xi^C, b \in F_\lambda\}$ for the set of highest weight vertices of weight λ in B_Ξ^C . Then we obtain:

$$\text{card}(F_\lambda^\Xi) = [V_n^C(\lambda), N_\Xi^C] = [V_n^C(\lambda), N_\Xi^A]$$

where $[V_n^C(\lambda), N_\Xi^A]$ is the multiplicity of $V_n^C(\lambda)$ in $N_\Xi^A \downarrow_{Sp_{2n}}^{GL_{2n}}$. Recall that $Y_{\lambda, (1^n)}(q) = \sum_{b \in F_\lambda} q^{H^C(b)}$ where F_λ is the set of highest weight vertices of weight λ in $B_{(1^n)}^C$. By Lemma 4.1.2 and the equality above we can write

$$Y_{\lambda, (1^n)}(q) = \sum_{\Xi \in \mathfrak{E}} \sum_{b \in F_\lambda^\Xi} q^{H^C(b)} = \sum_{\Xi \in \mathfrak{E}} \text{card}(F_\lambda^\Xi) q^{\theta_\Xi} = \sum_{\Xi \in \mathfrak{E}} [V_n^C(\lambda), N_\Xi^A] q^{\theta_\Xi} \quad (16)$$

where for any $\Xi = (\xi_1, \dots, \xi_{n-1})$ in \mathfrak{E} , $\theta_\Xi = \sum_{i=1}^{n-1} (n-i)\xi_i$.

By definition of the representations N_Ξ^A we have

$$\bigoplus_{\Xi \in \mathfrak{E}} N_\Xi^A \simeq V_{2n}^A(\Lambda_1)^{\otimes n} \quad (17)$$

as GL_{2n} -representations. The irreducible components of $B_{(1^n)}^A$ are indexed by standard tableaux with letters in $\{1, \dots, n\}$. For any $b \in B_{(1^n)}^A$ denote by $Q(b)$ the recording tableau (which is a standard tableau with letters in $\{1, \dots, n\}$) associated to b by the Robinson-Schensted correspondence. The Q -symbol yields a one to one correspondence between the highest weight vertices of $B_{(1^n)}^A$ (thus the irreducible components of $V_{2n}^A(\Lambda_1)^{\otimes n}$) and the set ST of standard tableaux with letters in $\{1, \dots, n\}$. For any $\tau \in ST$, denote by $V_n^A(\tau)$ the irreducible component of $V_{2n}^A(\Lambda_1)^{\otimes n}$ associated to τ . Then $V_{2n}^A(\tau) \simeq V_{2n}^A(\nu)$ where ν is the partition corresponding to the shape of τ . Write ST_Ξ for the set of standards tableaux $\tau \in ST$ such that there exists a highest weight vertex $b \in B_\Xi^A$ (thus for the A_{2n-1} -structure of graph) with $Q(b) = \tau$. Then from (17) the sets ST_Ξ , $\Xi \in \mathfrak{E}$ are disjoint and $\bigcup_{\Xi \in \mathfrak{E}} ST_\Xi = ST$. For any $\Xi \in \mathfrak{E}$, we have

$$N_\Xi^A = \bigoplus_{\tau \in ST_\Xi} V_{2n}^A(\tau).$$

Thus we derive from (16) that

$$Y_{\lambda, (1^n)}(q) = \sum_{\Xi \in \mathfrak{E}} \sum_{\tau \in ST_{\Xi}} [V_n^C(\lambda), V_{2n}^A(\tau)] q^{\theta_{\Xi}}.$$

Now the multiplicity $[V_n^C(\lambda), V_{2n}^A(\tau)]$ depends only on the shape ν of the standard tableau τ . This means that we have the equality $[V_n^C(\lambda), V_{2n}^A(\tau)] = [V_n^C(\lambda), V_{2n}^A(\nu)]$ for any τ of shape $\nu \in \mathcal{P}_n$. We deduce that

$$Y_{\lambda, (1^n)}(q) = \sum_{\Xi \in \mathfrak{E}} \sum_{\nu \in \mathcal{P}_n} \sum_{\tau \in ST_{\Xi}^{\nu}} [V_n^C(\lambda), V_{2n}^A(\nu)] q^{\theta_{\Xi}}$$

where $ST_{\Xi}^{\nu} = \{\tau \in ST_{\Xi} \text{ of shape } \nu\}$. So we obtain

$$Y_{\lambda, (1^n)}(q) = \sum_{\nu \in \mathcal{P}_n} [V_n^C(\lambda), V_{2n}^A(\nu)] \sum_{\Xi \in \mathfrak{E}} \text{card}(ST_{\Xi}^{\nu}) q^{\theta_{\Xi}}.$$

But we have $\sum_{\Xi \in \mathfrak{E}} \text{card}(ST_{\Xi}^{\nu}) q^{\theta_{\Xi}} = K_{\nu, (1^n)}^{A_{n-1}}(q)$ since $\theta_{\Xi} = H^A(b)$ for any highest weight vertex b of weight ν in B_{Ξ}^A . Thus

$$Y_{\lambda, (1^n)}(q) = \sum_{\nu \in \mathcal{P}_n} [V_n^C(\lambda), V_{2n}^A(\nu)] K_{\nu, (1^n)}^{A_{n-1}}(q).$$

Finally by Theorem 3.3.1, we obtain

$$q^{\frac{n-|\lambda|}{2}} Y_{\lambda, (1^n)}(q) = u_{\lambda, (1^n)}(q)$$

and the Theorem is proved. ■

Remark: When $\mu \in \mathcal{P}_n$ is not equal to (1^n) , it seem very difficult to define an analogue of the B_{Ξ} , that is, a set of vertices which can be naturally endowed with the double structure of a crystal for the types A_{2n-1} and C_n , and on which the two statistics H^A and H^C coincide. Indeed we have seen (see Remark following Theorem 4.1.1) that in this case there can exist some vertices $b = b_1 \otimes \cdots \otimes b_n$ such that $H^A(b) \neq H^C(b)$. Hence we can not use similar arguments to those of the above proof to establish the Conjecture 2.5.2 for any partition μ .

4.3 The $X = M = U$ conjecture when $\mu = (1, \dots, 1)$

We want to establish the equality $U_{\lambda, (1^n)}(q) = q^{n-|\lambda|} X_{\lambda, (1^n)}(q)$. By Proposition 3.3.2 we know that

$$U_{\lambda, (1^n)}(q) = q^{\frac{n(n-1)}{2} + n - |\lambda|} u_{\lambda', (1^n)}(q^{-1}).$$

Thus by Theorem 4.2.1 we obtain

$$U_{\lambda, (1^n)}(q) = q^{\frac{n(n-1)}{2} + \frac{n-|\lambda|}{2}} X_{\lambda', (1^n)}(q^{-1})$$

and it suffices to prove the equality

$$X_{\lambda', (1^n)}(q^{-1}) = q^{-\frac{n(n-1)}{2} + \frac{n-|\lambda|}{2}} X_{\lambda, (1^n)}(q)$$

which is equivalent to

$$X_{\lambda', (1^n)}(q) = q^{\frac{n(n-1)}{2} - \frac{n-|\lambda|}{2}} X_{\lambda, (1^n)}(q^{-1}). \quad (18)$$

In [15] we have introduced a Robinson-Schensted type correspondence for the vertices of $B_{(1^n)}^C$. In particular we have obtained a one to one correspondence between the highest weight vertices of $b \in B_{(1^n)}^C$ and the oscillating tableaux of length n . Recall that an oscillating tableau of length n is a sequence $Q = (Q_1, \dots, Q_n)$ of Young diagrams such that Q_i and Q_{i+1} differs by exactly one box (that is $Q_{k+1}/Q_k = \square$ or $Q_k/Q_{k+1} = \square$). We denote by $Q(b)$ the oscillating tableau associated to the highest weight vertex $b \in B_{(1^n)}^C$ under this correspondence. More precisely set $b = x_1 \otimes \dots \otimes x_n$. Then $Q(b)$ is defined recursively as follows:

$$Q_1 = \square \text{ and } Q_{i+1} = x_{i+1} \rightarrow Q_i \quad (19)$$

where $x_{i+1} \rightarrow Q_i$ is the Young diagram obtained from Q_i by adding a box on the k -th row of Q_i if $x_{i+1} = k \in \{1, \dots, n\}$ and by deleting a box on the k -th row of Q_i if $x_{i+1} = \bar{k} \in \{\bar{1}, \dots, \bar{n}\}$. Given any highest weight vertex b , it is easy to verify that $Q(b)$ is an oscillating tableau of length n . Suppose $Q(b) = (Q_1, \dots, Q_n)$. Then $Q' = (Q'_1, \dots, Q'_n)$, where for any $i \in \{1, \dots, n\}$ Q'_i is the conjugate diagram of Q_i , is also an oscillating tableau. There exists a highest weight vertex $b' \in B_{(1^n)}^C$ such that $Q(b') = Q'$. Moreover if the weight of b is equal to λ , then the weight of b' is equal to λ' . To prove our conjecture we need the two following technical lemmas:

Lemma 4.3.1 *Suppose that $b = x_1 \otimes \dots \otimes x_n$ is a highest weight vertex of $B_{(1^n)}^C$ and write $b' = x'_1 \otimes \dots \otimes x'_n$. Then:*

1. for any $i = 1, \dots, n$, the two letters x_i and x'_i are simultaneously barred or unbarred,
2. $H(x'_i \otimes x'_{i+1}) = \begin{cases} 1 - H(x_i \otimes x_{i+1}) & \text{if } x_i \text{ and } x_{i+1} \text{ are simultaneously barred or unbarred} \\ H(x_i \otimes x_{i+1}) & \text{otherwise} \end{cases}$.

Proof. We obtain 1 immediately from the definition (19) of Q and Q' .

As usual we enumerate the rows (resp. the columns) of the Young diagrams from top to bottom (resp. from left to right).

Suppose that $x_i = p$ and $x_{i+1} = q$ with $p, q \in \{1, \dots, n\}$. Then Q_{i+1} is obtained by adding first a box in the p -th row of Q_{i-1} to give Q_i , next a box in the q -th row of Q_i . Thus Q'_{i+1} is obtained by adding first a box in the p -th column of Q'_{i-1} , next a box in the q -th column of Q'_i if $p \neq q$, by adding two boxes in the p -th column of Q'_{i-1} otherwise. This implies that we have $x'_i \geq x'_{i+1}$ when $p < q$ (i.e. $x_i < x_{i+1}$) and $x'_i < x'_{i+1}$ when $p \geq q$ (i.e. $x_i \geq x_{i+1}$).

Now suppose that $x_i = \bar{p}$ and $x_{i+1} = \bar{q}$ with $p, q \in \{1, \dots, n\}$. Then Q_{i+1} is obtained by deleting first a box in the p -th row of Q_{i-1} to give Q_i , next a box in the q -th row of Q_i . Thus Q'_{i+1} is obtained by deleting first a box in the p -th column of Q'_{i-1} , next a box in the q -th column of Q'_i if $p \neq q$, by deleting two boxes in the p -th column of Q'_{i-1} otherwise. This implies that we have $x'_i \geq x'_{i+1}$ when $\bar{p} < \bar{q}$ (i.e. $x_i < x_{i+1}$) and $x'_i < x'_{i+1}$ when $\bar{p} \geq \bar{q}$ (i.e. $x_i \geq x_{i+1}$).

So in all cases we obtain $H(x'_i \otimes x'_{i+1}) = 1 - H(x_i \otimes x_{i+1})$ when x_i and x_{i+1} are simultaneously barred or unbarred.

The equality $H(x'_i \otimes x'_{i+1}) = H(x_i \otimes x_{i+1})$ when x_i and x_{i+1} are not simultaneously barred or unbarred follows from 1 since a barred letter is always greater than an unbarred one. ■

For any vertex $b = x_1 \otimes \dots \otimes x_n \in B_{(1^n)}^C$, set $Z_b = \{i \in \{1, \dots, n-1\}, x_i \text{ and } x_{i+1} \text{ are not simultaneously barred or unbarred}\}$

Lemma 4.3.2 *With the above notation we have*

$$\sum_{i \in Z_b} (n-i)(1-2H(x_i \otimes x_{i+1})) = \frac{n-|\lambda|}{2}$$

for any highest weight vertex $b = x_1 \otimes \cdots \otimes x_n \in B_{(1^n)}^C$ of weight λ .

Proof. Observe first that we have

$$1-2H(x_i \otimes x_{i+1}) = \begin{cases} 1 & \text{if } x_i \leq x_{i+1} \\ -1 & \text{otherwise} \end{cases}.$$

Since b is a highest weight vertex, it follows from the description of the action of the Kashiwara operators [8] on a tensor product of crystals that for any $i \in \{1, \dots, n\}$, $b_i = x_1 \otimes \cdots \otimes x_i$ is a highest weight vertex. In particular we must have $b_1 = x_1 = 1$. Thus we obtain

$$\sum_{i \in Z_b} (1-2H(x_i \otimes x_{i+1})) = \begin{cases} 0 & \text{if } x_n \text{ is unbarred} \\ 1 & \text{otherwise} \end{cases}. \quad (20)$$

To prove the lemma we proceed by induction on n . When $n = 1$, we have $b = 1$, $Z_b = \emptyset$ and $|\lambda| = 1$. Hence the lemma is true. Now suppose the lemma proved for any highest weight vertex of $B_{(1^{n-1})}^C$ and consider $b = x_1 \otimes \cdots \otimes x_n \in B_{(1^n)}^C$ a highest weight vertex of weight λ . Set $s_b = \sum_{i \in Z_b} (n-i)(1-2H(x_i \otimes x_{i+1}))$. We have $Z_b = Z_{b_{n-1}}$ if x_{n-1} and x_n are simultaneously barred or unbarred and $Z_b = Z_{b_{n-1}} \cup \{n-1\}$ otherwise. Write $\tilde{\lambda}$ for the weight of the highest weight vertex $b_{n-1} = x_1 \otimes \cdots \otimes x_{n-1}$. In the two cases $Z_b = Z_{b_{n-1}}$ and $Z_b = Z_{b_{n-1}} \cup \{n-1\}$ we derive by using the induction hypothesis

$$s_b = \frac{n-1-|\tilde{\lambda}|}{2} + \sum_{i \in Z_b} (1-2H(x_i \otimes x_{i+1})).$$

When x_n is unbarred, we have $|\tilde{\lambda}| = |\lambda| - 1$. Hence $\frac{n-1-|\tilde{\lambda}|}{2} = \frac{n-|\lambda|}{2}$ and we obtain $s_b = \frac{n-|\lambda|}{2}$ by (20). When x_n is barred, we have $|\tilde{\lambda}| = |\lambda| + 1$ thus $\frac{n-1-|\tilde{\lambda}|}{2} = \frac{n-|\lambda|}{2} - 1$ and by (20) we also derive $s_b = \frac{n-|\lambda|}{2}$ which proves the lemma. ■

Theorem 4.3.3 *(For any partition λ of length n we have*

$$U_{\lambda, (1^n)}(q) = q^{n-|\lambda|} X_{\lambda, (1^n)}(q).$$

Proof. Consider $b = x_1 \otimes \cdots \otimes x_n$ a highest weight vertex of $B_{(1^n)}^C$ of weight λ and set $b' = x'_1 \otimes \cdots \otimes x'_n$. Then

$$H(b') = \sum_{i=1}^{n-1} (n-i)H(x'_i \otimes x'_{i+1}) = \sum_{i \in Z_b} (n-i)H(x_i \otimes x_{i+1}) + \sum_{i \notin Z_b} (n-i)(1-H(x_i \otimes x_{i+1}))$$

by Lemma 4.3.1. Thus we have

$$H(b') = \sum_{i=1}^{n-1} (n-i)(1-H(x_i \otimes x_{i+1})) + \sum_{i \in Z_b} (n-i)(2H(x_i \otimes x_{i+1}) - 1) = \frac{n(n-1)}{2} - \frac{n-|\lambda|}{2} - H(b)$$

where the last equality follows from Lemma 4.3.2. Finally we derive

$$X_{\lambda',(1^n)}(q) = \sum_{b \in F_\lambda} q^{H(b')} = q^{\frac{n(n-1)}{2} - \frac{n-|\lambda|}{2}} \sum_{b \in F_\lambda} q^{-H(b')} = q^{\frac{n(n-1)}{2} - \frac{n-|\lambda|}{2}} X_{\lambda,(1^n)}(q^{-1})$$

which by (18) proves the Theorem. ■

Remark: Theorems 4.2.1 and 4.3.3 can be regarded as analogues for the affine root systems $C_n^{(1)}$ and $A_{2n-1}^{(2)}$ of Theorem 4.1.1 when $\mu = (1, \dots, 1)$.

4.4 Appendix on the one dimension sums

In [7], Kang, Kashiwara and Misra have defined crystals B_l^ϕ for quantum affine algebras associated to the affine root systems ϕ of types $A_n^{(1)}, C_n^{(1)}, A_{2n-1}^{(2)}, A_{2n}^{(2)}, D_{n+1}^{(2)}, B_n^{(1)}$ and $D_n^{(1)}$. We give below the decomposition of the crystals B_l^ϕ as classical crystals for each affine root system ϕ :

$C_n^{(1)}$	$A_{2n-1}^{(2)}$	$A_{2n}^{(2)}$	$D_{n+1}^{(2)}$	$B_n^{(1)}$	$D_n^{(1)}$	(21)
$\bigoplus_{k=0, k \equiv l \pmod 2}^l B^C(k\Lambda_1)$	$B^C(k\Lambda_1)$	$\bigoplus_{k=0}^l B^C(k\Lambda_1)$	$\bigoplus_{k=0}^l B^B(k\Lambda_1)$	$B^B(k\Lambda_1)$	$B^D(k\Lambda_1)$	

where $B^C(k\Lambda_1)$, $B^D(k\Lambda_1)$ and $B^B(k\Lambda_1)$ are the crystal graphs of the finite dimensional modules of highest weight $k\Lambda_1$ respectively for $U_q(\mathfrak{sp}_{2n})$, $U_q(\mathfrak{so}_{2n+1})$ and $U_q(\mathfrak{so}_{2n})$. Given any partition $\mu \in \mathcal{P}_n$, set $B_\mu^\phi = B_{\mu_1}^\phi \otimes \dots \otimes B_{\mu_n}^\phi$. Then, by using the energy functions explicited in [4] and [5], one can define a statistic H^ϕ on the vertices of B_μ^ϕ from which it is possible to calculate the one dimension sum $X_{\lambda,\mu}^\phi(q)$. In particular, with the notation of 2.5, we have

$$X_{\lambda,\mu}^{C_n^{(1)}}(q) = X_{\lambda,\mu}(q) \text{ and } X_{\lambda,\mu}^{A_{2n-1}^{(2)}}(q) = Y_{\lambda,\mu}(q)$$

since the polynomials $X_{\lambda,\mu}(q)$ and $Y_{\lambda,\mu}(q)$ are one dimension sums respectively for the affine root systems $C_n^{(1)}$ and $A_{2n-1}^{(2)}$. Denote by m the number of nonzero parts of μ . One proves that the highest weight vertices of weight λ in $B_\mu^{B_n^{(1)}}$ (considered as a B_n -crystal) are in one to one correspondence with those of $B_\mu^{A_{2n-1}^{(2)}}$ (considered as a C_n -crystal). Similarly when $m < n$ the highest weight vertices of weight λ in $B_\mu^{D_n^{(1)}}$ (considered as a D_n -crystal) are also in one to one correspondence with those of $B_\mu^{A_{2n-1}^{(2)}}$. Moreover the statistics $H^{A_{2n-1}^{(2)}}$, $H^{B_n^{(1)}}$ and $H^{D_n^{(1)}}$ are conserved via these correspondences. Thus we obtain by (21) the equality

$$X_{\lambda,\mu}^{A_{2n}^{(2)}}(q) = X_{\lambda,\mu}^{D_{n+1}^{(2)}}(q)$$

for any $\mu \in \mathcal{P}_n$. Similarly, when $m < n$, we have

$$X_{\lambda,\mu}^{A_{2n-1}^{(2)}}(q) = X_{\lambda,\mu}^{B_n^{(1)}}(q) = X_{\lambda,\mu}^{D_n^{(1)}}(q). \quad (22)$$

By Theorems 2.4.2, 4.2.1 and 4.3.3 we deduce the

Corollary 4.4.1

1. $X_{\lambda,\mu}^{A_{2n-1}^{(2)}}(q) = X_{\lambda,\mu}^{B_n^{(1)}}(q) = X_{\lambda,\mu}^{D_n^{(1)}}(q) = q^{-\frac{|\mu|+|\lambda|}{2}} u_{\lambda,\mu}(q) = q^{-\frac{|\mu|+|\lambda|}{2}} \tilde{K}_{\lambda,\hat{\mu}}^{D_n}(q)$ for $\mu = (1, \dots, 1, 0) \in \mathcal{P}_n$.
2. $X_{\lambda,\mu}^{C_n^{(1)}}(q) = q^{-\frac{|\mu|+|\lambda|}{2}} U_{\lambda,\mu}(q) = q^{-\frac{|\mu|+|\lambda|}{2}} \tilde{K}_{\lambda,\hat{\mu}}^{C_n}(q)$ for $\mu = (1, \dots, 1) \in \mathcal{P}_n$.

When ϕ is one of the two affine root systems $A_{2n}^{(2)}$ or $D_{n+1}^{(2)}$, the structure of classical crystal of B_μ^ϕ implies that the one dimension sum $X_{\lambda,\mu}^\phi(q)$ cannot be naturally related to the q -multiplicities $u_{\lambda,\mu}(q)$ or $U_{\lambda,\mu}(q)$ (see 2.4). However they can be expressed in terms of the polynomials $K_{\lambda,\mu}^{(1)}(q)$ [24].

Note: While revising this work, the author have been informed that, in a paper in preparation [24], Shimozono obtains a proof of the $X = M$ conjecture for tensor product of the “symmetric power” Kirilov-Reshetikin modules for nonexceptional affine algebras of type $D_{n+1}^{(2)}$, $A_{2n}^{(2)}$, $C_n^{(1)}$, $A_{2n}^{(2)}$ or $A_{2n}^{(2)\dagger}$. With our convention for the definition of the one dimension sums (which is that of [4]) this result can be reformulated by writing

$$K_{\lambda,\mu}^{(2)}(q) = q^{|\mu|-|\lambda|} X_{\lambda,\mu}^{C_n^{(1)}}(q^2) = q^{|\mu|-|\lambda|} X_{\lambda,\mu}^{A_{2n}^{(2)\dagger}}(q^2) \text{ and } K_{\lambda,\mu}^{(1)}(q) = q^{|\mu|-|\lambda|} X_{\lambda,\mu}^{A_{2n}^{(2)}}(q^2) = q^{|\mu|-|\lambda|} X_{\lambda,\mu}^{D_n^{(2)}}(q^2)$$

for any $\mu \in \mathcal{P}_n$. By (12) the first equality above establishes (ii) of Conjecture 2.5.2 since $X_{\lambda,\mu}(q^2)$ is a one dimension sum for $C_n^{(1)}$ -crystals. Thus assertion 2 of Corollary 4.4.1 holds for any $\mu \in \mathcal{P}_n$.

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