# $C^1$ Interpolatory Subdivision with Shape Constraints for Curves

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**Abstract**

We derive two reformulations of the $C^1$ Hermite subdivision scheme introduced in [12]. One where we separate computation of values and derivatives and one based of refinement of a control polygon. We show that the latter leads to a subdivision matrix which is totally positive. Based on this we give algorithms for constructing subdivision curves that preserve positivity, monotonicity, and convexity.

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## 1 Introduction

Subdivision is a technique for creating a smooth curve or surface out of a sequence of successive refinements of polygons, or grids see [1]. Subdivision has found applications in areas such as geometric design [6],[17], and in computer games and animation [4]. We consider here the two point Hermite scheme, the $HC^1$-algorithm, introduced in [12]. We start with values and derivatives at the endpoint of an interval and then compute values and derivatives at the midpoint. Repeating this on each subinterval we obtain in the limit a function with a certain smoothness. The scheme depends on two parameters $\alpha$ and $\beta$ and it has been shown that the limit function is $C^1$ for a range $C$ of these parameters. For more references to Hermite subdivision see [5, 11, 13, 14].

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The strong locality of the $HC^1$-algorithm was used in [14] to construct subdivision curves with shape constraints like positivity, monotonicity, and convexity. A notion of control points, control coefficients and a Bernstein basis for two subfamilies of the $HC^1$-interpolant were introduced in [16].

In this paper we continue the study of subdivision with shape constraints initiated in [14, 16]. Before detailing our results let us first describe the shape preserving subdivision process and give an example. Suppose we have values $y_1, \ldots, y_n$ and derivatives $y'_1, \ldots, y'_n$ at some abscissae $t_1 < t_2 < \cdots < t_n$. With each subinterval $[t_i, t_{i+1}]$ we associate parameters $(\alpha_i, \beta_i) \in C$ chosen so that the $HC^1$-interpolant using data $(y_i, y'_i, y_{i+1}, y'_{i+1})$ has the required shape on $[t_i, t_{i+1}]$. We then obtain a $C^1$-function on $[t_1, t_n]$. As an illustration consider the function in Figure 1.

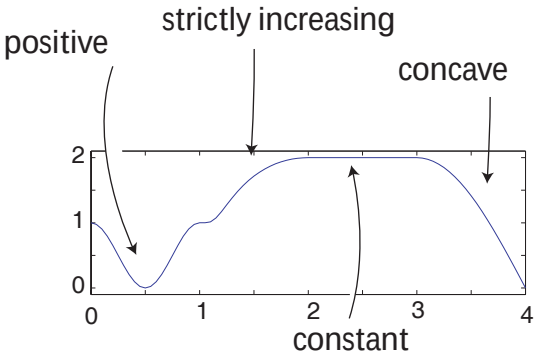


Figure 1: A given function.

This function is defined on the interval $[0, 4]$. It is positive on $[0, 1]$, strictly increasing on $[1, 2]$, constant on $[2, 3]$ and concave on $[3, 4]$. Suppose we want to use subdivision to construct a $C^1$-approximation to this function with the same shape characteristics and that all we know about the function are the function values $y_1, \ldots, y_n$ at some points $t_1 < \cdots < t_n$. We can achieve this with the $HC^1$-algorithm using only crude estimates for the derivatives $y'_1, \ldots, y'_n$ as long as the transition points $1, 2, 3$ are among the abscissae and the chosen derivatives are consistent with the required shapes. See Section 6 for details. For classical curve based shape preserving algorithms we refer to [7, 9, 10] and references therein.

Our paper can be detailed as follows. In Section 2, we recall the $HC^1$-algorithm and some properties which were proved in [14]. We give a new formulation of the $HC^1$-algorithm were we separate the computation of function values and derivatives. This formulation is useful for proving shape preserving properties and shows why the one parameter family given by $\alpha = \beta/(4(1-\beta))$ and $\beta \in [-1, 0)$ considered in [14, 16] really is an extension of the quadratic spline case. We will refer to this family as the *EQS-case*. We also give a new domain $C$ for $C^1$-convergence of the algorithm. In Section 3 we use control points to reformulate the $HC^1$ algorithm as

a stationary subdivision algorithm called $SC^1$. The control points depend on a third parameter $\lambda \geq 2$ and we show convergence of the stationary subdivision algorithm for $(\alpha, \beta) \in C$ and $\lambda \geq 2$. The $SC^1$ algorithm can also be used in the parametric case, but a discussion of this will be deferred to a future paper. Starting in Section 4, we restrict our attention to the EQS-case. By formulating the $SC^1$-algorithm as a corner cutting scheme we show that the subdivision matrix $\mathbf{S}$ is totally positive. We show this for an extended range of $\beta$ and $\lambda$ and also prove the total positivity of the $HC^1$-Bernstein basis. With this last property, the interpolant inherits shape properties of the control polygon such as nonnegativity, monotonicity or convexity. In Section 5, we give algorithms for interpolation with any of the previous shape constrains. An example based on Figure 1 is given in Section 6.

## 2 The $HC^1$ Algorithm

We recall the univariate version of the Hermite subdivision scheme for $C^1$ interpolation, given by Merrien [12] which we call here $HC^1$. We start with values $(f(a), p(a))$ and $(f(b), p(b))$ of a function $f$ and of its first derivative $p = f'$ at the endpoints $a, b$ of a bounded interval $I := [a, b]$ of $\mathbb{R}$. To build $f$ and $p$ on $I$, we proceed recursively. At step $n$ ($n \geq 0$), let us denote by $\mathcal{P}_n$ the regular partition of $I$ in $2^n$ subintervals and let us write $h_n := (b-a)/2^n$. If $c$ and $d$ are two consecutive points of $\mathcal{P}_n$, then we compute $f$ and $p$ at the midpoint $(c+d)/2$ according to the following scheme, which depends on two parameters $\alpha$ and $\beta$

$$\begin{aligned} f(\frac{c+d}{2}) :=& \frac{f(d)+f(c)}{2} + \alpha h_n[p(d)-p(c)], \\ p(\frac{c+d}{2}) :=& (1-\beta)\frac{f(d)-f(c)}{h_n} + \beta\frac{p(d)+p(c)}{2}. \end{aligned} \tag{1}$$

By applying these formulae on ever finer partitions, we define $f$ and $p$ on $\mathcal{P} = \cup\mathcal{P}_n$ which is a dense subset of $I$. We say that the scheme is $C^1$-*convergent* if, for any initial data, $f$ and $p$ can be extended from $\mathcal{P}$ to continuous functions on $I$ with $p = f'$. We call $f$ defined either on $I$ or on $\mathcal{P}$ the $HC^1$-*interpolant* to the data.

The $HC^1$-algorithm can also be formulated as follows. We start with Hermite data $f_0$, $p_0$, $f_1$, $p_1$ at the endpoints of a finite interval $[a, b]$ and set $f_0^0 = f_0$, $p_0^0 = p_0$, $f_1^0 = f_1$, and $p_1^0 = p_1$. For $n = 0, 1, 2, \ldots$, $h_n = 2^{-n}(b-a)$, and $k = 0, 1, \ldots, 2^n - 1$

$$f_{2k}^{n+1} := f_k^n, \quad f_{2k+1}^{n+1} := \frac{f_{k+1}^n + f_k^n}{2} + \alpha h_n\big(p_{k+1}^n - p_k^n\big), \tag{2}$$

$$p_{2k}^{n+1} := p_k^n, \quad p_{2k+1}^{n+1} := (1-\beta)\frac{f_{k+1}^n - f_k^n}{h_n} + \beta\frac{p_{k+1}^n + p_k^n}{2}, \tag{3}$$

and $f_{2^{n+1}}^{n+1} := f_{2^n}^n$, $p_{2^{n+1}}^{n+1} := p_{2^n}^n$. If the scheme is $C^1$-convergent with limit functions $f$ and $p$ then

$$f(t_k^n) = f_k^n, \ f'(t_k^n) = p(t_k^n) = p_k^n, \ t_k^n := a + kh_n, \ k = 0, 1, \ldots, 2^n. \tag{4}$$

## 2.1 The Vector Space of $HC^1$-interpolants

To each choice of $(\alpha, \beta)$ there is a vector space

$$VC^1_{\alpha,\beta}(\mathcal{P}) := \{f : \mathcal{P} \to \mathbb{R} : f, p \text{ computed by } (2) - (4)\}$$

of $HC^1$-interpolants. If the scheme is $C^1$-convergent we define

$$VC^1_{\alpha,\beta}(I) := \{f : I \to \mathbb{R} : f|_{\mathcal{P}} \in VC^1_{\alpha,\beta}(\mathcal{P})\}.$$

The *$HC^1$-Hermite basis functions* $\{\phi_0, \psi_0, \phi_1, \psi_1\}$ are defined by taking as initial data the four unit vectors $e_j = (\delta_{i,j})_{i=1}^4$, respectively. They are always defined on $\mathcal{P}$ and the $HC^1$-interpolant corresponding to initial data $(f_0, p_0, f_1, p_1)$ can be written $f = f_0\phi_0 + p_0\psi_0 + f_1\phi_1 + p_1\psi_1$. Since the Hermite basis functions are clearly linearly independent on $\mathcal{P}$ they form a basis for $VC^1_{\alpha,\beta}(\mathcal{P})$. Thus $VC^1_{\alpha,\beta}(\mathcal{P})$ and $VC^1_{\alpha,\beta}(I)$ are vector spaces of dimension 4.

Let us denote the $HC^1$-interpolant to initial data sampled from a function $g$ by $f = Hg$. By induction it is easy to see that for any $(\alpha, \beta)$ we have $g = Hg$ for all polynomials $g$ of degree at most one, while $g = Hg$ for all quadratic polynomials if and only if $\alpha = -1/8$. We also have $g = Hg$ for all cubic polynomials if and only if $\alpha = -1/8$ and $\beta = -1/2$ and it can be shown that $x^k \neq Hx^k$ for any integer $k \geq 4$. The fact that the scheme reproduces polynomials up to a certain degree can be used to give error bounds, see [14]. Assume $(\alpha, \beta)$ are chosen so that the scheme is $C^1$-convergent. Then there is a constant $C(\alpha, \beta)$ such that for all intervals $I = [a, b]$ and all $g \in C^k(I)$ we have

$$\|g - Hg\|_{L_\infty(I)} \leq C(\alpha, \beta)h^k\|g^{(k)}\|_{L_\infty(I)}, \tag{5}$$

where $h := b - a$ and $k = 2$ for most choices of $\alpha$ and $\beta$.

Notice some important choices of $(\alpha, \beta)$:

1. If $\alpha = -1/8$, $\beta = -1/2$, then $f$ is the cubic polynomial known as the Hermite cubic interpolant. For these parameters (5) holds with $k = 4$ and $C(\alpha, \beta) = 1/384$.

2. If $\alpha = -1/8$, $\beta = -1$, then $f$ is the Hermite quadratic interpolant, i.e. the quadratic $C^1$ spline interpolant with one knot at the midpoint of the initial interval. In this case (5) holds with $k = 3$ and $C(\alpha, \beta) = 1/96$, see [14].

3. The EQS-case $\alpha = \frac{\beta}{4(1-\beta)}$ with $\beta \in [-1, 0)$ is a one parameter extension of the quadratic spline case. It was introduced and studied in [14]. In this case (5) only holds with $k = 2$ and $C(\alpha, \beta) \leq 1/48$ unless $\beta = -1$, but as we will see this scheme has important shape preserving properties.

## 2.2 Direct computation of the function or the derivative

We can reformulate (3) so that only values of $p$ are involved and similarly (2) with $f$.

**Proposition 1** *For $\alpha, \beta \in \mathbb{R}$ and $n \in \mathbb{N}$, the function $f$ and the derivative $p$ of the $HC^1$-interpolant satisfy the following relations:*
*For $n = 1, 2, \ldots, \quad i = 0, 1, \ldots, 2^{n-1} - 1$,*

$$\begin{aligned} p_{4i+1}^{n+1} =& \mu p_{2i}^n + (1 + \frac{\beta}{2}) p_{2i+1}^n - \nu p_{2i+2}^n, \\ p_{4i+3}^{n+1} =& -\nu p_{2i}^n + (1 + \frac{\beta}{2}) p_{2i+1}^n + \mu p_{2i+2}^n, \end{aligned} \tag{6}$$

*and for $n = 1, 2, \ldots, \quad i = 0, 1, \ldots, 2^{n-2} - 1$,*

$$\begin{aligned} 4f_{8i+1}^{n+1} =& (1+\mu) f_{4i}^n + 2(2-\mu) f_{4i+1}^n + (\mu + \nu - 1) f_{4i+2}^n - 2\nu f_{4i+3}^n + \nu f_{4i+4}^n \\ 4f_{8i+3}^{n+1} =& -\mu f_{4i}^n + 2(1+\mu) f_{4i+1}^n + (2 - \mu - \nu) f_{4i+2}^n + 2\nu f_{4i+3}^n - \nu f_{4i+4}^n \\ 4f_{8i+5}^{n+1} =& -\nu f_{4i}^n + 2\nu f_{4i+1}^n + (2 - \mu - \nu) f_{4i+2}^n + 2(1+\mu) f_{4i+3}^n - \mu f_{4i+4}^n \\ 4f_{8i+7}^{n+1} =& \nu f_{4i}^n - 2\nu f_{4i+1}^n + (\mu + \nu - 1) f_{4i+2}^n + 2(2-\mu) f_{4i+3}^n + (1+\mu) f_{4i+4}^n \end{aligned} \tag{7}$$

*where $\mu := -2\alpha(1-\beta)$ and $\nu = \mu + \beta/2$.*

**Proof:** We will use the notation $\Delta p_k^n = p_{k+1}^n - p_k^n$, $\Delta f_k^n = f_{k+1}^n - f_k^n$ and $\Delta^2 f_k^n = \Delta(\Delta f_k^n) = f_{k+2}^n - 2f_{k+1}^n + f_k^n$.

Let us start by proving (6). Using (3) with $k = 2i$ and $k = 2i + 1$

$$\begin{aligned} p_{4i+1}^{n+1} &= (1-\beta)\frac{\Delta f_{2i}^n}{h_n} + \frac{\beta}{2}\big(p_{2i+1}^n + p_{2i}^n\big) \\ p_{4i+3}^{n+1} &= (1-\beta)\frac{\Delta f_{2i+1}^n}{h_n} + \frac{\beta}{2}\big(p_{2i+2}^n + p_{2i+1}^n\big). \end{aligned} \tag{8}$$

¿From (2) we obtain

$$\begin{aligned} \frac{\Delta f_{2i}^n}{h_n} &= \frac{\Delta f_i^{n-1}}{h_{n-1}} + 2\alpha \Delta p_i^{n-1} \\ \frac{\Delta f_{2i+1}^n}{h_n} &= \frac{\Delta f_i^{n-1}}{h_{n-1}} - 2\alpha \Delta p_i^{n-1}, \end{aligned} \tag{9}$$

The $f$ difference on the right can be eliminated by a reordering of (3) with $k = i$ and $n \to n - 1$

$$(1-\beta)\frac{\Delta f_i^{n-1}}{h_{n-1}} = p_{2i+1}^n - \frac{\beta}{2}\big(p_{i+1}^{n-1} + p_i^{n-1}\big). \tag{10}$$

Combining (8)-(10), we find

$$\begin{aligned} p_{4i+1}^{n+1} &= p_{2i+1}^n + \frac{\beta}{2}\big(p_{2i+1}^n - p_{i+1}^{n-1}\big) - \mu \Delta p_i^{n-1} \\ p_{4i+3}^{n+1} &= p_{2i+1}^n + \frac{\beta}{2}\big(p_{2i+1}^n - p_i^{n-1}\big) + \mu \Delta p_i^{n-1} \end{aligned}$$

and we obtain (6).

Now for (7), with $n \to n-1$ in (6), we obtain:

$$\begin{aligned}
\Delta p_{4i}^{n} &= (1-\mu)\Delta p_{2i}^{n-1} - \nu \Delta p_{2i+1}^{n-1} \\
\Delta p_{4i+1}^{n} &= \mu \Delta p_{2i}^{n-1} + \nu \Delta p_{2i+1}^{n-1} \\
\Delta p_{4i+2}^{n} &= \nu \Delta p_{2i}^{n-1} + \mu \Delta p_{2i+1}^{n-1} \\
\Delta p_{4i+3}^{n} &= -\nu \Delta p_{2i}^{n-1} + (1-\mu)\Delta p_{2i+1}^{n-1}.
\end{aligned} \tag{11}$$

Notice that with (2) , we get $\Delta^2 f_{2k}^{n+1} = -2\alpha h_n \Delta p_k^n$ since $f_{2k}^{n+1} = f_k^n$ and (11) can be written

$$\begin{aligned}
2\Delta^2 f_{8i}^{n+1} &= (1-\mu)\Delta^2 f_{4i}^{n} - \nu \Delta^2 f_{4i+2}^{n} \\
2\Delta^2 f_{8i+2}^{n+1} &= \mu \Delta^2 f_{4i}^{n} + \nu \Delta^2 f_{4i+2}^{n} \\
2\Delta^2 f_{8i+4}^{n+1} &= \nu \Delta^2 f_{4i}^{n} + \mu \Delta^2 f_{4i+2}^{n} \\
2\Delta^2 f_{8i+6}^{n+1} &= -\nu \Delta^2 f_{4i}^{n} + (1-\mu)\Delta^2 f_{4i+2}^{n}.
\end{aligned} \tag{12}$$

It remains to extract the values $f_{8i+j}^{n+1}, j = 1, 3, 5, 7$ from the previous formula using again $f_{2k}^{n+1} = f_k^n$ to obtain (7) $\square$

The previous formulae (6)-(7) can also be completed, then written in a vectorial way. For $n \geq 1$

$$\begin{bmatrix} p_{4i}^{n+1} \\ p_{4i+1}^{n+1} \\ p_{4i+2}^{n+1} \\ p_{4i+3}^{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \mu & 1+\beta/2 & -\nu \\ 0 & 1 & 0 \\ -\nu & 1+\beta/2 & \mu \end{bmatrix} \begin{bmatrix} p_{2i}^{n} \\ p_{2i+1}^{n} \\ p_{2i+2}^{n} \end{bmatrix}, \quad i = 0, 1, \ldots 2^{n-1} \tag{13}$$

and for $n \geq 2$

$$\begin{bmatrix} f_{8i}^{n+1} \\ f_{8i+1}^{n+1} \\ f_{8i+2}^{n+1} \\ f_{8i+3}^{n+1} \\ f_{8i+4}^{n+1} \\ f_{8i+5}^{n+1} \\ f_{8i+6}^{n+1} \\ f_{8i+7}^{n+1} \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 4 & 0 & 0 & 0 & 0 \\ 1+\mu & 2(2-\mu) & \mu+\nu-1 & -2\nu & \nu \\ 0 & 4 & 0 & 0 & 0 \\ -\mu & 2(1+\mu) & 2-\mu-\nu & 2\nu & -\nu \\ 0 & 0 & 4 & 0 & 0 \\ -\nu & 2\nu & 2-\mu-\nu & 2(1+\mu) & -\mu \\ 0 & 0 & 0 & 4 & 0 \\ \nu & -2\nu & \mu+\nu-1 & 2(2-\mu) & 1+\mu \end{bmatrix} \begin{bmatrix} f_{4i}^{n} \\ f_{4i+1}^{n} \\ f_{4i+2}^{n} \\ f_{4i+3}^{n} \\ f_{4i+4}^{n} \end{bmatrix}, \quad i = 0, 1, \ldots 2^{n-2} \tag{14}$$

¿From (6) it follows that the new $p$-values on level $n+1$ ($n \geq 1$) can be formed by an affine combination of three $p$ values on the previous level $n$. In the EQS-case we only need the two neighboring values. Moreover the derivatives will be sampled from a piecewise linear curve.

**Corollary 2** *In the EQS-case* $\alpha = \frac{\beta}{4(1-\beta)}$ *we have*

$$\begin{aligned} p_{4i+1}^{n+1} &= -\frac{\beta}{2} p_{2i}^n + (1+\frac{\beta}{2}) p_{2i+1}^n, \\ p_{4i+3}^{n+1} &= (1+\frac{\beta}{2}) p_{2i+1}^n - \frac{\beta}{2} p_{2i+2}^n. \end{aligned} \tag{15}$$

*and*

$$\begin{aligned} 4f_{8i+1}^{n+1} =& (1-\frac{\beta}{2}) f_{4i}^n + (4+\beta) f_{4i+1}^n - (1+\frac{\beta}{2}) f_{4i+2}^n \\ 4f_{8i+3}^{n+1} =& \frac{\beta}{2} f_{4i}^n + (2-\beta) f_{4i+1}^n + (2+\frac{\beta}{2}) f_{4i+2}^n \\ 4f_{8i+5}^{n+1} =& (2+\frac{\beta}{2}) f_{4i+2}^n + (2-\beta) f_{4i+3}^n + \frac{\beta}{2} f_{4i+4}^n \\ 4f_{8i+7}^{n+1} =& -(1+\frac{\beta}{2}) f_{4i+2}^n + (4+\beta) f_{4i+3}^n + (1-\frac{\beta}{2}) f_{4i+4}^n \end{aligned} \tag{16}$$

*If in addition* $\beta \in (-2, 0)$ *then there exist*

$$a = \tau_0^n < \tau_1^n < \cdots < \tau_{2^n}^n = b, \tag{17}$$

*with* $\tau_{2^{n-1}}^n = \frac{a+b}{2}$ *for* $n \geq 1$. *such that*

$$p_i^n = L(\tau_i^n), \quad i = 0, 1, \ldots, 2^n, \quad n = 0, 1, \ldots, \tag{18}$$

*where* $L$ *is the piecewise linear curve connecting the three points* $(a, p(a))$, $(\frac{a+b}{2}, p(\frac{a+b}{2}))$, $(b, p(b))$.

**Proof:** If $\alpha = \frac{\beta}{4(1-\beta)}$ then $\mu = -\beta/2$ and (15) follows from (6). Similarly, we obtain (16).

We claim that (18) holds with

$$\begin{aligned} \tau_{4i}^{n+1} = \tau_{2i}^n, \quad \tau_{4i+1}^{n+1} &= -\frac{\beta}{2} \tau_{2i}^n + (1+\frac{\beta}{2}) \tau_{2i+1}^n, \\ \tau_{4i+2}^{n+1} = \tau_{2i+1}^n, \quad \tau_{4i+3}^{n+1} &= -\frac{\beta}{2} \tau_{2i+2}^n + (1+\frac{\beta}{2}) \tau_{2i+1}^n. \end{aligned} \tag{19}$$

Since $p_0^n = p(a)$ and $p_{2^n}^n = p(b)$, we have $\tau_0^n = a$ and $\tau_{2^n}^n = b$ for all $n \geq 0$. Moreover, since $p_{2^{n-1}}^n = p(\frac{a+b}{2})$, we see that $\tau_{2^{n-1}}^n = \frac{a+b}{2}$ for all $n \geq 1$. Thus (17) will follow from (19) since the latter involves convex combinations for $\beta \in (-2, 0)$. (19) follows from (15) by induction. Suppose (18) holds for some $n$. Since $L$ is linear on the actual segment we obtain

$$p_{4i+1}^{n+1} = -\frac{\beta}{2} L(\tau_{2i}^n) + (1+\frac{\beta}{2}) L(\tau_{2i+1}^n) = L(\tau_{4i+1}^{n+1}),$$

where $\tau_{4i+1}^{n+1}$ is given by (19). The proof of the other $\tau$-relation is similar. □

## 2.3 $C^1$-convergence

To study convergence we observe that it is enough to consider the interval $[0,1]$. Indeed, if $I := [a,b]$ and $h := b-a$, defining the initial data $g(u) = f(a+hu), g'(u) = hf'(a+hu)$, for $u \in \{0,1\}$, the construction of $f$ on $[a,b]$ or $g$ on $[0,1]$ by (1) are equivalent and at step $n$, $g(u) = f(a+uh)$ and $g'(u) = hf'(a+hu)$ for $u \in \{0, 1/2^n, \ldots, \ell/2^n, \ldots, 1\}$.

In [13] it was shown that if there exist positive constants $c, \rho$ with $\rho < 1$ such that for each integer $n \geq 0$ we have $|\Delta p_i^n| \leq c\rho^n$ for $i = 0, 1, \ldots, 2^n - 1$, where

$$\Delta p_i^n := p\big(\frac{i+1}{2^n}\big) - p\big(\frac{i}{2^n}\big), \quad i = 0, 1, \ldots, 2^n - 1, \tag{20}$$

then $p$ has a unique continuous extension to $I$. Moreover, there is a positive constant $c_1$ such that for all $(x,y) \in [0,1]^2$

$$|p(x) - p(y)| \leq c_1 |x-y|^{-\log_2 \rho},$$

i.e. $p$ is Hölder continuous with exponent $-\log_2 \rho$.

Suppose $p$ is continuous and $\lim\limits_{n\to\infty} \max\limits_{0\leq i<2^n-1} |\Delta(f,p)_i^n| = 0$, where

$$\Delta(f,p)_i^n := 2^n \Delta f_i^n - \sigma p_i^n, \quad \sigma p_i^n := \frac{1}{2}\big(p\big(\frac{i+1}{2^n}\big) + p\big(\frac{i}{2^n}\big)\big), \tag{21}$$

and $\Delta f_i^n = f\big(\frac{i+1}{2^n}\big) - f\big(\frac{i}{2^n}\big)$. Then ([13]) $f$ has a unique continuous extension to $I := [0,1]$. Moreover $f \in C^1([0,1])$ with $f' = p$. From this discussion we have the following lemma.

**Lemma 3** *Let $U_i^n := [\Delta p_i^n, \Delta(f,p)_i^n]^T$ for $i = 0, 1, \ldots, 2^n$ and $n = 0, 1, 2, \ldots$. If we can find a vector norm $\|\cdot\|$ on $\mathbb{R}^2$ and positive constants $c, \rho$ with $\rho < 1$ such that*

$$\|U_i^n\| \leq c\rho^n, \ i = 0, 1, \ldots, 2^n \text{ and } n = 0, 1, \ldots$$

*then the $HC^1$-algorithm is $C^1$-convergent and $f' = p$ is Hölder continuous with exponent $-\log_2 \rho$.*

We can now show

**Proposition 4** *Algorithm $HC^1$ is $C^1$-convergent for $(\alpha, \beta) \in [-1/8, 0) \times [-2, 1)$.*

**Proof:** An immediate evaluation gives

$$U_{2i}^{n+1} = \Lambda_1 U_i^n \text{ and } U_{2i+1}^{n+1} = \Lambda_{-1} U_i^n \text{ for } i = 0, 1, \ldots, 2^n, n = 0, 1, \ldots,$$

where

$$\Lambda_\epsilon = \begin{bmatrix} \dfrac{1}{2} & \epsilon(1-\beta) \\ \epsilon\dfrac{8\alpha+1}{4} & \dfrac{1+\beta}{2} \end{bmatrix}, \quad \varepsilon = \pm 1. \tag{22}$$

Since the off-diagonal elements of $\Lambda_\varepsilon$ have the same sign for $\alpha \geq -1/8$ and $\beta \leq 1$, we can define a vector norm by $\|v\| := \|P^{-1}v\|_2$, where $\|\cdot\|_2$ is the usual Euclidian norm of a vector and $P$ is a suitable matrix. If $P := \begin{bmatrix} 2\sqrt{1-\beta} & 0 \\ 0 & \sqrt{8\alpha+1} \end{bmatrix}$ then $P^{-1}\Lambda_\varepsilon P$ is symmetric. The corresponding matrix operator norm is given by $\|\Lambda_\varepsilon\| := \|P^{-1}\Lambda_\varepsilon P\|_2$, where $\|A\|_2 := \sqrt{\rho(A^TA)}$ is the spectral norm of a matrix $A$. The eigenvalues of $\Lambda_\varepsilon$ or of $P^{-1}\Lambda_\varepsilon P$ are

$$\lambda_1 = \frac{1}{4}\big(2+\beta+\sqrt{(2-\beta)^2+32\alpha(1-\beta)}\big), \quad \lambda_2 = \frac{1}{4}\big(2+\beta-\sqrt{(2-\beta)^2+32\alpha(1-\beta)}\big)$$

Since $P^{-1}\Lambda_\varepsilon P$ is symmetric the eigenvalues are real with $\lambda_2 < \lambda_1$. Now for $\beta \in [-2,1)$ and $\alpha \in [-1/8, 0)$ we find $\lambda_1 < (2+\beta+\sqrt{(2-\beta)^2})/4 = 1$ and $\lambda_2 > (2+\beta-\sqrt{(2-\beta)^2})/4 = \beta/2 \geq -1$. Thus $\rho := \|\Lambda_\varepsilon\| = \max\{|\lambda_1|, |\lambda_2|\} < 1$ for $\varepsilon = \pm 1$ and we have shown that $\max\{\|U_{2i}^{n+1}\|, \|U_{2i+1}^{n+1}\|\} \leq \rho\|U_i^n\|$ or $\|U_i^n\| \leq \rho^n\|U_0^0\|$ for $i = 0, 1, \ldots, 2^n-1$ and $n = 0, 1, 2, \cdots$. The $C^1$-convergence for $(\alpha, \beta) \in [-1/8, 0)\times[-2, 1)$ now follows from Lemma 3 $\square$

By Proposition 4, the $HC^1$-algorithm converges for $\beta \in [-1, 0)$ if $\alpha = \frac{\beta}{4(1-\beta)}$. We can now extend this result.

**Proposition 5** *If $\alpha = \frac{\beta}{4(1-\beta)}$ then the $HC^1$-algorithm is $C^1$-convergent for $\beta \in (-2, 0)$.*

**Proof:** For $\varepsilon = \pm 1$ the matrices $\Lambda_\varepsilon$ in (22) take the form : $\Lambda_\varepsilon = \begin{pmatrix} \frac{1}{2} & \varepsilon(1-\beta) \\ \varepsilon\frac{\beta+1}{4(1-\beta)} & \frac{1+\beta}{2} \end{pmatrix}$.

Now, for any positive real number $\theta$, we define the norm $\|\cdot\|_\theta$ on $\mathbb{R}^2$ by $\|(x,y)\|_\theta = |x|+\theta|y|$. It is easy to prove that for any matrix $M = (m_{ij}) \in \mathbb{R}^{2\times 2}$, the corresponding matrix operator norm is given by $\|M\|_\theta := \max(|m_{11}|+\theta|m_{21}|, \frac{|m_{12}|}{\theta}+|m_{22}|)$. Choosing $\theta = 2(1-\beta)$ we find $\|\Lambda_1\|_\theta = \|\Lambda_{-1}\|_\theta = 1/2(1+|1+\beta|)$, which is stricly less than one for $-2 < \beta < 0$. Lemma 3 now gives the convergence.

$\square$

We define the convergence region $C$ by

$$C := \big\{(\alpha, \beta) : \text{ the scheme } HC^1 \text{ is } C^1 \text{ - convergent}\big\}. \tag{23}$$

We have shown that $[-1/8, 0) \times [-2, 1) \subset C$ and also that $\{(\frac{\beta}{4(1-\beta)}, \beta) : -2 < \beta < 0\} \subset C$.

The function $f' = p$ is Hölder continuous with exponent $-\log_2 \rho$. In the case where $\alpha = \frac{\beta}{4(1-\beta)}$ we have $\|\Lambda_1\|_\theta = \|\Lambda_{-1}\|_\theta = \rho = \rho(\beta) = 1/2(1+|1+\beta|)$which is piecewise linear with a minimum for $\beta = -1$ and we obtain the best regularity of the interpolant for $\beta = -1$ when $f$ is a quadratic spline.

To illustrate the smoothness properties of a $HC^1$-interpolant we show the Hermite basis with $\beta = -3/5$ and $\alpha = \frac{\beta}{4(1-\beta)} = -3/32$ in Figure 2. The spectral radius of

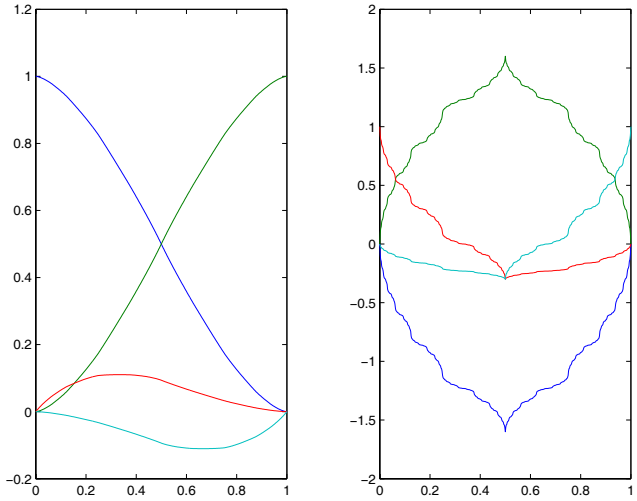


Figure 2: Hermite basis and derivatives, corresponding to $\alpha = -3/32$ and $\beta = -3/5$.

the matrices $\Lambda_\varepsilon$ is $7/10$ and hence the derivatives of the Hermite basis functions are Hölder continuous with exponent $\rho = -\log_2(7/10) \approx 0.5146$.
**Remark**: The data $f(a), p(a), f(b), p(b)$ can either have real values or vector values in $\mathbb{R}^s, s \geq 2$. In this second case, we look for vector continuous functions $f$ and $p$ with $f' = p$ from $I = [a, b]$ to $\mathbb{R}^s$. The $C^1$-convergence is guaranteed for all $(\alpha, \beta)$ in the convergence region $C$ since it suffices to study the convergence independently for each component of $f$ and $p$.

# 3 Control Polygons and Subdivision Algorithm

## 3.1 Control Coefficients and Control Polygons

Suppose we apply the subdivision scheme $HC^1$ to some real valued data $f(a)$, $p(a)$, $f(b)$, $p(b)$. In order to obtain a geometric formulation of the scheme we define *control coefficients* relative to the interval $[a, b]$ by

$$a_0 = f(a), \quad a_1 = f(a) + \frac{h}{\lambda}p(a), \quad a_2 = f(b) - \frac{h}{\lambda}p(b), \quad a_3 = f(b), \tag{24}$$

where $h := b - a$ and $\lambda \geq 2$ is a real number to be chosen. We define the *control points* $(A_0,\ A_1,\ A_2,\ A_3)$ on $[a, b]$ by

$$A_0 = (a, a_0), \quad A_1 = (a + \frac{h}{\lambda}, a_1), \quad A_2 = (b - \frac{h}{\lambda}, a_2), \quad A_3 = (b, a_3), \tag{25}$$

and the *control polygon* $\{A_0,\ A_1,\ A_2,\ A_3\}$ on $[a, b]$ by connecting the four control points by straight line segments. If $f$ is the $HC^1$-interpolant then the parametric

curve $(x, f(x))$ with $x \in [a, b]$ passes through $A_0$ with tangent directions $A_1 - A_0$ and $A_3$ with tangent direction $A_3 - A_2$. See Figure 3.

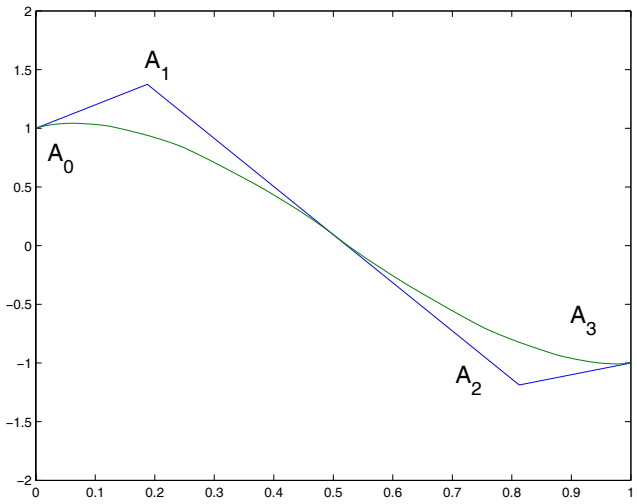


Figure 3: A $HC^1$-interpolant and its control polygon, $\beta = -3/5$, $\alpha = -3/32, \lambda = 16/3$.

We can also apply the subdivision scheme $HC^1$ to vector valued data $f_0, p_0, f_1, p_1$ in $\mathbb{R}^s$ for some $s \geq 2$. We pick an interval $[a, b]$ and use the $HC^1$-algorithm on each component of $f$ and $p$. To obtain a geometric formulation of this process we define control coefficients relative to $[a, b]$ by (24) and we define the control points to be the same as the control coefficients. The computed curve interpolates the first and last control coefficient and its tangent direction at $a_0$ is $a_1 - a_0$, and at $a_3$ the tangent direction is $a_3 - a_2$.

Note that if 4 points $a_0, a_1, a_2, a_3$ in $\mathbb{R}^s$ for $s \geq 1$ are given we can think of these as control coefficients of a $HC^1$-interpolant on some finite interval $[a, b]$ and apply the $HC^1$ algorithm to the data given by

$$f(a) := a_0, \;\; p(a) := \frac{\lambda}{h}\big(a_1 - a_0\big), \;\; f(b) := a_3, \;\; p(b) := \frac{\lambda}{h}\big(a_3 - a_2\big), \tag{26}$$

where $h := b - a$. We now derive a parameter independent formulation of this scheme. In particular suppose $(a_0, a_1, a_2, a_3)$ are points in $\mathbb{R}^s$ for some $s \geq 1$ which are distinct if $s \geq 2$ and let $[a, b]$ be any finite interval.

Using (24) and (26) we can compute new control coefficients $(\bar{a}_0, \bar{a}_1, \bar{a}_2, \bar{a}_3)$ for the interval $I_1$ and new control coefficients $(\bar{a}_3, \bar{a}_4, \bar{a}_5, \bar{a}_6)$ for $I_2$, and then join them into control coefficients $(\bar{a}_0, \bar{a}_1, \bar{a}_2, \bar{a}_3, \bar{a}_4, \bar{a}_5, \bar{a}_6)$ on $[a, b]$. In the following geometric formulation of the subdivision scheme we do this computation directly without picking an underlying interval $[a, b]$. The proposition is a generalization of Theorem 10 of [16]:

**Proposition 6** *Suppose $a_i \in \mathbb{R}^s$ for $i = 0, 1, 2, 3$ and some $s \geq 1$. After one subdivision of the control coefficients $(a_0, a_1, a_2, a_3)$ we obtain new control coefficients $(\bar{a}_0, \bar{a}_1, \bar{a}_2, \bar{a}_3, \bar{a}_4, \bar{a}_5, \bar{a}_6)$ given by*

$$\begin{bmatrix} \bar{a}_0 \\ \bar{a}_1 \\ \bar{a}_2 \\ \bar{a}_3 \\ \bar{a}_4 \\ \bar{a}_5 \\ \bar{a}_6 \end{bmatrix} = \mathbf{S} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} := \frac{1}{4} \begin{bmatrix} 4 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ \gamma & v-\beta & v+\beta & \delta \\ 2-v & v & v & 2-v \\ \delta & v+\beta & v-\beta & \gamma \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}, \tag{27}$$

*where*

$$\begin{aligned} v &= -4\alpha\lambda \\ \gamma &= 2 - v + (2 + \beta(\lambda - 2))/\lambda \\ \delta &= 2 - v - (2 + \beta(\lambda - 2))/\lambda. \end{aligned} \tag{28}$$

*Moreover,*

$$\bar{a}_3 = \frac{1}{2}(\bar{a}_2 + \bar{a}_4). \tag{29}$$

**Proof:** Pick any interval $[a, b]$ and let $h := b - a$. By (24)

$$\bar{a}_0 = f(a),\ \bar{a}_1 = f(a) + \frac{h}{2\lambda}p(a),\ \bar{a}_2 = f(\frac{a+b}{2}) - \frac{h}{2\lambda}p(\frac{a+b}{2}),\ \bar{a}_3 = f(\frac{a+b}{2}),$$
$$\bar{a}_6 = f(b),\ \bar{a}_5 = f(b) - \frac{h}{2\lambda}p(b),\ \bar{a}_4 = f(\frac{a+b}{2}) + \frac{h}{2\lambda}p(\frac{a+b}{2}).$$

¿From (1) and (26) we obtain on an interval $[a, b]$ the inverse relations

$$\begin{aligned} f(a) = a_0, \quad p(a) &= \frac{\lambda}{h}(a_1 - a_0) \\ f(b) = a_3, \quad p(b) &= \frac{\lambda}{h}(a_3 - a_2) \\ f(\frac{a+b}{2}) &= \frac{a_0 + a_3}{2} - \frac{v}{4}(a_3 - a_2 - a_1 + a_0) \\ \frac{h}{2\lambda}p(\frac{a+b}{2}) &= \frac{1-\beta}{2\lambda}(a_3 - a_0) + \frac{\beta}{4}(a_3 - a_2 + a_1 - a_0) \\ &= \frac{2 + \beta(\lambda - 2)}{\lambda}(a_3 - a_0) + \frac{\beta}{4}(a_1 - a_2). \end{aligned} \tag{30}$$

But then we see that $(\bar{a}_0, \bar{a}_1, \bar{a}_2, \bar{a}_3, \bar{a}_4, \bar{a}_5, \bar{a}_6)^T = \mathbf{S}(a_0, \ldots, a_3)^T$, where $\mathbf{S}$ is the matrix in equation (27). Since the sum of rows three and five in the matrix $\mathbf{S}$ equals twice row four the relation (29) follows. □

For $s \geq 2$ the control coefficients and control points are the same and the proposition also gives rules for subdividing the control polygon. The following corollary holds in general.

**Corollary 7** *Suppose* $(a_0, a_1, a_2, a_3) \in \mathbb{R}^s$ *for some* $s \geq 1$. *After one subdivision of the corresponding control polygon* $\{A_0, A_1, A_2, A_3\}$ *we obtain a new control polygon* $\{\bar{A}_0, \bar{A}_1, \bar{A}_2, \bar{A}_3, \bar{A}_4, \bar{A}_5, \bar{A}_6\}$ *given by*

$$\begin{bmatrix}\bar{A}_0 & \bar{A}_1 & \bar{A}_2 & \bar{A}_3 & \bar{A}_4 & \bar{A}_5 & \bar{A}_6\end{bmatrix}^T = \mathbf{S}\begin{bmatrix}A_0 & A_1 & A_2 & A_3\end{bmatrix}^T \tag{31}$$

*where* $\mathbf{S}$ *is given by* (27). *Moreover*

$$\bar{A}_3 = \frac{1}{2}(\bar{A}_2 + \bar{A}_4), \tag{32}$$

*which means that these control points always lie on a straight line.*

**Proof:** This has already been shown for $s \geq 2$ and for the control coefficients for $s = 1$. For the control point abscissas we obtain the relation $(a, a + h/(2\lambda), \bar{a} - h/(2\lambda), \bar{a}, \bar{a} + h/(2\lambda), b - h/(2\lambda), d)^T = \mathbf{S}(a, a + h/\lambda, b - h/\lambda, d)^T$, where $\bar{a} = (a+b)/2$, since the scheme $HC^1$ reproduces linear functions. Thus (31) and (32) also holds for $s = 1$. $\square$

## 3.2 A Stationary Subdivision Algorithm

By applying (27), we can reformulate the Hermite subdivision scheme $HC^1$ as a stationary subdivision scheme working on points in $\mathbb{R}^s$.

Starting with 4 points $a_0, a_1, a_2, a_3$ in $\mathbb{R}^s$, $s \geq 1$, $(\alpha, \beta)$ in the convergence region $C$, and $\lambda \geq 2$, we define Algorithm $SC^1$ as follows.

At step $n = 0$, we set $a_0^0 = a_0, a_1^0 = a_1, a_2^0 = a_2, a_3^0 = a_3$.

At step $n + 1, n \geq 0$, we define

$$\begin{bmatrix} a_{6i}^{n+1} \\ a_{6i+1}^{n+1} \\ a_{6i+2}^{n+1} \\ a_{6i+3}^{n+1} \\ a_{6i+4}^{n+1} \\ a_{6i+5}^{n+1} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ \gamma & v-\beta & v+\beta & \delta \\ 2-v & v & v & 2-v \\ \delta & v+\beta & v-\beta & \gamma \\ 0 & 0 & 2 & 2 \end{bmatrix}\begin{bmatrix} a_{3i}^n \\ a_{3i+1}^n \\ a_{3i+2}^n \\ a_{3i+3}^n \end{bmatrix}, \quad i = 0, 1, \ldots 2^n - 1 \tag{33}$$

and $a_{3.2^{n+1}}^{n+1} = a_{3.2^n}^n$. Here $v, \gamma, \delta$ are given by (28). The matrix $(s_{\ell,k})_{\ell=0,\ldots,5,k=0,\ldots,3}$ in (33) is formed from the first 6 rows of $\mathbf{S}$ given by (27).

**Lemma 8** *For all* $n \geq 1$ *and for all* $i = 1, \ldots, 2^n - 1$, *we have*

$$\begin{aligned} a_{6i}^{n+1} &= a_{3i}^n, \ i = 1, \ldots, 2^{n-1} \\ a_{6i+1}^{n+1} - a_{6i}^{n+1} &= \frac{1}{2}\big(a_{3i+1}^n - a_{3i}^n\big), \ i = 1, \ldots, 2^{n-1} - 1 \\ a_{3i+1}^n + a_{3i-1}^n &= 2a_{3i}^n, \ i = 1, \ldots, 2^n - 1 \end{aligned} \tag{34}$$

**Proof:** The first two equations follows immediately from (33). As in the proof of (29) it is clear that

$$a_{6i+2}^{n+1} + a_{6i+4}^{n+1} = 2a_{6i+3}^{n+1}, \quad i = 0, \ldots, 2^n - 1, \quad n = 0, 1, \ldots,$$

and in particular $a_2^1 + a_4^1 = 2a_3^1$. By (34) and induction on $n$

$$a_{6i+1}^{n+1} + a_{6i-1}^{n+1} = \frac{1}{2}(a_{3i}^n + a_{3i+1}^n) + \frac{1}{2}(a_{3i-1}^n + a_{3i}^n) = 2a_{3i}^n = 2a_{6i}^{n+1}.$$

□

If we define $a_i^0$ for $i < 0$ and $i > 3$ in any way, the subdivision scheme can be written $a_\ell^{n+1} = \sum_{k\in\mathbb{Z}} \sigma_{\ell,k} a_k^n, \ell \in \mathbb{Z}$ where $\sigma_{6i+\ell,3i+k} = s_{\ell,k}$ for $i \in \mathbb{Z}, \ell = 0, \ldots, 5, k = 0, \ldots, 3$ and $\sigma_{i,j} = 0$ otherwise. With the definitions recalled in [2], the scheme is *local* since $\sigma_{\ell,k} = 0$ for $|\ell - 2k| > 4$. Since $\sum_{k\in\mathbb{Z}} \sigma_{\ell,k} = 1$, it is *affine* but it is not interpolating in a classical sense since we generally have $a_{6i+2}^{n+1} \neq a_{3i+1}^n$.

## 3.3 Convergence of $SC^1$

The convergence of the subdivision schemes are usually established by studying the difference sequence. Alternatively convergence follows since $SC^1$ was derived from $HC^1$. Here are the details.

**Theorem 9** *Let $s \geq 1$ and $a_0, a_1, a_2, a_3$ be 4 points in $\mathbb{R}^s$. Suppose $\lambda \geq 2$ and that $(\alpha, \beta)$ is in the convergence region $C$ given by (23). We build the sequence of points $\{ a_i^n\}_{n\in\mathbb{N}, i=0,\ldots,3.2^n}$ by (33). Choose any interval $I := [a, b]$ with $h := b - a > 0$ and define $t_i^n := a + ih_n$, where $h_n := h2^{-n}$ for $n \in \mathbb{N}$ and $i = 0, \ldots, 2^n$. Then, there exists a $C^1$ function $f : I \to \mathbb{R}^s$ such that for all $n \in \mathbb{N}$:*

$$\begin{aligned} a_{3i}^n &= f(t_i^n), \quad i = 0, \ldots, 2^n, \\ a_{3i+1}^n - a_{3i}^n &= \frac{h_n}{\lambda} f'(t_i^n), \quad i = 0, \ldots, 2^n - 1, \\ a_{3i}^n - a_{3i-1}^n &= \frac{h_n}{\lambda} f'(t_i^n), \quad i = 1, \ldots, 2^n. \end{aligned}$$

*For $s \geq 2$, let $A_i^n = a_i^n$, $i = 0, 1, \ldots, 3\times 2^n$ and for $s = 1$, let $A_{3i}^n = (t_i^n, a_{3i}^n)$,$A_{3i+1}^n = (t_i^n + \frac{h_n}{\lambda}, a_{3i+1}^n)$, $A_{3i+2}^n = (t_{i+1}^n - \frac{h_n}{\lambda}, a_{3i+2}^n)$, $i = 0, 1, \ldots, 2^n - 1$, and $A_{3\times 2^n}^n = (b, a_{3\times 2^n}^n)$.*

*Then the sequence of polygons $\{A_0^n, \ldots, A_{3\times 2^n}^n\}$ converges to the curve $\{f(t), t \in I\}$.*

**Proof:** We will show that the scheme $SC^1$ generates sequences $\{f^n\}$ and $\{p^n\}$ of piecewise linear vector functions which interpolate values and derivatives at the points of $\mathcal{P}_n = \{t_0^n, \ldots, t_{2^n}^n\}$.

We define $f^n$ and $p^n$ to be linear on $[t_i^n, t_{i+1}^n], i = 0, \ldots, 2^n - 1$, and to interpolate the following values

$$\begin{aligned} f^n(t_i^n) = a_{3i}^n, \quad p^n(t_i^n) = \frac{\lambda}{h_n}(a_{3i+1}^n - a_{3i}^n), \quad i = 0, \ldots, 2^n - 1, \\ f^n(b) = a_{3.2^n}^n, \quad p^n(b) = \frac{\lambda}{h_n}(a_{3\times 2^n}^n - a_{3\times 2^n-1}^n). \end{aligned} \tag{35}$$

Since $t_i^n = t_{2i}^{n+1}$ we find from (34) and (35)

$$f^{n+1}(t_i^n) = f^n(t_i^n), \quad p^{n+1}(t_i^n) = p^n(t_i^n), \ i = 0, \ldots, 2^n. \tag{36}$$

Below we prove that, for $i = 0, \ldots, 2^n - 1$,

$$f^{n+1}(t_{2i+1}^{n+1}) = \frac{f^n(t_{i+1}^n) + f^n(t_i^n)}{2} + \alpha h_n(p^n(t_{i+1}^n) - p^n(t_i^n)), \tag{37}$$

$$p^{n+1}(t_{2i+1}^{n+1}) = (1-\beta)\frac{f^n(t_{i+1}^n) - f^n(t_i^n)}{h_n} + \beta\frac{p^n(t_{i+1}^n) + p^n(t_i^n)}{2}. \tag{38}$$

Comparing (36), (37) and (38) with (2)-(3) we conclude that $f^n = f$ and $p^n = p$ on $\mathcal{P}_n$ where $f$ and $p$ are the functions built on $\cup\mathcal{P}_n$ by $HC^1$ defined by (2)-(4) from the initial data $f(a) = a_0$, $p(a) = \frac{\lambda}{h}(a_1 - a_0)$, $f(b) = a_3$ and $p(b) = \frac{\lambda}{h}(a_3 - a_2)$, and then extended to $[a, b]$. So that, if $(\alpha, \beta) \in C$, then the sequences $f^n$ and $p^n$ defined from $SC^1$ by (35) converge uniformly to continuous vector functions $f$ and $p$ defined on $[a, b]$. Moreover $f \in C^1([a, b])$ and $f' = p$.

Now since $f'$ is bounded and $a_{3i+1}^n - a_{3i}^n = \frac{h}{\lambda 2^n} f'(t_i^n), i = 0, \ldots, 2^n - 1$, we deduce that $a_{3i+1}^n - a_{3i}^n$ tends uniformly to 0. We conclude that the sequence of polygons $\{A_0, \ldots, A_{3\times 2^n}\}$ tends to the curve $\{f(t), t \in I\}$ since $a_{3i}^n = f(t_i^n)$ for $i = 0, \ldots, 2^n$.

It remains to prove (37) and (38). Since $\alpha = -v/4\lambda$, for $i = 0, \ldots, 2^n - 1$ and using (35) and (33),

$$\begin{aligned} &\frac{1}{2}(f^n(t_{i+1}^n) + f^n(t_i^n)) + \alpha h_n(p^n(t_{i+1}^n) - p^n(t_i^n)) \\ &\quad = \frac{1}{2}(a_{3i+3}^n + a_{3i}^n) - \frac{v}{4}(a_{3i+3}^n - a_{3i+2}^n - a_{3i+1}^n + a_{3i}^n) \\ &\quad = a_{6i+3}^{n+1} = f^{n+1}(t_{2i+1}^{n+1}) \end{aligned}$$

so that (37) is proved.

Similarly, for (38), let $i \in \{0, \dots, 2^n - 1\}$. With the definitions of $\gamma$ and $\delta$ in (28) we find

$$\begin{aligned}
&\frac{1-\beta}{h_n}(f^n(t^n_{i+1}) - f^n(t^n_i)) + \frac{\beta}{2}(p^n(t^n_{i+1}) + p^n(t^n_i)) \\
&\qquad = \frac{1-\beta}{h_n}(a^n_{3i+3} - a^n_{3i}) + \frac{\beta\lambda}{2h_n}(a^n_{3i+3} - a^n_{3i+2} + a^n_{3i+1} - a^n_{3i}) \\
&\qquad = \frac{\lambda}{h_{n+1}}\big( - (\frac{1-\beta}{2\lambda} + \frac{\beta}{4})a^n_{3i} + \frac{\beta}{4}a^n_{3i+1} - \frac{\beta}{4}a^n_{3i+2} + (\frac{1-\beta}{2\lambda} + \frac{\beta}{4})a^n_{3i+3}\big) \\
&\qquad = \frac{1}{4}\frac{\lambda}{h_{n+1}}\big((\delta - 2 + v)a^n_{3i} + \beta a^n_{3i+1} - \beta a^n_{3i+2} + (\gamma - 2 + v)a^n_{3i+3}\big) \\
&\qquad = \frac{\lambda}{h_{n+1}}\big(a^{n+1}_{6i+4} - a^{n+1}_{6i+3}\big) = p^{n+1}(t^{n+1}_{2i+1}).
\end{aligned}$$

□

# 4 Total positivity and consequences

## 4.1 Corner Cutting and Total Positivity of the Subdivision Matrix

Consider now the subdivision process in the EQS-case when $\alpha = \frac{\beta}{4(1-\beta)}$ with $\beta \in (-2, 0)$. Since $v = -4\alpha\lambda = \frac{\beta}{\beta-1}\lambda$, or $\lambda = \frac{\beta-1}{\beta}v$ we find from (28)

$$\gamma = 2 - v + \frac{2 + \beta\lambda - 2\beta}{\lambda} = 2 - v + \frac{2 + (\beta-1)v - 2\beta}{(\beta-1)v}\beta = \frac{(2-v)(v-\beta)}{v},$$

and similarly

$$\delta = \frac{(2-v)(v+\beta)}{v}.$$

Thus the subdivision matrix (27) can be written

$$\mathbf{S} = \frac{1}{4}\begin{bmatrix} 4 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ \frac{(2-v)(v-\beta)}{v} & v-\beta & v+\beta & \frac{(2-v)(v+\beta)}{v} \\ 2-v & v & v & 2-v \\ \frac{(2-v)(v+\beta)}{v} & v+\beta & v-\beta & \frac{(2-v)(v-\beta)}{v} \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 4 \end{bmatrix}. \tag{39}$$

In this case, as soon as $1 \le v \le 2$ and $v \ge -\beta$, we can compute the subdivided control points

$$(\bar{A}_0, \bar{A}_1, \bar{A}_2, \bar{A}_3, \bar{A}_4, \bar{A}_5, \bar{A}_6)^T = \mathbf{S}(A_0, A_1, A_2, A_3)^T$$

by successive convex combinations starting with the polygon defined by ($A_0$, $A_1$, $A_2$, $A_3$). With 2 intermediate quantities $B$ and $C$ we have

$$
\begin{aligned}
&\bar{A}_0 = A_0, \quad \bar{A}_1 = \frac{1}{2}A_0 + \frac{1}{2}A_1, \quad \bar{A}_5 = \frac{1}{2}A_2 + \frac{1}{2}A_3, \quad \bar{A}_6 = A_3, \\
&B = (1 - \frac{v}{2})A_0 + \frac{v}{2}A_1 \\
&C = (1 - \frac{v}{2})A_3 + \frac{v}{2}A_2 \\
&\bar{A}_2 = \frac{v-\beta}{2v}B + \frac{v+\beta}{2v}C \\
&\bar{A}_4 = \frac{v+\beta}{2v}B + \frac{v-\beta}{2v}C \\
&\bar{A}_3 = \frac{1}{2}\bar{A}_2 + \frac{1}{2}\bar{A}_4.
\end{aligned}
\tag{40}
$$

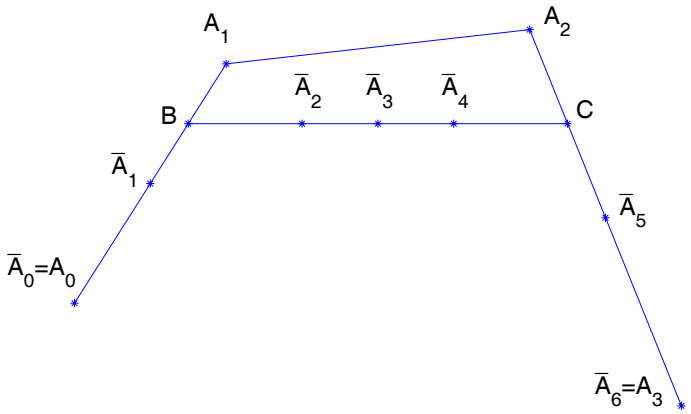


Figure 4: Corner Cutting with $\alpha = -3/32$, $\beta = -3/5$ and $v = 1.5$.

The equations (40) can be formulated as a corner cutting scheme in the following way. We start with the polygon $\{A_0, A_1, A_2, A_3\}$ and then either cut one of the previous corners or break an edge in a sequence of convex combinations.

1. $B = (1 - \frac{v}{2})A_0 + \frac{v}{2}A_1$ (replace $A_1$ by $B$ to obtain $\{A_0, B, A_2, A_3\}$)
2. $C = (1 - \frac{v}{2})A_3 + \frac{v}{2}A_2$ (replace $A_2$ by $C$ to obtain $\{A_0, B, C, A_3\}$)
3. $\bar{A}_1 = (1 - \frac{1}{v})A_0 + \frac{1}{v}B$ (break $[A_0, B]$ to obtain $\{A_0, \bar{A}_1, B, C, A_3\}$)
4. $\bar{A}_5 = \frac{1}{v}C + (1 - \frac{1}{v})A_3$ (break $[C, A_3]$ to obtain $\{A_0, \bar{A}_1, B, C, \bar{A}_5, A_3\}$ )
5. $\bar{A}_2 = \frac{v-\beta}{2v}B + \frac{v+\beta}{2v}C$ (replace $B$ by $\bar{A}_2$ to obtain $\{A_0, \bar{A}_1, \bar{A}_2, C, \bar{A}_5, A_3\}$)

6. $\bar{A}_4 = \frac{v+\beta}{v-\beta}\bar{A}_2 - \frac{2\beta}{v-\beta}C$ (replace $C$ by $\bar{A}_4$ to obtain $\{A_0, \bar{A}_1, \bar{A}_2, \bar{A}_4, \bar{A}_5, A_3\}$)

7. $\bar{A}_3 = (\bar{A}_2 + \bar{A}_4)/2$ (break $[\bar{A}_2, \bar{A}_4]$ to obtain $\{A_0, \bar{A}_1, \bar{A}_2, \bar{A}_3, \bar{A}_4, \bar{A}_5, A_3\}$)

Since $\bar{A}_0 = A_0$ and $\bar{A}_6 = A_3$ we have obtained the subdivided polygon $\{\bar{A}_0, \bar{A}_1, \bar{A}_2, \bar{A}_3, \bar{A}_4, \bar{A}_5, \bar{A}_6\}$ by carrying out a sequence of simple corner cuts (see for example [15, 9]) on the polygon defined by $\{A_0, A_1, A_2, A_3\}$.

We then obtain

**Theorem 10** *Suppose $-2 < \beta < 0$, $1 \le v := \frac{\lambda\beta}{\beta-1} \le 2$, and $\lambda \ge 1 - \beta$. Then the matrix $\mathbf{S}$ given by (39) is totally positive. For each $v \notin [1, 2]$ there is a $\beta \in [-1, 0[$ such that $\mathbf{S}$ is not totally positive.*

**Proof:** The sequence of simple corner cuts corresponds to a factorization of $\mathbf{S}$ into a product of 7 matrices as follows:

$$\mathbf{S} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{v+\beta}{v-\beta} & \frac{-2\beta}{v-\beta} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{v-\beta}{2v} & \frac{v+\beta}{2v} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{v} & 1 - \frac{1}{v} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 - \frac{1}{v} & \frac{1}{v} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{v}{2} & 1 - \frac{v}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 - \frac{v}{2} & \frac{v}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since these matrices are bidiagonal and the entries are nonnegative for the indicated values of the parameters it is well known that each of the 7 matrices are totaly positive (see for example [9]). Since a product of totally positive matrices is totally positive we conclude that $\mathbf{S}$ is totally positive.

If $v \notin [1, 2]$ then we can find $\beta \in [-1, 0)$ such that $\mathbf{S}$ has at least one negative entry. Hence $\mathbf{S}$ is not totally positive for these $v, \beta$. □

## 4.2 The $HC^1$-Bernstein Basis

Let $a, b$ be 2 real numbers with $a < b$. Let us define $h := b - a$. Recall that the $HC^1$-Hermite basis $\{\phi_0, \psi_0, \phi_1, \psi_1\}$ on $I := [a, b]$ forms a basis for the space $VC^1_{\alpha,\beta}(I)$ of all possible $HC^1$ interpolants on $I$. The *$HC^1$-Bernstein basis* $\{b_0, b_1, b_2, b_3\}$ on $I$ are defined as in [16] from the Hermite basis on $I$ by

$$b_0 := \phi_0 - \frac{\lambda}{h}\psi_0, \quad b_1 := \frac{\lambda}{h}\psi_0, \quad b_2 := -\frac{\lambda}{h}\psi_1, \quad b_3 := \phi_1 + \frac{\lambda}{h}\psi_1, \tag{41}$$

where $\lambda \geq 2$ is the parameter used to define the control points. These functions are clearly linearly independent and so, they form a basis for $VC^1_{\alpha,\beta}(I)$. The coefficients in terms of this basis are the control coefficients of $f$. This follows since

$$f := f(a)\phi_0 + p(a)\psi_0 + f(b)\phi_1 + p(b)\psi_1, \quad \Leftrightarrow \quad f = a_0b_0 + a_1b_1 + a_2b_2 + a_3b_3,$$

where $a_0, a_1, a_2, a_3$ are the control coefficients of $f$ on $I$ given by (24).

We note that $b_j(0) = \delta_{j,0}$ and $b_j(1) = \delta_{j,3}$.

For certain values of the parameters the $HC^1$-Benstein basis is totally positive.

**Theorem 11** *Suppose $-2 < \beta < 0$, $1 \leq v := \frac{\lambda\beta}{\beta-1} \leq 2$, and $\lambda \geq 1 - \beta$. Then the $HC^1$-Bernstein basis is totally positive.*

**Proof:** It is enough to prove the result for the interval $[0, 1]$. Consider for some integers $n, k$ with $n \geq 0$ and $0 \leq k \leq 2^n - 1$ the interval $I^n_k := [k/2^n, (k + 1)/2^n]$.

On $I^n_k$ the $HC^1$-Hermite basis $\{\phi^n_{0,k}, \psi^n_{0,k}, \phi^n_{1,k}, \psi^n_{1,k}\}$ can be expressed as

$$\begin{aligned}\phi^n_{0,k}(t) &= \phi_0(2^nt - k), \quad \psi^n_{0,k}(t) = 2^{-n}\psi_0(2^nt - k),\\ \phi^n_{1,k}(t) &= \phi_1(2^nt - k), \quad \psi^n_{1,k}(t) = 2^{-n}\psi_1(2^nt - k),\end{aligned}$$

where $\{\phi_0, \psi_0, \phi_1, \psi_1\}$ is the $HC^1$-Hermite basis on $[0, 1]$. From (41) with $h := 2^{-n}$, it then follows that the $HC^1$-Bernstein basis $\{b^n_{4k}, b^n_{4k+1}, b^n_{4k+2}, b^n_{4k+3}\}$ on $I^n_k$ can be expressed in terms of the $HC^1$-Bernstein basis $\{b_0, b_1, b_2, b_3\}$ on $[0, 1]$ as

$$b^n_{4k+j}(t) = \begin{cases} b_j(2^nt - k), & \text{if } t \in I^n_k \text{ and } j = 0, 1, 2, 3\\ 0 & \text{otherwise.}\end{cases} \tag{42}$$

We note that

$$b^n_{4k+j}(k/2^n) = \delta_{j,0}, \quad b^n_{4k+j}((k + 1)/2^n) = \delta_{j,3} \quad \text{for} \quad j = 0, 1, 2, 3. \tag{43}$$

Let $f \in C^1[0, 1]$ be a $HC^1$-interpolant to some initial data. We can then write

$$f = \sum_{i=0}^{m} a^n_i b^n_i,$$

where $m := 4 \times 2^n - 1$ and where for $k = 0, \ldots, 2^n - 1$ the numbers $a^n_{4k}$, $a^n_{4k+1}$, $a^n_{4k+2}$, $a^n_{4k+3}$ are the control points of $f$ on $I^n_k$. In vector form, we have $f = b^n a^n$ where $b^n = (b^n_0, \ldots, b^n_m)$ is a row vector and $a^n = (a^n_0, \ldots, a^n_m)^T$ a column vector. Note that $b^n$ is a vector of linearly independent functions on $[0, 1]$. They span a space containing $VC^1_{\alpha,\beta}[0, 1]$ as a 4-dimensional subspace. On level $n + 1$, we have $f = b^{n+1}a^{n+1}$ where from Proposition 6, it follows that $a^{n+1} = \mathbf{A}_n a^n$ for some matrix $\mathbf{A}_n$. The matrix $\mathbf{A}_n$ is a block diagonal with $2^n$ diagonal blocks $\hat{\mathbf{S}}$ of order $8 \times 4$. Indeed, $\hat{\mathbf{S}}$ is obtained from the matrix $\mathbf{S}$ in (27) by adding a copy of row 4 as a new

row 5. But then $f = b^{n+1}a^{n+1} = b^{n+1}\mathbf{A}_n a^n = b^n a^n$ and by linear independence, it follows that $b^n = b^{n+1}\mathbf{A}_n$. Thus we obtain

$$b^0 = b^n \mathbf{A}_{n-1} \cdots \mathbf{A}_0, \quad n \geq 1. \tag{44}$$

For distinct points $y_0, \ldots, y_p$ and functions $f_0, \ldots, f_q$ defined on the $y$'s, we use the standard notation

$$M \begin{bmatrix} y_1, \ldots, y_p \\ f_1, \ldots, f_q \end{bmatrix} := \begin{bmatrix} f_1(y_1) & \cdots & f_q(y_1) \\ \vdots & & \vdots \\ f_1(y_p) & \cdots & f_q(y_p) \end{bmatrix}$$

for a collocation matrix of order $p \times q$. In order to show total positivity of $b = b^0$ we choose $0 \leq x_0 < x_1 < x_2 < x_3 \leq 1$ and consider the collation matrix $M \begin{bmatrix} x_0, x_1, x_2, x_3 \\ b_0, b_1, b_2, b_3 \end{bmatrix}$. From (44) we immediatly obtain

$$M \begin{bmatrix} x_0, \ldots, x_3 \\ b_0, \ldots, b_3 \end{bmatrix} = M \begin{bmatrix} x_0, \ldots, x_3 \\ b_0^n, \ldots, b_m^n \end{bmatrix} \mathbf{A}_{n-1} \cdots \mathbf{A}_0, \quad n \geq 1. \tag{45}$$

Since the matrix $\mathbf{S}$ is totally positive, it follows that $\hat{\mathbf{S}}$ and hence each $\mathbf{A}_k$ is totally positive. We now show that the first matrix on the right of (45) is totally positive provided $x_j \in \mathcal{P}_n$ for $j = 0, 1, 2, 3$. For this, with $m = 2^{n-1} - 1$, we consider the bigger matrix

$$\mathbf{B} = M \begin{bmatrix} y_0, \ldots, y_{m+1} \\ b_0^n, \ldots, b_m^n \end{bmatrix}$$

using all points $y_i = i/2^n$, $i = 0, 1, \ldots, 2^n$ in $\mathcal{P}_n$. ¿From (43) it follows that $b_{4k-1}(y_k) = 1$ for $k = 1, \ldots, 2^n$, $b_{4k}(y_k) = 1$ for $k = 0, \ldots, 2^n - 1$ and $b_i^n(y_j) = 0$ otherwise. Thus the columns of $\mathbf{B}$ have the following form

$$\mathbf{B} = [e_1, 0, 0, e_2, e_2, 0, 0, e_3, e_3, 0, 0, e_4 \ldots, e_m, 0, 0, e_{m+1}],$$

where $e_j = (\delta_{i,j})_{i=0}^m$ is the $j$th unit vector in $\mathbb{R}^{m+1}$. From this explicit form we see that $\mathbf{B}$ is totally positive since each nonzero minor must be the determinant of the identity matrix. But then all matrices on the right in (45) are totally positive and we conclude that $M \begin{bmatrix} x_0, \ldots, x_3 \\ b_0, \ldots, b_3 \end{bmatrix}$ is totally positive provided $x_j \in \mathcal{P}_n$ for $j = 0, 1, 2, 3$. Since $\cup \mathcal{P}_n$ is dense in $[0, 1]$ we conclude that the $HC^1$-Bernstein basis is totally positive. □

**Corollary 12** *For $p \geq 0$ and $m = 4 \cdot 2^p - 1$, the basis $b^p = (b_0^p, \ldots, b_m^p)$ for the space span($b^p$) is totally positive on $[0, 1]$.*

**Proof:** Instead of (44) we use for $n > p$ the equation

$$b^p = b^n \mathbf{A}_{n-1} \cdots \mathbf{A}_p.$$

The argument now proceeds as in the proof of Theorem 11 replacing $x_0, \ldots, x_3$ by suitable $x_0, \ldots, x_m$. $\square$

It is well known that total positivity of the $HC^1$ Bernstein basis on $[0,1]$ implies that the $HC^1$-interpolant $f$ inherits properties of the control polygon $P^0$ defined by $\{a_0, a_1, a_2, a_3\}$, see for example ([9]). In particular if $P_0$ is positive (monotone, convex) then $f$ is positive (monotone, convex). We can use this to generalize Theorem 4 in ([16]).

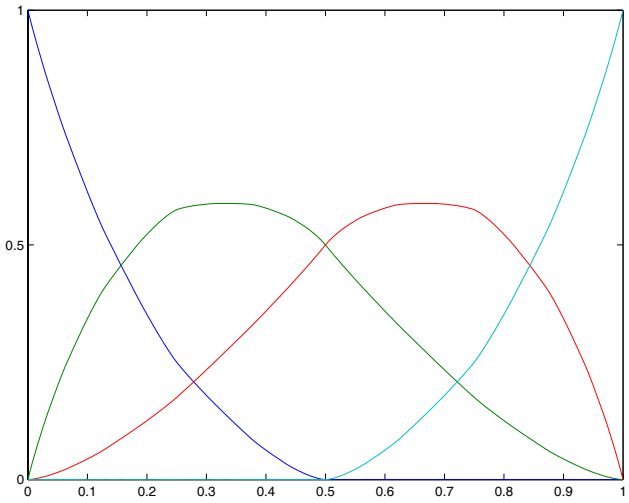


Figure 5: Bernstein basis, $\beta = -3/5$, $\alpha = -3/32, \lambda = 16/3$.

**Corollary 13** *Let $b_0, b_1, b_2, b_3$ be the $HC^1$ Bernstein basis on $[0,1]$ given by (41) with $\lambda = v(\beta - 1)/\beta > 2$. Suppose also $\alpha = \frac{\beta}{4(1-\beta)}$, $-1 \leq \beta < 0$ and $1 \leq v \leq 2$. Then*

1. *$b_0$ is nonnegative, decreasing, and convex on $[0,1]$. If $v = 2$ then $b_0(t) = 0$ for $t \in [1/2, 1]$.*

2. *$b_1$ is nonnegative and concave on $[0, 1/2]$ and nonnegative, decreasing and convex on $[1/2, 1]$*

3. *$b_2$ is nonnegative, increasing and convex on $[0, 1/2]$ and nonnegative and concave on $[1/2, 1]$.*

4. *$b_3$ is nonnegative, increasing, and convex on $[0,1]$. If $v = 2$ then $b_3(t) = 0$ for $t \in [0, 1/2]$.*

5. *$\sum_{j=0}^{3} b_j(t) = 1$ for $t \in [0,1]$*

**Proof:** ¿From (41) it follows that the control points of the function $b_j$ is the $j$th unit vector $e_{j+1}$ for $j = 0, 1, 2, 3$. Thus nonnegativity of $b_j$ follows from the nonnegativity of $e_{j+1}$ for $j = 0, 1, 2, 3$. Moreover the monotonicity and convexity properties of $b_0$ and $b_3$ follow. For the remaining properties of $b_1$ and $b_2$, we carry out one subdivision, then the proof is similar.

The refined points are given as the columns in the matrix **S** given by (39). When $v = 2$ the first column is given by $[1, 1/2, 0, 0, 0, 0, 0]$. Since the last four entries are zero it follows that $b_0(t) = 0$ for $t \in [1/2, 1]$. Similarly $b_3(t) = 0$ for $t \in [0, 1/2]$.

The interpolation of the constant function $f = 1$ with $p = f' = 0$ gives $a_0 = a_1 = a_2 = a_3 = 1$ in (24) so that 5. holds. □

# 5 Algorithms for local shape constraints

We base shape preserving algorithms on the extended quadratic spline case given by $\alpha = \frac{\beta}{4(1-\beta)}$. The control point subdivision matrix for this case is given by (39), where we have both $\beta$ and $\lambda$ as free parameters. The matrix simplifies when $v = \frac{\beta\lambda}{\beta-1} = 2$ and we will use this one parameter family of schemes in our algorithms. Using the parameter $\lambda$ to control the shape we thus have

$$\alpha = \frac{\beta}{4(1-\beta)} = -\frac{1}{2\lambda}, \quad \beta = \frac{2}{2-\lambda}. \tag{46}$$

We restrict our attention to $\lambda \geq 4$. We then have $\beta \in [-1, 0)$ and both algorithms $HC^1$ and $SC^1$ are convergent. In the limit when $n \to \infty$ we obtain a function $f \in C^1(I)$. This function is the quadratic spline interpolant with a knot at the midpoint of $I$ when $\lambda = 4$ , while $p = f'$ is Hölder continuous on $I$ with exponent

$$\log_2\left(1 + \frac{1}{\lambda - 3}\right) \approx \frac{1.44}{\lambda - 3}, \quad \lambda \to \infty.$$

Thus the derivative becomes less regular when $\lambda$ increases, but it is always $C^1$.

Given $s \geq 1$, points $a_j^0 = a_j \in \mathbb{R}^s$ for $j = 0, 1, 2, 3$, and $\lambda \geq 4$, the following algorithm computes sequences $\{a^n\}$ of control coefficients $a^n = (a_0^n, a_1^n, \ldots, a_{3\times 2^n}^n)$ in $\mathbb{R}^s$.

**Algorithm 14** ($CC^1$)

1. $\beta = 2/(2 - \lambda)$;
2. *For* $n = 0, 1, 2, 3, \ldots$
   *For* $i = 0, 1, \ldots 2^n - 1$
   (a) $a_{6i}^{n+1} = a_{3i}^n$;
   (b) $a_{6i+1}^{n+1} = \frac{1}{2}(a_{3i}^n + a_{3i+1}^n)$;

(c) $a_{6i+2}^{n+1} = (\frac{1}{2} - \frac{\beta}{4})a_{3i+1}^n + (\frac{1}{2} + \frac{\beta}{4})a_{3i+2}^n$;
(d) $a_{6i+3}^{n+1} = \frac{1}{2}(a_{3i+1}^n + a_{3i+2}^n)$;
(e) $a_{6i+4}^{n+1} = (\frac{1}{2} + \frac{\beta}{4})a_{3i+1}^n + (\frac{1}{2} - \frac{\beta}{4})a_{3i+2}^n$;
(f) $a_{6i+5}^{n+1} = \frac{1}{2}(a_{3i+2}^n + a_{3i+3}^n)$;

$a_{3\cdot 2^{n+1}} = a_{3\cdot 2^n}$;

The control points corresponding to the computed control coefficients converges to a $C^1$-curve. More specifically, pick any finite closed interval $[a, b]$ and define $h_n := (b-a)/2^n$ and $t_k^n := a + kh_n$ for $k = 0, \ldots, 2^n$, $n \geq 0$. By Theorem 9 the computed control points converge uniformly to a $C^1$-curve $f : [a, b] \to \mathbb{R}^s$. Moreover,

$$\begin{aligned} a_{3i}^n &= f(t_i^n), \quad i = 0, \ldots, 2^n, \\ a_{3i+1}^n - a_{3i}^n &= \frac{h_n}{\lambda} f'(t_i^n), \quad i = 0, \ldots, 2^n - 1, \\ a_{3i}^n - a_{3i-1}^n &= \frac{h_n}{\lambda} f'(t_i^n), \quad i = 1, \ldots, 2^n. \end{aligned}$$

We now discuss shape preservation in the scalar case $s = 1$ in more detail. We start by noting that if the initial control polygon is nonnegative (respectively increasing, convex) on an interval $I = [a, b]$, then the $HC^1$-interpolant computed in Algorithm 14 will be nonnegative (respectively increasing, convex) on the same interval $I$. This follows from the total positivity of the Bernstein basis. In addition to total positivity the main tool will be Corollary 2 which says that the $p$-values of the interpolant are located on the piecewise linear curve connecting the three points $(a, p(a)), (\frac{a+b}{2}, p(\frac{a+b}{2})), (b, p(b))$.

## 5.1 Nonnegative Interpolants

We already remarked that if the initial control coefficients are nonnegative then the $HC^1$-interpolant will be nonnegative. Notice that the converse is false. For example, the $HC^1$-interpolant to the function $f$ given on $[0, 1]$ by $f(x) := 16(x - 1/4)^2$ and using $\lambda = 4$ is $f$ itself. Note that $f$ is nonnegative, but the initial control coefficient $a_1 = -1$ is negative.

To give an algorithm for constructing a nonnegative interpolant we assume that

$$f(a) \geq 0, \ f(b) \geq 0, \ p(a) \geq 0 \text{ if } f(a) = 0, \text{ and } p(b) \leq 0 \text{ if } f(b) = 0. \tag{47}$$

Under these weak assumptions nonnegative initial control coefficients $a_0, \ldots, a_3$ can always be obtained by choosing $\lambda$ sufficiently large. Indeed, since $a_0 = f(a) \geq 0$ and $a_3 = f(b) \geq 0$ we only need to make sure that $a_1 = f(a) + hp(a)/\lambda \geq 0$ and $a_2 = f(b) - hp(b)/\lambda \geq 0$. If $f(a) = 0$ then $p(a) \geq 0$ and $a_1 \geq 0$ whenever $\lambda > 0$ .

Similarly $a_2 \geq 0$ if $f(b) = 0$. But then we can choose $\lambda = 4$ except possibly in the two cases $f(a) > 0$, $p(a) < 0$ and $f(b) > 0$, $p(b) > 0$. If (47) holds then the following algorithm will compute a nonnegative $HC^1$-interpolant on $[a, b]$.

**Algorithm 15 (Nonnegative Interpolant)**

1. *Compute* $\lambda$
   (a) $\lambda = 4$;
   (b) *if* $(f(a) > 0)$ & $(p(a) < 0)$ *then* $\lambda = \max(\lambda, -hp(a)/f(a))$;
   (c) *if* $(f(b) > 0)$ & $(p(b) > 0)$ *then* $\lambda = \max(\lambda, hp(b)/f(b))$;
2. *Compute initial control coefficients using* (24).
3. *Apply Algorithm 14 or Algorithm* $HC^1$ *with* $\alpha = -\frac{1}{2\lambda}, \quad \beta = \frac{2}{2-\lambda}$.

## 5.2 Monotone interpolants

The monotonicity of the $HC^1$-interpolant is completely determined by the monotonicity of the initial control polygon. If $f$ is decreasing then $-f$ is increasing and we restrict our discussion to increasing interpolants.

**Proposition 16** *Suppose that the parameters are chosen according to* (46). *Then the* $HC^1$*-interpolant* $f$ *is nondecreasing on an interval* $I = [a, b]$ *if and only if the control polygon on* $I$ *is nondecreasing.*

**Proof:** By Theorem 11 the Bernstein basis is totally positive and it follows that the $HC^1$-interpolant is nondecreasing if the control polygon is nondecreasing , see [9]. Conversely, suppose the $HC^1$-interpolant $f$ is nondecreasing. Since $\beta = 2/(2 - \lambda)$, we obtain from (1)

$$p(\frac{a+b}{2}) = \frac{1}{\lambda - 2}\left(\lambda \frac{f(b) - f(a)}{h} - (p(a) + p(b))\right). \tag{48}$$

From (24), we then find

$$a_1 - a_0 = \frac{h}{\lambda}p(a), \quad a_2 - a_1 = \frac{\lambda - 2}{\lambda}hp(\frac{a+b}{2}), \quad a_3 - a_2 = \frac{h}{\lambda}p(b). \tag{49}$$

Now $p \geq 0$ at all points if $f$ is nondecreasing. It follows that the control coefficients, and hence the control polygon is nondecreasing. □

Consider next the case of a strictly increasing interpolant.

**Proposition 17** *Suppose that the parameters are chosen according to* (46) *and that the* $HC^1$*-interpolant* $f$ *is nondecreasing on an interval* $I = [a, b]$. *Then* $f$ *is strictly increasing on* $[a, b]$ *if and only if the two middle control coefficients on* $I$ *satisfy* $a_2 > a_1$.

**Proof:** Since $f$ is nondecreasing, we have $p(a) \geq 0$, $p(\frac{a+b}{2}) \geq 0$ and $p(b) \geq 0$. By Corollary 2, it follows that $f$ is strictly increasing on $[a, b]$ if and only if $p(\frac{a+b}{2}) > 0$. By (49), this happens if and only if $a_2 > a_1$. $\square$

To give an algorithm to construct a nondecreasing interpolant we assume that

$$f(a) \leq f(b),\ p(a) \geq 0,\ p(b) \geq 0 \text{ and } p(a) = p(b) = 0 \text{ if } f(a) = f(b). \tag{50}$$

In the latter case the $HC^1$-interpolant is constant and we can set $\lambda = 4$.

Suppose $f(b) > f(a)$. With $h := b - a$ we then have

$$a_0 = f(a) \leq a_1 = f(a) + \frac{h}{\lambda}p(a) \leq a_2 = f(b) - \frac{h}{\lambda}p(b) \leq a_3 = f(b)$$

provided

$$a_2 - a_1 = f(b) - f(a) - \frac{h}{\lambda}\big(p(b) + p(a)\big) \geq 0$$

or

$$\lambda \geq \frac{(p(a) + p(b))h}{f(b) - f(a)}. \tag{51}$$

If (50) holds then the following algorithm will compute a nondecreasing $HC^1$-interpolant on $[a, b]$. It will be strictly increasing if $f(b) > f(a)$ and (51) holds with strict inequality.

**Algorithm 18 (Nondecreasing- or Strictly Increasing Interpolant)**

1. *Compute* $\lambda$
   - (a) $\lambda = 4$;
   - (b) *If* $f(a) < f(b)$ *then*
     - i. $\lambda_1 \geq \frac{(p(a)+p(b))h}{f(b)-f(a)}$
     - ii. $\lambda = \max(4, \lambda_1)$
2. *Compute initial control coefficients using* (24).
3. *Apply Algorithm 14 or Algorithm* $HC^1$ *with* $\alpha = -\frac{1}{2\lambda}, \quad \beta = \frac{2}{2-\lambda}$.

Note that if the initial control points are located on a straight line then the $HC^1$-interpolant is the line segment connecting the first and last control point. For if the initial control points are located on a straight line then

$$\frac{\lambda}{h}(a_1 - a_0) = \frac{\lambda}{(\lambda - 2)h}(a_2 - a_1) = \frac{\lambda}{h}(a_3 - a_2)$$

and by (49) the three slopes $p(a), p(\frac{a+b}{2}), p(b)$ are all equal. By Corollary 2, all slopes are equal and the function $f$ is a straight line.

In Figure 6 we interpolate three sets of data on $[0,1]$. In all cases $f(0) = -1$ and $f(1) = 1$. In the first case, with $p(0) = 3$ and $p(1) = 4$ we find $\frac{p(0)+p(1)}{f(1)-f(0)} = 7/2 < 4$. Suppose in Algorithm 18 we choose $7/2 \leq \lambda_1 \leq 4$ in Statement $(b)i.$ and apply Algorithm 14 with $\lambda = 4$. Then the $HC^1$-interpolant is the quadratic spline and it is strictly increasing since $\lambda > 7/2$. In the two other cases we use $p(0) = 8$ and $p(1) = 4$ giving $\frac{p(0)+p(1)}{f(1)-f(0)} = 6$. With $\lambda = 6$ we have $p(1/2) = 0$ and the interpolant is increasing, but not strictly increasing. We obtain a strictly increasing interpolant by using $\lambda = 10$. Note that choosing a bigger $\lambda$ decreases the regularity of the interpolant. In both cases the first derivative is Hölder continuous, but the exponent is $\log_2(4/3) \approx 0.415$ when $\lambda = 6$ and $\log_2(4/3) \approx 0.193$ when $\lambda = 10$.

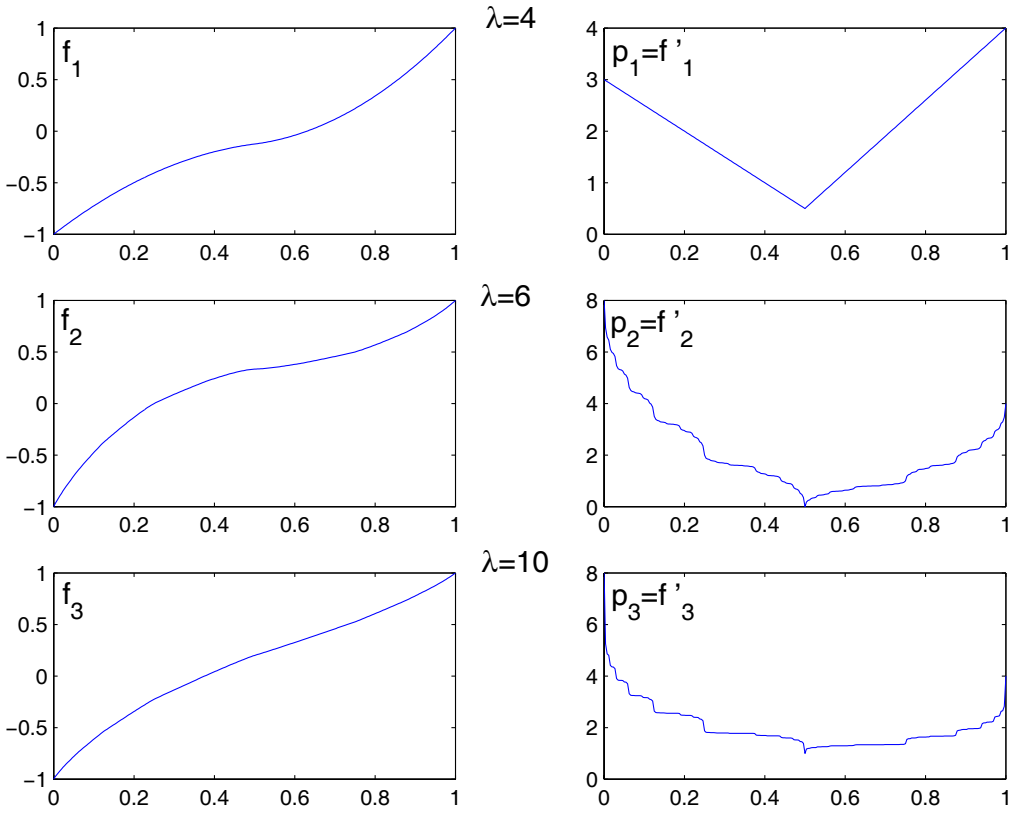


Figure 6: Monotone interpolants

## 5.3 Convex interpolants

The convexity of the $HC^1$-interpolant is also completely determined by the convexity of the initial control polygon.

**Proposition 19** *Suppose that the parameters are chosen according to* (46)*. Then $f$ is convex (concave) on an interval $I = [a, b]$ if and only if the control polygon on $I$ is convex (concave).*

**Proof:** Again by total positivity of the Bernstein basis the $HC^1$-interpolant is convex (concave) if the control polygon is convex (concave), see [9]. Conversely, suppose the $HC^1$-interpolant $f$ is convex (concave). Now the control polygon is convex if and only if the conditions

$$\frac{a_1 - a_0}{h/\lambda} \leq \frac{a_2 - a_1}{h - 2h/\lambda} \leq \frac{a_3 - a_2}{h/\lambda}$$

hold. But from (49) we find

$$\frac{a_1 - a_0}{h/\lambda} = p(a), \quad \frac{a_2 - a_1}{h - 2h/\lambda} = p(\frac{a+b}{2}), \quad \frac{a_3 - a_2}{h/\lambda} = p(b).$$

Since $f$ is convex (concave) the function $p$ is nondecreasing (nonincreasing) and hence the control polygon is convex(concave). □

To give an algorithm for constructing a convex (concave) $HC^1$-interpolant on an interval $I = [a, b]$ we first assume that

$$p(a) < \frac{f(b) - f(a)}{h} < p(b) \quad \big(p(a) > \frac{f(b) - f(a)}{h} > p(b)\big), \tag{52}$$

where $h := b - a$. We define

$$\lambda_1 := \frac{p(b) - p(a)}{p(b) - \frac{f(b)-f(a)}{h}}, \quad \lambda_2 := \frac{p(b) - p(a)}{\frac{f(b)-f(a)}{h} - p(a)} \tag{53}$$

and note that the tangents

$$t_c(x) := f(a) + (x - a)p(a), \quad t_d(x) := f(b) + (x - b)p(b)$$

of $f$ at $a$ and $b$ intersect at the point$(\bar{x}, \bar{y})$ given by

$$\frac{\bar{x} - a}{h} = \frac{1}{\lambda_1}, \quad \text{and} \quad \frac{b - \bar{x}}{h} = \frac{1}{\lambda_2}.$$

Moreover, the hypothesis (52) is equivalent to $a < \bar{x} < b$.

Under the assumption

$$p(a) \leq \frac{f(b) - f(a)}{h} \leq p(b) \quad \big(p(a) \geq \frac{f(b) - f(a)}{h} \geq p(b)\big) \tag{54}$$

the following algorithm will compute a convex (concave) interpolant.

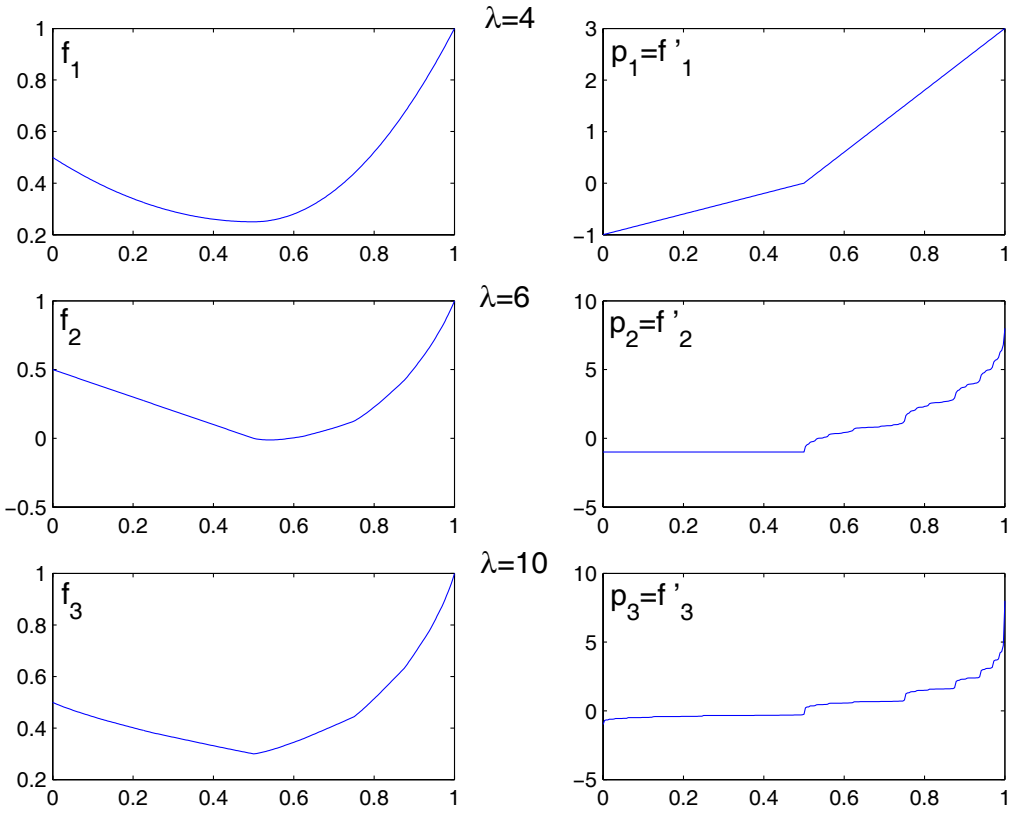


Figure 7: Convex interpolants

**Algorithm 20 (Convex or Concave interpolant)**

1. *(a) If* $p(a) = \frac{f(b)-f(a)}{h} \neq p(b)$, *choose* $\lambda \geq \max(4, \lambda_1)$
   *(b) If* $p(a) \neq \frac{f(b)-f(a)}{h} = p(b)$, *choose* $\lambda \geq \max(4, \lambda_2)$
   *(c) If* $p(a) \neq \frac{f(b)-f(a)}{h} \neq p(b)$, *choose* $\lambda \geq \max(4, \lambda_1, \lambda_2)$
2. *Compute initial control points using* (24)
3. *Apply Algorithm 14 or Algorithm* $HC^1$ *with* $\alpha = -\frac{1}{2\lambda}, \quad \beta = \frac{2}{2-\lambda}$.

In Figure 7, we have interpolated three sets of data on $[0, 1]$. In all cases $f(0) = 0.5$ and $f(1) = 1$.

In the first case, $p(0) = -1$ and $p(1) = 3$ so that $\lambda_1 = 8/5$ and $\lambda_2 = 8/3$. Then $\max(4, \lambda_1, \lambda_2) = 4$ and we have chosen $\lambda = 4$. In this case, the interpolant is the quadratic spline.

In the two other cases $p(0) = -1$ and $p(1) = 8$ so that $\lambda_1 = 18/5$ and $\lambda_2 = 6$. Then $\max(4, \lambda_1, \lambda_2) = 6$. With $\lambda = 6$ we have $p = -1$ on $[0, 1/2]$, while we obtain a

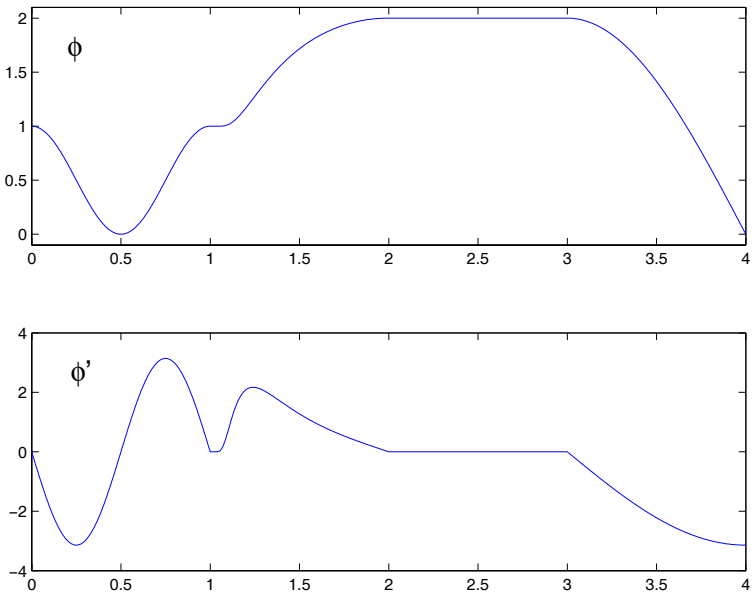


Figure 8: The function $\phi$ and its derivative

strictly convex interpolant by using $\lambda = 10$. Recall that choosing a bigger $\lambda$ decreases the regularity of the interpolant.

# 6 Example

Given data $(t_i, y_i, y_i')$ for $i = 1, \ldots, n$, where $t_1 < \cdots < t_n$ and the $y$'s are real numbers. We look for a function $f \in C^1([t_1, t_n])$ that satisfies

$$f(t_i) = y_i, \;\; f'(t_i) = y_i' \;\; \text{for } i = 1, \ldots, n. \tag{55}$$

In addition we would like $f$ to be positive, monotone, linear, or convex on some or all of the subintervals $I_i = [t_i, t_{i+1}], i = 1, \ldots n-1$. We assume that

(P) (47) holds for the subintervals where we want nonnegativity or positivity.

(M) (50) holds for the subintervals where we want a nondecreasing or a strictly increasing interpolant.

(L) $y_i' = y_{i+1}' = \frac{y_{i+1}-y_i}{t_{i+1}-t_i}$ for the subintervals where the interpolant should be linear.

(C) (54) holds for the subintervals where the interpolant should be convex or concave.

We also require that the given data is consistent with these shape requirements. We can compute $f$ locally by applying the $HC^1$-algorithm with parameters given by (46) on each subinterval $I_i = [t_i, t_{i+1}], i = 1, \ldots n-1$ using initial data $f(t_i) = y_i, f(t_{i+1}) = y_{i+1}$, $p(t_i) = y_i'$ and $p(t_{i+1}) = y_{i+1}'$. We obtain $C^1$-convergence and the desired shape locally by choosing the parameter $\lambda_i$ for the interval $I_i$ sufficiently large.

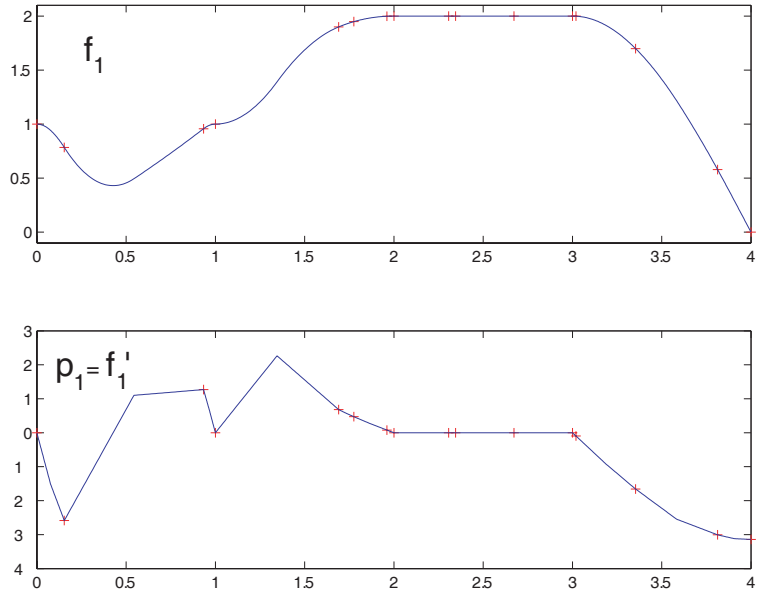


Figure 9: Interpolation with exact derivatives

Consider now (55) for the example illustrated in Figure 1. The data are sampled from the function $\phi \in C^1([0,4])$ given by

$$\phi(t) = \begin{cases} \frac{1}{2}\sin(2\pi t + \pi/2) + \frac{1}{2} & , \quad 0 \le t \le 1, \\ 1 + \exp(-\frac{1}{1-(t-2)^2} + 1) & , \quad 1 < t \le 2, \\ 2 & , \quad 2 < t \le 3, \\ 2\cos(\pi\frac{t-3}{2}) & , \quad 3 \le t \le 4. \end{cases} \tag{56}$$

The function and its first derivative are displayed in Figure 8 and it can be shown that $\phi$ is positive on $[0,1]$, strictly increasing on $[1,2]$, constant on $[2,3]$ and concave on $[3,4]$. Given $n$ and let $(t_1,\ldots,t_n)$ be a partition of $[0,4]$. The points $(t_2,\ldots,t_{n-1})$ are chosen randomly except that $1,2,3$ are among them. In the example, we used $t_1 = 0$, $t_{n_1} = t_5 = 1$, $t_{n_2} = t_9 = 2$, $t_{n_3} = t_{13} = 3$ and $t_n = t_{17} = 4$. We want an interpolant $f$ which is positive on $[t_1, t_{n_1}] = [0,1]$, strictly increasing on $[t_{n_1}, t_{n_2}] = [1,2]$, constant on $[t_{n_2}, t_{n_3} = [2,3]$ and concave on $[t_{n_3}, t_n] = [3,4]$.

In the first test we use $y_i = \phi(t_i)$ and exact derivatives $y_i' = \phi'(t_i), i = 1,\ldots,n$. In this case all $\lambda$'s become equal to 4 and the quadratic spline interpolant $f_1$ does the job. Plots of this function and its first derivative are shown in Figure 9. The first derivative appears continuous and piecewise linear.

For the second test shown in Figure 10, we kept the previous data $t_i$ and $y_i = \phi(t_i)$ for $i = 1,\ldots,n = 17$, but we used inexact derivatives given by crosses in the lower part of the figure. However the derivatives were chosen so that the relevant requirement (P),(M), (L), and (C) above are satisfied on each subinterval $[t_i, t_{i+1}]$. We obtain a $C^1$-interpolant $f_2$ satisfying the required shape constraints. The computed values of $\lambda_i$ are successively
$(4, 5.1425, 4, 4, 4, 12.8631, 55.8239, 4, 4, 4, 4, 17.6767, 20.0216, 4.4087, 11.3544)$. This ex-

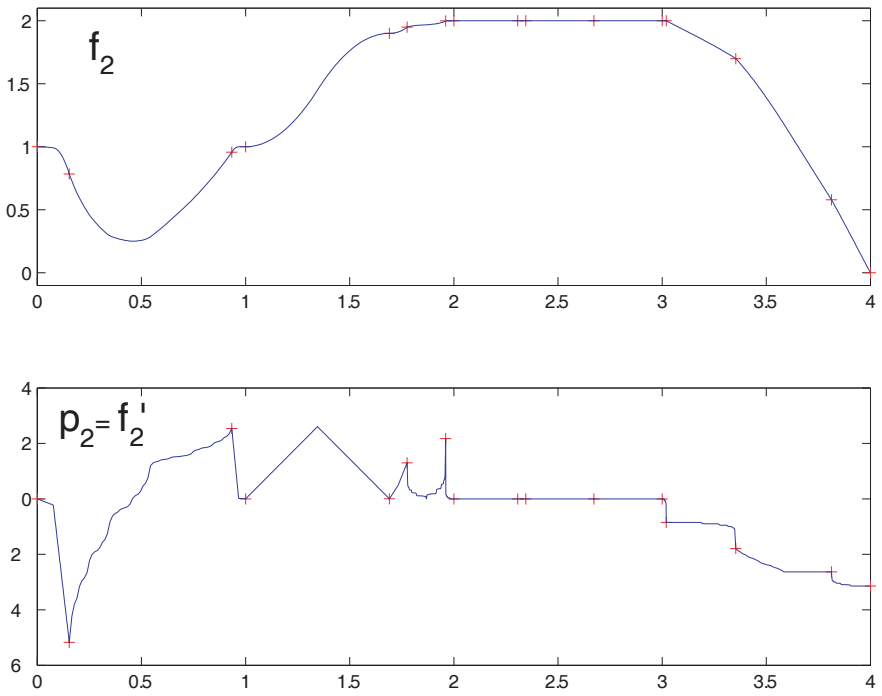


Figure 10: Interpolation with modified derivatives

ample shows that we can obtain a desired shape even with more or less random derivative values.

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