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D-instantons and matrix models

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Abstract. We discuss the matrix model aspect of configurations saturating a fixed number of fermionic zero-modes. This number is independent of the rank of the gauge group and the instanton number. This will allow us to define a large- N_c limit of the embedding of K D-instantons in the matrix model and make contact with the leading term (the measure factor) of the supergravity computations of D-instanton effects. We show that the connection between these two approaches is made through the Abelian modes of the matrix variables.

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1. Introduction

Over the past four years some tremendous progress and insights about the non-perturbative and global behaviour of supersymmetric gauge theory, superstring theory and supergravity have appeared. All these advances are founded on a web of consistent cross-checked conjectures culminating with the idea of M-theory as the mother of all theories. Most of the impressive and exact non-perturbative results were derived by considering BPS saturated amplitudes. Due to the saturation of the fermionic zero-modes these terms are protected by some non-renormalization theorems and can be computed both in the perturbative and non-perturbative regimes. This is the case of the eight-fermion terms in the three-dimensional super-Yang–Mills theory [1], the D-instantons of the type IIB string theory [2], or the wrapped D1-brane around tori of dimensions smaller than five in the type I theory [3]. In these cases, the D-instantons contributions belong to a half-BPS multiplet of the theory, and they come from amplitudes where 16 or eight fermionic zero-modes, respectively, for the type IIB and type I theory, have to be soaked up. The surprising aspect of these results is that even in a vacuum containing K D-instantons it is only necessary to saturate the same fixed number of fermionic zero-modes, independently of K . This is because of the existence of threshold bound states of D0-branes and the action of T-duality which exchange K D p -branes on top of each other singly wrapped around a $(p + 1)$ -torus with one D $(p - 1)$ -brane wrapped K -times around a p -torus, and the fact that the presence of winding modes does not break supersymmetry[†].

In the context of the correspondence between the supergravity results and the CFT computation, this independence of fermionic zero-modes with respect to the instanton number becomes much more obscure. Let us consider, for example, the case of a vacuum containing K D3-branes: it was claimed in [4] and impressively strengthened by the result of [5] that

[†] It should be noted that is not true for bound states of the D-particle and anti-D-particle.

the large- N_c limit of this theory is in correspondence with the four-dimensional super-Yang–Mills theory with gauge group $SU(N_c)$ in the large- N_c 't Hooft limit ($N_c g_{\text{YM}}^2 = \text{constant}$). This confirmation used a sector of the theory with a fixed number of fermionic zero-modes independently of N_c . The puzzle is that in the super-Yang–Mills case one would normally think that when the rank of the gauge group increases there are extra fermionic zero-modes and the result cannot match the supergravity ones. In fact, it was understood by [5] that an embedding of a configuration of K instantons in the $SU(N_c)$ group has a fixed number of fermionic zero-modes independent of the number of instantons and the rank of the group.

The main purpose of the present paper is to explain that in the context of matrix models, such a configuration of fermionic zero-modes can be realized, and can lead to a way of defining a large- N_c limit of the matrix model.

In section 2, we will introduce the supergravity aspect of the D-instanton effects, and in section 3 we will introduce the matrix model that we will use in the following. Section 4 contains a discussion of the dynamics of these models, emphasizing the importance of considering a gauge-invariant model. In section 5 we compute the partition function of the matrix model with various symmetries, and map these results to the supergravity results in section 6. Section 7 contains a discussion of this approach. Appendix A contains an explicit computation of the partition function of the supersymmetric matrix model with two real supercharges and appendix B summarizes our conventions for the Γ matrices used in the text.

2. The supergravity side

The type IIb chiral version of the ten-dimensional supergravity is peculiar in several respects. First, being chiral with a maximum number of supersymmetries, with two sets of 16 real component supercharges of the same spacetime chirality, defined with respect to the projector $(1 + \underline{\epsilon}^1)/2$, in ten dimensions it has a richer moduli space than its non-chiral counterpart, the type IIa supergravity. The superspace formalism of this chiral supergravity theory was worked out in [6] and will be used here. The superspace formalism uses a supermanifold with ten even coordinates, x^μ ($\mu = 0, \dots, 9$), and 16 odd complex coordinates[†], ϑ^α ($\alpha = 1, \dots, 16$), and their complex conjugates $(\vartheta^\alpha)^* \stackrel{\text{def}}{=} \vartheta^{\bar{\alpha}}$, the whole set being packaged in $z^M \stackrel{\text{def}}{=} (x^\mu, \vartheta^\alpha, \vartheta^{\bar{\alpha}})$. At each point in the superspace there are some local coordinates related to the 1-form dz^M by the vielbein

$$dE^A \stackrel{\text{def}}{=} dz^M E_M^A,$$

and as for usual Riemann manifolds the vielbein are invertible,

$$E_A^M E_M^B = \delta_A^B.$$

The tangent space is described by the covering of the group $SO(1, 9) \times U(1)_B$. The $U(1)_B$ factor is a local phase transformation on the fermionic coordinates by

$$\vartheta^\alpha \rightarrow \exp \frac{1}{2} i \Gamma^{11} \xi \vartheta^\alpha, \quad \vartheta^{\bar{\alpha}} \rightarrow \exp -\frac{1}{2} i \Gamma^{11} \xi \vartheta^{\bar{\alpha}}.$$

It was shown in [6] that this $U(1)_B$ factor is precisely the factor appearing in the coset space parametrization $SU(1, 1)/U(1)_B \times S(2, \mathbb{R})/U(1)_B$ of this theory. The $S(2, \mathbb{R})$ is a rigid transformation acting on the left of the fields and the $U(1)_B$ induced by the chiral nature of

[†] These coordinates will be put in correspondence with the zero-modes of the matricial fermions (see section 6.2).

the theory is a local transformation acting on the right of the coset. As in [7] we parametrize this coset with the matrix

$$V \stackrel{\text{def}}{=} \sqrt{\frac{1}{-2i\rho_2}} \begin{pmatrix} \rho e^{i\pm} & \bar{\rho} e^{-i\pm} \\ e^{i\pm} & e^{-i\pm} \end{pmatrix} .$$

The $S(2, \mathbb{R})$ group acts by a matrix multiplication on the left of this matrix and the scalar ρ transforms by a fractional linear transformation $\rho \rightarrow (a\rho + b)/(c\rho + d)$. The $U(1)_B$ acts on the right by

$$V \rightarrow V \begin{pmatrix} e^{i\xi} & 0 \\ 0 & e^{-i\xi} \end{pmatrix} ,$$

with

$$e^{i2\xi} = \frac{c\rho^- + d}{c\rho + d} .$$

Using the supergravity equations of motion allows us to relate the field ρ to the Ramond scalar field $C^{(0)}$ and the dilaton and the string coupling constant $g_s = \exp(\varphi)$ as

$$\rho = C^{(0)} + \frac{i}{g_s} .$$

The second special feature of this chiral theory resides in its peculiar point-like solitonic solution to the equations of supergravity. Despite the fact that this solution looks singular, because it is localized in spacetime, it belongs to the class of D-brane instantonic solutions [8]. The metric induced by the presence of N D-instantons is given (in Euclidean space) in the Einstein metric [9] by

$$g_{\mu\nu}^E = \delta_{\mu\nu}, \quad e^\varphi = g_s \left(1 + c_0 \frac{g_s N l_s^8}{r^8} \right) .$$

The presence of this D-instanton induces a correction of order $\alpha'^3 = l_s^6$ to the effective action expressed in the Einstein frame as [2]

$$S \sim \frac{1}{l_s^8} \int d^{10}x \det(E_\mu^m) \left[R + \frac{l_s^6}{2^{11}\pi^7} \mathbf{f}^{(0,0)}(\rho, \bar{\rho}) R^4 + \mathbf{f}^{(12,-12)}(\rho, \bar{\rho}) \Lambda^{16} + \dots \right] , \quad (2.1)$$

where Λ is a complex chiral $SO(9, 1)$ spinor which transforms under the $U_B(1)$ R-symmetry with charge $\frac{3}{2}$ (see [10] for detailed expressions). Quantum effects induced by looping around an arbitrary number of D-instantons are given by the functions $f^{(w,-w)}(\rho, \rho^-)$. These functions are modular forms of $Sl(2, \mathbb{Z})$ of indicated weight up to a phase [11] ($\gamma \in Sl(2, \mathbb{Z})$)

$$f^{(w,-w)}(\gamma \cdot \rho, \gamma \cdot \rho^-) = \frac{c\rho^- + d}{c\rho + d}{}^w f^{(w,-w)}(\rho, \rho^-) .$$

They are connected to the modular function $f^{(0,0)}$ by repeated action of the covariant derivative $D = (i\rho_2 \partial_\rho + w/2)$ which maps a (q, ρ) modular form into a $(q + 1, \rho - 1)$ form [10, 12]. The small-coupling expansion, $g_s \rightarrow 0$, of $f^{(w,-w)}$ reads

$$f^{(w,-w)} = 2\zeta(3) e^{-3\varphi/2} \left[\frac{1}{\Gamma} + \frac{2}{\Gamma} e^{\varphi/2} + \sum_{K=1}^{\infty} G_{K,w} e^{K\varphi/2} \right] ,$$

where the first two terms have the form of string tree-level and 1-loop terms and $G_{K,w}$ contains the charge- K D-instanton and anti-D-instanton terms. The instanton contribution $G_{K,w}$ has the asymptotic expansion in powers of $Ke^{-\varphi}$,

$$G_{k,w} = \mu(K) (4\pi K e^{-\varphi})^{-7/2} e^{2i\pi K \rho} S_w^D + e^{-2i\pi K \bar{\rho}} S_{-w}^D . \quad (2.2)$$

We will denote hereafter

$$\mu(K) = \sum_{0 < m | N} \frac{1}{m^2} \tag{2.3}$$

the measure factor, and the coefficients S^D_w for the D-instantons and S^{-D}_{-w} for the anti-D-instantons are given by

$$S^D_w = (4\pi K e^{-\varphi})^{w+4} \sqrt{2\Gamma - 1 - w} \times \sum_{p \geq 1} \frac{(-1)^p \Gamma(p - w - \frac{1}{2}) \Gamma(p - w + \frac{3}{2})}{p! \Gamma - w - \frac{1}{2} \Gamma - w + \frac{3}{2}} (4\pi K e^{-\varphi})^{-p} . \tag{2.4}$$

The phase of these modular functions is compensated by those of the fields multiplying them in the action. These fields carry a tensorial structure which has a well defined $U(1)_B$ weight induced by the coset space structure of the theory [12]. As was discovered in [6, 7], all these terms can be packaged in a chiral superfield $\Phi(z^M)$ for the type IIb supergravity, satisfying the constraints

$$D\Phi = 0, \quad D^4\Phi = 0 = \bar{D}^4\Phi.$$

These constraints imply that this superfield is independent of $(\vartheta^\alpha)^*$ and is a function $\Phi(x^\mu - \bar{\vartheta}^\mu \vartheta, \vartheta^\alpha)$, which can be expanded in terms of ϑ and $\bar{\vartheta}^* = \vartheta^T \Gamma^0$ as [6, 7]

$$\Phi = \tau_o + \hat{\Phi} = \tau - 2i\bar{\vartheta}^* \lambda - \frac{1}{24} \hat{G}_{\mu\nu\rho} \bar{\vartheta}^* \gamma^{\mu\nu\rho} \vartheta + \dots - \frac{1}{48} i R_{\mu\sigma\nu\tau} \bar{\vartheta}^* \gamma^{\mu\nu\rho} \vartheta \bar{\vartheta}^* \gamma^{\sigma\tau} \vartheta + \dots \tag{2.5}$$

and the correction to the action can be obtained by picking the terms containing 16 powers of ϑ in the Taylor expansion of some unknown function of the chiral superfield Φ [12]

$$S^{(3)} = \int d^{10}x d^{16}\vartheta \det(E_\mu^m) (F[\Phi] + \text{c.c.}).$$

The superfield Φ does not have a well defined $U(1)_B$ weight, because τ does not have a proper weight [13]. However, the fluctuations $\delta\tau$ around a classical value τ_o have weight +2 like the other terms in $\hat{\Phi}$. The superfield only depends on half of the fermionic coordinates. This is natural as it is related to the contribution of the D-instantons, which are half-BPS states. This welcome feature will enable us to find a correspondence to this field in the matrix model in section 6.2. We will now turn to an interpretation of the previous formulae in the matrix model setting.

3. The matrix model

The Dp -brane, and in particular the D-instantons ($p = -1$), have their dynamics described by open strings attached on their worldsheet [8]; thus the low-energy excitations of N Dp -branes on top of each other are described by a dimensional reduction to the $p + 1$ dimension of the ten-dimensional super-Yang–Mills theory with gauge group $U(N)$ [14] (see [15] for a comprehensive lecture on this subject). The ten-dimensional super-Yang–Mills theory has the Lagrangian

$$S_{[D=10]} = \frac{1}{g_{10}^2} \int d^{10}x \text{Tr} \left[-\frac{1}{4} F^2 + \frac{1}{2} i W^T \Gamma^0 \Gamma^\mu D_\mu W \right],$$

† $P_\mu = \partial_\mu \Phi$ with $P_\mu = -\epsilon_{\alpha\beta} V^\alpha \partial_\mu V^\beta$ has $U(1)_B$ weight +2. The weights normalized as in [7] are half those of [6, 13].

where the curvature field $F_{\mu\nu} = [D_\mu, D_\nu]$, the covariant derivative $D_\mu \stackrel{\text{def}}{=} \partial_\mu - i[A_\mu, \cdot]$, the Hermitian connection $A_\mu = A_\mu^a T_a$ and the fermion fields are 16-component Majorana–Weyl spinors of $SO(1, 9)$ Hermitian matrices in the adjoint of the gauge group $W = W^a T_a$. This theory is invariant under the supersymmetry transformation

$$\delta_\zeta A_\mu^a = \frac{1}{2} i \bar{\zeta} \Gamma_\mu W^a, \quad \delta_\zeta W^a = -\frac{1}{4} \Gamma^{\mu\nu} F_{\mu\nu}^a \zeta,$$

where ζ is a Majorana–Weyl spinor.

For later convenience, we rewrite the fields and the supersymmetry transformation by splitting the $SU(N)$ and the Abelian $U(1)$ part of the matrices: $W = \psi + \vartheta \mathbf{1}$, $A_\mu \stackrel{\text{def}}{=} X_\mu + X_\mu \mathbf{1}$,

$F_{\mu\nu} \stackrel{\text{def}}{=} F_{\mu\nu} + f_{\mu\nu} \mathbf{1}$ and

$$\delta_\zeta^1 X_\mu^a = \frac{1}{2} i \bar{\zeta} \Gamma_\mu \psi^a, \quad \delta_\zeta^1 \psi^a = -\frac{1}{4} \Gamma^{\mu\nu} F_{\mu\nu}^a \zeta,$$

for the $SU(N)$ part and

$$\delta_\zeta^2 X_\mu = \frac{1}{2} i \bar{\zeta} \Gamma_\mu \vartheta, \quad \delta_\zeta^2 \vartheta = -\frac{1}{4} \Gamma^{\mu\nu} f_{\mu\nu} \zeta,$$

for the $U(1)$ part.

In the particular case of the D-instanton we obtain a zero-dimensional model (i.e. the variables no longer depend on any coordinates) called the IKKT matrix model [14, 16]. The Lagrangian reduces to

$$S_{[10 \rightarrow 0]} = \frac{1}{g_0^2} \text{Tr} \frac{1}{4} \sum_{0 \leq \mu < \nu \leq 9} [X_\mu X_\nu]^2 + \frac{1}{2} \psi^T \Gamma^0 \Gamma^\mu [X_\mu \psi], \quad (3.1)$$

$$\delta_\zeta^1 X_\mu^a = \frac{1}{2} i \bar{\zeta} \Gamma_\mu \psi^a, \quad \delta_\zeta^1 \psi^a = -\frac{1}{4} \Gamma^{\mu\nu} F_{\mu\nu}^a \zeta,$$

where ζ is a Majorana–Weyl spinor. The model possesses an additional supersymmetry transformation which only involves the $U(1)$ part of the fields

$$\delta_\zeta^2 \vartheta = \zeta, \quad \delta_\zeta^2 X_\mu = \frac{1}{2} i \bar{\zeta} \Gamma^\mu \vartheta, \quad (3.2)$$

with ζ a Majorana–Weyl spinor. As all the fields are in the adjoint of the group, the Abelian piece of the coordinates has disappeared in the previous action, but as we will see in section 6.2 this part still plays a role in the dynamics of the model. It should be noted that this supersymmetry transformation is a superspace translation for the coordinate z^M .

As this model gives a description of the low-energy excitation of the open string with end points fixed on the D-instantons, the gauge coupling of this model is fixed to be $g_0^2 = g_s/l_s^4$.

Two other models relevant to this paper are: (a) the dimensional reduction to the quantum mechanical model in $1 + 0$ dimensions [17–19], known as the BFSS model after its revival by the paper [20],

$$S_{[10 \rightarrow 1]} = \frac{1}{2g_1^2} \int dt \text{Tr} \left[-\frac{1}{2} (\partial_t A^m)^2 + \frac{i}{2\pi l_s^2} W^T \Gamma^0 \partial_t W \right. \\ \left. + \frac{1}{4\pi^2 l_s^4} \sum_{1 \leq m < n \leq 9} [X_m X_n]^2 + \frac{1}{4\pi^2 l_s^4} \psi^T \Gamma^0 \Gamma^m [X_m \psi] \right], \quad (3.3)$$

where the coupling constant is given by $g_1^2 = g_s l_s$; and (b) the dimensional reduction to $1 + 1$ dimensions considered by [21],

$$S_{[10 \rightarrow 2]} = \frac{1}{g_2^2} \int d^2x \text{Tr} \left[-\frac{1}{4} F^2 - \frac{1}{8\pi^2 l_s^4} (D_\alpha A_\alpha)^2 + \frac{i}{4\pi l_s^2} W^T \Gamma^0 \Gamma^\alpha D_\alpha W \right. \\ \left. + \frac{1}{8\pi^2 l_s^4} \sum_{1 \leq i < j \leq 8} [X_i X_j]^2 + \frac{1}{8\pi^2 l_s^4} \psi^T \Gamma^0 \Gamma^i [X_i \psi] \right], \quad (3.4)$$

the coupling constant is given by $g^2 = g_s (2\pi l_s)^{-2}$.

In order to stress the importance of the supersymmetry we will consider more generally the models deduced by dimensional reduction from the $D = 3, 4, 6$ and 10 super-Yang–Mills theory. The amplitudes associated with these half-BPS contributions correspond to the vacuum expectation values of the same number of fermionic zero-modes as the non-broken supersymmetries. That is, 16 real fermions for $D = 10$, 8 real fermions for $D = 6$ and 4 real fermions for $D = 4$. This product of fermionic zero-modes is the fermion number operator of the theory $(-)^F$. So this amplitude is (a part of) the Witten index of the model.

4. The dynamics of the models

These matrix models all have in common a quartic bosonic potential $V_B = \text{Tr}[X, X]^2$ and a fermionic one $V_F = \text{Tr} W^\dagger \Gamma^0 \Gamma^\mu [X_\mu, W]$. As the coordinates are Hermitian matrices, the potential V_B is negative definite. The classical space of configurations is given by the vanishing of this potential $V_B = 0 = V_F$, and is described by the space parametrized by the eigenvalues of the matrices modulo permutations, so this is

$$M = (\mathbb{R}^d)^N / S_N,$$

if we have d -matrix coordinates X^i ($i = 1, \dots, d$) in the adjoint of $U(N)$ whose Weyl group is S_N . The potential is composed of valleys along the direction in the Cartan subalgebra of the group with a harmonic-oscillator-like shape $V_B \sim \omega^2 y^2$ for the coordinates orthogonal to these directions.

The description of this system in terms of harmonic oscillators shows that the spectrum of the *purely bosonic theory* is discrete (see [22] for a mathematical proof and [23] for numerical evidence). The physical argument given in [19], is that one chooses a flat direction by setting the coordinate X^d in the Cartan subalgebra of the group; then all transverse coordinates have a harmonic potential $V_B \sim |X^d|^2 Y^2$. The quantum model has a zero-point energy $E_0 \sim |X^d|/2$ which grows linearly along the flat direction, preventing the wavefunction from extending to infinity along X^d and localizing it. It then follows that the spectrum is discrete. As was shown rigorously by de Wit *et al* in [19], the spectrum of the supersymmetric theory is continuous. This is due to the cancellation of the zero energy between the transverse fluctuations of the bosonic coordinates orthogonal to the Cartan directions and their fermionic partners. Their result showing that there is continuous spectrum, does not prevent the theory from having a discrete spectrum sitting in the middle of it. A non-zero value for the Witten index of the theory will show that. The computation of the index is complicated by the presence of the continuous spectrum and the flat directions which could lead to infrared divergences. However, as noticed from explicit computation [24–26], this is not the case. We recall briefly how to perform the computation of the Witten index in order to justify the infrared finiteness of the model we will consider later on.

The Witten index of the $U(N)$ supersymmetric quantum mechanical model in $D = 1$ dimensions (3.3 is the particular case $D = 0$) is defined as the trace with insertion of the fermion counting operator

$$I_W \stackrel{\text{def}}{=} \lim_{\beta \rightarrow \infty} \text{tr} (-)^F e^{-\beta H}.$$

It was shown by Yi [25] and Sethi and Stern [26], that the computation can be reduced to computing the bulk part

$$I_W(\beta) \int \lim_{\beta \rightarrow 0} \frac{1}{\text{Vol } G} d\eta [dx]^{D-1} \text{tr} (-)^F \underline{\underline{e^{i\eta^A C^A}}} e^{-\beta H},$$

and a deficit part

$$I_{\text{deficit}} = \int_0^\infty d\beta \frac{d}{d\beta} \text{tr} (-)^F e^{-\beta H}$$

which can be rewritten as

$$I_{\text{deficit}} = - \int_0^\infty d\beta \text{tr} (-)^F H e^{-\beta H} = \int_{E>0} \{\rho_+(E) - \rho_-(E)\}.$$

From this expression it is obvious that the non-zero value of the deficit term is given by the continuous part of the spectrum, as the discrete parts cancels by supersymmetry. The total index can be rewritten as

$$I_W = I_W(0) + I_{\text{deficit}}.$$

Following the above references, we rescale the field by

$$\eta \rightarrow \beta X^0, \quad \chi^m \rightarrow X^m, \quad W \rightarrow W.$$

In the $\beta \rightarrow 0$ limit it is possible to expand the trace over the bosonic coordinates using the heat kernel expansion [25, 26], and the bulk term can easily be rewritten as $I_W(0) = \lim_{\beta \rightarrow 0} Z_{[D \rightarrow 0]}^{(N)}$ with

$$Z_{[D \rightarrow 0]}^{(N)} = \int \frac{1}{\beta^{(N^2-1)/(D-2+N/2)}} [dX][dW] e^{-\beta S_{[D \rightarrow 0]}} \tag{4.1}$$

where $S_{[D \rightarrow 0]}$ is the zero-dimensional matrix model action

$$S_{[D \rightarrow 0]} = \frac{\beta}{4\pi^2 g_s l_s^5} \text{Tr} \left\{ \frac{1}{4} X^\mu, X^\nu \right\}^2 + \frac{1}{2} W^T \Gamma^0 \Gamma^\mu [X^\mu, W] \tag{4.2}$$

obtained by dimensional reduction from the D -dimensional one. The upshot of expression (4.1) is that the overall power of β cancels due to identity of the bosonic transverse and the fermionic degrees of freedom, so the $\beta \rightarrow 0$ limit is well defined. In the previous path integral the integration over the fermion has to be done with *periodic* boundary condition as enforced by the insertion of the $(-)^F$ operator in the definition of the index†. In the final expression, we see that we have reduced the model to a zero-dimensional matrix model with an effective coupling constant given by $g_{\text{eff}}^2 = 4\pi^2 g_s l_s^5 / \beta$.

Of course, in a rigorous computation along the lines of [25, 26] or [27], all the elements of the matrices belonging to the non-Cartan part of the group have to be included, but as we will see in the following, putting this model on a 2-torus and sending β to zero *before* computing the integrals gives the correct answer. We want to use the non-trivial structure of the orbifold limit of two-dimensional gauge field theory [21]. When taking this limit, the scaling of the fields is crucial, as the important contributions have to come from fixed points of this orbifold space [28]. In our case the relevant scaling can be understood as follows.

We start from the zero-dimensional model (4.2) and we compactify the coordinates X^0 and X^9 on a 2-torus. Rewriting those fields in a Fourier transform basis [29] $FT (X^{0,D-1}) = 2\pi l_s^2 D_{0,D-1}$ gives

$$S_B = \frac{1}{4g_{\text{eff}}^2} \int d^2\sigma \text{Tr} \left((2\pi l_s^2)^4 F^2 + (2\pi l_s^2)^2 (DX^I)^2 + [X^I, X^J]^2 \right).$$

† Hereafter, we call this contribution a partition function, but one should keep in mind that the fermions satisfy periodic boundary conditions.

As the limit $\beta \rightarrow 0$ is the same as $g_{\text{eff}}^2 \rightarrow \infty$ in order to exhibit the non-trivial dynamics of this limit, we rescale the fields as

$$S_B = \int d\sigma \frac{4g}{g_{\text{eff}}^2} \text{Tr} F^2 + \frac{1}{4} \text{Tr}(DX)^2 + \frac{g_{\text{eff}}^2}{4(2\pi l_p^2)^2} \text{Tr}[X, X]^2.$$

We can now consider the infrared limit ($g_{\text{eff}}^2 \rightarrow \infty$). In this limit the gauge field decouples and we have to set the bosonic and fermionic potentials to zero. The existence of flat directions gives rise to a moduli space with an orbifold structure $(\mathbb{R}^{D-2})^N / S_N$ where the group of permutations S_N is the Weyl group of $U(N)$. This is a convenient way to separate the dynamics of these matrix models, which split naturally into the dynamics of two of the coordinates, the gauge field and an X^9 coordinate for the $0 + 1$ quantum mechanical models analysed in [19] or for the topological setting of [27].

Moreover, due to the supersymmetry of the model and the fact that we will subtract the zero-modes of the fields (see the definition of the measure of (5.1)), the computation will *not* suffer from any infrared divergences as the regulator cancels automatically between the $D - 2$ bosonic coordinates and the $\mathcal{N}/2$ fermionic ones[†]. This is intimately linked to the underlying gauge symmetry of the problem. A $U(N)$ matrix model *without* the gauge symmetry has no reason to be free of infrared divergences. In the end, we will be left with a two-dimensional gauge field theory, as we will see in equation (5.5)[‡].

5. The quasi-classical evaluation of the partition function

In order to perform the computation we put the previous matrix model on a two-dimensional torus, and adopt the language of the reduction to two dimensions of the ten-dimensional super-Yang–Mills theory. We will follow closely the notation and the logic of [30], except that the Yang–Mills coupling constant will be reintroduced. We will consider the generalized setting of the model deduced by dimensional reduction from the $D=3, 4, 6$ and 10 $U(N)$ super-Yang–Mills theories. A subscript $[D \rightarrow d]$ will indicate the dimension D of the mother theory and the dimension d of the model under consideration. The computation will be done after having projected all the fields onto the Higgs branch, described by the symmetric orbifold space $(\mathbb{R}^{D-2})^N / \mathcal{S}$, and the zero-size limit of the torus will be taken to return to the zero-dimensional model of equation (3.1).

The partition function under study is

$$Z_{[D \rightarrow 2]}^{(N)}(T, g) = \int [DA][DX][DW] e^{-S_{[D \rightarrow 2]}[A, X, W]}. \tag{5.1}$$

The action is that of the matrix string on a two-dimensional Minkowskian torus (3.4)

$$S_{[D \rightarrow 2]} = \frac{1}{g_{\text{YM}}^2} \int d^2z \text{Tr} \left(F_{ab}^2 + D_a X^I{}^2 + i\bar{W}\Gamma^a D_a W - [X^I, X^J]^2 + \bar{W}\Gamma^I [X^I, W] \right), \tag{5.2}$$

with the coupling constant $\hat{g}_{\text{YM}}^2 = g_{\text{eff}}^2 RT = 4\pi^2 g_s l_s^5 RT / \beta$. T-duality relates the radii to the string coupling g_s and the T-dual dilaton φ constant by $RTg_s = RT \exp(\varphi) = \exp(\varphi) l_s^2$; henceforth the gauge coupling is expressed as $\hat{g}_{\text{YM}}^2 = 4\pi^2 l_s^7 \exp(\varphi) / \beta$. Due to the invariance of the theory under area-preserving diffeomorphisms of the two-dimensional model, every quantity computed in this theory is a function only of the parameter $RT \hat{g}_{\text{YM}}^2 = 4\pi^2 l_s^9 \exp(2\varphi)$

[†] With the normalizations of [30], the contributions of the bosonic and fermionic Higgs fields cancel; this is, in particular, independent of the gauge coupling.

[‡] The finiteness of the matrix integral (4.1) for the $U(2)$ case was shown by the explicit computations of [23–26].

$\varphi)/\beta \sqrt{4\pi^2/g_s\beta}$. In the second equality we used the invariance under area-preserving diffeomorphisms to set to zero the value of the T-dual dilaton φ^\dagger . This will be important when we make contact with the supergravity theory in section 6.2. The integration measure has the zero-modes deleted [30]:

$$\begin{aligned}
 W_\alpha^{(0)} &= \frac{\text{Tr} \int_{T^2} d^2z W_\alpha}{NRT}, & X_I^{(0)} &= \frac{\text{Tr} \int_{T^2} d^2z X_I}{NRT}, \\
 \theta(\tau) &= \text{Tr} \int_0^R d\sigma A_\sigma(\sigma, \tau), & A_\sigma^{(0)} &= \frac{\int_0^{m\tau} d\tau \theta(\tau)}{NRT}, \\
 [DA] &= DA \delta \frac{A^{(0)}}{2\pi\hat{g}_{\text{YM}}}, & [DX] &= DX \delta \frac{X^{(0)}}{2\pi\hat{g}_{\text{YM}}}, \\
 [DW] &= DW \frac{W^{(0)}}{\hat{g}_{\text{YM}}}.
 \end{aligned}$$

The worldsheet coordinates are $z = (\sigma, \tau) \in [0, R] \times [0, T]$.

5.1. The $\hat{g}_{\text{YM}}^2 \rightarrow \infty$ limit

The limit $\hat{g}_{\text{YM}}^2 \rightarrow \infty$ (corresponding to $\beta \rightarrow 0$) for the bulk term of the Witten index projects the computation on the Higgs branch of the model. In [30] the field configuration in the infrared limit of the model was worked out. In this limit all matrices $\{D_\alpha, X_I, W_\alpha\}$ are simultaneously diagonalizable with a unitary matrix $V(\sigma, \tau)$ such that

$$f(\sigma, \tau) = V^{-1}(\sigma, \tau) f^D(\sigma, \tau) V(\sigma, \tau)$$

where $f^D = \text{diag}\{f_1, \dots, f_N\}$. We therefore have

$$f^D(R, \tau) = \hat{T}^{-1} f^D(0, \tau) \hat{T}, \quad f^D(T, \sigma) = \hat{S}^{-1} f^D(0, \sigma) \hat{S}$$

where $\hat{S} = V^{-1}(R, \tau) V(0, \tau)$ and $\hat{T} = V^{-1}(\sigma, T) V(\sigma, 0)$. By construction, $\hat{S}\hat{T} = \hat{T}\hat{S}$, and the matrices \hat{S} and \hat{T} represent two commuting permutations, s^\wedge and t^\wedge , of the diagonal elements. The permutations are given explicitly as commuting elements $s \mapsto s_i$ and $t \mapsto t_i$ of the symmetric group S_N

$$\hat{T}^{-1} f \hat{T} = f_{t_i}, \quad \hat{S}^{-1} f \hat{S} = f_{s_i}$$

The saturation of the fermionic zero-modes restricts the computation to a one-component covering of the torus, and the permutations are given by

$$s^\wedge = \{j \rightarrow i + m \pmod{N}\}, \quad \hat{t} = \begin{cases} \{j \rightarrow i + 1 \pmod{m}\} & \text{if } j = 0 \\ \{j \rightarrow i - j \pmod{N}\} & \text{if } j = 1, \dots, n - 1. \end{cases} \quad (5.3)$$

The topological sectors of the partition function are classified by the permutations

$$[DA][DX][DW] \rightarrow \frac{1}{N!} \sum_{s^\wedge t^\wedge = t^\wedge s^\wedge} [DA^D][DX^D][DW^D]$$

† This choice of unit corresponds to the limit $RT \rightarrow 0$ and $l_s \rightarrow 0$ with $l_s^2/(RT)$ kept fixed.

and the partition function after integration over the bosonic Higgs fields and the fermions can be written as

$$Z_{[D \rightarrow 2]}^{(N)}(RT \hat{g}_{\text{YM}}^2) = \frac{(N-1)!}{N!} \sum_{mn=N} \sum_{j=0, \dots, n-1} \delta_{m,n}^{\text{susy}} Z_{[m,n;j]} = \sum_{m|N} \frac{1}{m} \delta_{m,n}^{\text{susy}} Z_{mn}. \tag{5.4}$$

The function $\delta_{m,n}^{\text{susy}}$ is a function of the boundary conditions induced by the integration over the fermions, which will be discussed in the next subsection. It was shown in [30] that Z_{mn} reduces to the partition function of a $U(1)$ gauge theory defined on the torus of area NRT with periods (mT, jR) and $(0, nR)$ because, having subtracted the zero-modes, supersymmetry ensures that the bosonic and fermionic determinants cancel exactly, with the result†

$$Z_{[D \rightarrow 2]}^{(N)}(RT \hat{g}_{\text{YM}}^2) = \frac{1}{T} \sum_{m|N} \frac{1}{m^2} \delta_{m,n}^{\text{susy}} \sum_{E \in \mathbb{Z}} \exp \left[-\frac{E^2}{2\hat{g}_{\text{YM}}^2 RTN} \right]. \tag{5.5}$$

Decompactifying with the limit $RT \rightarrow 0$ and setting $V = \text{Vol}(SU(N)/Z_N) \stackrel{\text{def}}{=} (RT)^{7/2} N^2$ to be the overall volume factor, the partition function is

$$Z_{[D \rightarrow 0]}^{(N)} = V \times RT \hat{g}_{\text{YM}}^2 \sum_{m|N} \frac{1}{m^2} \delta_{m,n}^{\text{susy}}. \tag{5.6}$$

The factor $RT \hat{g}_{\text{YM}}^2$ comes from the normalizations of the $U(1)$ part of the matrices.

5.2. Constraints from supersymmetry

We have to compute the integral over the fermionic variables

$$Z[A] \Big|_{\text{b.c.}} \int [DW^D] \exp \bar{W}^D i DW^D \tag{5.7}$$

with boundary conditions (5.3). To compute the integral over the fermions it is crucial to define the measure. The measure $[DW^D]$ is defined using the mode expansion of the W^i with respect to the kinetic operator [33]. We consider

$$iDW\varphi_n = \lambda_n \varphi_n \quad W(\sigma, \tau) = \sum_n \varphi_n(\sigma, \tau) a_n, \quad \bar{W}(\sigma, \tau) = \sum_n \varphi_n^\dagger(\sigma, \tau) b_n.$$

Hence

$$[DW^D] \stackrel{\text{def}}{=} \prod_n da_n db_n.$$

In the case with $N = 16$ real supercharges the kinetic operator splits as $S_\alpha^t DS_\alpha + S_{\dot{\alpha}}^t \bar{D}S_{\dot{\alpha}}$, with respect to the two fermionic Spin(9) representation of a Majorana–Weyl spinor $W_{\mp\mp}(S_\alpha S_{\dot{\alpha}})$. The index α runs over the $\mathfrak{8}_s$ representation, and $\dot{\alpha}$ over the $\mathfrak{8}_c$ representation of Spin(9); see [34] and appendix B for representations of the Clifford algebra. So the mode expansion for the lightcone fermions S is $S = \sum_n \varphi_n(\sigma, \tau) a_n$ and all boundary conditions are satisfied.

For the theories with $N = 4, 8$ real supercharges, such a chiral factorization is not allowed as the fermions only satisfy the Majorana condition. The kinetic operator for the case with $N = 4$ real supercharges reads

$${}^t\chi_\alpha i\sigma^3 \partial_1 \chi_\alpha + \sigma^2 \partial_0 \bar{\chi}^{\dot{\alpha}} + {}^t\bar{\chi}^{\dot{\alpha}} i\sigma^3 \partial_1 \chi_\alpha + \sigma^2 \partial_0 \bar{\chi}^{\dot{\alpha}}.$$

† The fact that the $U(N)$ partition function can be decomposed as a sum over $U(1)$ partition functions with respect to S_N -cycle decomposition over a torus of size extended by the length of the cycle, was known by IK Kostov and the author and appeared in [31]; this was discovered independently by [32].

In the Higgs phase $\chi_\alpha = \text{diag}(\chi^l_d) (l = 1, \dots, N)$ the equations of motions read for each eigenvalue

$$\begin{aligned} \partial_1 \chi_1 - \partial_0 \bar{\chi}^2 &= 0, & -\partial_1 \chi_2 + \partial_0 \bar{\chi}^1 &= 0 \\ \partial_1 \bar{\chi}^1 - \partial_0 \chi_2 &= 0, & -\partial_1 \bar{\chi}^2 + \partial_0 \chi_1 &= 0. \end{aligned}$$

These equations imply that all fermions satisfy the Klein–Gordon equation $(\partial^2 - \partial^2)\chi = 0$, and due to the reality condition $\chi_\alpha = (\chi_\alpha)^*$ and $\bar{\chi}^\alpha = (\bar{\chi}^\alpha)^*$ their mode expansion is

$$\chi_\alpha = \sum_n e^{-n\tau} a_{n,\alpha} e^{-in\sigma} + a_{n,\alpha}^* e^{in\sigma}.$$

Therefore, only trivial boundary conditions in the σ direction are possible. This means that only the configuration with $n = 1$ (i.e. $S = 1$) and $m = N$ contribute to the partition function (5.6).

From this analysis, the constraints from the integration over the fermions are summarized by

$$\delta_{m,n}^{\text{susy}} = \begin{cases} n = 1, m = N & \text{when } N = 4, 8 \\ mn = N & \text{when } N = 16, \end{cases} \quad (5.8)$$

giving the final result

$$Z_{[D \rightarrow 0]}^{(N)} = V \times RT \hat{g}_{\text{YM}}^2 \sum_{m|N} \frac{1/N^2}{1/m^2} \begin{matrix} \text{for } N = 4, 8 \text{ i.e. } D = 4, 6 \\ \text{for } N = 16 \text{ i.e. } D = 10. \end{matrix} \quad (5.9)$$

The special case with two real super-charges ($D = 3$) is treated independently in appendix A.

6. Contact with threshold corrections

6.1. The heterotic/type I threshold corrections

There is another example where the matrix string setting is helpful to analyse the configuration of fermionic zero-modes, namely the case of the F^4 and R^4 terms in the effective action for the $Spin(32)/Z_2$ type I theory. The contribution of the complete effective action for the type I string on $R^{1,7} \times T^2$ consists of perturbative and non-perturbative terms induced by the wrapped Euclidean† D1-brane on a 2-torus included in the d -torus‡. These contributions can be completely evaluated because they are all mapped together to the 1-loop amplitudes of the heterotic string on $R^{1,7} \times T^2$. The non-perturbative part on the type I side can be written in the compact form as [3, 37]

$$I^{\text{inst}} = - \frac{V}{2^8 \pi^4} \sum_{\kappa=1}^{\infty} \frac{e^{-2i\pi\kappa T}}{T_2} H_\kappa[O\hat{A}](U) + \text{c.c.}$$

where $O = 1 + \dots$ is a differential operator, whose action is induced by the non-holomorphic terms in the elliptic genus \hat{z} and gives rise to a finite number of higher-loop effects around the D-instanton. Here

$$H_\kappa[\hat{A}](U) = \frac{1}{K} \sum_{\substack{mn \kappa \\ 0 \leq \bar{j} \leq n}} \hat{A} \frac{j+mU}{n}$$

† This wrapped D1-brane should be Euclidean, so a Wick rotation on the worldsheet of the previous two-dimensional model is necessary. The Majorana–Weyl fermions are now converted to complex Weyl fermions. See [35], for instance, for an explanation of how to handle this case.

‡ While this paper was being proof read the preprint [36] appeared with some related comments.

is the Hecke operator of rank K acting on the modular elliptic genus \hat{A} , whose gauge field part

$$\hat{A}(U) = t_8 \operatorname{tr} F^4 + \frac{1}{2^9 3^2} \frac{E^3}{\eta^{24}} + \frac{\hat{E}_2^2 E_4^2}{\eta^{24}} - 2 \frac{\hat{E}_2 E_4 E_6}{\eta^{24}} - 2 \sqrt{3}^2 t_8 (\operatorname{tr} F^2)^2$$

only will be needed for this discussion. The coefficient of the $\operatorname{tr} F^4$ terms is given explicitly by

$$H_K(1) = \sum_{m|K} \frac{1}{m}.$$

In this case the pertinent (T-dual) matrix model is the $(8, 0)$ quantum mechanical model considered in [38–40]

$$\begin{aligned} S_{[10, 1]} &= \frac{1}{2g} \operatorname{Tr} \left[\frac{1}{2} (DX^m)^2 - \frac{1}{2} (D\Phi)^2 + \frac{1}{4} X^m, X^m{}^2 + \frac{1}{2} i \textcircled{1} D \textcircled{1} \right. \\ &\rightarrow \left. - \frac{1}{2} i \textcircled{1} [\Phi, \textcircled{1}] - \frac{1}{2} [\Phi, X^m]{}^2 - \frac{1}{2} i \lambda D \lambda - \frac{1}{2} i \lambda [\Phi, \lambda] \right. \\ &\quad \left. + i X^m \nu_{\alpha\beta}^m \{ \textcircled{1}_\alpha, \lambda_\beta \} + i \chi' D \chi' + i \chi' \Phi \chi' + i m'{}^J \chi_I \chi_J \right] \end{aligned}$$

The coordinates X and $\textcircled{1}$ are in the symmetric representation of $SO(N)$, while Φ, λ and the gauge connection A_0 are in the antisymmetric representation. χ^I ($I = 1, \dots, 2N$) are in the real representation $(K, 2N)$ of $SO(K) \times SO(2N)$. We now put the coordinate Φ on a circle of radius l_s^2/R , and convert it into a second gauge connection component. We obtain a two-dimensional gauge model. This model presents several important differences relative to the one studied previously. Only the matrices X and $\textcircled{1}$ have Abelian degrees of freedom. The gauge coordinates do not have any Abelian quantum numbers because the group is $SO(K)$. Then the measure for the gauge connection is simply

$$[DA] = DA.$$

We assume[†] that we can project the theory onto the classical moduli space, $M = (\mathbb{R}^{1,7} \times T^2)^K / (S_{K/2} \times Z^{K/2})$ [42–44], and decouple the gauge degree of freedom from the Higgs field dynamics. Since we are only interested in the measure factor $\mu(K)$, the gauge field degrees of freedom are the only ones needed. A computation similar to the one given in section 5 gives

$$\mu_{\text{type 1}}(K) = H_K(1).$$

6.2. The D-instantons

In order to make contact with D-instanton corrections of the action (2.1) of section 2, it is necessary to specify the normalizations of the matrix model since for the D-instanton these are different from those of the gauge theory model of section 5. We identify the $U(1)$ part of the matrix coordinates with the superspace coordinates of section 2,

$$x_\mu = \operatorname{Tr} \frac{A_\mu}{N}, \quad \vartheta = \operatorname{Tr} \frac{W}{N}.$$

[†] The saturation of the fermionic zero-modes does not appear as easily as for the model (5.1). And the infrared finiteness of the model is not obvious. Equivalently, the $\beta \rightarrow 0$ limit of the bulk term for the Witten index of this type I[†] model is not as trivial as before. However, the existence of a sound limit is linked with the supersymmetry content of the model, and restricts the integral to be computed. A more rigorous derivation is a little more subtle and is deferred to a future publication [41].

Moreover, it is necessary to eliminate the volume of the gauge group, V ; then the partition function of the D-instanton matrix model is defined by

$$V Z_{\text{D-ins}}^{(N)} \stackrel{\text{def}}{=} \int_1 dX_0 \int_{\mu=1}^{\infty} dX_\mu \delta \frac{\text{Tr } X_\mu}{N} \int_{\alpha=1}^{\infty} dW_\alpha \delta \frac{\text{Tr } W_\alpha}{N} \exp(-S_{\text{D-ins}}), \tag{6.1}$$

$S_{\text{D-ins}} = \frac{1}{4} \text{Tr} [X_\mu, X_\nu]^2 + \frac{1}{2} W^T \Gamma_\mu [X^\mu, W]$
 where $g_{\text{ins}} \stackrel{\text{def}}{=} g_0 = g_s/l^4$. Using $\hat{g}_{\text{YM}}^2 = g_0 RT$ and $RT \hat{g}_{\text{YM}}^2 = g_s$ in the formulae of section 5, we obtain

$$Z_{\text{D-ins}}^{(N)} = \mu(N) N e^{-\varphi}^{-7/2}. \tag{6.2}$$

It should be remarked that this formula is independent of the string scale l_s . Depending on which component of the superfield (2.5) this D-instanton background couples to, a different power of $N \exp(-\varphi)$ appears, namely $w + \frac{1}{2}$, according to (2.2). This can be derived by explaining how to couple the supergravity field to the previous matrix model. We have seen that the computation of the measure factor $\mu(N)$ can be reduced to the contribution from the $U(1)$ part of the two-dimensional gauge field (see equation (5.5)), a feature particularly difficult to analyse in the zero-dimensional version of the model. The various supergravity states which appear in the decomposition of the superfield (2.5) are now seen as states of the Abelian part of the matrix model, and it is possible to construct them as eigenstates of the centre-of-mass Hamiltonian [45] by decomposing the representations of $SO(9)$ under $SO(7) \times U(1)$ †.

It is natural to represent the external supergravity states by Wilson point observables [47]

$$W_{\text{Wilson point}} \stackrel{\text{def}}{=} \text{Tr}_{\text{adjoint}} \exp \int \frac{1}{\hat{g}_{\text{YM}}^2} d^2z \wedge F.$$

This operator is the only observable which decompactifies correctly in the limit $R, T \rightarrow 0$. We promote the $U(1)$ part of this curvature to a superfield $F^{U(1)}(x^\mu, \vartheta^\alpha)$ for the Abelian supersymmetry transformation (3.2) and identify the superfield Φ of (2.5) with $F^{U(1)} = \Phi$. We assign $U(1)_B$ weight $+\frac{1}{2}$ to the supercharges associated with the supersymmetry transformations (3.2) and weight $-\frac{1}{2}$ to the supercharges associated with the transformations δ^1 of the $SU(N)$ part of the coordinates. Units are as before: the supergravity coordinates X have dimension l_s^2 times the gauge field connection ones. The coupling of the matrix states with the supergravity external states is given by insertions of

$$W_{\text{Wilson point}} = N \exp \frac{1}{g_s} \Phi + \text{c.c.} \tag{6.3}$$

The term with $U(1)_B$ -weight w is now given by a correlation of $w + 4$ insertions of the operator $W_{\text{Wilson point}}$ where we pick the $16-\vartheta$ s term. The gauge symmetry fixes the power of the string coupling constant in (6.3) in such a way that the $16-\vartheta$ s term is proportional to $(N/g_s)^{w+4}$, which multiplies (6.2). This is exactly what is needed to reproduce the result of (2.2).

7. Discussion

7.1. The large- N_c limit

The conjecture about the equivalence between the supergravity theory and the super-conformal Yang–Mills theory on the boundary space relies heavily on the large- N_c limit on the Yang–Mills side. Here we have only discussed finite- N_c computations but we can see from these

† See [46] for a clear lecture on the subject.

that a large- N_c limit can be defined. The guideline for this is that we have to keep a *fixed number of fermionic zero-modes* irrespective of the rank of the gauge group. If we consider a configuration of K long strings, restricted to join forming long strings of size a multiple of N_c , but *not to split*, all the dynamics is embedded in a $U(N_c K)$ group. From the previous analysis, of the zero-dimensional model for the D-instantons and the computation of section 6.2, we deduce that the measure factor for the interactions is $\mu(K)$, and the factor of N_c appears as an overall power. It is now safe to take the large- N_c limit keeping K finite.

If we start from a configuration of fields decomposed as $A_{N_c K} \stackrel{\text{def}}{=} A_K \otimes I_{N_c}$ for a $U(N_c K)$ model, the action $S_{\text{D-ins}}$ gets an overall N_c factor, so for such a field configuration

$$\begin{aligned} S_{\text{D-ins}}'_{N_c K} &= N_c S_{\text{D-ins}}'_{K} \\ &= \frac{N_c^4}{g_s} \text{Tr}_{K \times K} \left[\frac{1}{4} [X_\mu X_\nu]^2 + \frac{1}{2} W^T \Gamma_\mu [X^\mu, W] \right] \end{aligned} \quad (7.1)$$

and since we do not obtain any extra factor of N_c from the measure, it follows that

$$Z_{\text{D-ins}}^{(N_c K)} = N_c^{-7/2} Z_{\text{D-ins}}^{(K)} = N_c^{-7/2} \mu(K) (K e^{-\varphi})^{-7/2}. \quad (7.2)$$

The Wilson point observables are now given by

$$W_{\text{Wilson point}} = K \exp \frac{1}{g_s} \Phi + \text{c.c.} ;$$

because the interaction occurs only between long strings of length which is a multiple of N_c , we do not obtain any extra power of N_c by inserting them:

$$(W_{\text{Wilson point}})^{w+4} = N^{1/2} (1/N_c)^4 \mu(K) (K e^{-\varphi})^{-7/2+w+4}. \quad (7.3)$$

This means that we can take a large- N_c limit, with the instanton number K fixed, in a well defined way. This splitting of the fields means that we restrict the integration in the matrix model to occur only between long strings with length *at least* N_c . This can be made much more rigorous by embedding a $U(K)$ instanton constructed by Giddings, Verlinde and Hacquebord [48, 49], tensored with the diagonal matrices doing cycles of length N_c along the lines of [50]†. Finally, the scaling (7.1) shows that the combination $N_c \alpha'^2$ appears naturally, as a reminiscence of the scaling $\alpha'^{-1} \propto N^{-1/2}$ of the AdS/CFT correspondence [4]. The overall power of α' in (7.2) is not correct, due to our inability to derive the relative normalization with respect to the kinetic Einstein–Hilbert term of the supergravity theory.

7.2. D-instantons loop expansion

That we were not able to obtain the full expansion of the modular form (2.2) and (2.4) from our matrix model analysis is certainly due to the fact that we are only getting the physics of the linearized version of the supergravity theory, which gives only the dominant term in the instantons expansion. This is not surprising since the $Sl(2, \mathbb{Z})$ symmetry of the type IIb theory is not explicit in the model. It may be possible to understand how this symmetry can appear along the lines of [13] by considering the constraints from the $U(1)_B$ weight of the fields. Moreover, it would certainly be worth analysing how much information we can obtain about these α'^3 corrections by deforming the superspace analysis. We hope to return to this problem in a future publication.

† This configuration smoothes the gauge interactions out of the infrared limit at finite \hat{g}_{YM}^2

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Appendix A. The $N = 2$ real supercharges case

For the case of $N = 2$ real supercharges the two-dimensional model (5.2) is composed of a gauge field $A_{\sigma,\tau}$ and one bosonic Higgs X , and its real bispinor partner. The only potential is the one induced by the covariant derivative so it will not be necessary in that case to take the infrared limit of the model. The measures are defined as in section 5.

The field $X(\sigma, \tau)$ is conjugate to an element in the Cartan subalgebra of $U(N)$

$$X(\sigma, \tau) = V^{-1}(\sigma, \tau)X^D(\sigma, \tau)V(\sigma, \tau);$$

likewise for the fermion due to spacetime supersymmetry

$$W(\sigma, \tau) = V^{-1}(\sigma, \tau)W^D(\sigma, \tau)V(\sigma, \tau).$$

Using the gauge transform

$$A_\alpha(\sigma, \tau) \rightarrow V^{-1}(\sigma, \tau) \hat{A}_\alpha(\sigma, \tau) + \partial_\alpha V(\sigma, \tau),$$

we can now integrate the diagonalized Higgs fields. Once again, the determinants (without the zero-modes) cancel due to supersymmetry, leaving just a constraint on the gauge field configurations from the equations of motion of the fermions

$$Z_{[3 \rightarrow 2]}^{(N)} = \int [D\hat{A}] \delta_{[\hat{A}]}^{\text{susy}} \exp \int - (1/4\hat{g}_{\text{YM}}^2) F^2 .$$

The equations of motion for the fermions reads

$$\not{\partial} W^D + [\hat{A}, W^D] = 0. \tag{A.1}$$

Because the fermions are in the Cartan torus the group indices of the connection in $[\hat{A}_\alpha, W^D]$ can be restricted to the orthogonal (with respect to the Cartan metric) complementary, n , of the Cartan subalgebra of $u(N)$, $[\hat{A}_\alpha^n, W^D]$. Moreover, $[A_\alpha^n, W^D]$ belongs to n , so from (A.1) we deduce that

$$\not{\partial} W^D = 0, \quad [A_\alpha^n, W^D] = 0.$$

The second equation implies that $\hat{A}_\alpha^n = 0$. Henceforth, the configurations of the fields are classified as before (see equation (5.4) and section 5.2) with the result

$$Z_{[3 \rightarrow 0]}^{(N)} = V \times \frac{1}{RT} \hat{g}_{\text{YM}}^2 \frac{7/2}{N^2}. \tag{A.2}$$

Appendix B. Representation of the real Clifford algebra

We list the $d \times d$ irreducible representations of the real Clifford algebras. The spinors are chosen real, $\psi = \psi^*$, the charge conjugation matrix is $C = \Gamma^0$ and $(\Gamma^\mu)^* = \Gamma^\mu$,

$$\{\Gamma^\mu, \Gamma^\nu\} = 2g^{\mu\nu} \quad (\mu, \nu) \in \{1, 2, \dots, D\}^2, \quad \text{sign}(g) = (-1, +1, \dots, +1).$$

The Pauli matrices are

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and we define $\kappa = i\sigma_2$.

B.1. $N = 2, d = 3$

$$\Gamma^0 = \kappa, \quad \Gamma^1 = \sigma_3, \quad \Gamma^2 = \sigma_1.$$

B.2. $N = 4, d = 4$

The basis used by [34] is

$$\begin{aligned} \Gamma^0 &= \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix} = \sigma_3 \otimes \kappa, & \Gamma^1 &= -1 \otimes \sigma_1, \\ \Gamma^2 &= -\kappa \otimes \kappa, & \Gamma^3 &= 1 \otimes \sigma_3, & \Gamma_5 &= -i\sigma_1 \otimes \kappa. \end{aligned}$$

B.3. $N = 16, d = 16$

We choose a basis well adapted for the $\mathfrak{8}_s \oplus \mathfrak{8}_c$ decomposition of the representations of Spin(9). All gamma matrices are 16-dimensional square matrices,

$$\Gamma^9 = 1 \otimes 1 \otimes 1 \otimes \sigma_3 = \text{diag}(1^8, -1^8), \quad \Gamma^i = \begin{pmatrix} 0 & \gamma_{aa}^i \\ \gamma_{bb}^i & 0 \end{pmatrix} \quad i = 1, \dots, 8.$$

In this basis there is no Γ^0 matrix. The $8 \times 8 \gamma^i$ matrices are defined by

$$\begin{aligned} \gamma^1 &= \kappa \otimes 1 \otimes 1, & \gamma^2 &= \sigma_3 \otimes \kappa \otimes \sigma_3, & \gamma^3 &= \sigma_1 \otimes \sigma_3 \otimes \kappa, & \gamma^4 &= \sigma_3 \otimes \kappa \otimes \sigma_1 \\ \gamma^5 &= 1 \otimes \kappa \otimes 1, & \gamma^6 &= \sigma_3 \otimes 1 \otimes \kappa, & \gamma^7 &= 1 \otimes \sigma_1 \otimes \kappa, & \gamma^8 &= 1 \otimes 1 \otimes 1. \end{aligned} \tag{B.1}$$

We have $\Gamma^i = \gamma^i \otimes \kappa$ for $i = 1, \dots, 7$ and $\Gamma^8 = \gamma^8 \otimes \sigma_1$. The full 32-dimensional gamma matrices are obtained by $\Gamma^\mu = \Gamma^\mu \otimes \sigma_1$.

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