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Some Connections Between (Sub)Critical Branching Mechanisms and Bernstein Functions

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Abstract

We describe some connections, via composition, between two functional spaces: the space of (sub)critical branching mechanisms and the space of Bernstein functions. The functions $e_\alpha : x \rightarrow x^\alpha$ where $x \geq 0$ and $0 < \alpha \leq 1/2$, and in particular the critical parameter $\alpha = 1/2$, play a distinguished role.

1 Introduction

This note is a prolongation of [8] where the following remarkable property of the function $e_\alpha : x \rightarrow x^\alpha$ was pointed at for $\alpha = 1/2$: if Ψ is a (sub)critical branching mechanism, then $\Psi \circ e_{1/2}$ is a Bernstein function (see the next section for the definition of these notions). In the present work, we first show that this property extends to every $\alpha \in]0, 1/2]$. Then we characterize the class of so-called internal functions, i.e. that of Bernstein functions Φ such that the compound function $\Psi \circ \Phi$ is again a Bernstein function for every (sub)critical branching mechanism Ψ . In the final section, we gather classical results on transformations of completely monotone functions, Bernstein functions and (sub)critical branching mechanisms which are used in our analysis.

2 Some functional spaces

2.1 Completely monotone functions

For every Radon measure μ on $[0, \infty[$, we associate the function $\mathcal{L}_\mu :]0, \infty[\rightarrow [0, \infty]$ defined by

$$\mathcal{L}_\mu(q) := \int_{]0, \infty[} e^{-qx} \mu(dx), \quad (1)$$

i.e. \mathcal{L}_μ is the Laplace transform of μ . We denote by

$$\text{CM} := \{ \mathcal{L}_\mu : \mathcal{L}_\mu(q) < \infty \text{ for all } q > 0 \}, \quad (2)$$

which is an algebraic convex cone (i.e. a convex cone which is further stable under inner product). The celebrated theorem of Bernstein (see for instance Theorem 3.8.13 in [6]) identifies \mathbf{CM} with the space of completely monotone functions, i.e. functions $f :]0, \infty[\rightarrow [0, \infty[$ of class \mathcal{C}^∞ such that for every integer $n \geq 1$, the n -th derivative $f^{(n)}$ of f has the same sign as $(-1)^n$. Recall from monotone convergence that \mathcal{L}_μ has a (possibly infinite) limit at $0+$ which coincides with the total mass of μ .

We shall focus on two natural sub-cones of \mathbf{CM} :

$$\mathbf{B}_1 := \left\{ \mathcal{L}_\mu : \int_{]0, \infty[} (1 \wedge x^{-1}) \mu(dx) < \infty \right\} \quad (3)$$

We further denote by \mathbf{B}_1^\downarrow the sub-space of functions in \mathbf{B}_1 which are the Laplace transforms of absolutely continuous measures with a decreasing density :

$$\mathbf{B}_1^\downarrow := \left\{ \mathcal{L}_\mu : \mu(dx) = g(x)dx, g \text{ decreasing and } \int_0^\infty (1 \wedge x^{-1})g(x)dx < \infty \right\}. \quad (4)$$

Note that the density g then has limit 0 at infinity.

2.2 Bernstein functions

For every triple (a, b, Λ) with $a, b \geq 0$ and Λ a positive measure on $]0, \infty[$ such that

$$\int_{]0, \infty[} (x \wedge 1) \Lambda(dx) < \infty, \quad (5)$$

we associate the function $\Phi_{a,b,\Lambda} :]0, \infty[\rightarrow [0, \infty[$ defined by

$$\Phi_{a,b,\Lambda}(q) := a + bq + \int_{]0, \infty[} (1 - e^{-qx}) \Lambda(dx), \quad (6)$$

and call $\Phi_{a,b,\Lambda}$ the Bernstein function with characteristics (a, b, Λ) . We denote the convex cone of Bernstein functions by

$$\mathbf{B}_2 := \{ \Phi_{a,b,\Lambda} : a, b \geq 0 \text{ and } \Lambda \text{ positive measure fulfilling (5)} \}. \quad (7)$$

It is well-known that \mathbf{B}_2 can be identified with the space of real-valued \mathcal{C}^∞ functions $f :]0, \infty[\rightarrow [0, \infty[$ such that for every integer $n \geq 1$, the n -th derivative $f^{(n)}$ of f has the same sign as $(-1)^{n-1}$. See Definition 3.9.1 and Theorem 3.9.4 in [6].

Bernstein functions appear as Laplace exponents of subordinators, see e.g. Chapter 1 in [3], Chapter 6 in [9], or Section 3.9 in [6]. This means that $\Phi \in \mathbf{B}_2$ if and only if there exists an increasing process $\sigma = (\sigma_t, t \geq 0)$ with values in $[0, \infty]$ (∞ serves as absorbing state) with independent and stationary increments as long as $\sigma_t < \infty$, such that for every $t \geq 0$

$$\mathbb{E}(\exp(-q\sigma_t)) = \exp(-t\Phi(q)), \quad q > 0.$$

In this setting, a is known as the killing rate, b as the drift coefficient, and Λ as the Lévy measure.

We shall further denote by \mathbf{B}_2^\downarrow the subspace of Bernstein functions for which the Lévy measure is absolutely continuous with a monotone decreasing density, viz.

$$\mathbf{B}_2^\downarrow := \left\{ \Phi_{a,b,\Lambda} : a, b \geq 0 \text{ and } \Lambda(dx) = g(x)dx, g \geq 0 \text{ decreasing and } \int_0^\infty (x \wedge 1)g(x)dx < \infty \right\}.$$

2.3 (Sub)critical branching mechanisms

For every triple (a, b, Π) with $a, b \geq 0$ and Π positive measure on $]0, \infty[$ such that

$$\int_{]0, \infty[} (x \wedge x^2) \Pi(dx) < \infty \quad (8)$$

we associate the function $\Psi_{a,b,\Pi} :]0, \infty[\rightarrow [0, \infty[$ defined by

$$\Psi_{a,b,\Pi}(q) := aq + bq^2 + \int_{]0, \infty[} (e^{-qx} - 1 + qx) \Pi(dx), \quad (9)$$

and denote the convex cone of such functions by

$$\mathbf{B}_3 := \{ \Psi_{a,b,\Pi} : a, b \geq 0 \text{ and } \Pi \text{ a positive measure such that (8) holds} \} \quad (10)$$

Functions in \mathbf{B}_3 are convex increasing functions of class \mathcal{C}^∞ that vanish at 0; they coincide with the class of branching mechanisms for (sub)critical continuous state branching processes, where (sub)critical means critical or sub-critical. See Le Gall [7] on page 132.

Alternatively, functions in the space \mathbf{B}_3 can also be viewed as Laplace exponents of Lévy processes with no positive jumps that do not drift to $-\infty$ (or, equivalently, with nonnegative mean). In this setting, a is the drift coefficient, $2b$ the Gaussian coefficient, and Π the image of the Lévy measure by the map $x \rightarrow -x$. See e.g. Chapter VII in [2].

3 Composition with \mathbf{e}_α

Stable subordinators correspond to a remarkable one-parameter family of Bernstein functions denoted here by $(\mathbf{e}_\alpha, 0 < \alpha < 1)$, where

$$\mathbf{e}_\alpha(q) := q^\alpha = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty (1 - e^{-qx}) x^{-1-\alpha} dx, \quad q > 0.$$

Theorem 1 *The following assertions are equivalent:*

- (i) $\alpha \in]0, 1/2]$.
- (ii) For every $\Psi \in \mathbf{B}_3$, $\Psi \circ \mathbf{e}_\alpha \in \mathbf{B}_2$.

The implication (ii) \Rightarrow (i) is immediate. Indeed, $\Psi_{0,1,0} : q \rightarrow q^2$ belongs to \mathbf{B}_3 , but $\mathbf{e}_{2\alpha} = \Psi_{0,1,0} \circ \mathbf{e}_\alpha$ is in \mathbf{B}_2 if and only if $2\alpha \leq 1$. However, the converse (i) \Rightarrow (ii) is not straightforward and relies on the following technical lemma, which appears as Lemma VI.1.2 in [8]. Here, for the sake of completeness, we provide a proof.

Lemma 2 *For $\alpha \in]0, 1/2]$, let $\sigma^{(\alpha)} = (\sigma_x^{(\alpha)}, x \geq 0)$ be a stable subordinator with index α with Laplace transform*

$$\mathbb{E} \left(\exp \left(-q \sigma_x^{(\alpha)} \right) \right) = \exp(-xq^\alpha), \quad x, q > 0.$$

Denote by $p^{(\alpha)}(x, t)$ the density of the law of $\sigma_x^{(\alpha)}$. Then for every $x, t > 0$, we have

$$p^{(\alpha)}(x, t) \leq \frac{\alpha}{\Gamma(1-\alpha)} x t^{-(1+\alpha)}.$$

Remark : The bound in Lemma 2 is sharp, as it is well-known that for any $0 < \alpha < 1$ and each fixed $t > 0$

$$p^{(\alpha)}(x, t) \sim \frac{\alpha}{\Gamma(1-\alpha)} x t^{-(1+\alpha)}, \quad x \rightarrow \infty.$$

More precisely, there is a series representation of $p^{(\alpha)}(x, t)$, see Formula (2.4.7) on page 90 in Zolotarev [10]:

$$p^{(\alpha)}(x, 1) = \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\Gamma(n\alpha + 1)}{\Gamma(n + 1)} \sin(\pi n\alpha) x^{-n\alpha-1}.$$

Using the identity

$$\Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin(\alpha\pi)},$$

this agrees of course with the above estimate. It is interesting to note that the second leading term in the expansion,

$$-\frac{\Gamma(2\alpha + 1)}{2\pi} \sin(2\pi\alpha) x^{-2\alpha-1},$$

is negative for $\alpha < 1/2$, but positive for $\alpha > 1/2$. So the bound in Lemma 2 would fail for $\alpha > 1/2$.

Proof: In the case $\alpha = 1/2$, there is an explicit expression for the density

$$p^{(1/2)}(x, t) = \frac{x}{2\sqrt{\pi t^3}} \exp\left(-\frac{x^2}{4t}\right),$$

from which the claim is obvious (recall that $\Gamma(1/2) = \sqrt{\pi}$).

In the case $\alpha < 1/2$, we start from the identity

$$\exp(-xq^\alpha) = \int_0^\infty e^{-qt} p^{(\alpha)}(x, t) dt,$$

and take the derivative in the variable q to get

$$\alpha q^{\alpha-1} \exp(-xq^\alpha) = \int_0^\infty e^{-qt} \frac{t}{x} p^{(\alpha)}(x, t) dt,$$

and then

$$\alpha q^{\alpha-1} (1 - \exp(-xq^\alpha)) = \int_0^\infty e^{-qt} \left(\frac{\alpha}{\Gamma(1-\alpha)} t^{-\alpha} - \frac{t}{x} p^{(\alpha)}(x, t) \right) dt.$$

Denote the left hand-side by $g(x, q)$, and take the derivative in the variable x . We obtain

$$\frac{\partial g(x, q)}{\partial x} = \alpha q^{2\alpha-1} e^{-xq^\alpha} = \alpha q^{2\alpha-1} \int_0^\infty e^{-qt} p^{(\alpha)}(x, t) dt.$$

On the other hand, since $1 - 2\alpha > 0$,

$$q^{2\alpha-1} = \frac{1}{\Gamma(1-2\alpha)} \int_0^\infty e^{-qs} s^{-2\alpha} ds,$$

and hence

$$\frac{\partial g(x, q)}{\partial x} = \frac{\alpha}{\Gamma(1-2\alpha)} \int_0^\infty \frac{ds}{s^{2\alpha}} \int_0^\infty dt e^{-q(s+t)} p^{(\alpha)}(x, t).$$

The change of variables $u = t + s$ yields

$$\frac{\partial g(x, q)}{\partial x} = \frac{\alpha}{\Gamma(1-2\alpha)} \int_0^\infty du \int_0^u \frac{ds}{s^{2\alpha}} e^{-qu} p^{(\alpha)}(x, u-s);$$

and since $g(0, t) = 0$, we finally obtain the identity

$$\begin{aligned} & \int_0^\infty e^{-qt} \left(\frac{\alpha}{\Gamma(1-\alpha)} t^{-\alpha} - \frac{t}{x} p^{(\alpha)}(x, t) \right) dt \\ &= \frac{\alpha}{\Gamma(1-2\alpha)} \int_0^x dy \int_0^\infty du \int_0^u \frac{ds}{s^{2\alpha}} e^{-qu} p^{(\alpha)}(x, u-s). \end{aligned}$$

Inverting the Laplace transform, we conclude that

$$\frac{\alpha}{\Gamma(1-\alpha)} t^{-\alpha} - \frac{t}{x} p^{(\alpha)}(x, t) = \frac{\alpha}{\Gamma(1-2\alpha)} \int_0^x dy \int_0^t \frac{ds}{s^{2\alpha}} p^{(\alpha)}(x, t-s),$$

which entails our claim. ■

We are now able to prove Theorem 1.

Proof: Let $\Psi_{a,b,\Pi} \in \mathbf{B}_3$. Since both $a\mathbf{e}_\alpha$ and $b\mathbf{e}_{2\alpha}$ are Bernstein functions, there is no loss of generality in assuming that $a = b = 0$. Set for $t > 0$

$$\nu_\alpha(t) := \frac{\alpha}{\Gamma(1-\alpha)t^{1+\alpha}} \int_0^\infty \Pi(dx) x \left(1 - \frac{\Gamma(1-\alpha)t^{1+\alpha}}{\alpha x} p^{(\alpha)}(x, t) \right).$$

It follows from Lemma 2 that $\nu_\alpha(t) \geq 0$. We have for every $q > 0$

$$\begin{aligned} & \int_0^\infty (1 - e^{-qt}) \nu_\alpha(t) dt \\ &= \int_0^\infty \Pi(dx) x \int_0^\infty dt \left(\frac{\alpha(1 - e^{-qt})}{\Gamma(1-\alpha)t^{1+\alpha}} - \frac{p^{(\alpha)}(x, t)}{x} + e^{-qt} \frac{p^{(\alpha)}(x, t)}{x} \right) \\ &= \int_0^\infty \Pi(dx) x \left(q^\alpha - \frac{1}{x} + \frac{e^{-q^\alpha x}}{x} \right) \\ &= \Psi_{0,0,\Pi}(\mathbf{e}_\alpha(q)). \end{aligned}$$

As this quantity is finite for every $q > 0$, this shows that $\Psi_{0,0,\Pi} \circ \mathbf{e}_\alpha \in \mathbf{B}_2$. ■

Remark : The proof gives a stronger result than that stated in Theorem 1. Indeed, we specified the Lévy measure ν_α of $\Psi_{0,0,\Pi} \circ \mathbf{e}_\alpha$. Furthermore, in the case $\alpha = 1/2$, this expression shows that $\Psi_{0,0,\Pi} \circ \mathbf{e}_{1/2} \in \mathbf{B}_2^\downarrow$. It is interesting to combine this observation with the forthcoming Proposition 7 : for every $\Psi \in \mathbf{B}_3$, $\Psi \circ \mathbf{e}_{1/2} \in \mathbf{B}_2^\downarrow$, thus $\text{Id} \times (\Psi \circ \mathbf{e}_{1/2}) : q \rightarrow q\Phi(\sqrt{q})$ is again in \mathbf{B}_3 , and in turn $\mathbf{e}_{1/2} \times (\Psi \circ \mathbf{e}_{1/4}) \in \mathbf{B}_2^\downarrow$. More generally, we have by iteration that for every integer n

$$\mathbf{e}_{2^{-2^{1-n}}} \times (\Psi \circ \mathbf{e}_{2^{-n}}) \in \mathbf{B}_3,$$

and

$$\mathbf{e}_{1-2^{-n}} \times (\Psi \circ \mathbf{e}_{2^{-n-1}}) \in \mathbf{B}_2^\downarrow.$$

4 Internal functions

It is well-known that the cone \mathbf{CM} of completely monotone functions and the cone \mathbf{B}_2 of Bernstein functions are both stable by right composition with a Bernstein function; see Proposition 8 below. Theorem 1 incites us to consider also compositions of (sub)critical branching mechanisms and Bernstein functions; we make the following definition :

Definition 3 *A Bernstein function $\Phi \in \mathbf{B}_2$ is said internal if $\Psi \circ \Phi \in \mathbf{B}_2$ for every $\Psi \in \mathbf{B}_3$.*

Theorem 1 shows that the functions \mathbf{e}_α are internal if and only if $\alpha \in]0, 1/2]$. The critical parameter $\alpha = 1/2$ plays a distinguished role. Indeed, we could also prove Theorem 1 using the following alternative route. First, we check that $\mathbf{e}_{1/2}$ is internal (see [8]), and then we deduce by subordination that for every $\alpha < 1/2$ that $\Psi \circ \mathbf{e}_\alpha = \Psi \circ \mathbf{e}_{1/2} \circ \mathbf{e}_{2\alpha}$ is again a Bernstein function for every $\Psi \in \mathbf{B}_3$. Developing this argument, we easily arrive at the following characterization of internal functions :

Theorem 4 *Let $\Phi = \Phi_{a,b,\Lambda} \in \mathbf{B}_2$ be a Bernstein function. The following assertions are then equivalent:*

- (i) Φ is internal,
- (ii) $\Phi^2 \in \mathbf{B}_2$,
- (iii) $b = 0$ and there exists a subordinator $\sigma = (\sigma_t, t \geq 0)$ such that

$$\Lambda(dx) = c \int_0^\infty t^{-3/2} \mathbb{P}(\sigma_t \in dx) dt.$$

Proof: (i) \Rightarrow (ii) is obvious as $\Psi_{0,1,0} \circ \Phi = \Phi^2$.

(ii) \Rightarrow (i). We know from Theorem 1 or [8] that for every $\Psi \in \mathbf{B}_3$, $\Psi \circ \mathbf{e}_{1/2} \in \mathbf{B}_2$. It follows by subordination that for every Bernstein function $\kappa \in \mathbf{B}_2$, $\Psi \circ \mathbf{e}_{1/2} \circ \kappa \in \mathbf{B}_2$. Take $\kappa = \Phi^2$, so $\mathbf{e}_{1/2} \circ \kappa = \Phi$, and hence Φ is internal.

(iii) \Rightarrow (ii) Let κ denote the Bernstein function of σ . We have

$$\begin{aligned} \Phi(q) &= a + \int_{]0, \infty[} (1 - e^{-qx}) \Lambda(dx) \\ &= a + c \int_{]0, \infty[} \int_0^\infty dt (1 - e^{-qx}) t^{-3/2} \mathbb{P}(\sigma_t \in dx) \\ &= a + c \int_0^\infty dt (1 - e^{-t\kappa(q)}) t^{-3/2}. \end{aligned}$$

The change of variables $t\kappa(q) = u$ yields

$$\Phi(q) = a + c' \sqrt{\kappa(q)}$$

and hence

$$\Phi^2(q) = a^2 + 2ac' \sqrt{\kappa(q)} + c'^2 \kappa(q).$$

Since $\kappa^{1/2} = \mathbf{e}_{1/2} \circ \kappa$ is again a Bernstein function, we thus see that $\Phi^2 \in \mathbf{B}_2$.

(ii) \Rightarrow (iii) Recall that the drift coefficient b of $\Phi_{a,b,\Lambda}$ is given by

$$\lim_{q \rightarrow \infty} \Phi_{a,b,\Lambda}(q)/q = b;$$

see e.g. page 7 in [3]. It follows immediately that $b = 0$ whenever $\kappa := \Phi_{a,b,\Lambda}^2 \in \mathbf{B}_2$. Recall from Sato [9] on page 197-8 that if $\tau^{(1)}$ and $\tau^{(2)}$ are two independent subordinators with respective Bernstein functions $\Phi^{(1)}$ and $\Phi^{(2)}$, then the compound process $\tau^{(1)} \circ \tau^{(2)} := \tau^{(3)}$ is again a subordinator with Bernstein function $\Phi^{(3)} := \Phi^{(2)} \circ \Phi^{(1)}$; moreover its Lévy measure $\Lambda^{(3)}$ is given by

$$\Lambda^{(3)}(dx) = \int_0^\infty \mathbb{P}(\tau_t^{(1)} \in dx) \Lambda^{(2)}(dt),$$

where $\Lambda^{(2)}$ denotes the Lévy measure of $\tau^{(2)}$. As $\Phi_{a,b,\Lambda} = \mathbf{e}_{1/2} \circ \kappa$, and the Lévy measure of $\mathbf{e}_{1/2}$ is $ct^{-3/2}dt$ with $c = 1/(2\sqrt{\pi})$, we deduce that

$$\Lambda(dx) = c \int_0^\infty \mathbb{P}(\sigma_t \in dx) t^{-3/2} dt.$$

The proof of Theorem 4 is now complete. ■

It is noteworthy that if $\Phi_{a,b,\Lambda}$ is internal and $\Lambda \neq 0$, then

$$\int_{]0,\infty[} x \Lambda(dx) = \infty.$$

Indeed,

$$\int_{]0,\infty[} x \Lambda(dx) = c \int_0^\infty \int_{]0,\infty[} x \mathbb{P}(\sigma_t \in dx) t^{-3/2} dt = c \int_0^\infty \mathbb{E}(\sigma_1) t^{-1/2} dt = \infty.$$

For instance, the Bernstein function $q \rightarrow \log(1+q)$ of the gamma subordinator is not internal.

Corollary 5 *For every $\Psi \in \mathbf{B}_3$, we write Φ for the inverse function of Ψ and then Φ' for its derivative. Then $1/\Phi'$ is internal.*

Proof: It is known (see Corollary 10 below) that $1/\Phi'$ is a Bernstein function; let us check that its square is also a Bernstein function.

We know that $\Psi'' \in \mathbf{B}_1$ (Proposition 6 below) and $\Phi \in \mathbf{B}_2$ (Proposition 9 below); we deduce from Proposition 8 that $\Psi'' \circ \Phi \in \mathbf{B}_1$. If we write $I(f) : x \rightarrow \int_0^x f(y) dy$ for every locally integrable function f , then again by Proposition 6, we get that $I(\Psi'' \circ \Phi)$ is a Bernstein function.

Now

$$\Psi'' = -\frac{\Phi'' \circ \Psi}{(\Phi' \circ \Psi)^3},$$

so

$$\Psi'' \circ \Phi = -\frac{\Phi''}{(\Phi')^3},$$

and we conclude that

$$\frac{1}{2(\Phi')^2} = I(\Psi'' \circ \Phi) \in \mathbf{B}_2. \quad \blacksquare$$

5 Some classical results and their consequences

For convenience, this section gathers some classical transformations involving \mathbf{B}_j , $j \in \{1, 2, 3\}$ and related subspaces, which have been used in the preceding section. We start by considering derivatives and indefinite integrals. The following statement is immediate.

Proposition 6 *Let $j = 2, 3$ and $f :]0, \infty[\rightarrow [0, \infty[$ be a C^∞ -function with derivative f' . For $j = 3$, we further suppose that $\lim_{q \rightarrow 0} f(q) = 0$. There is the equivalence*

$$f \in \mathbf{B}_j \iff f' \in \mathbf{B}_{j-1}.$$

The next statement is easily checked using integration by parts.

Proposition 7 *Let $j = 2, 3$ and consider two functions $f, g :]0, \infty[\rightarrow [0, \infty[$ which are related by the identity $f(q) = qg(q)$. Then there is the equivalence*

$$f \in \mathbf{B}_j \text{ and } \lim_{q \rightarrow 0} f(q) = 0 \iff g \in \mathbf{B}_{j-1}^\dagger.$$

Proposition 7 has well-known probabilistic interpretations. First, let σ be a subordinator with Bernstein function $f \in \mathbf{B}_2$ with unit mean, viz. $\mathbb{E}(\sigma_1) = 1$, which is equivalent to $f'(0+) = 1$. Then the completely monotone function $g(q) := f(q)/q$ is the Laplace transform of a probability measure on \mathbb{R}_+ . The latter appears in the renewal theorem for subordinators (see e.g. [4]); in particular it describes the weak limit of the so-called age process $A(t) = t - g_t$ as $t \rightarrow \infty$, where $g_t := \sup \{\sigma_s : \sigma_s < t\}$. Second, let X be a Lévy process with no positive jumps and Laplace exponent $f \in \mathbf{B}_3$. The Lévy process reflected at its infimum, $X_t - \inf_{0 \leq s \leq t} X_s$, is Markovian; and if τ denotes its inverse local time at 0, then $\sigma = -X \circ \tau$ is a subordinator called the descending ladder-height process. The Bernstein function of the latter is then given by $g(q) = f(q)/q$; see e.g. Theorem VII.4(ii) in [2].

We next turn our attention to composition of functions; here are some classical properties

Proposition 8 *Consider two functions $f, g :]0, \infty[\rightarrow [0, \infty[$. Then we have the implications*

$$\begin{aligned} f, g \in \mathbf{B}_2 &\implies f \circ g \in \mathbf{B}_2, \\ f \in \mathbf{CM} \text{ and } g \in \mathbf{B}_2 &\implies f \circ g \in \mathbf{CM}, \\ f \in \mathbf{B}_1 \text{ and } g \in \mathbf{B}_2 &\implies f \circ g \in \mathbf{B}_1. \end{aligned}$$

The first statement in Proposition 8 is related to the celebrated subordination of Bochner (see, e.g. Section 3.9 in [6] or Chapter 6 in [9]); more precisely if σ and τ are two independent subordinators with respective Bernstein functions f_σ and f_τ , then $\sigma \circ \tau$ is again a subordinator whose Bernstein function is $f_\tau \circ f_\sigma$. The second statement is a classical result which can be found as Criterion 2 on page 441 in Feller [5]; it is also related to Bochner's subordination.

Finally we turn our attention to inverses.

Proposition 9 Consider a function $f :]0, \infty[\rightarrow]0, \infty[$. Then

$$f \in \mathbf{B}_2 \cup \mathbf{B}_3 \implies 1/f \in \mathbf{CM}.$$

Further, if f^{-1} denotes the inverse of f when the latter is a bijection, then

$$f \in \mathbf{B}_3, f \neq 0 \implies f^{-1} \in \mathbf{B}_2.$$

We mention that if $f \in \mathbf{B}_3$, the completely monotone function $1/f$ is the Laplace transform of the so-called scale function of the Lévy process X with no positive jumps which has Laplace exponent f . See Theorem VII.8 in [2]. On the other hand, f^{-1} is the Bernstein function of the subordinator of first-passage times $T_t := \inf \{s \geq 0 : X_s > t\}$; see e.g. Theorem VII.1 in [2]. Finally, in the case when $f \in \mathbf{B}_2$ is a Bernstein function, the completely monotone function $1/f$ is the Laplace transform of the renewal measure $U(dx) = \int_0^\infty \mathbb{P}(\sigma_t \in dx) dt$, where σ is a subordinator with Bernstein function f .

Corollary 10 Let $\Psi \neq 0$ be a function in \mathbf{B}_3 , and denote by $\Phi = \Psi^{-1} \in \mathbf{B}_2$ its inverse bijection. Then $q \rightarrow 1/\Phi'(q)$ and $\text{Id}/\Phi : q \rightarrow q/\Phi(q)$ are Bernstein functions. Furthermore $\frac{1}{\Phi\Phi'} : q \rightarrow 1/(\Phi(q)\Phi'(q))$ is completely monotone.

Proof: We know from Propositions 6 and 9 that both Φ and Ψ' are Bernstein functions. We conclude from Proposition 8 that $1/\Phi' = \Psi' \circ \Phi$ is again in \mathbf{B}_2 .

Similarly, we know from Proposition 7 that $q \rightarrow \Psi(q)/q$ is a Bernstein function, and composition on the right by the Bernstein function Φ yields Id/Φ that is again in \mathbf{B}_2 .

Finally, we can write $1/(\Phi\Phi') = f \circ \Phi$ where $f(q) = \Psi'(q)/q$. We know from Proposition 6 that $\Psi' \in \mathbf{B}_2$, so $f \in \mathbf{CM}$ by Proposition 7. Since $\Phi \in \mathbf{B}_2$, we conclude from Proposition 8 that $f \circ \Phi \in \mathbf{CM}$. ■

If $\Phi = \Psi^{-1}$ is the Bernstein function given by the inverse of a function $\Psi \in \mathbf{B}_3$, the Bernstein function $1/\Phi'$ is the exponent of the subordinator L^{-1} defined as the inverse of the local time at 0 of the Lévy process with no positive jumps and Laplace exponent Ψ . See e.g. Exercise VII.2 in [2]. On the other hand, Id/Φ is then the Bernstein function of the decreasing ladder times, see Theorem VII.4(ii) in [2]. The interested reader is also referred to [1] for further factorizations for Bernstein functions which arise naturally for Lévy processes with no positive jumps, and their probabilistic interpretations.

Next, recall that a function $f :]0, \infty[\rightarrow \mathbb{R}_+$ is called a Stieltjes transform if it can be expressed in the form

$$f(q) = b + \int_{[0, \infty[} \frac{\nu(dt)}{t + q}, \quad q > 0,$$

where $b \geq 0$ and ν is a Radon measure on \mathbb{R}_+ such that $\int_{[0, \infty[} (1 \wedge t^{-1}) \nu(dt) < \infty$. Equivalently, a Stieltjes transform is the Laplace transform of a Radon measure μ on \mathbb{R}_+ of the type $\mu(dx) = b\delta_0(dx) + h(x)dx$, where $b \geq 0$ and h is a completely monotone function which belongs to $L^1(e^{-qx} dx)$ for every $q > 0$; see e.g. Section 3.8 in [6].

Corollary 11 *Let $f \in \mathbf{B}_2$ be a Bernstein function such that its derivative f' is a Stieltjes transform. Then for every Bernstein function $g \in \mathbf{B}_2$, the function $f \circ \frac{1}{g}$ is completely monotone.*

Proof: We can write

$$f(q) = a + bq + \int_0^q dr \int_0^\infty dx e^{-rx} h(x), \quad q > 0,$$

where $a, b \geq 0$ and $h \in \mathbf{B}_1$. Thus

$$f(q) = a + bq + \int_0^\infty dx (1 - e^{-qx}) \frac{h(x)}{x}, \quad q > 0,$$

and then

$$f \circ \frac{1}{g}(q) = a + \frac{b}{g(q)} + \int_0^\infty dx (1 - e^{-x/g(q)}) \frac{h(x)}{x}.$$

We already know from Proposition 9 that $a + b/g \in \mathbf{CM}$. The change of variable $y = x/g(q)$ yields

$$\int_0^\infty dx (1 - e^{-x/g(q)}) \frac{h(x)}{x} = \int_0^\infty (1 - e^{-y}) h(yg(q)) \frac{dy}{y}.$$

For each fixed $y > 0$, yg is a Bernstein function, so by Proposition 8, the function $q \rightarrow h(yg(q))$ is completely monotone.

We conclude that for every integer $n \geq 0$,

$$(-1)^n \frac{\partial^n}{\partial q^n} (f \circ \frac{1}{g})(q) = \int_0^\infty (-1)^n \frac{\partial^n}{\partial q^n} (h(yg(\cdot)))(q) (1 - e^{-y}) \frac{dy}{y} \geq 0,$$

which establishes our claim. ■

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