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► **To cite this version:**

Antonio Bucciarelli, Benjamin Leperchey. Hypergraphs and degrees of parallelism, a completeness result. Foundations of Software Science and Computation Structures 7th International Conference, FOSSACS 2004, 2004, Barcelone, Spain. pp.58-71. hal-00003584

HAL Id: hal-00003584

<https://hal.science/hal-00003584>

Submitted on 15 Dec 2004

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Hypergraphs and degrees of parallelism: a completeness result

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Abstract. In order to study relative PCF-definability of boolean functions, we associate a hypergraph H_f to any boolean function f (following [2, 4]).

We introduce the notion of timed hypergraph morphism and show that it is:

- Sound: if there exists a timed morphism from H_f to H_g then f is PCF-definable relatively to g .
- Complete for subsequential functions: if f is PCF-definable relatively to g , and g is subsequential, then there exists a timed morphism from H_f to H_g .

1 Introduction

PCF is a simple, paradigmatic functional programming language, defined by D. Scott in his seminal paper [10], a milestone in the area of denotational semantics.

Following Scott, Plotkin studied in [7] the relationship between operational and denotational semantics of PCF. The main results of [7] may be summarized as follows:

- The Scott model of PCF is adequate with respect to contextual equivalence.
- The model is not complete, due to the presence of non-definable, “parallel” functions.
- All the (algebraic) elements of the model become definable if a *parallel conditional statement* is added to the language.

Since then, a lot of work has been devoted to the search of a satisfactory semantic characterization of the notion of PCF-definable function (see [1] for a survey). We have now a number of different notions of sequentiality, and all of them characterize exactly PCF definability for first order functions.

In this paper, we study the relative definability problem for Finitary PCF (FPCF) with respect to its Scott model. FPCF is the finitary fragment of PCF: it has a single ground type \mathbf{B} , the corresponding constants \perp , \mathbf{tt} , \mathbf{ff} , and just one more constant, the **if – then – else**.

The Scott model of FPCF is the finite type hierarchy where $\llbracket \mathbf{bool} \rrbracket$ is the flat domain of boolean values, and $\llbracket \sigma \rightarrow \tau \rrbracket$ is the set of monotonic functions from

$\llbracket \sigma \rrbracket$ to $\llbracket \tau \rrbracket$, ordered pointwise. FPCF-terms are interpreted in the standard way in this model, and in particular, for every closed term $M : \sigma$, $\llbracket M \rrbracket \in \llbracket \sigma \rrbracket$.

An instance of the relative definability problem is a pair $f \in \llbracket \sigma \rrbracket$, $g \in \llbracket \tau \rrbracket$, and a solution is either a term $M : \tau \rightarrow \sigma$ such that $\llbracket M \rrbracket g = f$, or a proof that such a term does not exist (when M does exist, we say that f is *less parallel* than g , and we write $f \leq_{\text{par}} g$).

Conceptually, the relative definability problem for the finitary fragment of PCF is settled: we know that it is undecidable in general [6] and decidable for functions of order 1 or 2 [11].

Nevertheless, decidability results may be not completely satisfactory: from a theoretical point of view, we still lack a characterization of the poset of degrees of parallelism (i.e. equivalence classes of inter-definable functions, noted $[f]$) which, even in the decidable case, is rich and complex [2, 8].

In this paper, we give a complete, geometric characterization of relative definability for “subsequential”, first-order functions; the exact correspondence we establish between geometric objects (a particular kind of hypergraph morphisms) and computational ones (the terms solving relative definability problems), is, we believe, interesting in itself.

Moreover, our analysis of relative definability problems provide a simple way of choosing, among the terms solving a given instance, an “optimal” one (for instance, a term defining f with as few calls of g as possible).

1.1 Related works

The study of degrees of parallelism was pioneered by Sazonov and Trakhtenbrot [9, 13] who singled out some finite subposets of degrees. Some results on degrees are corollaries of well known facts: for instance Plotkin’s full abstraction result for PCF+por implies that this poset has a top. The bottom of degrees is the set of PCF-definable functions which is fully characterized, for first order functions, by the notion of sequentiality (in any of its formulations). Moreover Sieber’s sequentiality relations [11] provide a characterization of first-order degrees of parallelism and this characterization is effective: given f and g one can decide if $f \leq_{\text{par}} g$. A. Stoughton [12] has implemented an algorithm which solves this decision problem. R. Loader has shown that the problem of deciding if a given continuous function(al) is PCF-definable, is undecidable [6]. As a consequence, the relation \leq_{par} is undecidable in general (at higher-order), since, if g is PCF-definable and f continuous, then f is PCF-definable if and only if $f \leq_{\text{par}} g$.

In [2], the first author investigates the poset of degrees of parallelism using categories of hypergraphs for representing boolean function. The starting point of the investigation was the observation that the trace of a function f (i.e. the subset of the graph of f whose first projection is the set of minimal points on which f is defined) can be turned into a hypergraph H_f , in such a way that hypergraph morphisms from H_f to H_g are “witnesses” of the inequality $f \leq_{\text{par}} g$. In particular, a rich subposet of degrees for which the hypergraph representation is sound and complete is singled out in [2]. If $[f], [g]$ belong to that subposet $f \leq_{\text{par}} g$ holds if and only if there exists a morphism from H_f to H_g . In [4] P.

Malacaria and the first author showed a general result about hypergraphs and degrees: if there exists a morphism from H_f to H_g , then $f \leq_{\text{par}} g$. However, for the notion of hypergraph morphism they used (the standard one, based on the preservation of hyperarcs), no general completeness result seems to hold.

1.2 Plan of the paper

In this paper we introduce a weaker notion of hypergraph morphism (the timed morphisms) and we show that it is sound in general, and complete for subsequential functions (i.e. for functions which have a sequential upper bound). The proof of soundness presented in [4] goes through the framework of timed morphisms with some very minor changes. The proof of completeness is an application of Sieber’s sequentiality relations.

In Section 2 we introduce the notions of hypergraphs representing boolean functions and of h -morphisms between them (h -morphisms were called “weak” in [4]; since timed morphisms are weaker, we change the terminology here). In Section 3 the “timed” hypergraph morphisms are defined, and we show by some examples how they behave as boolean function transformers. In Section 4, we recall some useful properties of subsequential functions. Sections 5,6 and 7 are devoted to the proof of soundness and completeness of timed morphisms w.r.t. the relation \leq_{par} .

2 Hypergraphs and h -morphisms

We denote by \mathbf{B} the flat domain of boolean values $\{\perp, \mathbf{tt}, \mathbf{ff}\}$. Tuples of boolean values are ordered componentwise. Given a monotone function $f : \mathbf{B}^n \rightarrow \mathbf{B}$, the *trace* of f is defined by

$$\text{tr}(f) = \{(v, b) \mid f(v) = b \neq \perp \text{ and } v \text{ minimal}\}$$

We note the first and second projection π_1 and π_2 . In particular, $\pi_1(\text{tr}(f))$ is the set of minimal points where f is defined.

A subset $A = \{v_1, \dots, v_k\}$ of \mathbf{B}^n is *linearly coherent* (or simply *coherent*) if for all $1 \leq i \leq n$ either $\exists 1 \leq j \leq k, v_j^i = \perp$, or $\forall 1 \leq j, j' \leq k, v_j^i = v_{j'}^i$. The set of coherent subsets of \mathbf{B}^n is denoted $\mathcal{C}(\mathbf{B}^n)$. The coherence is related to sequentiality: if f is a n -ary boolean function, and $\pi_1(\text{tr}(f))$ is coherent, then f has no sequentiality index and it is not PCF-definable. Actually f is definable if and only if no subset of $\pi_1(\text{tr}(f))$ is coherent.

The following easy property of the coherence will be useful:

Fact 1 *If $A \in \mathcal{C}(\mathbf{B}^n)$ and B is an Egli-Milner lower bound of A (that is if $\forall x \in A \exists y \in B y \leq x$ and $\forall y \in B \exists x \in A y \leq x$) then $B \in \mathcal{C}(\mathbf{B}^n)$.*

Definition 1. *A colored hypergraph $H = (V_H, A_H, C_H)$ is given by:*

- a finite set V_H of vertices,

- a set $A_H \subseteq \{A \subseteq V_H \mid \#A \geq 2\}$ of (hyper)arcs,
- a coloring function $C_H : V_H \rightarrow \{b, w\}$.

Definition 2. Let $f : \mathcal{B}^n \rightarrow \mathcal{B}$ be the n -ary function defined by $\mathbf{tr}(f) = \{(v_1, b_1), \dots, (v_k, b_k)\}$. The hypergraph H_f is defined by

- $V_{H_f} = \pi_1(\mathbf{tr}(f))$,
- A_{H_f} contains the coherent subsets of $\pi_1(\mathbf{tr}(f))$ with at least two elements,
- $C_{H_f}(v_i) = \begin{cases} w & \text{if } b_i = \mathbf{tt} \\ b & \text{if } b_i = \mathbf{ff} \end{cases}$

One can check that the hypergraphs associated to monotone functions by the definition above (*functional hypergraph*) verify the following conditions:

H1 : If $\{x, y\} \in A_H$ then $C_H(x) = C_H(y)$.

H2 : If X_1, X_2 are hyperarcs and $X_1 \cap X_2 \neq \emptyset$ then $X_1 \cup X_2$ is a hyperarc.

Definition 3. A h -morphism from a hypergraph H to a hypergraph K is a function $m : V_H \rightarrow V_K$ such that:

- For all $A \subseteq V_H$, if $A \in A_H$ then $m(A) \in A_K$.
- for all $X \in A_H$, if $x, x' \in X$ and $C_H(x) \neq C_H(x')$ then $C_K(m(x)) \neq C_K(m(x'))$.

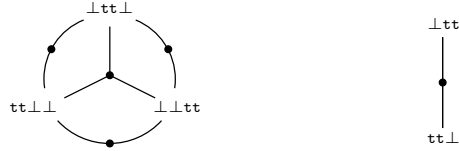
Colored hypergraphs and h -morphisms form a category, \mathcal{H} . In [4], it has been proved that, if there exists a h -morphism from H_f to H_g , then $f \leq_{\text{par}} g$. The problem of finding a weaker notion of hypergraph morphism, for which some sort of completeness result would hold, was left open.

We give here the motivating example for the definition of timed morphisms. Let $\text{por}_2 : \mathcal{B}^2 \rightarrow \mathcal{B}$ and $\text{por}_3 : \mathcal{B}^3 \rightarrow \mathcal{B}$ be defined by

$$\text{por}_2(x, y) = \begin{cases} \mathbf{tt} & \text{if one of } x, y \text{ is } \mathbf{tt} \\ \perp & \text{otherwise} \end{cases}$$

$$\text{por}_3(x, y, z) = \begin{cases} \mathbf{tt} & \text{if one of } x, y, z \text{ is } \mathbf{tt} \\ \perp & \text{otherwise} \end{cases}$$

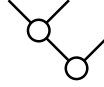
The associated hypergraphs are:



It is easy to see that there exists no h -morphism $m : H_3 \rightarrow H_2$. Nevertheless $\text{por}_3 \leq_{\text{par}} \text{por}_2$, since for instance $\text{por}_3 = \llbracket M \rrbracket \text{por}_2$ where

$$M = \lambda f \lambda x_1 x_2 x_3 \text{ if } f(f(x_1, x_2), x_3) \text{ then } \mathbf{tt} \text{ else } \perp$$

The tree of nested calls to f in M (the *nesting tree* of M), where the nodes are the occurrences of f , and the links are the arguments of f , is:

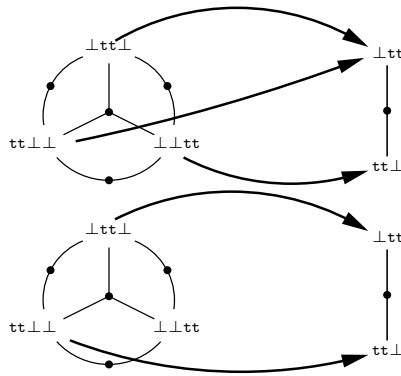


Actually, the nesting of calls to f in the term which defines por_3 with respect to por_2 is necessary. By looking at the way M “maps” the minimal points of por_3 onto the ones of por_2 , we realize that at the outermost level $(\text{tt}, \perp, \perp)$ and $(\perp, \text{tt}, \perp)$ are both mapped on (tt, \perp) , while $(\perp, \perp, \text{tt})$ is mapped on (\perp, tt) . The internal call of f maps $(\text{tt}, \perp, \perp)$ on (tt, \perp) and $(\perp, \text{tt}, \perp)$ on (\perp, tt) .

3 Timed Morphisms

The idea is the following: we want to allow morphisms to “collapse” an hyperarc on a singleton, provided that we have another morphism mapping this hyperarc on a hyperarc. More precisely, we want a finite sequence of morphisms $m_1 \dots m_l$ with domains $D_i \in A_H$, such that if m_i collapses an hyperarc B , there exists m_{i+k} with domain B . In the proof of soundness, each step in the sequence will appear as a nesting in the term.

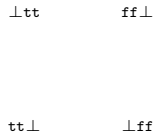
For our example, the sequence corresponding to M is :



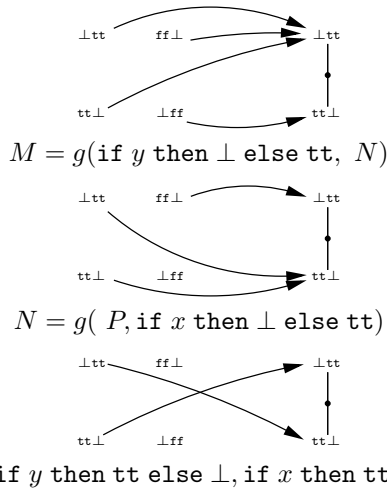
In general, by looking at the morphism, one can easily see the nesting of calls to the defining function (and then build a term quite easily). First, we spot the vertices of H_g corresponding to each argument of g^1 : tt, \perp for the first argument, \perp, tt for the second. Then, we know how to organize the nested calls to g : if we collapse an hyperarc X on the vertex corresponding to the argument i , we put a call to g at argument i , which will be defined by the morphism with domain X .

For example, let $f(x, y)$ be tt whenever x or y is defined, and \perp elsewhere. H_f is

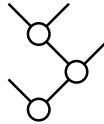
¹ In the general case, one cannot associate vertices to an argument. In our example por_2 , this is obvious. For more details, see the proof of soundness.



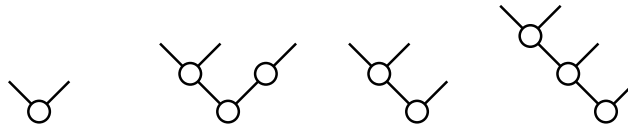
The only subsets that are not coherent are $\{tt\perp, ff\perp\}$ and $\{\perp tt, \perp ff\}$. In the following, we will not put the hyperarcs again. Here is a timed morphism from H_f to H_2 , and the corresponding term $\lambda g \lambda x \lambda y M$ defining f with por_2 :



The corresponding tree is:



but one can also easily find morphisms (and terms) for these nesting trees (and for some others, too):



The smallest nesting tree correspond to the “natural” solutions to this relative definability problem, namely:

$$\lambda g \lambda x \lambda y g(\text{if } x \text{ then } tt \text{ else } \perp, \text{if } y \text{ then } tt \text{ else } \perp)$$

Timed morphisms are sequences. For a given problem, shorter sequences correspond to terms with smaller depth, w.r.t. the nesting of calls of g . Timed morphisms provide a handy tool for constructing these optimal solution.

Actually, we give a more abstract, equivalent definition of timed morphisms. We will argue that the two notions coincide after the following couple of definitions.

Definition 4. Let $H = (V_H, A_H, C_H)$ be a (functional) hypergraph.

- The timed image of H , \overline{H} is defined by: $V_{\overline{H}} = V_H$, $C_{\overline{H}} = C_H$ and $A_{\overline{H}} = A_H \cup \{\{v\} \mid v \in V_H\}$.
- Let $B \subseteq V_H$. $H|_B$ is the sub-hypergraph of H defined by:

$$\begin{cases} V_{H|_B} = B \\ A_{H|_B} = \{X \in A_H \mid X \subseteq B\} \\ C_{H|_B} = (C_H)|_B \end{cases}$$

Given two functional hypergraphs H, K , we say that a morphism $\alpha \in \mathcal{H}(H, \overline{K})$ is non-trivial if $\#\alpha(V_H) > 1$.

Definition 5. Let H, K be functional hypergraphs; a timed morphism $\alpha \in \mathcal{TH}(H, K)$ is a collection

$$\{\alpha_X \in \mathcal{H}(H|_X, \overline{K})\}_{X \in A_H}$$

where all the α_A 's are non-trivial, and non-redundant in the following sense:

$$\forall X \subseteq Y \in A_H \quad \alpha_Y|_X \text{ is non trivial} \Rightarrow \alpha_X = \alpha_Y|_X$$

The intuitive description of timed morphisms in terms of sequences, given in the examples of this section coincides with the definition above. Given a sequence $m = m^1, \dots, m^k$ of h -morphisms from H to \overline{K} , and a hyperarc $X \in A_H$, define $\alpha_X^m = m^j|_X$, where j is the smallest index such that $m^j|_X$ is non trivial. Conversely, given $\{\alpha_X\}_{X \in A_H}$ we have to construct a sequence of morphisms m^1, \dots, m^k from (restrictions of) H to \overline{K} , such that if m^i collapses an hyperarc B , there exists m^{i+k} non-trivial of domain B . Let $\{A_i\}_{i \in I}$ be the set of maximal elements of V_H (note that these are disjoint, H being functional); m^1 is obtained by "gluing" all the α_{A_i} , $i \in I$. Now, letting $\{A_i\}_{i \in J}$ $J = \{j^1, \dots, j^l\}$ be the set of maximal elements of V_H which are "collapsed" by m^1 , we define $m^2 = \alpha_{A_{j^1}}, \dots, m^{l+1} = \alpha_{A_{j^l}}$, and we proceed by considering the hyperarcs collapsed by m^2, \dots, m^{l+1} . By finiteness of H , iterating this construction we obtain a sequence m^1, \dots, m^k obeying the definition of timed morphism in terms of sequences.

Timed morphisms compose componentwise (i.e. $(\alpha \circ \beta)_A = \alpha_{\beta(A)} \circ \beta_A$). To any h -morphism $m : H \rightarrow K$ corresponds canonically the timed morphism defined by $\alpha_A = m|_A$.

4 Subsequential Functions

A monotone function $f : \mathcal{B}^n \rightarrow \mathcal{B}$ is *subsequential* if it is extensionally upper bounded by a sequential (i.e. PCF-definable) function. As shown in proposition 6 subsequential functions correspond to hypergraphs with monochromatic hyperarcs and to functions preserving linear coherence. Such a class of functions admits hence a natural characterization in order theoretic, graph theoretic and algebraic terms.

Proposition 6. *Let $f : \mathcal{B}^n \rightarrow \mathcal{B}$ be a monotone function. The following are equivalent:*

1. f is subsequential.
2. For all $A \in \mathcal{C}(\mathcal{B}^n)$, $f(A) \in \mathcal{C}(\mathcal{B})$. (i.e. f preserves the linear coherence of \mathcal{B}^n .)
3. If $X \in A_{H_f}$ then for all $x, y \in X$ $C_{H_f}(x) = C_{H_f}(y)$ (i.e. X is monochromatic).

A proof can be found in [4].

Given a set $A = \{v_1, \dots, v_k\} \subseteq \mathcal{B}^n$, there exist in general a number of functions whose minimal points are exactly the elements of A . For instance, if the v_i are pairwise unbounded, there exist 2^k such functions. The following lemma states that, among these functions, the subsequential ones are those whose degree of parallelism is minimal.

Lemma 7. *Let $f, g : \mathcal{B}^n \rightarrow \mathcal{B}$ be such that g is subsequential and $\pi_1(\text{tr}(f)) = \pi_1(\text{tr}(g))$. Then $g \leq_{\text{par}} f$.*

Proof. Let M be a PCF term which defines a sequential upper bound \bar{g} of g . Let us define $g_0 : \mathcal{B}^n \rightarrow \mathcal{B}$ by:

$$g_0 = \llbracket \lambda f \lambda x. \text{if } f x \text{ then } M x \text{ else } M x \rrbracket f$$

If we prove that $g_0 = g$ we are done. Let $\bar{a} = (a_1, \dots, a_n) \in \mathcal{B}^n$, and suppose $g(\bar{a}) = b \neq \perp$; then $f(\bar{a}) \neq \perp$ and $\bar{g}(\bar{a}) = b$, hence $g_0(\bar{a}) = b$. If $g(\bar{a}) = \perp$, then $f(\bar{a}) = \perp$, hence $g_0(\bar{a}) = \perp$ too. Conversely if $g_0(\bar{a}) = b \neq \perp$ then $f(\bar{a}) \neq \perp$ and hence $g(\bar{a}) \neq \perp$ as well. Since $g(\bar{a}) \leq \bar{g}(\bar{a}) = b$, we get $g(\bar{a}) = b = g_0(\bar{a})$. If $g_0(\bar{a}) = \perp$, then $f(\bar{a}) = \perp$ or $\bar{g}(\bar{a}) = \perp$, and in each case $g(\bar{a}) = \perp$.

In section 5, we prove that if there exists a timed morphism $\alpha : H_f \rightarrow H_g$, then $f \leq_{\text{par}} g$. The following lemma introduces a key notion toward that result, namely that of *slice function*. The idea is the following: in order to reduce $f : \mathcal{B}^m \rightarrow \mathcal{B}$ to $g : \mathcal{B}^n \rightarrow \mathcal{B}$ we start by transforming the minimal points of f into the ones of g . This amounts to defining a function from \mathcal{B}^m to \mathcal{B}^n , that we describe as a set of functions $f_1, \dots, f_n : \mathcal{B}^m \rightarrow \mathcal{B}$. If these functions are g -definable, then we can already g -define a function which is defined (that is, not equal to \perp) if and only if f is defined, namely

$$h = \lambda x. g(f_1 x) \dots (f_n x)$$

and we are left with the problem of forcing h to agree with f whenever it converges.

For the time being we show that, if the f_i 's are defined via a timed morphism $\alpha : H_f \rightarrow H_g$, then they are subsequential, hence “relatively simple”.

Lemma 8. *Let $f : \mathcal{B}^m \rightarrow \mathcal{B}$, $g : \mathcal{B}^n \rightarrow \mathcal{B}$ be monotone functions and $\alpha : H_f \rightarrow H_g$ be a timed morphism. For $B \in A_{H_f}$, $1 \leq i \leq n$ let $f_i^B : \mathcal{B}^m \rightarrow \mathcal{B}$ be the function defined by*

$$\text{tr}(f_i^B) = \{(v, \alpha_B(v)_i) \mid v \in B, \alpha_B(v)_i \neq \perp\}$$

Then f_i^B is subsequential. We will call f_i^B the i th-slice of α_B .

Proof. Given $A \subseteq \text{tr}(f_i^B)$ such that $\pi_1(A) \in \mathcal{C}(\mathcal{B}^m)$, if $\pi_2(A) \notin \mathcal{C}(\mathcal{B})$ then α_B maps the coherent set $\pi_1(A)$ onto a non coherent set. This cannot be the case by definition of timed morphism.

5 Soundness

Timed morphisms are sound with respect to \leq_{par} , in the sense expressed by the following theorem:

Theorem 9. *Let $f : \mathcal{B}^l \rightarrow \mathcal{B}$, $g : \mathcal{B}^m \rightarrow \mathcal{B}$ be monotone functions such that $\mathcal{TH}(H_f, H_g) \neq \emptyset$. Then $f \leq_{\text{par}} g$.*

The proof, which can be found in App. A is essentially the same as in [4]. The key point lies in the restriction of morphisms to a hypergraph. In [4], the hypothesis was too strong: we only need a morphism from this hyperarc to H_g , we do not need it to be a part of the initial morphism from H_f to H_g . This generalization allow us to prove a completeness result.

This soundness result allow to derive easily corollaries on degrees of parallelism; for instance, in order to check that $[\text{por}_2]$ (from our motivating example) is the top of subsequential degrees (i.e. of degrees of subsequential functions; note that a subsequential function and a non-subsequential one cannot have the same degree of parallelism) it is sufficient to remark that, if f is subsequential, a timed morphism from H_f to por_2 is simply a (non-trivial and non redundant) partition of any hyperarc of H_f .

6 Sequentiality relations

Definition 10 (Sieber). *For each $n \geq 0$ and each pair of sets $A \subseteq B \subseteq \{1, \dots, n\}$ let $S_{A,B}^n \subseteq \mathbf{B}^n$ be defined by*

$$S_{A,B}^n(b_1, \dots, b_n) \Leftrightarrow (\exists i \in A \ b_i = \perp) \vee (\forall i, j \in B \ b_i = b_j)$$

An n -ary logical relation R is called a sequentiality relation if it is an intersection of relations of the form $S_{A,B}^n$.

We define $S_{n,n+1} = S_{\{1, \dots, n\}, \{1, \dots, n+1\}}^{n+1}$.

We write

$$\begin{pmatrix} x_{11} & \dots & x_{1n} \\ \vdots & & \vdots \\ x_{m1} & \dots & x_{mn} \end{pmatrix} \in R$$

meaning that each row is in R . A function $f : \mathbf{B}^m \rightarrow \mathbf{B}$ is invariant under the logical relation R of arity n whenever the matrix $(x_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}$ is in R :

$$(f(x_{11}, \dots, x_{m1}), \dots, f(x_{1n}, \dots, x_{mn})) \in R$$

Proposition 11. *For any $f : \mathbf{B}^n \rightarrow \mathbf{B}$ and $g : \mathbf{B}^m \rightarrow \mathbf{B}$ continuous functions, $f \leq_{\text{par}} g$ if and only if for any sequentiality relation R , if g is invariant under R then f is invariant too.*

Actually this is a relativized version of the main theorem of [11]: a continuous function of first or second order is PCF-definable if and only if it is invariant under all sequentiality relations.

Coherence is tightly related to sequentiality relations:

Lemma 12. *Let $A = \{\mathbf{x}_1, \dots, \mathbf{x}_n\} \subseteq \mathbf{B}^m$, and B be a subset of $\{1, \dots, n\}$. $\{\mathbf{x}_i\}_{i \in B}$ is coherent iff $(x_{ij}) \in S_{B,B}^n$. Moreover, A is coherent iff:*

$$\begin{pmatrix} x_{11} & \dots & x_{1n} & \bigwedge_{1 \leq i \leq n} x_{1i} \\ \vdots & & \vdots & \vdots \\ x_{m1} & \dots & x_{mn} & \bigwedge_{1 \leq i \leq n} x_{mi} \end{pmatrix} \in S_{n,n+1}$$

These sequentiality relations are closely related to strong stability at first order (see [3] from an overview on strong stability): f is strongly stable if it preserves linear coherence (that is, f is invariant by the relations $S_{B,B}^n$), and f is conditionally multiplicative: if A is coherent $f(\bigwedge A) = \bigwedge_{a \in A} f(a)$, (that is, f is invariant for the relation $S_{n,n+1}$).

7 Completeness

Theorem 13. *Let $f : \mathcal{B}^n \rightarrow \mathcal{B}$ and $g : \mathcal{B}^m \rightarrow \mathcal{B}$ be subsequential functions, such that $\mathcal{TH}(H_f, H_g) = \emptyset$. Then $f \not\leq_{\text{par}} g$.*

Proof. The first remark is that $\mathcal{TH}(H_f, H_g) = \emptyset$ if and only if there exists $A \in A_{H_f}$ such that there is no non-trivial morphism from $H_{f|_A}$ to $\overline{H_g}$. Throughout this proof, we restrict our attention to $H_{f|_A}$, for such an $A = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$. Let A_1, \dots, A_l be the arcs of $H|_A$, and, for $1 \leq j \leq l$, let B_j be the corresponding set of the indices: $A_j = \{\mathbf{v}_i\}_{i \in B_j}$.

We consider the $(k+1)$ -ary sequential logical relation

$$S_A = \left(\bigcap_{1 \leq j \leq l} S_{B_j, B_j}^{k+1} \right) \bigcap S_{k, k+1}$$

If we prove that g is invariant with respect to S_A and f is not, we are done.

Let us start by proving that f is not invariant. Let $V = (\mathbf{v}_1, \dots, \mathbf{v}_k, \bigwedge_{1 \leq j \leq k} \mathbf{v}_j)$: by lemma 12, for $1 \leq j \leq l$, $V \in S_{B_j, B_j}^{k+1}$ and $V \in S_{k, k+1}$, *i.e.* $V \in S_A$. On the contrary:

$$\left(f(\mathbf{v}_1), f(\mathbf{v}_2), \dots, f(\mathbf{v}_k), f\left(\bigwedge_{1 \leq j \leq k} \mathbf{v}_j\right) \right) \notin S_{k, k+1}$$

since the first k components of this vector are defined (the \mathbf{v}_j are in the trace of f), and the last is \perp ($\bigwedge \mathbf{v}_i$ can't be *above* a \mathbf{v}_j). Therefore, this tuple does not belong to S_A .

It remains to show that that $g \in S_A$. Let us suppose by *reductio ad absurdum* that there exists a matrix $W = (\mathbf{w}_1, \dots, \mathbf{w}_{k+1}) \in \mathcal{B}^{m \times (k+1)}$ such that:

$$W \in S_A \text{ and } g(W) = (g(\mathbf{w}_1), \dots, g(\mathbf{w}_{k+1})) \notin S_A$$

First, we note that, since $W \in S_A$, for all $1 \leq j \leq l$, $W \in S_{B_j, B_j}^{k+1}$, that is $\{\mathbf{w}_i\}_{i \in B_j}$ is coherent, so $\{g(\mathbf{w}_i)\}_{i \in B_j}$ is coherent, and g is subsequential, which entails, by proposition 6 and lemma 12, that $g(W)$ is invariant by S_{B_j, B_j}^{k+1} . Therefore, $g(W) \notin S_A$ means that $g(W) \notin S_{k, k+1}$, that is, $\forall j \leq k, g(\mathbf{w}_j) \neq \perp$ and $\exists j, j' \leq k+1, g(\mathbf{w}_j) \neq g(\mathbf{w}_{j'})$. Since g is subsequential and $\{\mathbf{w}_1, \dots, \mathbf{w}_k\}$ is coherent (lemma 12), $\forall j, j' \leq k, g(\mathbf{w}_j) = g(\mathbf{w}_{j'})$: there exists $b \in \{\mathbf{tt}, \mathbf{ff}\}$ such that

$$\forall j \leq k, g(\mathbf{w}_j) = b \text{ and } g(\mathbf{w}_{k+1}) = \perp$$

Hence any w_j , for $1 \leq j \leq k$, has at least a lower bound in $\pi_1(\text{tr}(g))$, which we denote by z_j . We have:

- the set $\{z_1, \dots, z_k\}$ is not a singleton, otherwise $g(w_{k+1}) = b$, being $w_{k+1} \geq \bigwedge_{1 \leq j \leq k} w_j$, by definition of $S_{k, k+1}$.
- for all $1 \leq j \leq l$ the set $\{z_i\}_{i \in B_j}$ is coherent, being an Egli-Milner lower bound of the coherent set $\{w_i\}_{i \in B_j}$ (see fact 1).
- Last, by proposition 6, f being subsequential, C_{H_f} is constant on A .

Hence the function $\alpha : A \rightarrow H_g$ defined by $\alpha_A(\mathbf{v}_i) = z_i$ is in $\mathcal{H}(H_{f|_A}, \overline{H_g})$, and it is not trivial, a contradiction.

Remark that, if g is subsequential and f is not, then $f \not\prec_{\text{par}} g$, hence the hypothesis of Theorem 13 could be weakened.

In order to see that completeness of timed morphisms fails in general, let us consider the following monotone functions:

$$\begin{cases} f(\perp, \mathbf{tt}, \mathbf{tt}, \mathbf{ff}) = \mathbf{tt} \\ f(\mathbf{ff}, \perp, \mathbf{tt}, \mathbf{tt}) = \mathbf{tt} \\ f(\mathbf{tt}, \mathbf{ff}, \perp, \mathbf{tt}) = \mathbf{tt} \\ f(\mathbf{tt}, \mathbf{tt}, \mathbf{ff}, \perp) = \mathbf{tt} \end{cases}$$

$$\begin{cases} g(\perp, \mathbf{tt}, \mathbf{ff}) = \mathbf{tt} \\ g(\mathbf{ff}, \perp, \mathbf{tt}) = \mathbf{tt} \\ g(\mathbf{tt}, \mathbf{ff}, \perp) = \mathbf{ff} \end{cases}$$

Since all subsets of H_f with at least three elements are hyperarcs, and H_g is composed by a single ternary hyperarc, it is easy to see that there is no non trivial h -morphism from the maximal hyperarc of H_f to $\overline{H_g}$, and hence no timed morphism from H_f to H_g . On the other hand $f \leq_{\text{par}} g$, since the degree of g (the “B-K function”) is the top of stable degrees ([5], p. 334), and f is stable.

8 Conclusion

For a wide class of boolean functions (the subsequential ones) we are able to solve relative definability problems in a geometric way, using a suitable representation of functions as hypergraphs and PCF-terms as hypergraphs morphisms.

We can also list all the (sensible) terms solving a given problem f, g , by enumerating the timed morphisms from H_f to H_g , and choose, for instance, the one which uses as few calls of g as possible (but other notion of optimality could be considered).

A natural question is wether this approach can be extended to non subsequential boolean functions and/or to higher-order functions. We do not know at present, but probably a combination of more complex representations of functions as hypergraphs and of more involved notions of morphisms is required.

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A Proof of the soundness theorem

First, we state a straightforward, yet useful, lemma:

Lemma 14. *One can restrict a morphism $\alpha : H \rightarrow K$ to $B \subseteq V_H$ by $(\alpha|_B)_X = \alpha_X$ whenever X is a hyperarc of $H|_B$. Moreover, if α is a timed morphism from H_f to H_g , and $\text{tr}(f') \subseteq \text{tr}(f)$, $\alpha|_{\text{tr}(f')}$ is a morphism from $H_{f'}$ to H_g .*

Now we can proceed with the proof of the soundness theorem:

Theorem 15. *Let $f : \mathcal{B}^l \rightarrow \mathcal{B}$, $g : \mathcal{B}^m \rightarrow \mathcal{B}$ be monotone functions such that $\mathcal{TH}(H_f, H_g) \neq \emptyset$. Then $f \leq_{\text{par}} g$.*

Proof. Let $\alpha \in \mathcal{TH}(H_f, H_g)$. We prove the theorem by induction on $k = \#\text{tr}(f)$. If $k = 1$ f is sequential, hence PCF-definable, and $f \leq_{\text{par}} g$ holds trivially. Suppose now $k = n + 1$; we reason by cases on the structure of H_f .

First, let us assume that $V_{H_f} \notin A_{H_f}$. This means that there exists an $1 \leq i \leq l$ such that of $\pi_i(\pi_1(\text{tr}(f))) = \{\mathbf{tt}, \mathbf{ff}\}$ (i is a sequentiality index of f). Splitting the trace of f in two subsets according to the value of this component, we define the functions $f_{\mathbf{tt}}$ and $f_{\mathbf{ff}}$.

For $\rho = \mathbf{tt}, \mathbf{ff}$, $\#\text{tr}(f_\rho) < \#\text{tr}(f)$. By lemma 14, there exists a timed morphism from H_{f_ρ} to H_g . By inductive hypothesis, there exists M_ρ g -defining f_ρ . Define:

$$M = \lambda g \lambda x. \text{if } x_i \text{ then } M_{\mathbf{tt}}g x \text{ else } M_{\mathbf{ff}}g x$$

It is easy to check that M g -defines f .

Now, let us assume that $V = V_{H_f} \in A_{H_f}$. Let f_i , $1 \leq i \leq m$, be the i th-slice of α_V , and define \hat{f}_i as

$$\hat{f}_i = \begin{cases} f_i & \text{if } \#\text{tr}(f_i) < \#\text{tr}(f) \\ \lambda x.v & \text{for } v \in \pi_2(\text{tr}(f_i)) \text{ otherwise} \end{cases}$$

First, let us prove that the \hat{f}_i 's are well defined. If $\#\text{tr}(f_i) = \#\text{tr}(f)$ then $\pi_1(\text{tr}(f_i)) = \pi_1(\text{tr}(f))$. Since V_{H_f} is a hyperarc of H_f , it is also a hyperarc of H_{f_i} . Since f_i is subsequential, its hyperarcs are monochromatic. We get $\#\pi_2(\text{tr}(f_i)) = 1$.

Let us prove that the \hat{f}_i 's are g -definable. The only case to be checked is $\hat{f}_i = f_i$ in the previous definition, since $\lambda \mathbf{x}.v$ is PCF-definable. Since the f_i 's are subsequential, by lemma 7 $f_i \leq_{\text{par}} \overline{f}_i$, where $\text{tr}(\overline{f}_i) = \{v \in \text{tr}(f) \mid \pi_1(v) \in \pi_1(\text{tr}(f_i))\}$ ($\overline{f}_i(\mathbf{x})$ is equal to $f(\mathbf{x})$ whenever $f_i(\mathbf{x}) \neq \perp$, to \perp otherwise). Now $\#\text{tr}(\overline{f}_i) < \#\text{tr}(f)$, and, by lemma 14, $\mathcal{TH}(H_{\overline{f}_i}, H_g) \neq \emptyset$. Hence by inductive hypothesis $\overline{f}_i \leq_{\text{par}} g$, and finally $f_i \leq_{\text{par}} g$ by transitivity of \leq_{par} . Let M_i be a term g -defining \hat{f}_i .

Before constructing a term M g -defining f let us prove that we can already g -define a “convergence test” for f , i.e. that for all $\mathbf{x} = (x_1, \dots, x_l) \in \mathcal{B}^l$:

$$f(\mathbf{x}) \neq \perp \Leftrightarrow g(\llbracket M_1 \rrbracket g\mathbf{x}, \dots, \llbracket M_m \rrbracket g\mathbf{x}) \neq \perp$$

The direction \Rightarrow is trivial, since the \hat{f}_i 's are upper bounds of the f_i 's, hence if there exists $\mathbf{v} \in \pi_1(\text{tr}(f))$ such that $\mathbf{v} \leq \mathbf{x}$, then $(\llbracket M_1 \rrbracket g\mathbf{x}, \dots, \llbracket M_m \rrbracket g\mathbf{x}) \geq \alpha_V(\mathbf{v})$.

For the opposite direction, let us suppose that $f(\mathbf{x}) = \perp$, and hence for all $\mathbf{v} \in \pi_1(\text{tr}(f))$, $\mathbf{x} \not\geq \mathbf{v}$. By definition of the \hat{f}_i 's we know that for all $\mathbf{w} \in \alpha_V(V_{H_f})$, $(\llbracket M_1 \rrbracket g\mathbf{x}, \dots, \llbracket M_m \rrbracket g\mathbf{x}) \leq \mathbf{w}$, since, under the hypothesis $f(\mathbf{x}) = \perp$, we have that for all $1 \leq j \leq m$, if $\llbracket M_j \rrbracket g\mathbf{x} = b > \perp$ then $\hat{f}_j = \lambda \mathbf{x}. b$, and hence for all $\mathbf{w} \in \alpha(V_{H_f})$, $w_j = b$. Since $V = V_{H_f}$ is a hyperarc, we know that $\#\alpha_V(V) \geq 2$, and by minimality of the elements of $\pi_1(\text{tr}(g))$ we conclude that for all $w \in \pi_1(\text{tr}(g))$ $(\llbracket M_1 \rrbracket g\mathbf{x}, \dots, \llbracket M_m \rrbracket g\mathbf{x}) \not\geq w$, and hence $g(\llbracket M_1 \rrbracket g\mathbf{x}, \dots, \llbracket M_m \rrbracket g\mathbf{x}) = \perp$.

We can now conclude the proof, again by case reasoning on the structure of H_f .

If V_{H_f} is a monochromatic hyperarc (w.l.o.g. assume that all vertices are white). Then it is easy to check that f is g -defined by the term:

$$M = \lambda g \lambda \mathbf{x}. \text{if } g(M_1 g\mathbf{x}) \dots (M_m g\mathbf{x}) \text{ then tt else tt}$$

If $V = V_{H_f}$ is not monochromatic, whenever $C(x) = C(y)$, there exists $z \in V_{H_f}$ such that $C(z) \neq C(x)$ (resp. y). Then $C(\alpha_V(z)) \neq C(\alpha_V(x))$ (resp. y), which means:

$$\forall x, y \in V_{H_f} \ C(x) = C(y) \Leftrightarrow C(\alpha_V(x)) = C(\alpha_V(y))$$

i.e. α_V acts as the identity or the “negation” on colors. We define then

$$M = \lambda g \lambda \mathbf{x}. \epsilon(g(M_1 g\mathbf{x}) \dots (M_m g\mathbf{x}))$$

where ϵ is the boolean identity or the boolean negation according to how α_V acts on colors. Then again it is easily checked that M g -defines f .