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RECURRENCE RATE IN RAPIDLY MIXING DYNAMICAL SYSTEMS

BENOIT SAUSSOL

ABSTRACT. For measure preserving dynamical systems on metric spaces we study the time needed by a typical orbit to return back close to its starting point. We prove that when the decay of correlation is super-polynomial the recurrence rates and the pointwise dimensions are equal. This gives a broad class of systems for which the recurrence rate equals the Hausdorff dimension of the invariant measure.

1. INTRODUCTION

1.1. **Decay of correlations.** Let (X, f, μ) be a measure preserving dynamical system. Recall that the system is said to be *mixing* if for any functions φ, ψ in L^2 the covariance

$$\text{Cov}(\varphi \circ f^n, \psi) := \int \varphi \circ f^n \psi d\mu - \int \varphi d\mu \int \psi d\mu \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (1)$$

The decay of the correlation function is, in great generality, arbitrarily slow. The notion of rapid mixing needs a little more structure.

Assume that X is a metric space with metric d , and consider the space $\text{Lip}(X)$ of real Lipschitz functions on X . For many dynamical systems an upper bound for (1) of the form $\|\varphi\| \|\psi\| \theta_n$ has been computed, where $\theta_n \rightarrow 0$ with some rate, and $\|\cdot\|$ is a norm on a space of functions with some regularity. Without loss of generality we are considering in this paper the rate of decay of correlations for Lipschitz observables¹.

A broad class of systems enjoy exponential decay of correlations. The main result of the paper (Theorem 3) applies to systems with super-polynomial decay of correlation. This includes for example Axiom A systems with equilibrium states, hyperbolic systems with singularities with their SBR measures such as those considered by Chernov in [7], many systems with a Young tower [16, 17], expanding maps with singularities such as in [13], some non-uniformly expanding maps [1], etc. The main reference for these questions is certainly the book by Baladi [2]. The reader will also find in the review by Luzzatto [11] an exposition of the recent methods for non-uniformly expanding systems and an extensive bibliography on this active field.

1.2. **Recurrence rate and dimensions.** The return time of a point $x \in X$ under the map f in its r -neighborhood is

$$\tau_r(x) = \inf\{n \geq 1 : d(f^n x, x) < r\}.$$

We are interested in the behavior as $r \rightarrow 0$ of the return time. We define the recurrence rate as the limits

$$\underline{R}(x) = \liminf_{r \rightarrow 0} \frac{\log \tau_r(x)}{\log(1/r)} \quad \text{and} \quad \overline{R}(x) = \limsup_{r \rightarrow 0} \frac{\log \tau_r(x)}{\log(1/r)}.$$

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¹For example an immediate approximation argument allows easily to go from Holder or class C^k to Lipschitz.

Whenever $\underline{R}(x) = \overline{R}(x)$ we denote by $R(x)$ the value of the limit.

From now on we assume that X is a finite dimensional Euclidean space. Denote by $HD(Y)$ the Hausdorff dimension of a set $Y \subset X$. We define the Hausdorff dimension of a probability measure μ by

$$HD(\mu) = \inf\{HD(Y) : \mu(Y) = 1\}$$

We also define a local version of the dimension, namely

$$\underline{d}_\mu(x) = \liminf_{r \rightarrow 0} \frac{\log \mu(B(x, r))}{\log r} \quad \text{and} \quad \overline{d}_\mu(x) = \limsup_{r \rightarrow 0} \frac{\log \mu(B(x, r))}{\log r} \quad (2)$$

It is well known that the Hausdorff dimension satisfies the relation

$$HD(\mu) = \text{ess-sup } \underline{d}_\mu. \quad (3)$$

Barreira and Saussol established in [4] the following relation

Proposition 1. *Let f be a measurable map and μ be an invariant measure for f . The recurrence rates are bounded from above by the pointwise dimensions :*

$$\underline{R} \leq \underline{d}_\mu \quad \text{and} \quad \overline{R} \leq \overline{d}_\mu \quad \mu\text{-a.e.}$$

We refer to the works by Boshernitzan [6] and Ornstein and Weiss [12] for pioneering related results.

In this paper we are giving conditions under which the opposite inequalities will hold, establishing the equalities

$$\underline{R} = \underline{d}_\mu \quad \text{and} \quad \overline{R} = \overline{d}_\mu \quad \mu\text{-a.e.} \quad (4)$$

1.3. Statement of the results.

Definition 2. *We say that (X, f, μ) has super-polynomial decay of correlations if we have*

$$\left| \int \varphi \circ f^n \psi d\mu - \int \varphi d\mu \int \psi d\mu \right| \leq \|\varphi\| \|\psi\| \theta_n \quad (5)$$

with $\lim_n \theta_n / n^p = 0$ for all $p > 0$, where $\|\cdot\|$ is the Lipschitz norm.

We say that the local decay of correlations is super-polynomial if there exists a partition (modulo μ) into open sets V_i and sequences θ_n^i such that (5) holds whenever $\text{supp } \varphi \subset V_i$ and $\text{supp } \psi \subset V_i$, where $\lim_n \theta_n^i / n^p = 0$ for all $p > 0$.

The main result of the paper is the following.

Theorem 3. *Let (X, f, μ) be a measure preserving dynamical system. If the entropy $h_\mu(f) > 0$, f is Lipschitz (or piecewise Lipschitz with finite average Lipschitz exponent ; see Definition 15) and the (local) decay of correlation is super-polynomial then*

$$\underline{R} = \underline{d}_\mu \quad \text{and} \quad \overline{R} = \overline{d}_\mu \quad \mu\text{-a.e.}$$

We postpone the proof at the end of Section 3. This extends some results by Barreira and Saussol in [4, 5], including the case of Axiom A systems with equilibrium states. The theorem also applies to loosely Markov dynamical systems and we recover Urbanski's result in [15]. The hypotheses in Theorem 3 are satisfied in a number of systems such as those already quoted in the introduction. All these systems have in common some hyperbolic behavior. We now give an example of a relatively different nature, due to the possibility of zero Lyapunov exponents, where one can still apply Theorem 3.

Example 4 (Ergodic toral automorphisms). Recall that any matrix $A \in Sl(k, \mathbb{Z})$ (i.e. the entries of A are in \mathbb{Z} and $|\det A| = 1$) gives rise to an automorphism f of the torus \mathbb{T}^k by $f(x) = Ax \pmod{\mathbb{Z}^k}$ which preserves the Lebesgue measure. The map f is ergodic if and only if the matrix A has no eigenvalue root of unity. Lind's established [10] the exponential decay of correlations (using the algebraic nature and Fourier transform) which is more than enough to apply Theorem 3 and get

$$R(x) = k \quad \text{for Lebesgue a.e. } x \in \mathbb{T}^k.$$

for any ergodic automorphism of the torus, even non-hyperbolic.

Let f be a diffeomorphism of a compact manifold M and μ be an invariant measure. By Oseledec's multiplicative ergodic Theorem the Lyapunov exponents

$$\lambda(x, v) = \lim_{n \rightarrow \infty} \frac{1}{n} \log |d_x f^n v|$$

are well defined for all nonzero $v \in T_x M$ for a.e. $x \in M$. Recall that a measure μ is said to be hyperbolic if none of its Lyapunov exponents are zero. Barreira, Pesin and Schmeling [3] prove the following.

Proposition 5. *Let f be a diffeomorphism of a compact manifold and μ be an ergodic hyperbolic measure. Then we have*

$$\underline{d}_\mu = \overline{d}_\mu = HD(\mu) \quad \mu\text{-a.e.}$$

The case of an hyperbolic measure with zero entropy is completely understood.

Proposition 6. *let f be a diffeomorphism of a compact manifold and μ be an hyperbolic invariant measure. If $h_\mu(f) = 0$ then $R = 0 = HD(\mu)$ μ -a.e.*

Proof. Barreira and Saussol established in [4] the inequality $\overline{R} \leq \overline{d}_\mu$ μ -a.e. and it follows from Ledrappier and Young's work [9] that $HD(\mu) = 0$ if $h_\mu(f) = 0$, which allows to conclude by Proposition 5. \square

Corollary 7. *Let f be a diffeomorphism of a compact manifold and μ be an hyperbolic measure with super-polynomial rate of decay of correlation. Then we have*

$$R = HD(\mu) \quad \mu\text{-a.e.}$$

Proof. If the entropy is zero then this is the content of Proposition 6. If the entropy is non-zero then this is the content of Theorem 3. \square

We point out that in the case of interval maps with nonzero Lyapunov exponent, Saussol, Troubetzkoy and Vaienti prove that $R = HD(\mu)$ μ -a.e. for ergodic measures, under very weak regularity conditions [14]. See Remark 17-(i) for related results.

We now give a sketch of the strategy adopted in this paper.

Theorem 8 states that under sufficiently rapid mixing the recurrence rates equal the pointwise dimensions a.e. on the set where $\underline{R} > 0$. Indeed, mixing implies that $\mu(B \cap f^{-n}B) \rightarrow \mu(B)^2$ as $n \rightarrow \infty$. Thus we have $\mu(B \cap f^{-n}B) \leq 2\mu(B)^2$ for large n . If now we consider the set $B \cap f^{-n}B \cap f^{-n-1}B \cap \dots \cap f^{-n-\ell}B$ then its measure is bounded by $2\ell\mu(B)^2$. If $\ell \leq \mu(B)^{-1+\varepsilon}$ then we get that the proportion of points inside B that never enter in B in the time interval $[n, n + \ell]$ is bounded by $2\mu(B)^\varepsilon$. Using the decay of correlations we are able to prove that this

last statement is true for n of the order $\text{diam}(B)^{-\delta}$ for some small $\delta > 0$, whenever B is a ball. This is what we call the *long fly property*. A Borel Cantelli argument then shows that typical points do have long flies (see Lemma 9 for precise statement). If in addition we also have $\underline{R} > \delta$ then it immediately shows that the return time into small neighborhoods B cannot be much less (at an exponential scale) than $\mu(B)^{-1}$, establishing Equation (4).

On the other hand, for systems which are not too wild (e.g. finite Lyapunov exponents, see Lemma 16) and with nonzero metric entropy, a symbolic coding (see Lemma 14) allows to use Ornstein-Weiss' theorem on repetition time of symbolic sequences to prove that the return time of a typical point in a ball B is not less than $\text{diam}(B)^{-\delta}$; see Lemma 12.

The structure of the paper is as follows. We state and prove in Section 2 the core result, Theorem 8. In Section 3 we provide some conditions under which the recurrence rate is nonzero.

2. RAPID MIXING IMPLIES LONG FLIES

Theorem 8. *Assume that the local rate of decay of correlations is super-polynomial. Then on the set $\{\underline{R} > 0\}$ we have*

$$\underline{R} = \underline{d}_\mu \quad \text{and} \quad \overline{R} = \overline{d}_\mu \quad \mu\text{-a.e.}$$

Proof. By Proposition 1 we know that $\underline{R} \leq \underline{d}_\mu$ and $\overline{R} \leq \overline{d}_\mu$. Furthermore, the first inequality implies that $\{\underline{R} > a\} \subset \{\underline{d}_\mu > a\}$ μ -a.e. But on the set $\{\underline{R} > a\}$ we have $\tau_r(x) \geq r^{-a}$ provided r is sufficiently small. By Lemma 9 below with $\delta = a$ and $\varepsilon > 0$ we get that $\tau_r(x) \geq \mu(B(x, r))^{-1+\varepsilon}$ provided r is sufficiently small, for μ -a.e. $x \in \{\underline{R} > a\}$. Thus $\underline{R} \geq (1 - \varepsilon)\underline{d}_\mu$ and $\overline{R} \geq (1 - \varepsilon)\overline{d}_\mu$ μ -a.e. on $\{\underline{R} > a\}$. The conclusion follows by taking $\varepsilon > 0$ arbitrary small. \square

The following lemma expresses that the orbit of a typical point has the long fly property.

Lemma 9. *Let $X_a = \{\underline{d}_\mu > a\}$ for some $a > 0$. For any $\delta, \varepsilon > 0$, for μ -a.e. $x \in X_a$ there exists $r(x) > 0$ such that for any $r \in (0, r(x))$ and any integer n in $[r^{-\delta}, \mu(B(x, r))^{-1+\varepsilon}]$ we have $d(f^n x, x) \geq r$.*

Proof. For clarity we assume that the (global) rate of decay of correlation is super-polynomial. The obvious modifications in the proof would consist essentially in considering separately each sets $G \cap \{x \in V_i : d(x, \partial V_i) > \nu\}$ for arbitrarily small $\nu > 0$ in place of the unique set G defined below.

Let $D = \dim(X)$. Fix $b > 0$, $c = a\varepsilon/3$ and consider for $r_0 > 0$ the set $G = G_1 \cap G_2 \cap G_3$ where

$$\begin{aligned} G_1 &= \{x \in X_a : \forall r \leq r_0, \mu(B(x, r)) \leq r^a\} \\ G_2 &= \{x \in X : \forall r \leq r_0, \mu(B(x, r)) \geq r^{D+b}\} \\ G_3 &= \{x \in X : \forall r \leq r_0, \mu(B(x, r/2)) \geq \mu(B(x, 4r))r^c\}. \end{aligned}$$

We claim that $\mu(G) \rightarrow \mu(X_a)$ as $r_0 \rightarrow 0$. Indeed, by definition of the lower pointwise dimension we have $\mu(G_1) \rightarrow \mu(X_a)$. In addition since $\overline{d}_\mu \leq D$ a.e. we have $\mu(G_2) \rightarrow 1$ and since X is Euclidean the measure μ is weakly diametrically regular (see Lemma 1 in [4]), thus $\mu(G_3) \rightarrow 1$ as well. Let $r \leq r_0$ and define the set

$$A_\varepsilon(r) = \{y \in X : \exists n \in [r^{-\delta}, \mu(B(y, 3r))^{-1+\varepsilon}], d(f^n y, y) < r\}.$$

Let $x \in G$. By the triangle inequality we get the inclusions

$$\begin{aligned} B(x, r) \cap A_\varepsilon(r) &\subset \{y \in B(x, r) : \exists n \in [r^{-\delta}, \mu(B(x, 2r))^{-1+\varepsilon}], d(f^n y, x) < 2r\} \\ &\subset \bigcup_{r^{-\delta} \leq n \leq \mu(B(x, 2r))^{-1+\varepsilon}} B(x, r) \cap f^{-n} B(x, 2r). \end{aligned}$$

Let $\eta_r: [0, \infty) \rightarrow \mathbb{R}$ be the r^{-1} -Lipschitz map such that $1_{[0, r]} \leq \eta_r \leq 1_{[0, 2r]}$ and set $\varphi_{x, r}(y) = \eta_r(d(x, y))$. Clearly $\varphi_{x, r}$ is also r^{-1} -Lipschitz. By the assumption on the decay of correlation we obtain

$$\begin{aligned} \mu(B(x, r) \cap f^{-n} B(x, 2r)) &\leq \int \varphi_{x, 2r} \varphi_{x, 2r} \circ f^n d\mu \\ &\leq \|\varphi_{x, 2r}\|^2 \theta_n + \left(\int \varphi_{x, 2r} d\mu \right)^2 \\ &\leq r^{-2} \theta_n + \mu(B(x, 4r))^2. \end{aligned}$$

Choose $p > 1$ such that $\delta(p-1) - 2 \geq D + 2b$ and take r_0 so small that $n \geq r_0^{-\delta}$ implies $\theta_n \leq (p-1)n^{-p}$. Since $\sum_{n \geq q} n^{-p} \leq \frac{1}{p-1} q^{1-p}$ we obtain

$$\begin{aligned} \mu(B(x, r) \cap A_\varepsilon(r)) &\leq r^{\delta(p-1)-2} + \mu(B(x, 2r))^{-1+\varepsilon} \mu(B(x, 4r))^2 \\ &\leq \mu(B(x, r/2)) (r^b + r^{\varepsilon a - 2c}). \end{aligned}$$

Let $B \subset G$ be a maximal r -separated set². Since $(B(x, r))_{x \in B}$ covers G we have

$$\begin{aligned} \mu(G \cap A_\varepsilon(r)) &\leq \sum_{x \in B} \mu(B(x, r) \cap A_\varepsilon(r)) \\ &\leq \sum_{x \in B} \mu(B(x, r/2)) (r^b + r^{\varepsilon a - 2c}) \\ &\leq r^b + r^{\varepsilon a - 2c} \end{aligned}$$

since by the balls $(B(x, r/2))_{x \in B}$ are disjoint. This implies that

$$\sum_m \mu(A_\varepsilon(e^{-m})) < \infty,$$

thus by Borel-Cantelli Lemma we obtain that for μ -a.e. $y \in G$ there exists $m(y)$ such for every $m > m(y)$ there exists no $n \in [e^{-\delta m}, \mu(B(y, 3e^{-m}))^{-1+\varepsilon}]$ such that $d(f^n y, y) < e^{-m}$. By weak diametric regularity (and changing slightly if necessary the values of ε and δ), this proves the lemma. \square

Remark 10. Observe that we only use that the decay of correlation is at least n^{-p} for some $p > \frac{D+2}{\delta} + 1$. If in addition (5) holds with the first norm $\|\varphi\|$ taken to be the $L^1(\mu)$ norm (e.g. expanding maps) then $p > \frac{D+1}{\delta} + 1$ suffices.

²that is if $x \neq x' \in B$ then $d(x, x') \geq r$ and maximal in the sense that for any $y \in G$ there exists $x \in B$ such that $d(x, y) < r$.

3. NON-ZERO RECURRENCE RATE

We proceed now to find conditions under which the recurrence rate does not vanish. Denote by $\xi(x)$ the unique element of a partition ξ containing the point x and by $\xi^n = \xi \vee f^{-1}\xi \vee \dots \vee f^{-n+1}\xi$ the dynamical partition, for any integer n .

3.1. Coding by symbolic systems : partitions with large interior.

Definition 11. *We say that a partition ξ has large interior if for μ -a.e. x there exists $\chi = \chi(x) < \infty$ such that $B(x, e^{-\chi n}) \subset \xi^n(x)$ for all n sufficiently large.*

Next lemma, which proof is fairly simple, is the key-observation which gives to Theorem 8 all its interest.

Lemma 12. *If there exists a partition with large interior and nonzero entropy then $\underline{R} > 0$ μ -a.e.*

Proof. Let ξ be such a partition. Define

$$R_n(x, \xi) = \min\{k > 0 : f^k x \in \xi^n(x)\}.$$

Ornstein and Weiss [12] prove that if ξ is a finite partition with entropy $h_\mu(f, \xi)$ then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log R_n(x, \xi) = h_\mu(f, \xi) \quad \mu\text{-a.e.}$$

Since ξ has large interior, for μ -a.e. $x \in X$ there exists a number $\chi = \chi(x)$ such that $B(x, e^{-\chi n}) \subset \xi^n(x)$. Thus

$$\underline{R}(x) = \liminf_{n \rightarrow \infty} \frac{\log \tau_{e^{-n}}(x)}{n\chi(x)} \geq \liminf_{n \rightarrow \infty} \frac{\log R_n(x, \xi)}{n\chi(x)} = \frac{h_\mu(f, \xi)}{\chi(x)} > 0 \quad \mu\text{-a.e.}$$

□

Combining Lemma 12 and Theorem 8 we get that if we have local super-polynomial decay of correlations and a partition of positive entropy with large interior then $\underline{R} = \underline{d}_\mu$ and $\overline{R} = \overline{d}_\mu$. The rest of the section consists in finding sufficient conditions for the existence of such a partition.

3.2. Reasonable dependence on initial condition.

Definition 13. *We say that a system (X, f, μ) is reasonably sensitive if for μ -a.e. x there exists $\gamma, \lambda > 0$ such that f^n is $e^{\lambda n}$ -Lipschitz on the ball $B(x, e^{-\gamma n})$ for all n sufficiently large.*

Lemma 14. *If the system (X, f, μ) is reasonably sensitive and the entropy $h_\mu(f) > 0$ then there exists a partition with large interior and nonzero entropy.*

Proof. Claim : For any $x \in X$, $s > 0$ there exists $\rho \in (s, 2s)$ such that

$$\mu(\{y \in X : \rho - 4^{-n}s < d(x, y) < \rho + 4^{-n}s\}) \leq \frac{1}{2^{n-1}} \mu(B(x, 2s)). \quad (6)$$

Indeed, let m be the measure on the interval $(0, 2)$ defined by $m([0, t]) = \mu(B(x, st))$. We construct a sequence of open intervals I_n starting from $I_0 = (1, 2)$. If I_n is an interval of length 4^{-n} we divide it into 4 pieces of equal length and choose I_{n+1} the left of the right central piece of smallest measure. We have $m(I_{n+1}) \leq \frac{1}{2}m(I_n)$. I_n is a decreasing sequence of intervals with $\overline{I}_{n+1} \subset I_n$ thus $\cap_n I_n$ contains one point, say $\bar{\rho}$. Since $\bar{\rho} \in I_n$ we have $\bar{\rho} \pm 4^{-n} \in I_{n-1}$ thus

$m((\bar{\rho} - 4^{-n}, \bar{\rho} + 4^{-n})) \leq m(I_{n-1}) \leq \frac{1}{2^{n-1}}m(I_0)$. Proving the claim with $\rho = s\bar{\rho}$.

Fix $s > 0$ so small that any partition made by sets of diameter less than $2s$ has nonzero entropy. Choose a maximal s -separated set E . For any $x \in E$ take $\rho_x \in (s, 2s)$ such that (6) in the claim holds. Let $E = \{x_1, x_2, \dots\}$ be an enumeration of the (at most) countable set E . Put $B_i = B(x_i, \rho_{x_i})$ and define $Q_1 = B_1$, $Q_2 = B_2 \setminus Q_1$, $Q_3 = B_3 \setminus (Q_1 \cup Q_2)$, \dots . By maximality the collection of sets $\xi = \{Q_1, Q_2, \dots\}$ is a partition of X (modulo μ) and since $\partial\xi \subset \cup_i \partial B_i$ we get

$$\begin{aligned} \mu(\{x \in X : d(x, \partial\xi) < 4^{-n}s\}) &\leq \mu(\cup_i \{x \in X : \rho_{x_i} - 4^{-n} < d(x_i, x) < \rho_{x_i} + 4^{-n}\}) \\ &\leq \frac{1}{2^{n-1}} \sum_i \mu(B(x_i, 2s)). \end{aligned}$$

Since the x_i are s -separated and X is Euclidean there are at most $c(X) = c(\dim X)$ balls of radius $2s$ that can intersect, thus the last sum is bounded by $\frac{c(X)}{2^{n-1}}$. This proves that for some constants $a, c > 0$ and all $\varepsilon > 0$

$$\mu(x \in X : d(x, \partial\xi) < \varepsilon) < c\varepsilon^a.$$

Thus for any $b > 0$ we have by the invariance of μ

$$\sum_n \mu(\{x \in X : d(f^n x, \xi) < e^{-bn}\}) \leq \sum_n ce^{-abn} < \infty.$$

This implies by Borel-Cantelli Lemma that for μ -a.e. x there exists $n(x) < \infty$ such that $d(f^n x, \partial\xi) \geq e^{-bn}$, hence $B(f^n x, e^{-bn}) \subset \xi(f^n x)$, for any $n \geq n(x)$. Taking $c(x) \in (0, 1)$ sufficiently small we have $B(f^n x, c(x)e^{-bn}) \subset \xi(f^n x)$ for all integer n .

Fix $x \in X$ where the reasonable sensitivity condition holds. Without loss of generality, and changing if necessary $c(x)$ into a smaller constant we assume that f^n is $e^{\lambda n}$ -Lipschitz on the ball $B(x, c(x)e^{-\gamma n})$ for all integer n and that $\lambda > \gamma + b$.

We show then by induction that $B(x, c(x)e^{-\lambda n}) \subset \xi^k(x)$ for any $k \leq n$. Indeed, this is trivially true for $k = 1$, and if this holds for some $k \leq n - 1$ then we have

$$f^k(B(x, c(x)^2 e^{-\gamma n})) \subset B(f^k x, c(x)e^{\lambda k - \gamma n}) \subset B(f^k x, e^{-bn}) \subset \xi(f^k x).$$

Hence $B(x, c(x)^2 e^{-\gamma n}) \subset \xi^{k+1}(x)$. □

We finally provide a sufficient condition for reasonable sensitivity.

Definition 15. *If there exists a partition \mathcal{A} (modulo μ) into open sets such that on each $A \in \mathcal{A}$ the map f is Lipschitz with constant $L_f(A)$ and the singularity set $\partial\mathcal{A} = \cup_{A \in \mathcal{A}} \partial A$ is such that $\mu(\{x \in X : d(x, \partial\mathcal{A}) < \epsilon\}) \leq c\epsilon^a$ for some constants $c > 0$ and $a > 0$ then we say that f is piecewise Lipschitz with average Lipschitz exponent $\log L_f = \int \log^+ L_f(A(x)) d\mu(x) = \sum_{A \in \mathcal{A}} \log^+ L_f(A) \mu(A)$.*

Lemma 16. *If f is Lipschitz, or piecewise Lipschitz with finite Lipschitz exponent then (X, f, μ) is reasonably sensitive.*

Proof. We prove the piecewise case, the other one is obvious. Let $\lambda > \log L_f$. By the Birkhoff Ergodic Theorem, for μ -a.e. x there exists $m(x)$ such that

$$L_f(A(x))L_f(A(fx)) \cdots L_f(A(f^{n-1}x)) \leq e^{\lambda n}$$

for all $n \geq m(x)$. Replacing if necessary the upper bound by $e^{\lambda n}/c(x)$ for some constant $c(x) \geq 1$ the inequality will hold for any integer n . Proceeding as in the last part of the proof of Lemma 14 we get that for any $b > 0$, changing $c(x)$ if necessary, we have $B(f^n x, c(x)e^{-bn}) \subset \mathcal{A}(f^n x)$ for any integer n . We then conclude similarly that $B(x, c(x)^2 e^{-bn} e^{-\lambda n}) \subset \mathcal{A}^n(x)$. This concludes the proof taking $\gamma = b + \lambda$. \square

The proof of Theorem 3 follows now easily from the preceding results.

Proof of Theorem 3. By Lemma 16 the map is reasonably sensitive. This implies by Lemma 14 the existence of a partition with large interior. By Lemma 12 we find that $\underline{R} > 0$ a.e. and the conclusion follows from Theorem 8. \square

Remark 17. (i) We remark that if f is C^1 on a compact manifold, or more generally if f is piecewise $C^{1+\alpha}$ with reasonable singularity set such as in [8], then the exponents λ and γ in Definition 13 can be taken arbitrarily close to the largest Lyapunov exponent³ λ_μ^+ . Thus the exponent χ in Lemma 12 may also be taken arbitrarily close to λ_μ^+ . This readily implies that $\underline{R} \geq h_\mu/\lambda_\mu^+$. This is optimal in dimension one or more generally for conformal maps, where under mild assumptions we have $HD(\mu) = h_\mu/\lambda_\mu$.

(ii) Combining the above observation with Remark 10 shows that the assumption on the super-polynomial decay of correlations in Theorem 8 may be reduced to a decay at a rate n^{-p} for some $p > \frac{D+2}{h_\mu} \lambda_\mu^+ + 1$.

REFERENCES

- [1] J. F. Alves, SRB measures for non-hyperbolic systems with multidimensional expansion, *Annales Scientifiques de l'ENS* **33** (2000) 1–32.
- [2] V. Baladi, *Positive transfer operators and decay of correlations*, Advances Series in Nonlinear Dynamics, vol. 16, World Scientific Publishing Co. Inc., River Edge, 2000.
- [3] L. Barreira, Y. Pesin, J. Schmeling, Dimension and product structure of hyperbolic measures, *Annals of Mathematics* **149** (1999) 755–783.
- [4] L. Barreira, B. Saussol, Hausdorff dimension of measures via Poincaré recurrence, *Communication in Mathematical Physics* **219** (2001) 443–463.
- [5] L. Barreira, B. Saussol, Product structure of Poincaré recurrence, *Ergodic Theory and Dynamical Systems* **22** (2002) 33–61.
- [6] M. Boshernitzan, Quantitative recurrence results, *Inventiones Mathematicæ* **113** (1993) 617–631.
- [7] N. Chernov, Statistical properties of piecewise smooth hyperbolic systems in high dimensions. *Discrete and Continuous Dynamical Systems* **5** (1999) 425–448.
- [8] A. Katok, J.-M. Strelcyn, F. Ledrappier, and F. Przytycki, *Invariant manifolds, entropy and billiards; smooth maps with singularities*, Lecture Notes in Mathematics, vol. 1222, Springer-Verlag, Berlin, 1986.
- [9] F. Ledrappier, L.-S. Young, The metric entropy of diffeomorphisms: I. Characterization of measures satisfying Pesin’s formula, II. Relations between entropy, exponents and dimension, *Annals of Mathematics* **122** (1985) 509–539.
- [10] D. A. Lind, Dynamical properties of quasi-hyperbolic toral automorphisms, *Ergodic Theory and Dynamical Systems* **2** (1982) 49–68.
- [11] S. Luzzatto, Mixing and decay of correlations in non-uniformly expanding maps: a survey of recent results, preprint 2004.

³to see this, consider a Lyapunov chart whose local chart at x has a diameter $\rho(x)$, where ρ is η -slowly varying. A choice like $\lambda = \lambda_\mu^+ + 2\eta$ and $\gamma = \lambda + \eta$ would do the job.

- [12] D. Ornstein, B. Weiss, Entropy and data compression schemes, *IEEE Transaction on Information Theory* **39** (1993) 78–83.
- [13] B. Saussol, Absolutely continuous invariant measures for multidimensional expanding maps, *Israel Journal of Mathematics* **116** (2000) 223–248.
- [14] B. Saussol, S. Troubetzkoy, S. Vaienti, Recurrence, dimensions and Lyapunov exponents, *Journal of Statistical Physics* **106** (2002) 623–634.
- [15] M. Urbanski, Recurrence rates for loosely Markov dynamical systems, *Journal of the Australian Mathematical Society*, to appear.
- [16] L.-S. Young, Statistical properties of dynamical systems with some hyperbolicity, *Annals of Mathematics* **147** (1998) 585–650.
- [17] L.-S. Young, Recurrence times and rates of mixing, *Israel Journal of Mathematics* **110** (1999) 153–188.
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