



HAL
open science

Recurrence rate in rapidly mixing dynamical systems

Benoit Saussol

► **To cite this version:**

| Benoit Saussol. Recurrence rate in rapidly mixing dynamical systems. 2004. hal-00003516

HAL Id: hal-00003516

<https://hal.science/hal-00003516v1>

Preprint submitted on 10 Dec 2004

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

RECURRENCE RATE IN RAPIDLY MIXING DYNAMICAL SYSTEMS

BENOIT SAUSSOL

ABSTRACT. For measure preserving dynamical systems on metric spaces we study the time needed by a typical orbit to return back close to its starting point. We prove that when the decay of correlation is super-polynomial the recurrence rates and the pointwise dimensions are equal. This gives a broad class of systems for which the recurrence rate equals the Hausdorff dimension of the invariant measure.

1. INTRODUCTION

1.1. Decay of correlations. Let (X, f, μ) be a measure preserving dynamical system. Recall that the system is said to be *mixing* if for any functions φ, ψ in L^2 the covariance

$$\text{Cov}(\varphi \circ f^n, \psi) := \int \varphi \circ f^n \psi d\mu - \int \varphi d\mu \int \psi d\mu \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (1)$$

The decay of the correlation function is, in great generality, arbitrarily slow. The notion of rapid mixing needs a little more structure.

Assume that X is a metric space with metric d , and consider the space $\text{Lip}(X)$ of real Lipschitz functions on X . For many dynamical systems an upper bound for (1) of the form $\|\varphi\| \|\psi\| \theta_n$ has been computed, where $\theta_n \rightarrow 0$ with some rate, and $\|\cdot\|$ is a norm on a space of functions with some regularity. Without loss of generality we are considering in this paper the rate of decay of correlations for Lipschitz observables¹.

A broad class of systems enjoy exponential decay of correlations. The main result of the paper (Theorem 3) applies to systems with super-polynomial decay of correlation. This includes for example Axiom A systems with equilibrium states, hyperbolic systems with singularities with their SBR measures such as those considered by chernov in [7], many systems with a Young tower [16, 17], expanding maps with singularities such as in [13], some non-uniformly expanding maps [1], etc. The main reference for these questions is certainly the book by Baladi [2]. The reader will also find in the review by Luzzatto [11] an exposition of the recent methods for non-uniformly expanding systems and an extensive bibliography on this active field.

1.2. Recurrence rate and dimensions. The return time of a point $x \in X$ under the map f in its r -neighborhood is

$$\tau_r(x) = \inf\{n \geq 1 : d(f^n x, x) < r\}.$$

We are interested in the behavior as $r \rightarrow 0$ of the return time. We define the recurrence rate as the limits

$$\underline{R}(x) = \liminf_{r \rightarrow 0} \frac{\log \tau_r(x)}{\log(1/r)} \quad \text{and} \quad \overline{R}(x) = \limsup_{r \rightarrow 0} \frac{\log \tau_r(x)}{\log(1/r)}.$$

Date: November 2004.

¹For example an immediate approximation argument allows easily to go from Holder or class C^k to Lipschitz.

Whenever $\underline{R}(x) = \overline{R}(x)$ we denote by $R(x)$ the value of the limit.

From now on we assume that X is a finite dimensional Euclidean space. Denote by $HD(Y)$ the Hausdorff dimension of a set $Y \subset X$. We define the Hausdorff dimension of a probability measure μ by

$$HD(\mu) = \inf\{HD(Y) : \mu(Y) = 1\}$$

We also define a local version of the dimension, namely

$$\underline{d}_\mu(x) = \liminf_{r \rightarrow 0} \frac{\log \mu(B(x, r))}{\log r} \quad \text{and} \quad \overline{d}_\mu(x) = \limsup_{r \rightarrow 0} \frac{\log \mu(B(x, r))}{\log r} \quad (2)$$

It is well known that the Hausdorff dimension satisfies the relation

$$HD(\mu) = \text{ess-sup } \underline{d}_\mu. \quad (3)$$

Barreira and Saussol established in [4] the following relation

Proposition 1. *Let f be a measurable map and μ be an invariant measure for f . The recurrence rates are bounded from above by the pointwise dimensions :*

$$\underline{R} \leq \underline{d}_\mu \quad \text{and} \quad \overline{R} \leq \overline{d}_\mu \quad \mu\text{-a.e.}$$

We refer to the works by Boshernitzan [6] and Ornstein and Weiss [12] for pioneering related results.

In this paper we are giving conditions under which the opposite inequalities will hold, establishing the equalities

$$\underline{R} = \underline{d}_\mu \quad \text{and} \quad \overline{R} = \overline{d}_\mu \quad \mu\text{-a.e.} \quad (4)$$

1.3. Statement of the results.

Definition 2. *We say that (X, f, μ) has super-polynomial decay of correlations if we have*

$$\left| \int \varphi \circ f^n \psi d\mu - \int \varphi d\mu \int \psi d\mu \right| \leq \|\varphi\| \|\psi\| \theta_n \quad (5)$$

with $\lim_n \theta_n / n^p = 0$ for all $p > 0$, where $\|\cdot\|$ is the Lipschitz norm.

We say that the local decay of correlations is super-polynomial if there exists a partition (modulo μ) into open sets V_i and sequences θ_n^i such that (5) holds whenever $\text{supp } \varphi \subset V_i$ and $\text{supp } \psi \subset V_i$, where $\lim_n \theta_n^i / n^p = 0$ for all $p > 0$.

The main result of the paper is the following.

Theorem 3. *Let (X, f, μ) be a measure preserving dynamical system. If the entropy $h_\mu(f) > 0$, f is Lipschitz (or piecewise Lipschitz with finite average Lipschitz exponent ; see Definition 15) and the (local) decay of correlation is super-polynomial then*

$$\underline{R} = \underline{d}_\mu \quad \text{and} \quad \overline{R} = \overline{d}_\mu \quad \mu\text{-a.e.}$$

We postpone the proof at the end of Section 3. This extends some results by Barreira and Saussol in [4, 5], including the case of Axiom A systems with equilibrium states. The theorem also applies to loosely Markov dynamical systems and we recover Urbanski's result in [15]. The hypotheses in Theorem 3 are satisfied in a number of systems such as those already quoted in the introduction. All these systems have in common some hyperbolic behavior. We now give an example of a relatively different nature, due to the possibility of zero Lyapunov exponents, where one can still apply Theorem 3.

Example 4 (Ergodic toral automorphisms). Recall that any matrix $A \in Sl(k, \mathbb{Z})$ (i.e. the entries of A are in \mathbb{Z} and $|\det A| = 1$) gives rise to an automorphism f of the torus \mathbb{T}^k by $f(x) = Ax \pmod{\mathbb{Z}^k}$ which preserves the Lebesgue measure. The map f is ergodic if and only if the matrix A has no eigenvalue root of unity. Lind's established [10] the exponential decay of correlations (using the algebraic nature and Fourier transform) which is more than enough to apply Theorem 3 and get

$$R(x) = k \quad \text{for Lebesgue a.e. } x \in \mathbb{T}^k.$$

for any ergodic automorphism of the torus, even non-hyperbolic.

Let f be a diffeomorphism of a compact manifold M and μ be an invariant measure. By Oseledec's multiplicative ergodic Theorem the Lyapunov exponents

$$\lambda(x, v) = \lim_{n \rightarrow \infty} \frac{1}{n} \log |d_x f^n v|$$

are well defined for all nonzero $v \in T_x M$ for a.e. $x \in M$. Recall that a measure μ is said to be hyperbolic if none of its Lyapunov exponents are zero. Barreira, Pesin and Schmeling [3] prove the following.

Proposition 5. *Let f be a diffeomorphism of a compact manifold and μ be an ergodic hyperbolic measure. Then we have*

$$\underline{d}_\mu = \overline{d}_\mu = HD(\mu) \quad \mu\text{-a.e.}$$

The case of an hyperbolic measure with zero entropy is completely understood.

Proposition 6. *let f be a diffeomorphism of a compact manifold and μ be an hyperbolic invariant measure. If $h_\mu(f) = 0$ then $R = 0 = HD(\mu)$ μ -a.e.*

Proof. Barreira and Saussol established in [4] the inequality $\overline{R} \leq \overline{d}_\mu$ μ -a.e. and it follows from Ledrappier and Young's work [9] that $HD(\mu) = 0$ if $h_\mu(f) = 0$, which allows to conclude by Proposition 5. \square

Corollary 7. *Let f be a diffeomorphism of a compact manifold and μ be an hyperbolic measure with super-polynomial rate of decay of correlation. Then we have*

$$R = HD(\mu) \quad \mu\text{-a.e.}$$

Proof. If the entropy is zero then this is the content of Proposition 6. If the entropy is non-zero then this is the content of Theorem 3. \square

We point out that in the case of interval maps with nonzero Lyapunov exponent, Saussol, Troubetzkoy and Vaienti prove that $R = HD(\mu)$ μ -a.e. for ergodic measures, under very weak regularity conditions [14]. See Remark 17-(i) for related results.

We now give a sketch of the strategy adopted in this paper.

Theorem 8 states that under sufficiently rapid mixing the recurrence rates equal the pointwise dimensions a.e. on the set where $\underline{R} > 0$. Indeed, mixing implies that $\mu(B \cap f^{-n}B) \rightarrow \mu(B)^2$ as $n \rightarrow \infty$. Thus we have $\mu(B \cap f^{-n}B) \leq 2\mu(B)^2$ for large n . If now we consider the set $B \cap f^{-n}B \cap f^{-n-1}B \cap \dots \cap f^{-n-\ell}B$ then its measure is bounded by $2\ell\mu(B)^2$. If $\ell \leq \mu(B)^{-1+\varepsilon}$ then we get that the proportion of points inside B that never enter in B in the time interval $[n, n + \ell]$ is bounded by $2\mu(B)^\varepsilon$. Using the decay of correlations we are able to prove that this

last statement is true for n of the order $\text{diam}(B)^{-\delta}$ for some small $\delta > 0$, whenever B is a ball. This is what we call the *long fly property*. A Borel Cantelli argument then shows that typical points do have long flies (see Lemma 9 for precise statement). If in addition we also have $\underline{R} > \delta$ then it immediately shows that the return time into small neighborhoods B cannot be much less (at an exponential scale) than $\mu(B)^{-1}$, establishing Equation (4).

On the other hand, for systems which are not too wild (e.g. finite Lyapunov exponents, see Lemma 16) and with nonzero metric entropy, a symbolic coding (see Lemma 14) allows to use Ornstein-Weiss' theorem on repetition time of symbolic sequences to prove that the return time of a typical point in a ball B is not less than $\text{diam}(B)^{-\delta}$; see Lemma 12.

The structure of the paper is as follows. We state and prove in Section 2 the core result, Theorem 8. In Section 3 we provide some conditions under which the recurrence rate is nonzero.

2. RAPID MIXING IMPLIES LONG FLIES

Theorem 8. *Assume that the local rate of decay of correlations is super-polynomial. Then on the set $\{\underline{R} > 0\}$ we have*

$$\underline{R} = \underline{d}_\mu \quad \text{and} \quad \overline{R} = \overline{d}_\mu \quad \mu\text{-a.e.}$$

Proof. By Proposition 1 we know that $\underline{R} \leq \underline{d}_\mu$ and $\overline{R} \leq \overline{d}_\mu$. Furthermore, the first inequality implies that $\{\underline{R} > a\} \subset \{\underline{d}_\mu > a\}$ μ -a.e. But on the set $\{\underline{R} > a\}$ we have $\tau_r(x) \geq r^{-a}$ provided r is sufficiently small. By Lemma 9 below with $\delta = a$ and $\varepsilon > 0$ we get that $\tau_r(x) \geq \mu(B(x, r))^{-1+\varepsilon}$ provided r is sufficiently small, for μ -a.e. $x \in \{\underline{R} > a\}$. Thus $\underline{R} \geq (1 - \varepsilon)\underline{d}_\mu$ and $\overline{R} \geq (1 - \varepsilon)\overline{d}_\mu$ μ -a.e. on $\{\underline{R} > a\}$. The conclusion follows by taking $\varepsilon > 0$ arbitrary small. \square

The following lemma expresses that the orbit of a typical point has the long fly property.

Lemma 9. *Let $X_a = \{\underline{d}_\mu > a\}$ for some $a > 0$. For any $\delta, \varepsilon > 0$, for μ -a.e. $x \in X_a$ there exists $r(x) > 0$ such that for any $r \in (0, r(x))$ and any integer n in $[r^{-\delta}, \mu(B(x, r))^{-1+\varepsilon}]$ we have $d(f^n x, x) \geq r$.*

Proof. For clarity we assume that the (global) rate of decay of correlation is super-polynomial. The obvious modifications in the proof would consist essentially in considering separately each sets $G \cap \{x \in V_i : d(x, \partial V_i) > \nu\}$ for arbitrarily small $\nu > 0$ in place of the unique set G defined below.

Let $D = \dim(X)$. Fix $b > 0$, $c = a\varepsilon/3$ and consider for $r_0 > 0$ the set $G = G_1 \cap G_2 \cap G_3$ where

$$\begin{aligned} G_1 &= \{x \in X_a : \forall r \leq r_0, \mu(B(x, r)) \leq r^a\} \\ G_2 &= \{x \in X : \forall r \leq r_0, \mu(B(x, r)) \geq r^{D+b}\} \\ G_3 &= \{x \in X : \forall r \leq r_0, \mu(B(x, r/2)) \geq \mu(B(x, 4r))r^c\}. \end{aligned}$$

We claim that $\mu(G) \rightarrow \mu(X_a)$ as $r_0 \rightarrow 0$. Indeed, by definition of the lower pointwise dimension we have $\mu(G_1) \rightarrow \mu(X_a)$. In addition since $\overline{d}_\mu \leq D$ a.e. we have $\mu(G_2) \rightarrow 1$ and since X is Euclidean the measure μ is weakly diametrically regular (see Lemma 1 in [4]), thus $\mu(G_3) \rightarrow 1$ as well. Let $r \leq r_0$ and define the set

$$A_\varepsilon(r) = \{y \in X : \exists n \in [r^{-\delta}, \mu(B(y, 3r))^{-1+\varepsilon}], d(f^n y, y) < r\}.$$

Let $x \in G$. By the triangle inequality we get the inclusions

$$\begin{aligned} B(x, r) \cap A_\varepsilon(r) &\subset \{y \in B(x, r) : \exists n \in [r^{-\delta}, \mu(B(x, 2r))^{-1+\varepsilon}], d(f^n y, x) < 2r\} \\ &\subset \bigcup_{r^{-\delta} \leq n \leq \mu(B(x, 2r))^{-1+\varepsilon}} B(x, r) \cap f^{-n} B(x, 2r). \end{aligned}$$

Let $\eta_r: [0, \infty) \rightarrow \mathbb{R}$ be the r^{-1} -Lipschitz map such that $1_{[0, r]} \leq \eta_r \leq 1_{[0, 2r]}$ and set $\varphi_{x, r}(y) = \eta_r(d(x, y))$. Clearly $\varphi_{x, r}$ is also r^{-1} -Lipschitz. By the assumption on the decay of correlation we obtain

$$\begin{aligned} \mu(B(x, r) \cap f^{-n} B(x, 2r)) &\leq \int \varphi_{x, 2r} \varphi_{x, 2r} \circ f^n d\mu \\ &\leq \|\varphi_{x, 2r}\|^2 \theta_n + \left(\int \varphi_{x, 2r} d\mu \right)^2 \\ &\leq r^{-2} \theta_n + \mu(B(x, 4r))^2. \end{aligned}$$

Choose $p > 1$ such that $\delta(p-1) - 2 \geq D + 2b$ and take r_0 so small that $n \geq r_0^{-\delta}$ implies $\theta_n \leq (p-1)n^{-p}$. Since $\sum_{n \geq q} n^{-p} \leq \frac{1}{p-1} q^{1-p}$ we obtain

$$\begin{aligned} \mu(B(x, r) \cap A_\varepsilon(r)) &\leq r^{\delta(p-1)-2} + \mu(B(x, 2r))^{-1+\varepsilon} \mu(B(x, 4r))^2 \\ &\leq \mu(B(x, r/2)) (r^b + r^{\varepsilon a - 2c}). \end{aligned}$$

Let $B \subset G$ be a maximal r -separated set². Since $(B(x, r))_{x \in B}$ covers G we have

$$\begin{aligned} \mu(G \cap A_\varepsilon(r)) &\leq \sum_{x \in B} \mu(B(x, r) \cap A_\varepsilon(r)) \\ &\leq \sum_{x \in B} \mu(B(x, r/2)) (r^b + r^{\varepsilon a - 2c}) \\ &\leq r^b + r^{\varepsilon a - 2c} \end{aligned}$$

since by the balls $(B(x, r/2))_{x \in B}$ are disjoint. This implies that

$$\sum_m \mu(A_\varepsilon(e^{-m})) < \infty,$$

thus by Borel-Cantelli Lemma we obtain that for μ -a.e. $y \in G$ there exists $m(y)$ such for every $m > m(y)$ there exists no $n \in [e^{-\delta m}, \mu(B(y, 3e^{-m}))^{-1+\varepsilon}]$ such that $d(f^n y, y) < e^{-m}$. By weak diametric regularity (and changing slightly if necessary the values of ε and δ), this proves the lemma. \square

Remark 10. Observe that we only use that the decay of correlation is at least n^{-p} for some $p > \frac{D+2}{\delta} + 1$. If in addition (5) holds with the first norm $\|\varphi\|$ taken to be the $L^1(\mu)$ norm (e.g. expanding maps) then $p > \frac{D+1}{\delta} + 1$ suffices.

²that is if $x \neq x' \in B$ then $d(x, x') \geq r$ and maximal in the sense that for any $y \in G$ there exists $x \in B$ such that $d(x, y) < r$.

3. NON-ZERO RECURRENCE RATE

We proceed now to find conditions under which the recurrence rate does not vanish. Denote by $\xi(x)$ the unique element of a partition ξ containing the point x and by $\xi^n = \xi \vee f^{-1}\xi \vee \dots \vee f^{-n+1}\xi$ the dynamical partition, for any integer n .

3.1. Coding by symbolic systems : partitions with large interior.

Definition 11. *We say that a partition ξ has large interior if for μ -a.e. x there exists $\chi = \chi(x) < \infty$ such that $B(x, e^{-\chi n}) \subset \xi^n(x)$ for all n sufficiently large.*

Next lemma, which proof is fairly simple, is the key-observation which gives to Theorem 8 all its interest.

Lemma 12. *If there exists a partition with large interior and nonzero entropy then $\underline{R} > 0$ μ -a.e.*

Proof. Let ξ be such a partition. Define

$$R_n(x, \xi) = \min\{k > 0 : f^k x \in \xi^n(x)\}.$$

Ornstein and Weiss [12] prove that if ξ is a finite partition with entropy $h_\mu(f, \xi)$ then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log R_n(x, \xi) = h_\mu(f, \xi) \quad \mu\text{-a.e.}$$

Since ξ has large interior, for μ -a.e. $x \in X$ there exists a number $\chi = \chi(x)$ such that $B(x, e^{-\chi n}) \subset \xi^n(x)$. Thus

$$\underline{R}(x) = \liminf_{n \rightarrow \infty} \frac{\log \tau_{e^{-n}}(x)}{n\chi(x)} \geq \liminf_{n \rightarrow \infty} \frac{\log R_n(x, \xi)}{n\chi(x)} = \frac{h_\mu(f, \xi)}{\chi(x)} > 0 \quad \mu\text{-a.e.}$$

□

Combining Lemma 12 and Theorem 8 we get that if we have local super-polynomial decay of correlations and a partition of positive entropy with large interior then $\underline{R} = \underline{d}_\mu$ and $\overline{R} = \overline{d}_\mu$. The rest of the section consists in finding sufficient conditions for the existence of such a partition.

3.2. Reasonable dependence on initial condition.

Definition 13. *We say that a system (X, f, μ) is reasonably sensitive if for μ -a.e. x there exists $\gamma, \lambda > 0$ such that f^n is $e^{\lambda n}$ -Lipschitz on the ball $B(x, e^{-\gamma n})$ for all n sufficiently large.*

Lemma 14. *If the system (X, f, μ) is reasonably sensitive and the entropy $h_\mu(f) > 0$ then there exists a partition with large interior and nonzero entropy.*

Proof. Claim : For any $x \in X$, $s > 0$ there exists $\rho \in (s, 2s)$ such that

$$\mu(\{y \in X : \rho - 4^{-n}s < d(x, y) < \rho + 4^{-n}s\}) \leq \frac{1}{2^{n-1}} \mu(B(x, 2s)). \quad (6)$$

Indeed, let m be the measure on the interval $(0, 2)$ defined by $m([0, t]) = \mu(B(x, st))$. We construct a sequence of open intervals I_n starting from $I_0 = (1, 2)$. If I_n is an interval of length 4^{-n} we divide it into 4 pieces of equal length and choose I_{n+1} the left of the right central piece of smallest measure. We have $m(I_{n+1}) \leq \frac{1}{2}m(I_n)$. I_n is a decreasing sequence of intervals with $\overline{I}_{n+1} \subset I_n$ thus $\cap_n I_n$ contains one point, say $\bar{\rho}$. Since $\bar{\rho} \in I_n$ we have $\bar{\rho} \pm 4^{-n} \in I_{n-1}$ thus

$m((\bar{\rho} - 4^{-n}, \bar{\rho} + 4^{-n})) \leq m(I_{n-1}) \leq \frac{1}{2^{n-1}}m(I_0)$. Proving the claim with $\rho = s\bar{\rho}$.

Fix $s > 0$ so small that any partition made by sets of diameter less than $2s$ has nonzero entropy. Choose a maximal s -separated set E . For any $x \in E$ take $\rho_x \in (s, 2s)$ such that (6) in the claim holds. Let $E = \{x_1, x_2, \dots\}$ be an enumeration of the (at most) countable set E . Put $B_i = B(x_i, \rho_{x_i})$ and define $Q_1 = B_1$, $Q_2 = B_2 \setminus Q_1$, $Q_3 = B_3 \setminus (Q_1 \cup Q_2)$, \dots . By maximality the collection of sets $\xi = \{Q_1, Q_2, \dots\}$ is a partition of X (modulo μ) and since $\partial\xi \subset \cup_i \partial B_i$ we get

$$\begin{aligned} \mu(\{x \in X : d(x, \partial\xi) < 4^{-n}s\}) &\leq \mu(\cup_i \{x \in X : \rho_{x_i} - 4^{-n} < d(x_i, x) < \rho_{x_i} + 4^{-n}\}) \\ &\leq \frac{1}{2^{n-1}} \sum_i \mu(B(x_i, 2s)). \end{aligned}$$

Since the x_i are s -separated and X is Euclidean there are at most $c(X) = c(\dim X)$ balls of radius $2s$ that can intersect, thus the last sum is bounded by $\frac{c(X)}{2^{n-1}}$. This proves that for some constants $a, c > 0$ and all $\varepsilon > 0$

$$\mu(x \in X : d(x, \partial\xi) < \varepsilon) < c\varepsilon^a.$$

Thus for any $b > 0$ we have by the invariance of μ

$$\sum_n \mu(\{x \in X : d(f^n x, \xi) < e^{-bn}\}) \leq \sum_n ce^{-abn} < \infty.$$

This implies by Borel-Cantelli Lemma that for μ -a.e. x there exists $n(x) < \infty$ such that $d(f^n x, \partial\xi) \geq e^{-bn}$, hence $B(f^n x, e^{-bn}) \subset \xi(f^n x)$, for any $n \geq n(x)$. Taking $c(x) \in (0, 1)$ sufficiently small we have $B(f^n x, c(x)e^{-bn}) \subset \xi(f^n x)$ for all integer n .

Fix $x \in X$ where the reasonable sensitivity condition holds. Without loss of generality, and changing if necessary $c(x)$ into a smaller constant we assume that f^n is $e^{\lambda n}$ -Lipschitz on the ball $B(x, c(x)e^{-\gamma n})$ for all integer n and that $\lambda > \gamma + b$.

We show then by induction that $B(x, c(x)e^{-\lambda n}) \subset \xi^k(x)$ for any $k \leq n$. Indeed, this is trivially true for $k = 1$, and if this holds for some $k \leq n - 1$ then we have

$$f^k(B(x, c(x)e^{-\gamma n})) \subset B(f^k x, c(x)e^{\lambda k - \gamma n}) \subset B(f^k x, e^{-bn}) \subset \xi(f^k x).$$

Hence $B(x, c(x)e^{-\gamma n}) \subset \xi^{k+1}(x)$. □

We finally provide a sufficient condition for reasonable sensitivity.

Definition 15. *If there exists a partition \mathcal{A} (modulo μ) into open sets such that on each $A \in \mathcal{A}$ the map f is Lipschitz with constant $L_f(A)$ and the singularity set $\partial\mathcal{A} = \cup_{A \in \mathcal{A}} \partial A$ is such that $\mu(\{x \in X : d(x, \partial\mathcal{A}) < \epsilon\}) \leq c\epsilon^a$ for some constants $c > 0$ and $a > 0$ then we say that f is piecewise Lipschitz with average Lipschitz exponent $\log L_f = \int \log^+ L_f(A(x)) d\mu(x) = \sum_{A \in \mathcal{A}} \log^+ L_f(A) \mu(A)$.*

Lemma 16. *If f is Lipschitz, or piecewise Lipschitz with finite Lipschitz exponent then (X, f, μ) is reasonably sensitive.*

Proof. We prove the piecewise case, the other one is obvious. Let $\lambda > \log L_f$. By the Birkhoff Ergodic Theorem, for μ -a.e. x there exists $m(x)$ such that

$$L_f(A(x))L_f(A(fx)) \cdots L_f(A(f^{m(x)-1}x)) \leq e^{\lambda m(x)}$$

for all $n \geq m(x)$. Replacing if necessary the upper bound by $e^{\lambda n}/c(x)$ for some constant $c(x) \geq 1$ the inequality will hold for any integer n . Proceeding as in the last part of the proof of Lemma 14 we get that for any $b > 0$, changing $c(x)$ if necessary, we have $B(f^n x, c(x)e^{-bn}) \subset \mathcal{A}(f^n x)$ for any integer n . We then conclude similarly that $B(x, c(x)^2 e^{-bn} e^{-\lambda n}) \subset \mathcal{A}^n(x)$. This concludes the proof taking $\gamma = b + \lambda$. \square

The proof of Theorem 3 follows now easily from the preceding results.

Proof of Theorem 3. By Lemma 16 the map is reasonably sensitive. This implies by Lemma 14 the existence of a partition with large interior. By Lemma 12 we find that $\underline{R} > 0$ a.e. and the conclusion follows from Theorem 8. \square

Remark 17. (i) We remark that if f is C^1 on a compact manifold, or more generally if f is piecewise $C^{1+\alpha}$ with reasonable singularity set such as in [8], then the exponents λ and γ in Definition 13 can be taken arbitrarily close to the largest Lyapunov exponent³ λ_μ^+ . Thus the exponent χ in Lemma 12 may also be taken arbitrarily close to λ_μ^+ . This readily implies that $\underline{R} \geq h_\mu/\lambda_\mu^+$. This is optimal in dimension one or more generally for conformal maps, where under mild assumptions we have $HD(\mu) = h_\mu/\lambda_\mu$.

(ii) Combining the above observation with Remark 10 shows that the assumption on the super-polynomial decay of correlations in Theorem 8 may be reduced to a decay at a rate n^{-p} for some $p > \frac{D+2}{h_\mu} \lambda_\mu^+ + 1$.

REFERENCES

- [1] J. F. Alves, SRB measures for non-hyperbolic systems with multidimensional expansion, *Annales Scientifiques de l'ENS* **33** (2000) 1–32.
- [2] V. Baladi, *Positive transfer operators and decay of correlations*, Advances Series in Nonlinear Dynamics, vol. 16, World Scientific Publishing Co. Inc., River Edge, 2000.
- [3] L. Barreira, Y. Pesin, J. Schmeling, Dimension and product structure of hyperbolic measures, *Annals of Mathematics* **149** (1999) 755–783.
- [4] L. Barreira, B. Saussol, Hausdorff dimension of measures via Poincaré recurrence, *Communication in Mathematical Physics* **219** (2001) 443–463.
- [5] L. Barreira, B. Saussol, Product structure of Poincaré recurrence, *Ergodic Theory and Dynamical Systems* **22** (2002) 33–61.
- [6] M. Boshernitzan, Quantitative recurrence results, *Inventiones Mathematicæ* **113** (1993) 617–631.
- [7] N. Chernov, Statistical properties of piecewise smooth hyperbolic systems in high dimensions. *Discrete and Continuous Dynamical Systems* **5** (1999) 425–448.
- [8] A. Katok, J.-M. Strelcyn, F. Ledrappier, and F. Przytycki, *Invariant manifolds, entropy and billiards; smooth maps with singularities*, Lecture Notes in Mathematics, vol. 1222, Springer-Verlag, Berlin, 1986.
- [9] F. Ledrappier, L.-S. Young, The metric entropy of diffeomorphisms: I. Characterization of measures satisfying Pesin’s formula, II. Relations between entropy, exponents and dimension, *Annals of Mathematics* **122** (1985) 509–539.
- [10] D. A. Lind, Dynamical properties of quasi-hyperbolic toral automorphisms, *Ergodic Theory and Dynamical Systems* **2** (1982) 49–68.
- [11] S. Luzzatto, Mixing and decay of correlations in non-uniformly expanding maps: a survey of recent results, preprint 2004.

³to see this, consider a Lyapunov chart whose local chart at x has a diameter $\rho(x)$, where ρ is η -slowly varying. A choice like $\lambda = \lambda_\mu^+ + 2\eta$ and $\gamma = \lambda + \eta$ would do the job.

- [12] D. Ornstein, B. Weiss, Entropy and data compression schemes, *IEEE Transaction on Information Theory* **39** (1993) 78–83.
- [13] B. Saussol, Absolutely continuous invariant measures for multidimensional expanding maps, *Israel Journal of Mathematics* **116** (2000) 223–248.
- [14] B. Saussol, S. Troubetzkoy, S. Vaienti, Recurrence, dimensions and Lyapunov exponents, *Journal of Statistical Physics* **106** (2002) 623–634.
- [15] M. Urbanski, Recurrence rates for loosely Markov dynamical systems, *Journal of the Australian Mathematical Society*, to appear.
- [16] L.-S. Young, Statistical properties of dynamical systems with some hyperbolicity, *Annals of Mathematics* **147** (1998) 585–650.
- [17] L.-S. Young, Recurrence times and rates of mixing, *Israel Journal of Mathematics* **110** (1999) 153–188.
E-mail address: `benoit.saussol@u-picardie.fr`
URL: `http://lamfa.u-picardie.fr/saussol`

LAMFA - CNRS UMR 6140, UNIVERSITÉ DE PICARDIE JULES VERNE, 33 RUE ST LEU, 80039 AMIENS
CEDEX 1, FRANCE