HAL
open science

On Lorentz dynamics : From group actions to warped products via homogeneous spaces

Abdelouahab Arouche, Mohamed Deffaf, Abdelghani Zeghib

To cite this version:

Abdelouahab Arouche, Mohamed Deffaf, Abdelghani Zeghib. On Lorentz dynamics : From group actions to warped products via homogeneous spaces. 2004. hal-00003480

HAL Id: hal-00003480

https://hal.science/hal-00003480v1

Preprint submitted on 9 Dec 2004

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

ccsd-00003480, version 1 - 9 Dec 2004

On Lorentz dynamics : From group actions to warped products via homogeneous spaces

A. Arouche, M. Deffaf and A. Zeghib

December 9, 2004

Abstract

We show a geometric rigidity of isometric actions of non compact (semisimple) Lie groups on Lorentz manifolds. Namely, we show that the manifold has a warped product structure of a Lorentz manifold with constant curvature by a Riemannian manifold.

1 Introduction

Recall the following result of [9], which shows how homogeneous spaces are rare in Lorentz geometry (in comparison with the Riemannian case, say)

Theorem 1.1 *[9]Let (M, g) be a homogeneous Lorentz space of dimension ≥ 3, with* irreducible *isotropy group, then it has constant sectional curvature.*

Observe that the statement in [9] seems weaker, since the isotropy group is assumed to satisfy a supplementary condition: non-precompactness. However, this follows from irreducibility. Indeed, in the same vein as [9], the principal result of [3] says how irreducibility is strong in the Lorentz setting:

Theorem 1.2 *[3] A Lie subgroup (not assumed a priori to be closed) of $O(1, n)$, which does not preserve any one dimensional isotropic subspace of $\mathbb{R}^{1+n}$, is up to conjugacy, a union of some components of some $O(1, p) \subset O(1, n)$.*

Our goal in the present article is to relax homogeneity by considering (non-transitive) isometric group actions. This work is actually motivated by the study of isometric Lie group actions on *non-compact* Lorentz manifolds, for instance in the same vein as [1, 7, 6]...

1.1 Warped product structure versus partial homogeneity

We ask firstly if there is an adaptation of Theorem 1.1 to non-transitive isometric actions. In this situation, we consider a group G acting isometrically on a Lorentz manifold (M, g). Each orbit is a homogeneous space. However, the causal type of the orbit may be, timelike, spacelike or lightlike, that is, the induced metric is Lorentzian, Riemannian or degenerate, respectively. The following generalization of Theorem 1.1 relies on the existence of orbits of Lorentz type satisfying irreducibility. It says roughly that the space is partially of constant curvature.

Theorem 1.3 *Let G be a Lie group acting isometrically on a Lorentz manifold (M, g) of dimension ≥ 3. Suppose there exists an orbit N which is a (homogeneous) Lorentz space with irreducible isotropy.*

Then, N has constant (sectional) curvature, and a neighborhood of it is a warped product $L \times_w N$, where L is some Riemannian manifold. Furthermore, the factor N corresponds to the orbits of G.

Definition and fundamental properties of warped products are in §2. It follows that the group G is a subgroup of the isometry group of a constant curvature manifold N (acting transitively on it). It is a non-difficult algebraic matter to classify them. Conversely, any such group acts isometrically on any warped product $L \times_w N$.

1.2 Non-properness versus Irreducibility

Let us go a step further, and try to get rid of the irreducibility hypothesis. In fact, irreducibility is an algebraic condition which looks somehow non-adapted to our dynamico-geometrical setting here. We want to substitute for it a more natural dynamical condition. Our theory is that non-properness is good enough for this role.

1.2.1 Recalls

We find it worthwhile to make some order around the concept of non-properness of actions. This will be useful in the sequel (statements and proofs).

Recall that an action of a group G on a space M is called **proper**, if for any sequences (x_n) of M, and (g_n) of G, if (x_n) and $(g_n x_n)$ converge in M, then some subsequence of (g_n) converges in G.

For our purpose here the following variant will be useful. We say that the action of G is **locally equicontinuous**, if keeping the notations as above, a subsequence of (g_n) is (locally) equicontinuous (or say it is equicontinuous

in a neighborhood of the limit of (x_n)). Therefore, a subsequence of (g_n) is converging in the group of homeomorphisms of M, but, the limit does not necessarily belong to G. Obviously, a non locally-equicontinuous action is non-proper. The converse is not true. The standard example of a non-proper but equicontinuous action is the usual linear action of $\mathbb{R}$ on the torus with dense orbits. This is in general the case of any non-closed Lie group of the isometry group of a compact Riemannian manifold. Another example is the action of the universal cover $\tilde{G}$ on G (via the canonical projection). It is always locally equicontinuous, but proper only if G has a finite fundamental group. Observe nevertheless:

Fact 1.4 *Let G be a Lie group acting by preserving a pseudo-Riemannian structure on a manifold M. If G is the full isometry group, or G is semi-simple with finite center, then its action is non-proper iff it is non-equicontinuous.*

For the proof, recall the well known fact that a $C0$-limit of pseudo-Riemannian (smooth) isometries is a smooth isometry, and that the Lie group topology coincides with the $C0$ topology. This is equivalent to saying that the isometry group is closed in the group of homeomorphisms. (Actually, this fact is general for all rigid geometric structures). For G a simple Lie group with finite center, recall that its image under a homomorphism into any Lie group is closed, and that G is a finite cover of it. An analogous argument applies to the semi-simple case with finite center. $\diamondsuit$

Therefore, in statements (essentially inside proofs) below,which involve semi-simple Lie groups, we will not worry to sway from compactness to pre-compactness.

A G-homogeneous space G/H is non-proper if the G-left action on it is. This is equivalent, in the general case, to the fact that H is not precompact, and to that H is not compact in the semi-simple case.

1.2.2 Semi-simple group actions with non-proper orbits

Without a priori irreducibility hypothesis, we have the following generalization of Theorem 1.3, assuming the orbits are non-proper, and the group G is semi-simple (a kind of intrinsic irreducibility).

Theorem 1.5 *Let G be a* semi-simple*Lie group of finite center acting isometrically on a Lorentz manifold (M, g) of dimension ≥ 3. Suppose that no (local) factor of G is locally isomorphic to $SL(2, \mathbb{R})$ and that there exists a non-proper orbit N of Lorentz type, (that is N has a non-compact isotropy).*

Then, up to a finite cover, G factorizes $G = G_1 \times G_2$, where:

- G_2 acts properly on a neighborhood of N, with spacelike orbits of constant dimension.

- On a G-invariant neighborhood U of N, G_1 acts isometrically on the Lorentz quotient U/G_2 which is a warped product $L \times_w N_1$, where N_1 has constant (sectional) curvature and corresponds to the G_1-orbits. In particular, if G is simple, then U itself is a warped product.

Looking at semi-simple Lie groups acting transitively, with non-compact isotropy on a constant curvature Lorentz space, on can prove in a standard way the following:

Corollary 1.6 *In the case above, up to a (central) cover, G_1 is $O(1,n)$ (resp. $O(2,n)$) the isometry group of the de Sitter space (resp. anti de Sitter space), that is the universal Lorentz space of positive (resp. negative) curvature.*

Remarks 1.7

1) In both theorems above, the warped product is local, i.e. not the whole space is a warped product. To see this, one considers the $O(1,n)$-action on the Minkowski space $\mathbb{R}^{1,n}$. If the Lorentz quadratic form is $q = -x_0 2 + x_1 2 + \ldots + x_n 2$, then the warped product is defined exactly on the region $q > 0$.

2) The result does not seem to be optimal, that is, it might be generalized to other groups.

3) Warped product structures on universal covers of *compact* Lorentz manifolds with strong dynamics, are obtained, for instance, in [4, 10, 11].

1.3 From non-proper actions to non-proper homogeneous spaces

Let us go another step, by asking how to get such non-proper orbits from a global condition on the action? For instance, is orbital non-properness inherited from non-properness of the (ambient) action?

Theorem 1.8 *Let G be a semi-simple Lie group of finite center acting isometrically and* nonproperly *on a Lorentz manifold M. Suppose that no (local) factor of G is locally isomorphic to $SL(2,\mathbb{R})$. Then, there is a point with a non-compact stabilizer. In particular, the restriction of the action of G to its orbit is nonproper. More exactly, the stabilizer of some point contains a non-trivial unipotent one-parameter group. (In other words, a non proper Lorentz G-space contains a non-proper G-homogeneous orbit)*

This result allows one to get from (non-transitive) actions to homogeneous (i.e. transitive) ones. This is a common philosophy for actions with

strong dynamics and a geometric background. The result here is in particular reminiscent to the so-called Zimmer's embedding Theorem (see for instance [12]). "Unfortunately", there is a damper to put on : the orbit is a non-proper homogeneous space, but not necessarily Lorentz! The nuisance is that it can be lightlike (degenerate); another story.

2 Proof of Theorem 1.3

2.1 An algebraic lemma

Lemma 2.1 *Let E (resp. F) be a Lorentz (resp. Euclidean) vector space. Denote by $O(E)$ and $O(F)$ their respective orthogonal groups. Let H' be a Lie subgroup of $O(E) \times O(F)$, whose projection on $O(E)$ acts irreducibly on E.*

Then, H' contains a subgroup $H \subset O(E) \times \{1\}$, which contains the identity component of $O(E)$. In particular:

- Any linear H-invariant mapping $f : E \to F$ ($f \circ h = f$, for any $h \in H$) is trivial.

- The same is true for any H-invariant bilinear antisymmetric mapping $E \times E \to F$.

Proof. The crucial fact follows from Theorem 1.2 applied it to H, the projection of H' on $O(E)$. It is a finite union of components of $O(1,p)$. Say $H = O(1,p)$ to simplify notation. Since $O(F)$ is compact, H is isomorphic to the non-compact semi-simple Levi factor of H'. Therefore, (up to a cover...) H' contains a subgroup isomorphic to H, that is, there exists a homomorphism $\rho : H = O(1,p) \to O(F)$, such that the graph $\{(h, \rho(h)), h \in O(1,p)\}$ is contained in H'.

Next, one checks ρ is trivial. This uses a basic fact of Lie groups theory: a semi-simple Lie group of non-compact type has no non-trivial homomorphism into a compact group. The idea, in this case, is that in $O(1,p)$ there are one parameter groups (the unipotent ones) having all their non-trivial elements conjugate (this is easy to see in the case of $O(1,2)$ which is essentially $PSL(2,\mathbb{R})$). Such a conjugacy is impossible in a compact group.

For the last two conclusions of the lemma, one can assume $F = \mathbb{R}$. The kernel of th linear mapping f is $O(1,p)$-invariant; hence it is trivial by irreducibility. A similar argument yields triviality of invariant antisymmetric bilinear mappings. $\diamondsuit$

2.2 Group actions

Lemma 2.2 *Let G be a Lie group acting isometrically on a Lorentz manifold (M, g). Let N be an orbit of G, which is of Lorentz type and has an irreducible isotropy group (inside G).*

Then, the same is true for all orbits in a neighborhood of N. In particular, the orbits of G determine a foliation (i.e. have a constant dimension). Furthermore, the orthogonal distribution of this foliation is integrable.

Proof. Consider $x_0 \in N$, and denote by H its isotropy group. The orthogonal space L_{x_0} of $T_{x_0}N$ in $T_{x_0}M$ is spacelike (the metric on it is definite positive). We are in position to apply Lemma 2.1 with $E = T_{x_0}N$ and $F = L_{x_0}$. It then follows that the action of H on L_{x_0} is trivial. Let $\exp_{x_0}$ denote the exponential of the Lorentz metric and consider the (local) submanifold $\mathcal{L}_{x_0} = \exp_{x_0}(L_{x_0})$. Then $\exp_{x_0}$ conjugates the infinitesimal action of H on L_{x_0}, with its action on $\mathcal{L}_{x_0}$. In particular, H acts trivially on this latter submanifold. That is H is contained in the isotropy group of any point of $\mathcal{L}_{x_0}$. An obvious semi-continuity argument implies that isotropy groups can not be bigger. Therefore, we have a foliation by G-orbits, all satisfying the same irreducibility condition for their isotropy groups. Let us denote this foliation by $\mathcal{N}$ and its tangent bundle by $T\mathcal{N}$. Let L be the orthogonal distribution. The obstruction to integrability of L can measured by means of a tensor $T : L \times L \to T\mathcal{N}$. It is defined by $T(X, Y) =$ the orthogonal projection on $T\mathcal{N}$ of the bracket $[X, Y]$, where X and Y are sections of L. Since the isotropy group acts trivially on L and irreducibly on $T\mathcal{N}$, T is trivial, that is L is integrable. $\diamondsuit$

2.3 Warped product, end of the proof

Let (L, h) and (N, m) be two pseudo-Riemannian manifolds and $w : L \to \mathbb{R}^+ - \{0\}$ a *warping* function. The warped product $M = L \times_w N$, is the topological product $L \times N$, endowed with the pseudo-Riemannian metric $g = h \bigoplus wm$.

Our goal now is to prove that M is a warped product. So far, we have the orthogonal foliations $\mathcal{N}$ and $\mathcal{L}$. One can say that De Rham decomposition theorem is a criterion for a couple of such foliations in order that they determine a (local) *direct* pseudo-Riemannian product. The condition is that (the tangent bundles of) $\mathcal{N}$ and $\mathcal{L}$ are parallel, or a priori more weakly, that leaves of $\mathcal{N}$ and $\mathcal{L}$ are geodesic. There is a similar, but more complicated, criteria for warped products [5, 8]. We will not use this criterion, but rather give a brief proof in our case. Our terminology here is close to that of [11], which may be consulted for a more complete exposition. Let N and L be

(local) leaves of a point x_0 for the foliations $\mathcal{N}$ and $\mathcal{L}$ respectively. So, locally, M has an adapted topological product $L \times N$. The metric can be written

$$g_{(l,n)} = h_{(l,n)} \bigoplus m_{(l,n)}$$

• Let us show that $h_{(l,n)} = h$, that is, it does not depend on n. This is clear since G acts isometrically: if $k \in G$, then it sends $\mathcal{L}_{(l,n)}$ to $\mathcal{L}_{k(l,n)}$, where $k(l,n)$ has the form (l,n') (orbits of G correspond to $\mathcal{N}$). Therefore $g = h \bigoplus m_{(l,n)}$ (the geometric meaning of this fact is that $\mathcal{N}$ is a geodesic foliation [11]).

• In order to understand the variation of $m_{(l,n)}$ as a function of (l,n), write $x_0 = (l_0, n_0)$, fix $l_1 \in L$ and consider the mapping

$$S : (l_0, n) \in N = \mathcal{N}_{(l_0,n_0)} \to (l_1, n) \in \mathcal{N}_{(l_1,n_0)}$$

S commutes with the G action on the G-orbits of (l_0, n_0) and (l_1, n_0). In particular it commutes with the isotropy actions at these two points. As showed previously these isotropy groups are the full orthogonal groups of the Lorentz scalar products on their tangent spaces. In particular, they preserve, up to a multiplicative constant, only one Lorentz scalar product. This means that S is a homothety at (l_0, n_0) : the image metric equals the metric at (l_1, n_0) (along $\mathcal{N}_{(l_1,n_0)}$) up to a multiplicative factor $w(l_0, n_0)$. Now, since S commutes with the (full) action on orbits, it follows that w does not depend of n. That is, if $m = m_{(l_0,n)}$ is the metric on N, then $m_{(l,n)} = w(l)m$. In sum, $g = h \bigoplus w(l)m$, that is, M is a warped product.

• Finally, it remains to see that N has constant curvature. This is exactly the content of Theorem 1.1 since N has a non-precompact irreducible isotropy. $\diamondsuit$

3 Proof of Theorem 1.5

We will in fact prove Theorem 1.5 under the homogeneity assumption, that is G acts transitively on M. This will be a generalization Theorem 1.1, where one keeps non-precompactness assumption together with semi-simplicity of the group, and gives up the irreducibility one. The proof in the non transitive case will be just a variation of that of Theorem 1.3 using the transitive statement which is:

Theorem 3.1 *Let (M, g) be an* irreducible *G-homogeneous Lorentz space of dimension ≥ 3, with* non-precompact *isotropy group, and G a semi-simple Lie group with no (local) factor locally isomorphic to $SL(2, \mathbb{R})$.*

• *Then, the isotropy group is irreducible and M has constant sectional curvature.*

- In the general case where M is not assumed to be irreducible, we have:

• M is locally a direct product $M = L \times N$ where L is a Riemannian homogeneous manifold, and N is Lorentz and has constant curvature. To this splitting corresponds an analogous (local) one for G.

Proof. For $x \in M$, and a lightlike (i.e. isotropic) vector $u \in T_xM$, consider the orthogonal hyperplane $u^\perp$. Let C_x the set of those u, for which $u^\perp$ is tangent to a totally geodesic (lightlike) hypersurface, that is, $\exp_x(u^\perp)$ is a totally geodesic hypersurface (near x). The crucial fact, proved in [9] is that non-precompactness of the isotropy group H_x implies C_x is non-empty.

- Suppose C_x is finite. One can (locally) define only finitely many continuous sections $x \to u(x) \in C_x$. In particular, one can suppose these sections invariant under the G-action. In fact, to simplify notation in the following argument, one is allowed to suppose that C_x has (everywhere) cardinality 1. Therefore, we have a G-invariant distribution of hyperplanes $x \to u(x)^\perp$. It is integrable, the leaf at x being the geodesic hypersurface

$$\mathcal{H}_u = \exp_x(u^\perp)$$

We get from this that M possesses a codimension one G-invariant foliation. The quotient space is a 1-manifold. But a simple Lie group acting (non-trivially) on a 1-manifold must be locally isomorphic to $SL(2, \mathbb{R})$. This is impossible because of our assumption on G.

– It then follows that C_x is infinite. One then shows in a standard way that there must exist a subspace E_x which is: generated by $E_x \cap C_x$, spacelike, and on which the isotropy group is irreducible.

• Therefore, we have a distribution E on which the isotropy group acts irreducibly. As in the proof of Theorem 1.3, one defines an integrability obstruction tensor for E, which must vanish, by Lemma 2.1 since it is anti-symmetric and invariant under the isotropy group. Therefore E is integrable. We denote by $\mathcal{N}$ its tangent foliation.

• Also $L = E^\perp$ is integrable. Indeed, the leaf of L at x is nothing but the intersection

$$\mathcal{L}_x = \cap_{u \in C_x} \mathcal{H}_u$$

In addition, as all the hypersurfaces $\mathcal{H}_u$ are geodesic, the foliation $\mathcal{L}$ is geodesic. We recalled in §2.3 the interpretation of being geodesic by the fact that the holonomy mappings of the orthogonal foliation $\mathcal{N}$, defined as mappings between (local) leaves $\mathcal{L}$, are *isometric*.

• A leaf N of $\mathcal{N}$ is a Lorentz manifold with a big, in fact maximal isotropy group (at each point), that is (essentially) $O(1, p)$ (where $p + 1 = \dim M$). At this stage, we don't know if M is homogeneous. However, remember the proof in [9]: non-precompacteness and irreducibility of isotropy groups

lead to existence of geodesic hypersurfaces, which leads in turn to that the sectional curvature is constant for all 2-planes tangent to a same point. Then, using Schur's lemma one proves N has (everywhere the same) constant curvature.

• The isotropy group of any point $x \in N$ acts trivially on the leaf $\mathcal{L}_x$. It follows that the group R generated by all the isotropy groups of points of N, preserves individually the leaves of $\mathcal{N}$. It is known, that for constant curvature spaces, the isotropy group of two different points generate the full isometry group. In particular R acts transitively on the leaves of $\mathcal{N}$. In fact, the foliation $\mathcal{N}$ is defined by the R-action. We are thus exactly in the situation of Theorem 1.3 (where R plays the role of G). Therefore, we deduce that M is (locally) a warped product $L \times_w N$.

• The quotient $M/\mathcal{L}$ has a similarity Lorentz structure, that is, a Lorentz metric up to a (global) constant, preserved by G. On other words, G acts by homothety on the constant curvature N. It is easy to describe the similarity group of the Minkowski space. One can in particular see that a non semi-simple Lie group acts *transitively* by homothety.

• We infer from this that N has a non-vanishing curvature. Since G acts transitively on $M = L \times_w N$ by preserving the warped product structure, all the leaves $\{l\} \times N$ are isometric, and hence have a same curvature. However, metrics at two levels l_1 and l_2 are related by a factor $\frac{w(l_1)}{w(l_2)}$. Curvature are related by the inverse ratio. From constancy of curvature, we infer that w is a constant function, that is $M = L \times N$ is a *direct* product. This finishes the proof of Theorem 3.1. $\diamondsuit$

4 Proof of Theorem 1.8

The following method has become a standard ingredient in the study of "geometric" G-actions, see for instance [1, 2, 7, 6]... One considers the action of the group G on the space $S^2(\mathcal{G})$ of symmetric bilinear forms on its Lie algebra $\mathcal{G}$. There is a Gauss G-equivariant map $\Phi : M \to S^2(\mathcal{G})$. Non-properness of the G-action on M translates to a non-properness of the action of G on the image $\Phi(M)$. This latter is "algebraic", it has a poor dynamics, easy to understand. From this, one hopes to get information about the G-action on M.

In our case here, one shows there exists a point $q \in M$ such that $\mathcal{G}$ admits an isotropic subspace with respect to the symmetric bilinear form $\Phi(q)$, of dimension ≥ 2. Then the non-precompactness of the stabilizer $stab(q)$ follows.

4.1 The Gauss map

Let G be a Lie group acting by isometries on a Lorentz manifold (M, h). For each $X \in \mathcal{G}$ let $\overline{X}$ be the vector field on M given by:

$$\overline{X}_x = \frac{d}{dt}(exp(tX).x)|_{t=0}.$$

Let $\Phi : M \to S^2(\mathcal{G})$ be the so-called Gauss map given by

$$x \mapsto \Phi_x : (X, Y) \mapsto h_x(\overline{X_x}, \overline{Y_x}).$$

Recall the definition of the G-action on $S^2\mathcal{G}$. It is given by :

$$(g.q)\,(X_1, X_2) = q\left(Ad_{g^{-1}}X_1,\ Ad_{g^{-1}}X_2\right)$$

. for q in $S2\,(\mathcal{G})$ and g in G, and $X_1, X_2 \in \mathcal{G}$.

Then Φ is equivariant, that is :

$$g.\Phi_x = \Phi_{g.x} \forall g \in G.$$

Indeed, for $g \in G$ and $X \in \mathcal{G}$, we have :

$$\overline{Ad_g X_{gx}} = \frac{d}{dt}(exptAd_g X.gx)\ |_{t=0} = \frac{d}{dt}(g.exptX.g^{-1}.gx)\ |_{t=0} = dg_x(\overline{X_x}).$$

Hence, for $X, Y \in \mathcal{G}$,$x \in M$ and $g \in G$, we get (using the fact that G acts on M by isometries) :

$$\begin{aligned} g.\Phi_x(X, Y) &= h_x(\overline{Ad_{g^{-1}}X}_x, \overline{Ad_{g^{-1}}Y}_x) = h_x(dg_{gx}^{-1}\overline{X}_{gx}, dg_{gx}^{-1}\overline{Y}_{gx}) \\ &= h_{gx}(\overline{X}_{gx}, \overline{Y}_{gx}) \\ &= \Phi_{gx}(X, Y). \end{aligned}$$

Observe that if G acts non-properly on M, then so does it on $\Phi(M)$.

4.2 Root decomposition

Let $\mathcal{A}$ be a Cartan subalgebra, that is, a maximal abelian $\mathbb{R}$-split subalgebra of $\mathcal{G}$ and A the associated Cartan group. Let $\Phi = \Phi\,(\mathcal{A},\ \mathcal{G})$ be the root system of $(\mathcal{A},\ \mathcal{G})$ and

$$\mathcal{G} = \mathcal{G}_0 \oplus \bigoplus_{\alpha \in \Phi} \mathcal{G}_\alpha$$

the root space decomposition where

$$\mathcal{G}_\alpha = \{X \in \mathcal{G} : adA.X = \alpha(A).X, \forall A \in \mathcal{A}\}$$

$$\mathcal{G}_0 = \{X \in \mathcal{G} : adA.X = 0, \forall A \in \mathcal{A}\}.$$

Then A acts on $\mathcal{G}$ by diagonal matrices, since

$$Ad_{g^{-1}} = Ad_{expH} = e^{adH} = diag(e^{\alpha(H)})_{\alpha \in \phi \cup \{0\}}$$

where $g^{-1} = exp(H)$, $H \in \mathcal{A}$. It follows that A acts by diagonal matrices, and $S^2(\mathcal{G})$ admits the following decomposition

$$S^2(\mathcal{G}) = \bigoplus_{\lambda \in \Phi \cup \{0\} + \Phi \cup \{0\}} V_\lambda.$$

where V_λ is the set of symmetric bilinear forms q on $\mathcal{G}$ which satisfy:

$$q(exp(H).X_1, exp(H).X_2) = e^{\lambda(H)}.q(X_1, X_2),$$

for all $H \in \mathcal{A}$ and all $X_1, X_2 \in \mathcal{G}$. Keeping in mind that for $X_1 \in \mathcal{G}_\alpha$ and $X_2 \in \mathcal{G}_\beta$ we have :

$$q(exp(H).X_1, exp(H).X_2) = e^{(\alpha+\beta)(H)}.q(X_1, X_2).$$

It follows that the forms q in V_λ satisfy :

$$\alpha + \beta \neq \lambda \Rightarrow \mathcal{G}_\alpha \perp \mathcal{G}_\beta.$$

4.3 Properness of abelian actions

The following is a criterion for the non-properness of linear actions of abelian Lie groups

Lemma 4.1 *Let $\{\lambda_1, \cdots \lambda_n\}$ be a generating system in $\mathbb{R}^d$. Let $\mathbb{R}^d$ act faithfully on $\mathbb{R}^n$ by diagonal matrices as follows. For $t \in \mathbb{R}^d$, set $M(t) = diag(e^{\langle \lambda_i, t \rangle})_{1 \leq i \leq n}$, where $\langle ., . \rangle$ is the usual inner product in $\mathbb{R}^d$. Assume V is an invariant (topological) subspace of $\mathbb{R}^n$ on which the action is nonproper. Then there exists a nonzero vector $t_0 \in \mathbb{R}^d$ and an element $x \in \mathbb{R}^n$ such that $x_i = 0$ if $\lambda_i(t_0) < 0$ or an element $y \in \mathbb{R}^n$ such that $y_i = 0$ if $\lambda_i(t_0) > 0$.*

Proof. Since the action on V is nonproper, there exists a sequence (t_p) with $t_p \to +\infty$ in $\mathbb{R}^d$ and a sequence (x_p) in V such that $x_p \to x$ in V and $y_p = t_p.x_p \to y$ in V. Consider the sequence $\frac{t_p}{\|t_p\|}$. Up to taking a subsequence, we may assume it has a limit t_0. Since the action is faithful, and $t_0 \neq 0$, there exists $i \in 1, \cdots, d$ such that $\lambda_i(t_0) \neq 0$. Note that $\lambda_i(t_p) \to +\infty$ if $\lambda_i(t_0) > 0$ and $\lambda_i(t_p) \to -\infty$ if $\lambda_i(t_0) < 0$. Hence $x^i = 0$ if $\lambda_i(t_0) > 0$ and $y^i = 0$ if $\lambda_i(t_0) < 0$. $\diamondsuit$

4.4 End of the proof

As we mentioned above, G acts nonproperly on $\Phi(M)$. Let $G = KAK$ be the cartan decomposition of G. Since G has finite center K is compact. So A acts also nonproperly on $\Phi(M)$. From this, it follows that there exists $t \neq 0$, $q \in \Phi(M)$ and $\lambda_0 \in \Phi$ such that $\lambda_0(t) < 0$ and $q_\lambda = 0$ for all $\lambda \in \Phi$ with $\lambda(t) < 0$. Put $q = \Phi_x$. Then $\bigoplus_{\alpha(t)<0} \mathcal{G}_\alpha$ is isotropic with respect to Φ_x. Hence the image of $\bigoplus_{\alpha(t)<0} \mathcal{G}_\alpha$ by the map $X \mapsto \overline{X}_x$ is an isotropic subspace of T_xM, so its dimension is less or equal to 1. However:

Fact 4.2 *For any $t \in \mathcal{A}$, the dimension of $\bigoplus_{\alpha(t)<0} \mathcal{G}_\alpha$ is at least 2 (where G is assumed to have no local factor locally isomorphic to $SL(2,\mathbb{R})$).*

Proof. This dimension can not be 0, since G is semi-simple. If it equals 1, then, the subalgebra $\bigoplus_{\alpha(t)\geq 0} \mathcal{G}_\alpha$ has codimension 1 in $\mathcal{G}$. This contradicts the non existence of $SL(2,\mathbb{R})$ factor condition (only simple groups locally isomorphic to $SL(2,\mathbb{R})$ act on 1-manifolds). $\diamondsuit$

We infer from this the existence of a nonzero element $X \in \bigoplus_{\alpha(t)<0} \mathcal{G}_\alpha$ such that $\overline{X}_x = 0$, which yields : $exp(tX) \in stab(x), \forall t \in \mathbb{R}$. But elements of $\bigoplus_{\alpha(t)<0} \mathcal{G}_\alpha$ are nilpotent, and thus generate non-compact groups. This finishes the proof of Theorem 1.8. $\diamondsuit$

References

[1] S. Adams, Dynamics of simple Lie groups on Lorentz manifolds. Geom. Dedicata 105 (2004), 1–12.

[2] S. Adams, G. Stuck, Garrett, The isometry group of a compact Lorentz manifold. I, II. Invent. Math. 129 (1997), no. 2, 239–261, 263–287.

[3] C. Boubel, A. Zeghib, Isometric actions of Lie subgroups of the Moebius group. Nonlinearity 17 (2004), no. 5, 1677–1688.

[4] M. Gromov, Rigid transformations groups. Géométrie différentielle (Paris, 1986), 65–139, Travaux en Cours, 33, Hermann, Paris, 1988.

[5] S. Hiepko, Eine innere Kennzeichnung der verzerrten Produkte. Math. Ann. 241 (1979) 209–215.

[6] N. Kowalsky, Actions of non-compact simple groups of Lorentz manifolds. C. R. Acad. Sci. Paris Sér. I Math. 321 (1995), no. 5, 595–599.

[7] N. Kowalsky, Noncompact simple automorphism groups of Lorentz manifolds. Ann. Math. 144 (1997), 611–640.

[8] R. Ponge; H. Reckziegel, Twisted products in pseudo-Riemannian geometry. Geom. Dedicata 48 (1993), no. 1, 15–25.

[9] A. Zeghib, Remarks on Lorentz symmetric spaces, Compositio Math. 140 (2004) 1675-1678.

[10] A. Zeghib, Sur les espaces-temps homogènes. The Epstein birthday schrift, 551–576, Geom. Topol. Monogr., 1, Geom. Topol. Publ., Coventry, 1998.

[11] A. Zeghib, Isometry groups and geodesic foliations of Lorentz manifolds. Part II: Geometry of analytic Lorentz manifolds with large isometry groups. GAFA, 9 (1999) 823–854.

[12] R. Zimmer, On the automorphism group of a compact Lorentz manifold and other geometric manifolds. Invent. Math. 83 (1986) 411-426.

A. Arouche, M. Deffaf,
Faculté des mathématiques, USTHB
BP 32 El'Alia, Bab Ezzouar, Alger, Algeria.

A. Zeghib,
CNRS, UMPA, École Normale Supérieure de Lyon
46, allée d'Italie, 69364 Lyon cedex 07, France

arouche@math.usthb.dz
deffaf1@yahoo.fr
Zeghib@umpa.ens-lyon.fr, www.umpa.ens-lyon.fr/˜zeghib/