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► **To cite this version:**

| Frederic Dambreville. Definition of a Deterministic Bayesian Logic. 2004. hal-00003388v1

**HAL Id: hal-00003388**

**<https://hal.science/hal-00003388v1>**

Preprint submitted on 28 Nov 2004 (v1), last revised 29 Jun 2007 (v2)

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# Definition of a Deterministic Bayesian Logic

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## Abstract

The Bayesian logic is generally associated to the definition of a prior probabilistic law. Conditional algebra have been investigated by some authors though, but somehow the background framework is still probabilistic and the entire logic is not specified. In this paper, the definition of a *Deterministic* Bayesian Logic is proposed. This logic is completely independent of any notion of probability. The coherence of this logic is proven and various logical theorems are derived. It is shown that this logic is probabilizable and avoids the *negative result of Lewis*. At last the probabilistic Bayesian rule is recovered by *posteriorly* probabilizing our logic.

**Keywords:** Probability, Bayesian inference, Logic, Bayesian Logic, Probabilized Logic

## Notations

- NEG , AND and OR are the meta-logical negation, conjunction and disjunction in the meta-language,
- $\wedge, \vee, \neg, \rightarrow$  are the logical conjunction, disjunction, negation and classical inference,
- $\perp$  is the *ever false* proposition;  $\top$  is the *ever true* proposition,
- $(\cdot|\cdot)$  is the logical Bayesian operator to be defined,
- A (one-objective) sequent is a meta-proposition of the form  $\vdash \phi$ . Intuitively,  $\vdash \phi$  means “ $\phi$  is proven”, but  $\vdash \phi$  may be TRUE or FALSE: in this paper, *the notation  $\vdash \phi$  is not exactly related to a proven proposition*,
- The meta-proposition  $\not\vdash \phi = \text{NEG } \vdash \phi$  means that “ $\phi$  is not proven”,
- A multi-objective sequent is a meta-proposition of the form:

$$\left( \bigvee_{i=1}^m \text{OR } \vdash \phi_i \right) \text{ OR } \left( \bigvee_{j=1}^n \text{OR } \not\vdash \psi_j \right).$$

Multi-objective sequents generalize sequents, and the two terms will be identified,

- The meta-relation  $\phi \equiv \psi$  means  $\vdash \phi \rightarrow \psi$  AND  $\vdash \psi \rightarrow \phi$ ,
- The meta-relation  $\phi \not\equiv \psi$  means  $\not\vdash \phi \rightarrow \psi$  OR  $\not\vdash \psi \rightarrow \phi$ .

This definition of *sequent* will be useful in the sequel, but is not customary. In particular, it is not related to the definition of a sequent calculus.

# 1 Introduction

This work originates from a contribution to the book of Dezert and Smarandache [1] about the DSmT, a theory related to the Dempster-Shafer Theory of Evidence. Evidence Theory [2] is a theoretical and practical tool for manipulating non deterministic informations. In particular, it is used for fusing information. Among other non deterministic theories, Evidence Theory is considered by some as a possible alternative to the Bayesian Probability.

A question then arises. Why using a theory instead another theory? This question implies another sub-question: what is the logic behind a non deterministic theory? For example, when is known the logic behind the DSmT or the Bayesian Probability, then it may be possible to make a comparison between these two theories. It has been shown in [1, 3] that such a logic can be found for the DSmT. More precisely, the DSmT may be deduced by probabilizing a particular (deterministic) multi-Modal logic. The next issue is to find a similar result for the Bayesian Probability:

- Construct a *deterministic* logic,
- Derive the Bayesian Probability by probabilizing this logic.

In [3], a possible construction of a deterministic Bayesian logic was suggested. This work was incomplete and some fundamental issues were discarded. This paper intends to correct and finalize this construction in a much rigorous way. In particular, the coherence of the logic is discussed; the logical coherence and the probabilistic coherence.

It is interesting to locate this construction in the general field of probability logic. This field involves at least three viewpoints, which are however quite related. First should be mentioned Cox's axiomatic derivation of the Bayesian probability ([4] but also [5, 6]). On the other hand, the theory of Probabilized Logic [7, 8, 9, 10] explains how probabilities and the probabilistic Bayesian rule could be an approach for reasoning with uncertain propositions. Quite related to this last topic, the notion of conditional Event algebras (and the possible enrichment of the Bayesian rule) have been investigated by some authors [11, 12, 13, 14]. This last viewpoint is not far from ours. But the notion of probability remains the essential basis of these works, and the logical abstraction is still partial. Now, the logic behind the conditional operator is not so obvious: Lewis [15] has shown some critical issues related to the notion of conditional probability; refer also to [16, 17, 18]; this notion has to be manipulated with care. In this paper, a *self-sufficient* deterministic Bayesian logic is constructed *without any reference to probabilistic laws*. Purely logical properties are then derived, and the probabilistic Bayesian rule is recovered by posteriorly probabilizing this logic. Moreover, it is shown that this logic avoids Lewis' negative result. At last, it appears that the Bayesian logic could be handled independently of the notion of probability.

Section 2 is dedicated to the construction of the Deterministic Bayesian Logic. A *weak* coherence theorem is proven for the logic.

Section 3 expresses several theorems of the logic. A discussion about Lewis' negative result is opened here. A purely logical viewpoint is considered in this section.

Section 4 is defining *what is a probability over logical propositions*. This is a really minimal definition and the notion of conditional probability is not defined.

In section 5, it is proven that the Deterministic Bayesian Logic is probabilizable. As a consequence, it is shown that our construction avoids Lewis' negative result and it is explained why. A *strong* coherence theorem is deduced. At last, the (probabilistic) Bayesian rule is recovered by probabilizing the Deterministic Bayesian Logic.

And then, the conclusion.

## 2 Defining the Deterministic Bayesian Logic (DBL)

A Bayesian logic is defined now, which is independent from any definition of a probability. As it has been introduced, this work was motivated by some reflexions about the logical meaning of the DSmT, a non deterministic framework for manipulating informations. In our previous publication [3], it has been shown that the DSmT could be deduced by probabilizing a particular multi-Modal logic, denoted mM. In other words, DSmT appeared as the implementation of a particular logic for manipulating the information in a probabilistic world. A question arised then: *Is the probabilistic Bayesian rule,  $p(\phi \wedge \psi) = p(\phi)p(\psi|\phi)$ , also obtained by probabilizing a particular logic?* If the answer is *yes*, then the Bayesian reasoning could be seen as purely logical and not related to a notion of probability. In [3], such a Bayesian logic has been introduced and the probabilistic Bayesian rule has been recovered by probabilizing this logic. This logic has been denoted bM, for *Bayesian multi-Modal logic*. The terminology is confusing. Indeed, this logic is not exactly modal ([19] for an introduction to modal logics; also refer to appendix A), although it has some flavor: it is essentially classical, but with “Bayesian modalities”... Moreover, the description of bM was partial and it was somewhat erroneous: it failed Lewis’ “negative result”. Consequently, a probability over bM was necessarily trivial... But bM will still be an essential source of inspiration for the logic we are constructing now.

In this section, the main ideas of bM are kept with slight enhancements and changes. The coherence of this new logic is proven. We are also changing the name of the logic for DBL, Deterministic Bayesian Logic.

### 2.1 Introducing the Deterministic Bayesian Logic

**Casual definition.** A logic of the Bayesian inference, denoted DBL, will be defined accordingly to the following principles:

- DBL contains the classical logic,
- Any proposition  $\phi$  induces a sub-universe, entirely characterized by the Bayesian operator  $(\cdot|\phi)$ . For this reason,  $(\cdot|\phi)$  could be seen as a conditional modality. But this modality possesses a strange quality: the implied sub-universe is essentially *classical*,
- Since sub-universes are essentially classical, the negation has a classical behavior. Thus, it is assumed that a sub-universe will always decide if a proposition is true or false: there is no doubt and no contradiction within the modality  $(\cdot|\phi)$ . This property is expressed by:

$$\vdash (\psi|\phi) \vee (\neg\psi|\phi) \quad \text{and} \quad \vdash \neg((\psi|\phi) \wedge (\neg\psi|\phi))$$

or equivalently:

$$\neg(\neg\psi|\phi) \equiv (\psi|\phi) .$$

Nevertheless, truth may change depending on the proposition of reference  $\phi$ , unless the proposition  $\psi$  is absolutely true or absolutely false,

- The sub-universe  $(\cdot|\phi)$  is a projection of the sup-universe into  $\phi$ . In particular, the theorems of  $(\cdot|\phi)$  result from the propositions which are theorems within the range  $\phi$  in the sup-universe:

$$\not\vdash \neg\phi \quad \text{and} \quad \vdash \phi \rightarrow \psi \quad \text{implies} \quad \vdash (\psi|\phi) .$$

Notice the condition  $\not\vdash \neg\phi$ . This condition is required, because of the singularity  $(\cdot|\perp)$ . More precisely, it is known that  $\vdash \perp \rightarrow \perp$  and  $\vdash \perp \rightarrow \neg\perp$ . Without the condition, it would be proven  $\vdash (\perp|\perp)$  and  $\vdash (\neg\perp|\perp)$ . There would be a contradiction with the negation rule  $\neg(\neg\perp|\perp) \equiv (\perp|\perp)$ .

Could this rule be made into an axiom like “ $\not\vdash \neg\phi$  implies  $\vdash (\phi \rightarrow \psi) \rightarrow (\psi|\phi)$ ”? The answer is no! Notice first that  $(\psi|\phi)$  just describes the viewpoint of the universe  $(\cdot|\phi)$  about  $\psi$ . It depends on the relationship between  $\psi$  and  $\phi$ , but not exactly on  $\phi$ . Assume that this relationship is fixed; for instance take  $\psi \equiv \perp$ . Then, the proposition  $(\perp|\phi) \equiv \perp$  is invariant (always false is of course false from any viewpoint). On the contrary,  $\phi \rightarrow \perp$  is fully dependent on the proposition  $\phi$ : the hypothesized axiom becomes  $\vdash (\phi \rightarrow \perp) \rightarrow \perp$  that is  $\vdash \phi$ , which is false unless  $\phi \equiv \top$ ,

- Any sub-proposition  $(\psi|\phi)$  implies that  $\psi$  is true within the range  $\phi$  in the sup-universe:

$$\vdash (\psi|\phi) \rightarrow (\phi \rightarrow \psi) .$$

This last point is not exactly the converse of the previous point. The previous point concerns theorems, while any possible propositions are considered here. This (modal-like) difference is necessary and makes the distinction between  $(\cdot|\cdot)$  and  $\rightarrow$ ,

- The modus ponens should work in the sub-universes (sub-universes are “logical”):

$$\vdash (\psi \rightarrow \eta|\phi) \rightarrow ((\psi|\phi) \rightarrow (\eta|\phi)) .$$

In association with the Bayesian inference is the notion of independence between propositions. *This independence meta-relation will be denoted  $\times$* . More precisely, a proposition  $\psi$  may be considered as logically independent of a proposition  $\phi$ , also denoted  $\psi \times \phi$ , if it does not make any difference to observe  $\psi$  in the sub-universe  $(\cdot|\phi)$  or in the sup-universe:

$$\psi \times \phi \quad \text{means} \quad (\psi|\phi) \equiv \psi .$$

*Example: it is the night (the context). I drop a stone; it will fall, ie.  $\psi$ . Now the light is on, ie.  $\phi$ . It appears that the stone will also fall the same way, if I drop it when the light is on, ie.  $\psi \equiv (\psi|\phi)$ . As a conclusion, the proposition a dropped stone falls is independent of the proposition the light is on.*

Of course, this logical independence matches the probabilistic characterization of the independence, ie.  $p(\psi|\phi) = p(\psi)$ . But it is stronger, as noticed in section 5. *Propositions which are recognized as probabilistically independent are not necessarily logically independent!*

Deciding whether the meta-operator  $\times$  is symmetric or not is probably another philosophical issue. *This symmetry hypothesis is optional in our logic.*

## 2.2 Definition of the Bayesian logic DBL

**Toward multi-objective axioms.** In the previous section, some rules have been established of the form:

$$\text{“} \not\vdash \phi \text{ implies } \vdash \psi \text{” } (*)$$

The problem with such rules is that the resulting logic is self-referring. Self-reference does not suit a practicable and recursive definition of the provable propositions. It is necessary to change this viewpoint.

It is generally admitted that a proof will be concluded by a proved proposition. But this is a strong requirement. It is still possible to imagine proofs with weaker conclusions. For example the sketched rule  $(*)$  is equivalent to the “multi-objective axiom”:

$$\text{“} \vdash \phi \text{ OR } \vdash \psi \text{”} ,$$

where OR is the meta-logical disjunction.

In the sequel, the logic DBL will be constructed from such multi-objective axioms. It is noteworthy that the resulting logic:

- is straightly related to the intuitive rules defined previously,
- is not self-referring,
- has a valid recursive definition.

**Preliminary definitions.** The notation  $\vdash \phi$  is now a meta-proposition which means “ $\phi$  has a proof”. It may be TRUE or FALSE. The meta-proposition  $\not\vdash \phi$  designates the negation of  $\vdash \phi$ . A *sequent* is a meta-proposition of the form:<sup>1</sup>

$$\left( \bigvee_{i=1}^m \vdash \phi_i \right) \text{ OR } \left( \bigvee_{j=1}^n \not\vdash \psi_j \right).$$

A (multi-objective) axiom will be a sequent hypothesized TRUE in the meta-language.

### 2.2.1 Definition of the langage of DBL

Let  $\Theta = \{\theta_i/i \in I\}$  be a set of atomic propositions. The set of the *Bayesian* propositions,  $DBL(\Theta)$ , is constructed recursively:

- $\Theta \subset DBL(\Theta)$ ,
- $\perp, \top \in DBL(\Theta)$ ,
- If  $\phi, \psi \in DBL(\Theta)$ , then  $\neg\phi \in DBL(\Theta)$ ,  $\phi \wedge \psi \in DBL(\Theta)$ ,  $\phi \vee \psi \in DBL(\Theta)$ ,  $\phi \rightarrow \psi \in DBL(\Theta)$  and  $(\psi|\phi) \in DBL(\Theta)$ .

For simplicity, the notation  $DBL$  will be used instead of  $DBL(\Theta)$ .

### 2.2.2 Definition of the axioms of DBL

The Deterministic Bayesian Logic obeys the following axioms (c\* for classical axioms, b\* for Bayesian axioms) which are sequents assumed to be TRUE for any  $\phi, \psi, \eta \in DBL$ :

- c1.**  $\vdash \top$ ,
- c2.**  $\vdash \phi \rightarrow (\psi \rightarrow \phi)$ ,
- c3.**  $\vdash (\eta \rightarrow (\phi \rightarrow \psi)) \rightarrow ((\eta \rightarrow \phi) \rightarrow (\eta \rightarrow \psi))$ ,
- c4.**  $\vdash (\neg\phi \rightarrow \neg\psi) \rightarrow ((\neg\phi \rightarrow \psi) \rightarrow \phi)$ ,
- c5.**  $\perp \equiv \neg\top$ , (in fact two axioms:  $\vdash \perp \rightarrow \neg\top$  and  $\vdash \neg\top \rightarrow \perp$ )
- c6.**  $\phi \rightarrow \psi \equiv \neg\phi \vee \psi$ , (two axioms)
- c7.**  $\phi \wedge \psi \equiv \neg(\neg\phi \vee \neg\psi)$ , (two axioms)
- Modus ponens.**  $\not\vdash \phi$  OR  $\not\vdash \phi \rightarrow \psi$  OR  $\vdash \psi$ ,  
or recursively:  $\vdash \phi$  and  $\vdash \phi \rightarrow \psi$  implies  $\vdash \psi$ ,
- b1.**  $\not\vdash \phi \rightarrow \psi$  OR  $\vdash \neg\phi$  OR  $\vdash (\psi|\phi)$ ,  
or recursively:  $\vdash \phi \rightarrow \psi$  implies  $\vdash \neg\phi$  OR  $\vdash (\psi|\phi)$ ,
- b2.**  $\vdash (\psi \rightarrow \eta|\phi) \rightarrow ((\psi|\phi) \rightarrow (\eta|\phi))$ ,
- b3.**  $\vdash (\psi|\phi) \rightarrow (\phi \rightarrow \psi)$ ,
- b4.**  $\neg(\neg\psi|\phi) \equiv (\psi|\phi)$ , (two axioms)

<sup>1</sup>NB: these sequents are different from the customary sequents  $\psi_j|_{j=1}^n \vdash \phi_i|_{i=1}^m$

**b5 (definition).**  $\psi \times \phi$  means  $(\psi|\phi) \equiv \psi$ ,

**b6 (optional).** ( $\times$  is symmetric) :  $(\psi|\phi) \not\equiv \psi$  OR  $(\phi|\psi) \equiv \phi$ , (two axioms)  
or recursively:  $(\psi|\phi) \equiv \psi$  implies  $(\phi|\psi) \equiv \phi$ ,

**Customized axioms.** Axioms introduced for the specification of a particular sub-theory of DBL. In particular the customized axioms may be used to characterize the properties of the atomic propositions: axioms like  $\vdash \theta_i$ ,  $\not\vdash \theta_i$  and  $\theta_i \times \theta_j$  are possible.

The *free DBL logic* is obtained when there is no customized axiom and no axiom b6. In the sequel, the optional axiom b6 is almost never used or never needed. When this axiom is involved, the notation DBL\* will be used for referring to the logic.

**The notion of proof.** A *proof* of DBL is a sequence of sequents, where each sequent is either:

- an axiom,
- deduced from previously sequenced sequents by applying a meta-logical rule.

Consequently, when the DBL sequent  $(\text{OR}_{i=1}^m \vdash \phi_i) \text{ OR } (\text{OR}_{j=1}^n \not\vdash \psi_j)$  is sequenced in a proof of DBL, then it is TRUE.

The converse is false: a sequent never sequenced in any proof of DBL is *not necessarily* FALSE (refer also to the paragraph **Precision about  $\not\vdash \phi$** ).

Moreover, sequents like  $\not\vdash \phi$  cannot be sequenced in any proof of the free DBL logic: *Indeed, in its recursive formulation, the axiomatization does not manipulate any meta-proposition of the form  $\not\vdash \phi$ .*

By definition, a proposition  $\phi$  has a proof if and only if  $\vdash \phi$  is TRUE.

*Definition.* A DBL logic is coherent (weak definition) if and only if  $\vdash \perp$  is **not sequenced** in any proof of DBL.

The classical axioms c\* and the modus ponens guarantee that DBL contains the classical logic: *for any classical theorem  $\vdash \phi$  (ie.  $\phi$  is proven in classical logic), there is a proof of DBL where  $\vdash \phi$  is sequenced.*

**An example of proof.** The following sequence may be constructed:

1.  $\vdash \psi \rightarrow (\phi \rightarrow \psi)$  (axiom c2)
2.  $\vdash (\psi \rightarrow (\phi \rightarrow \psi)) \rightarrow ((\psi \rightarrow \phi) \rightarrow (\psi \rightarrow \psi))$  (axiom c3)
3.  $\not\vdash \psi \rightarrow (\phi \rightarrow \psi)$  OR  $\not\vdash (\psi \rightarrow (\phi \rightarrow \psi)) \rightarrow ((\psi \rightarrow \phi) \rightarrow (\psi \rightarrow \psi))$   
OR  $\vdash (\psi \rightarrow \phi) \rightarrow (\psi \rightarrow \psi)$  (modus ponens)
4.  $\vdash (\psi \rightarrow \phi) \rightarrow (\psi \rightarrow \psi)$  ( AND of the three previous meta-propositions)
- 4-bis.  $\vdash (\psi \rightarrow (\eta \rightarrow \psi)) \rightarrow (\psi \rightarrow \psi)$  (by defining  $\phi = \eta \rightarrow \psi$ )
5.  $\vdash \psi \rightarrow (\eta \rightarrow \psi)$  (axiom c2)
6.  $\not\vdash \psi \rightarrow (\eta \rightarrow \psi)$  OR  $\not\vdash (\psi \rightarrow (\eta \rightarrow \psi)) \rightarrow (\psi \rightarrow \psi)$  OR  $\vdash \psi \rightarrow \psi$  (modus ponens)
7.  $\vdash \psi \rightarrow \psi$  ( AND of the three previous meta-propositions)

Conclusion:  $\psi \rightarrow \psi$  is proven.

Notice that this proof could be derived in *english*; the two formulations are quite equivalent:

**Proof.**

From axiom c2, it is deduced  $\vdash \psi \rightarrow (\phi \rightarrow \psi)$ .

From axiom c3, it is deduced  $\vdash (\psi \rightarrow (\phi \rightarrow \psi)) \rightarrow ((\psi \rightarrow \phi) \rightarrow (\psi \rightarrow \psi))$ .

Applying the modus-ponens, it comes  $\vdash (\psi \rightarrow \phi) \rightarrow (\psi \rightarrow \psi)$ .

It follows  $\vdash (\psi \rightarrow (\eta \rightarrow \psi)) \rightarrow (\psi \rightarrow \psi)$ .

Applying c2 again, it comes  $\vdash \psi \rightarrow (\eta \rightarrow \psi)$ .

Applying the modus ponens, it is deduced  $\vdash \psi \rightarrow \psi$ .

□□□

In the sequel and particularly in section 3, this *english* formulation will be of constant use, since it is concise and more didactic. Moreover, some classical results will be assumed without proof, in order to reduce the size of the proofs when needed.

**Precision about  $\not\vdash \phi$ .** Assume that  $\vdash \phi$  is not sequenced in any proof of DBL. Does that means “ $\phi$  is not proven”? In fact, NO! It has even been noticed that  $\not\vdash \phi$  is not sequenced in any proof of the free DBL. This is counterintuitive, but has an explanation. Although  $\vdash \phi$  is not sequenced, some weaker sequents, *eg.*  $\vdash \phi$  OR  $\vdash \psi$ , may be sequenced. Such sequents contain a partial knowledge about the proof of  $\phi$  and may infer a contradiction with  $\not\vdash \phi$ .

*Example.* Assume  $\vdash \phi$  OR  $\vdash \psi$ . Assume also that neither  $\vdash \phi$  nor  $\vdash \psi$  are sequenced in a proof of DBL. Inferring then  $\not\vdash \phi$  and  $\not\vdash \psi$  will induce a contradiction since  $\not\vdash \phi$  AND  $\not\vdash \psi$  is the negation of  $\vdash \phi$  OR  $\vdash \psi$ .

**Noteworthy deduction processes.** The following deductions may be noticed:

- Any deduction of the classical logic may be done in DBL.  
*Obvious.*
- From  $\vdash \phi$ , it is deduced  $\vdash \phi$  OR  $\vdash \psi$ .
- From  $\not\vdash \phi$  and  $\vdash \phi$  OR  $\vdash \psi$ , it is deduced  $\vdash \psi$ .
- From  $\vdash \perp$  OR  $\vdash \phi$ , it is deduced  $\vdash \phi$ .  
*Proof.* From the classical theorem  $\vdash \perp \rightarrow \phi$  and the modus ponens, it is deduced  $\not\vdash \perp$  OR  $\vdash \phi$ . Combined with  $\vdash \perp$  OR  $\vdash \phi$ , this meta-proposition implies  $\vdash \phi$ .
- (not encountered when DBL is coherent) From  $\vdash \perp$ , it is deduced  $\vdash \phi$ .  
*Proof.* It is a consequence of the *modus ponens* and the classical theorem  $\vdash \perp \rightarrow \phi$ .

**2.2.3 Coherence**

**The free DBL (or DBL\*) is not stronger than the classical logic.** Assume that there is no customized axiom. Let  $\phi$  be a classical proposition, *ie.*  $\phi$  does not contain the logical operator  $(\cdot|\cdot)$ , such that  $\vdash \phi$  is sequenced in a proof of DBL (or DBL\*). Then,  $\vdash \phi$  can be deduced from a *classical proof*.

**Proof.** Construct recursively the function  $f : DBL \rightarrow \{0, 1\}$  by:

- $f(\perp) = 0$ ,  $f(\top) = 1$  and  $\forall i \in I$ ,  $f(\theta_i) \in \{0, 1\}$ ,
- $f(\neg\phi) = 1 - f(\phi)$ ,
- $f(\phi \wedge \psi) = f(\phi)f(\psi)$ ,
- $f(\phi \vee \psi) = f(\phi) + f(\psi) - f(\phi)f(\psi)$ ,
- $f(\phi \rightarrow \psi) = 1 - f(\phi) + f(\phi)f(\psi)$ ,
- $f((\psi|\phi)) = f(\psi)$ .

It is proved in appendix B:



- $f \circ \mathbf{c1.}$   $f(\top) = 1$ ,  
 $f \circ \mathbf{c2.}$   $f(\phi \rightarrow (\psi \rightarrow \phi)) = 1$ ,  
 $f \circ \mathbf{c3.}$   $f((\eta \rightarrow (\phi \rightarrow \psi)) \rightarrow ((\eta \rightarrow \phi) \rightarrow (\eta \rightarrow \psi))) = 1$ ,  
 $f \circ \mathbf{c4.}$   $f((\neg\phi \rightarrow \neg\psi) \rightarrow ((\neg\phi \rightarrow \psi) \rightarrow \phi)) = 1$ ,  
 $f \circ \mathbf{c5.}$   $f(\perp \rightarrow \neg\top) = 1$  and  $f(\neg\top \rightarrow \perp) = 1$ ,  
 $f \circ \mathbf{c6.}$   $f((\phi \rightarrow \psi) \rightarrow (\neg\phi \vee \psi)) = 1$  and  $f((\neg\phi \vee \psi) \rightarrow (\phi \rightarrow \psi)) = 1$ ,  
 $f \circ \mathbf{c7.}$   $f((\phi \wedge \psi) \rightarrow \neg(\neg\phi \vee \neg\psi)) = 1$  and  $f(\neg(\neg\phi \vee \neg\psi) \rightarrow (\phi \wedge \psi)) = 1$ ,  
 $f \circ \{\mathbf{modus ponens}\}.$   $f(\phi) = 1$  and  $f(\phi \rightarrow \psi) = 1$  implies  $f(\psi) = 1$ ,  
 $f \circ \mathbf{b1.}$   $f(\phi \rightarrow \psi) = 1$  implies  $f(\neg\phi) = 1$  OR  $f((\psi|\phi)) = 1$ ,  
 $f \circ \mathbf{b2.}$   $f((\psi \rightarrow \eta|\phi) \rightarrow ((\psi|\phi) \rightarrow (\eta|\phi))) = 1$ ,  
 $f \circ \mathbf{b3.}$   $f((\psi|\phi) \rightarrow (\phi \rightarrow \psi)) = 1$ ,  
 $f \circ \mathbf{b4.}$   $f(\neg(\neg\psi|\phi) \rightarrow (\psi|\phi)) = 1$  and  $f((\psi|\phi) \rightarrow \neg(\neg\psi|\phi)) = 1$ ,  
 $f \circ \mathbf{b6.}$   $f((\psi|\phi) \rightarrow \psi) = 1$  and  $f(\psi \rightarrow (\psi|\phi)) = 1$  implies  $f((\phi|\psi) \rightarrow \phi) = 1$  and  $f(\phi \rightarrow (\phi|\psi)) = 1$ .

These meta-propositions are exactly the axioms of the free DBL (or DBL\*), where  $\vdash *$  has been replaced by  $f(*) = 1$ . Consequently,  $\vdash \phi$  is sequenced in a proof of the free DBL (or DBL\*) implies  $f(\phi) = 1$ .

*Conclusion.* When restricted to classical propositions, the function  $f$  describes exactly the construction of the truth values of the *classical propositions* from the *initial* truth setting  $f(\theta_i) \in \{0, 1\}$ .

Now, assume  $f(\phi) = 1$  for any choice  $f(\theta_i) \in \{0, 1\}$ ; it is in particular the case when  $\vdash \phi$  is sequenced in DBL (or DBL\*). Then, it is known that  $\phi$  has a classical proof.

□□□

*Corollary.* The free DBL (or DBL\*) is coherent: the sequent  $\vdash \perp$  is not sequenced in any proof of the free DBL (or DBL\*).

**Proof.** The classical logic is coherent:  $\vdash \perp$  cannot be deduced from the classical logic.

The corollary is obtained by applying the previous property.

□□□

*Comment.* These coherence results are weak. In particular, (1) it does not prevent Lewis' negative result (refer to the next sections). Moreover, (2) an erroneous definition of the axioms could even imply the deduction  $\vdash \phi$  OR  $\vdash \neg\phi$  for any  $\phi$ , which is irrelevant! In section 5, the justification of the coherence will be completed by proving that the free DBL is probabilizable. This property will give a refutation of (1) and (2) for the free DBL.

**The DBL is increasing.** Let  $\text{DBL}_1$  and  $\text{DBL}_2$  be two DBL defined over the same set of atomic propositions  $\Theta$ . Denote  $C_i$  the set of customized axioms of  $\text{DBL}_i$ , for  $i = 1, 2$ . Assume  $C_1 \subset C_2$ . Then, any sequent sequenced in (a proof of)  $\text{DBL}_1$  is sequenced in  $\text{DBL}_2$ .

**Proof.** Since  $C_1 \subset C_2$ , any proof of  $\text{DBL}_1$  is a proof of  $\text{DBL}_2$ .

□□□

### 3 Theorems of DBL

*Unless specified, these theorems are proven without the optional axiom, symmetry of  $\times$ .* The proofs are made in the english formulation.

**Preliminary properties.**  $\psi \equiv \eta$  implies  $\vdash \neg\phi$  OR  $(\psi|\phi) \equiv (\eta|\phi)$ .

**Proof (in english).** From  $\vdash \psi \rightarrow \eta$ , it is deduced  $\vdash \phi \rightarrow (\psi \rightarrow \eta)$ .

By applying axiom b1, it comes  $\vdash \neg\phi$  OR  $\vdash (\psi \rightarrow \eta|\phi)$ .

Axiom b2 then implies  $\vdash \neg\phi$  OR  $\vdash (\psi|\phi) \rightarrow (\eta|\phi)$ .

From  $\vdash \eta \rightarrow \psi$ , it is similarly proven  $\vdash \neg\phi$  OR  $\vdash (\eta|\phi) \rightarrow (\psi|\phi)$ .

□□□

However, the property “ $\phi \equiv \psi$  implies  $(\eta|\phi) \equiv (\eta|\psi)$ ” needs probably an additional axiom. Moreover, from axiom b4, “ $\vdash \phi \rightarrow \psi$  does not necessarily imply  $\vdash (\eta|\phi) \rightarrow (\eta|\psi)$ ” and “ $\vdash \phi \rightarrow \psi$  does not necessarily imply  $\vdash (\eta|\psi) \rightarrow (\eta|\phi)$ ” as well.

### 3.1 The full universe

$\vdash \phi$  implies  $\psi \times \phi$ , ie.  $(\psi|\phi) \equiv \psi$ . In particular  $(\psi|\top) \equiv \psi$ .

Interpretation: the sub-universe of the *full* proposition is the whole universe.

**Proof.** From axiom b3, it comes  $\vdash (\psi|\phi) \rightarrow (\phi \rightarrow \psi)$ .

Applying the hypothesis  $\vdash \phi$ , it is deduced  $\vdash (\psi|\phi) \rightarrow \psi$ .

Also holds  $\vdash (\neg\psi|\phi) \rightarrow \neg\psi$ , and then  $\vdash \psi \rightarrow \neg(\neg\psi|\phi)$ .

By applying b4, it comes  $\vdash \psi \rightarrow (\psi|\phi)$ .

□□□

### 3.2 Sub-universes are classical

- $(\neg\psi|\phi) \equiv \neg(\psi|\phi)$ ,
- $\vdash \neg\phi$  OR  $(\psi \wedge \eta|\phi) \equiv (\psi|\phi) \wedge (\eta|\phi)$ ,
- $\vdash \neg\phi$  OR  $(\psi \vee \eta|\phi) \equiv (\psi|\phi) \vee (\eta|\phi)$ ,
- $\vdash \neg\phi$  OR  $(\psi \rightarrow \eta|\phi) \equiv (\psi|\phi) \rightarrow (\eta|\phi)$ .

**Proof.** The first theorem is a consequence of axiom b4.

From now on, assume  $\not\vdash \neg\phi$ .

From axiom b2, it is deduced  $\vdash (\neg\psi \vee \eta|\phi) \rightarrow (\neg(\psi|\phi) \vee (\eta|\phi))$ .

Applying the first theorem, it is deduced  $\vdash (\neg\psi \vee \eta|\phi) \rightarrow ((\neg\psi|\phi) \vee (\eta|\phi))$ .

At last  $\vdash (\neg\neg\psi \vee \eta|\phi) \rightarrow ((\neg\neg\psi|\phi) \vee (\eta|\phi))$  and  $\vdash (\psi \vee \eta|\phi) \rightarrow ((\psi|\phi) \vee (\eta|\phi))$ .

It is deduced  $\vdash ((\psi|\phi) \vee (\eta|\phi)) \rightarrow \neg(\psi \vee \eta|\phi)$  and, by applying the first theorem,

$$\vdash ((\neg\psi|\phi) \wedge (\neg\eta|\phi)) \rightarrow (\neg\psi \wedge \neg\eta|\phi).$$

At last:

$$\vdash ((\psi|\phi) \wedge (\eta|\phi)) \rightarrow (\psi \wedge \eta|\phi).$$

Now, it is deduced from  $\vdash \phi \rightarrow ((\psi \wedge \eta) \rightarrow \psi)$  and axiom b1 that (assuming  $\not\vdash \neg\phi$ ):

$$\vdash ((\psi \wedge \eta) \rightarrow \psi|\phi).$$

By applying the axioms b2 and modus ponens:

$$\vdash (\psi \wedge \eta|\phi) \rightarrow (\psi|\phi).$$

It is similarly proved that  $\vdash (\psi \wedge \eta|\phi) \rightarrow (\eta|\phi)$  and finally:

$$\vdash (\psi \wedge \eta|\phi) \rightarrow ((\psi|\phi) \wedge (\eta|\phi)).$$

The second theorem is then proved.

Third theorem is a consequence of the first and second theorems.

Last theorem is a consequence of the first and third theorems.

□□□

### 3.3 Evaluating $(\top|\cdot)$ and $(\perp|\cdot)$

Assume  $\not\vdash \neg\phi, \vdash \psi$ . Then  $(\phi|\phi) \equiv (\psi|\phi) \equiv (\top|\phi) \equiv \top$  and  $(\neg\phi|\phi) \equiv (\neg\psi|\phi) \equiv (\perp|\phi) \equiv \perp$ .

**Proof.** The second row is obtained from the first by negation.

Assume now  $\not\vdash \neg\phi$ .

From  $\vdash \phi \rightarrow \phi$ , it is deduced  $\vdash (\phi|\phi)$ .

From  $\vdash \phi \rightarrow \top$ , it is deduced  $\vdash (\top|\phi)$ .

From  $\vdash \psi$ , it is deduced  $\vdash \phi \rightarrow \psi$  and then  $\vdash (\psi|\phi)$ .

□□□

### 3.4 Inference property

$(\psi|\phi) \wedge \phi \equiv \phi \wedge \psi$ ; this is a quite classical result in conditional algebra [11].

**Proof.** From b3 it comes  $\vdash (\psi|\phi) \rightarrow (\phi \rightarrow \psi)$ .

Then  $\vdash \neg(\phi \rightarrow \psi) \rightarrow \neg(\psi|\phi)$  and  $\vdash (\phi \wedge \neg\psi) \rightarrow (\neg\psi|\phi)$ .

It follows  $\vdash (\phi \wedge \psi) \rightarrow (\psi|\phi)$  and finally:

$$\vdash (\phi \wedge \psi) \rightarrow ((\psi|\phi) \wedge \phi).$$

The converse is more simple. From  $\vdash (\psi|\phi) \rightarrow (\phi \rightarrow \psi)$ , it follows:

$$\vdash ((\psi|\phi) \wedge \phi) \rightarrow ((\phi \rightarrow \psi) \wedge \phi).$$

Since  $(\phi \rightarrow \psi) \wedge \phi \equiv \phi \wedge \psi$ , the converse is proved.

□□□

### 3.5 Introspection

$\vdash \neg\phi$  OR  $\vdash (\phi|\phi)$ .

Interpretation: a non empty proposition sees itself as ever true.

**Proof.** Obvious from  $\vdash \phi \rightarrow \phi$  and b1.

□□□

### 3.6 Inter-independence

$\vdash \neg\phi$  OR  $(\psi|\phi) \times \phi$ .

Interpretation: a proposition is co-independent of its sub-universe.

**Proof.** Assume  $\not\vdash \neg\phi$ .

Then  $\vdash (\phi|\phi)$  and consequentlty:

$$((\psi|\phi)|\phi) \equiv ((\psi|\phi)|\phi) \wedge (\phi|\phi) \equiv ((\psi|\phi) \wedge \phi|\phi) \equiv (\phi \wedge \psi|\phi) \equiv (\psi|\phi) \wedge (\phi|\phi) \equiv (\psi|\phi).$$

□□□

### 3.7 Independence invariance

Assume  $\not\vdash \neg\phi$ ,  $\psi \times \phi$  and  $\eta \times \phi$ . Then  $\neg\psi \times \phi$  and  $\psi \wedge \eta \times \phi$ .

Assume  $\not\vdash \neg\phi$ ,  $\psi \times \phi$  and  $\eta \equiv \psi$ . Then  $\eta \times \phi$ .

**Proof.**  $(\psi|\phi) \equiv \psi$  implies  $\neg(\psi|\phi) \equiv \neg\psi$  and finally  $(\neg\psi|\phi) \equiv \neg\psi$ .

Assume now  $\not\vdash \neg\phi$ ,  $(\psi|\phi) \equiv \psi$  and  $(\eta|\phi) \equiv \eta$ .

Then  $(\psi|\phi) \wedge (\eta|\phi) \equiv \psi \wedge \eta$  and finally  $(\psi \wedge \eta|\phi) \equiv \psi \wedge \eta$ .

Assume  $\not\vdash \neg\phi$  and  $\eta \equiv \psi$ .

Then  $(\psi|\phi) \equiv (\eta|\phi)$ .

And from  $(\psi|\phi) \equiv \psi$ , it is deduced  $(\eta|\phi) \equiv \eta$ .

□□□

### 3.8 Narcissistic independence

$\phi \times \phi$  implies  $\vdash \neg\phi$  OR  $\vdash \phi$ .

**Proof.**

From  $\vdash \phi \rightarrow \phi$ , it is deduced  $\vdash \neg\phi$  OR  $\vdash (\phi|\phi)$ .

Since  $\phi \times \phi$ , it comes  $(\phi|\phi) \equiv \phi$ .

At last  $\vdash \neg\phi$  OR  $\vdash \phi$ .

□□□

### 3.9 The link between $((\eta|\psi)|\phi)$ and $(\eta|\phi \wedge \psi)$

There is a really critical point here. A main argument against *Bayesian propositions* relies on *Lewis' negative result*. This result will be explained later, when speaking about probabilities. It will be shown that it does not work with our construction. However it has yet a strictly logical counterpart, when assuming the additional but *tempting axiom* " $((\eta|\psi)|\phi) \equiv (\eta|\phi \wedge \psi)$ ".

*Let  $((\eta|\psi)|\phi) \equiv (\eta|\phi \wedge \psi)$  be assumed as axiom, for any  $\phi, \psi, \eta$ . Then  $\psi \times \phi$  for any  $\phi, \psi$  such that  $\not\vdash \neg(\psi \wedge \phi)$  and  $\not\vdash \neg(\neg\psi \wedge \phi)$ .*

This deduction is of course irrelevant and forbids the use of such axiom.

**Proof.** It is assumed  $\not\vdash \neg(\psi \wedge \phi)$  and  $\not\vdash \neg(\neg\psi \wedge \phi)$ , which implies  $\not\vdash \neg\phi$ ,  $\not\vdash \psi$  and  $\not\vdash \neg\psi$ .

From the hypothesized axiom, it follows:

$$((\psi|\phi)|\psi) \equiv (\psi|\psi \wedge \phi) \equiv (\psi|\psi \wedge \phi) \wedge (\psi \wedge \phi|\psi \wedge \phi) \equiv (\psi \wedge \phi|\psi \wedge \phi) \equiv \top$$

and

$$((\psi|\phi)|\neg\psi) \equiv (\psi|\neg\psi \wedge \phi) \equiv (\psi|\neg\psi \wedge \phi) \wedge (\neg\psi \wedge \phi|\neg\psi \wedge \phi) \equiv (\perp|\neg\psi \wedge \phi) \equiv \perp.$$

It is deduced then:

$$\begin{aligned} (\psi|\phi) &\equiv ((\psi|\phi) \wedge \psi) \vee ((\psi|\phi) \wedge \neg\psi) \\ &\equiv (((\psi|\phi)|\psi) \wedge \psi) \vee (((\psi|\phi)|\neg\psi) \wedge \neg\psi) \equiv (\top \wedge \psi) \vee (\perp \wedge \neg\psi) \equiv \psi. \end{aligned}$$

□□□

However, it is proven without any additional axiom:

$$((\eta|\psi)|\phi) \wedge (\phi \wedge \psi) \equiv (\eta|\phi \wedge \psi) \wedge (\phi \wedge \psi).$$

**Proof.**

$$((\eta|\psi)|\phi) \wedge \phi \wedge \psi \equiv (\eta|\psi) \wedge \phi \wedge \psi \equiv \phi \wedge \psi \wedge \eta \equiv (\eta|\phi \wedge \psi) \wedge (\phi \wedge \psi).$$

□□□

### 3.10 Independence and proof

Assume  $\psi \times \neg\phi$  and  $\vdash \phi \vee \psi$ . Then  $\vdash \phi$  OR  $\vdash \psi$ .

Interpretation: independence implies links between disjunction proofs and proof disjunctions.

**Proof.** Assume  $\psi \times \neg\phi$  and  $\vdash \phi \vee \psi$ .

Then  $\psi \equiv (\psi|\neg\phi)$  and  $\vdash \neg\phi \rightarrow \psi$ .

From axiom b1, it is deduced  $\vdash \phi$  OR  $\vdash (\psi|\neg\phi)$ .

Then  $\vdash \phi$  OR  $\vdash \psi$ .

□□□

### 3.11 Independence and regularity

$\vdash (\phi \wedge \eta) \rightarrow (\psi \wedge \eta)$ ,  $\phi \times \eta$  and  $\psi \times \eta$  imply  $\vdash \neg\eta$  OR  $\vdash \phi \rightarrow \psi$ .

Interpretion: unless it is empty, a proposition may be removed from a logical equation, when it appears in the both sides and is (co-)independent with the equation components.

**Proof.** From now on, assume  $\not\vdash \neg\eta$ .

Assume  $\vdash (\phi \wedge \eta) \rightarrow (\psi \wedge \eta)$ .

Then  $\vdash \neg\phi \vee \neg\eta \vee (\psi \wedge \eta)$  and follows  $\vdash \neg\phi \vee ((\neg\eta \vee \psi) \wedge (\neg\eta \vee \eta))$ .

Then  $\vdash \neg\phi \vee (\neg\eta \vee \psi)$  and follows  $\vdash \eta \rightarrow (\phi \rightarrow \psi)$ .

By applying axiom b1, it is deduced  $\vdash (\phi \rightarrow \psi|\eta)$ .

Now, since  $\phi \times \eta$  and  $\psi \times \eta$ , it follows from the independence invariance that  $(\phi \rightarrow \psi) \times \eta$ , which means  $(\phi \rightarrow \psi|\eta) \equiv \phi \rightarrow \psi$ .

□□□

*Corollary.*  $\phi \times \eta$ ,  $\psi \times \eta$  and  $\phi \wedge \eta \equiv \psi \wedge \eta$  implies  $\vdash \neg\eta$  OR  $\phi \equiv \psi$ .

### 3.12 Consequences of the independence symmetry

In this subsection, *the optional axiom b.6 is hypothesized.*

**Evaluating**  $(\cdot|\perp)$ .  $\not\vdash \neg\phi$  OR  $(\psi|\phi) \equiv \psi$ .

In particular,  $(\psi|\perp) \equiv \psi$ .

**Proof.** Assume first  $\not\vdash \neg\psi$ .

Then  $\vdash \neg\phi$  implies  $(\neg\phi|\psi) \equiv \top \equiv \neg\phi$ .

Applying b.4, it comes  $(\phi|\psi) \equiv \phi$ .

From b.6, it is deduced  $(\psi|\phi) \equiv \psi$ .

Case  $\vdash \neg\psi$ .

Then  $(\phi|\neg\psi) \equiv \phi$  and  $(\neg\psi|\phi) \equiv \neg\psi$ .

Then  $(\psi|\phi) \equiv \psi$ .

□□□

**Reduction rule.** The axiom b.6 implies  $(\phi|(\psi|\phi)) \equiv \phi$ .

**Proof.** From  $\vdash \neg\phi$  OR  $(\psi|\phi) \times \phi$  and b.6, it comes  $\vdash \neg\phi$  OR  $\phi \times (\psi|\phi)$ .

Then  $\vdash \neg\phi$  OR  $(\phi|(\psi|\phi)) \equiv \phi$ .

The previous results have shown that  $\phi \equiv \perp$  implies  $(\phi|\eta) \equiv \perp$  for any  $\eta$ ; it is thus deduced  $(\phi|(\psi|\phi)) \equiv \phi$ .

□□□

**Independence invariance revised.**

Assume  $\psi \times \phi$  and  $\eta \times \phi$ . Then  $\neg\psi \times \phi$  and  $\psi \wedge \eta \times \phi$ .

Assume  $\psi \times \phi$  and  $\eta \equiv \psi$ . Then  $\eta \times \phi$ .

Assume  $\phi \times \psi$  and  $\phi \times \eta$ . Then  $\phi \times \neg\psi$  and  $\phi \times \psi \wedge \eta$ .

Assume  $\phi \times \psi$  and  $\eta \equiv \psi$ . Then  $\phi \times \eta$ .

**Proof.** The two last propositions are obtained from the two first, by using the symmetry.

The two first propositions are obtained from the original property *independence invariance* by removing the condition  $\not\vdash \neg\phi$ .

It is thus necessary to prove that this condition could be removed.

This is obvious: it has been proven from b.6 that  $\vdash \neg\phi$  implies  $\psi \times \phi$  for any  $\psi$  (paragraph *evaluating* ( $\cdot|\perp$ )).

□□□

**Independence and proof 2.** Assume  $\psi \times \phi$  and  $\vdash \phi \vee \psi$ . Then  $\vdash \phi$  OR  $\vdash \psi$ .

**Proof.** From  $\psi \times \phi$ , it is deduced  $\psi \times \neg\phi$ .

Applying the property 3.10 concludes the proof.

□□□

**Markov Property.** Assume  $\forall t \geq 3, \forall \tau \in \{1, \dots, t-2\}, (\phi_t|\phi_{t-1}) \times \phi_\tau$ . Then:

$$\forall t \geq 2, (\phi_t|\phi_{t-1}) \equiv \left( \phi_t \left| \bigwedge_{\tau=1}^{t-1} \phi_\tau \right. \right).$$

**Proof.** Assume  $\forall \tau \in \{1, \dots, t-2\}, (\phi_t|\phi_{t-1}) \times \phi_\tau$ .

It is known that  $(\phi_t|\phi_{t-1}) \times \phi_{t-1}$  and it is thus deduced  $(\phi_t|\phi_{t-1}) \times \bigwedge_{\tau=1}^{t-1} \phi_\tau$ .

Now,  $(\phi_t|\phi_{t-1}) \wedge \bigwedge_{\tau=1}^{t-1} \phi_\tau \equiv \bigwedge_{\tau=1}^t \phi_\tau \equiv \left( \phi_t \left| \bigwedge_{\tau=1}^{t-1} \phi_\tau \right. \right) \wedge \bigwedge_{\tau=1}^{t-1} \phi_\tau$ .

Since  $\left( \phi_t \left| \bigwedge_{\tau=1}^{t-1} \phi_\tau \right. \right) \times \bigwedge_{\tau=1}^{t-1} \phi_\tau$ , the proof is achieved by applying the regularity.

□□□

## 4 Probability over propositions: a minimalist<sup>†</sup> definition

<sup>†</sup> This definition is related to finite probabilities and excludes any Bayesian consideration.

Probabilities are classically defined over measurable sets. However, this is only a manner to modelize the notion of probability, which is essentially a measure of the belief of logical propositions [10]. Probability could be defined without reference to the measure theory, at least when the number of propositions is finite. The notion of probability is explained now within a strict propositional formalism. Conditional probabilities are excluded from this definition, but the notion of independence is considered.

Intuitively, a probability over a set of logical propositions is a measure of belief which is additive (disjoint propositions are adding their chances) and increasing with the propositions. This measure should be zeroed for the ever-false propositions and full for the ever-true propositions. Moreover, *a probability is a multiplicative measure for independent propositions*. The independence of propositions is a meta-relation between propositions, which generally depends on the problem setting. These intuitions are now formalized.

## 4.1 Definition

Let  $L$  be at least an extension of the classical logic of propositions; in particular,  $L$  contains the operators  $\wedge, \vee, \neg$  (and, or, negation) and the propositions  $\perp, \top$  (always false, always true). Assume moreover that some propositions pairs of  $L$  are *recognized as independent propositions* (this is a meta-relation not necessarily related to the logic itself).

A probability  $p$  over  $L$  is a  $\mathbb{R}^+$  valued function such that for any proposition  $\phi$  and  $\psi$  of  $L$ :

- $\phi \equiv \psi$  implies  $p(\phi) = p(\psi)$  (equivalent propositions have the same probability),

**Additivity.**  $p(\phi \wedge \psi) + p(\phi \vee \psi) = p(\phi) + p(\psi)$ ,

**Coherence.**  $p(\perp) = 0$ ,

**Finiteness.**  $p(\top) = 1$ ,

**Multiplicativity.** When  $\phi$  and  $\psi$  are *independent* propositions, then  $p(\phi \wedge \psi) = p(\phi)p(\psi)$ .

When a probability  $p$  is defined over the logic  $L$ , this logic is said to be *probabilized*.

## 4.2 Property

The coherence and additivity implies the increaseness of  $p$ :

**Increaseness.**  $p(\phi \wedge \psi) \leq p(\phi)$ .

**Proof.** Since  $\phi \equiv (\phi \wedge \psi) \vee (\phi \wedge \neg \psi)$  and  $(\phi \wedge \psi) \wedge (\phi \wedge \neg \psi) \equiv \perp$ , it follows from the additivity:

$$p(\phi) + p(\perp) = p(\phi \wedge \psi) + p(\phi \wedge \neg \psi).$$

From the coherence  $p(\perp) = 0$ , it is deduced  $p(\phi) = p(\phi \wedge \psi) + p(\phi \wedge \neg \psi)$ .

Since  $p$  is non negatively valued,  $p(\phi) \geq p(\phi \wedge \psi)$ .

□□□

## 5 Probabilizing DBL

**Probability over DBL.** Any probability  $P$  over DBL is defined according to the definition of section 4. In particular, since the meta-operator  $\times$  characterizes an independence between propositions, it is naturally hypothesized that:

$$\phi \times \psi \text{ implies } P(\phi \wedge \psi) = P(\phi)P(\psi).$$

**Free DBL is probabilizable.** Let  $p$  be a probability defined over the Boolean algebra generated by  $\Theta$ . Then, there is a probability  $P$  (at least one) defined over the free DBL, such that  $P(\phi) = p(\phi)$  for any proposition  $\phi$  which is classical.

**Proof.** Refer to appendix C. At this time, the proof does not hold for the free DBL\*.

□□□

*Corollary (strong coherence).* Let  $\phi$  be a classical proposition. Assume that  $\vdash \phi$  OR  $\vdash \neg \phi$  is sequenced in the free DBL. Then either  $\phi$  or  $\neg \phi$  is proven in classical logic. Consequently, either  $\vdash \phi$  or  $\vdash \neg \phi$  is sequenced in the free DBL.

**Proof.** Assume that neither  $\phi$  nor  $\neg \phi$  is proven in classical logic.

Then it is possible to construct a probability  $p$  of the Boolean algebra such that  $0 < p(\phi) < 1$ .

Then it is possible to construct a probability  $P$  of the free DBL such that  $0 < P(\phi) < 1$ . But the hypothesis  $\vdash \phi$  OR  $\vdash \neg \phi$  implies  $P(\phi) = 0$  OR  $P(\phi) = 1$ , which is contradictory.

□□□

**Bayes inference.** Assume a probability  $P$  defined over DBL. Define  $P(\psi|\phi)$  as an abbreviation for  $P((\psi|\phi))$ . Then:

$$P(\psi|\phi)P(\phi) = P(\phi \wedge \psi) .$$

**Proof.** Assume first  $\not\vdash \neg\phi$ .

The result is implied by the theorems  $(\psi|\phi) \wedge \phi \equiv \phi \wedge \psi$  and  $\vdash \neg\phi \text{ OR } (\psi|\phi) \times \phi$ .

Assume now  $\vdash \neg\phi$ , *ie.*  $\phi \equiv \perp$ .

Then  $\phi \wedge \psi \equiv \perp$  and  $P(\phi) = P(\phi \wedge \psi) = 0$ .

□□□

**About Lewis' negative result.** This argument is often used as a refutation of the propositional nature of the Bayesian relation. Since we have shown that DBL is probabilizable, it is interesting to see why Lewis' result does not work here. Lewis' argument stands as follows:

$$\begin{aligned} \frac{P(\phi \wedge \psi)}{P(\phi)} &= P(\psi|\phi) = P((\psi|\phi)|\psi)P(\psi) + P((\psi|\phi)|\neg\psi)P(\neg\psi) = \\ &P(\psi|\psi \wedge \phi)P(\psi) + P(\psi|\neg\psi \wedge \phi)P(\neg\psi) = 1 \times P(\psi) + 0 \times P(\neg\psi) = P(\psi) . \end{aligned}$$

The problem here comes from the justification of  $P((\eta|\psi)|\phi) = P(\eta|\phi \wedge \psi)$ . This equality is not derived from the rules of DBL; in particular, the theorem  $((\eta|\psi)|\phi) \equiv (\eta|\phi \wedge \psi)$  is not implied by DBL. That is why DBL is probabilizable.

Now, there is another possible "justification" [15] by writing:

$$P((\eta|\psi)|\phi) = \frac{P_\phi(\psi \wedge \eta)}{P_\phi(\psi)} = \frac{\frac{P(\phi \wedge \psi \wedge \eta)}{P(\phi)}}{\frac{P(\psi \wedge \phi)}{P(\phi)}} = \frac{P(\phi \wedge \psi \wedge \eta)}{P(\psi \wedge \phi)} = P(\eta|\phi \wedge \psi) ,$$

where  $P_\phi$  is the probabilistic law defined by  $P_\phi(\psi) = P(\phi \wedge \psi)/P(\phi)$ . However, the use of  $P_\phi$  is questionable. It is noteworthy that  $P_\phi$  is not exactly the same as  $P(\cdot|\phi)$ , since the first is an unconditional probability while the latter is  $P$  with the conditional information  $|\phi$ . In particular,  $P((\eta|\psi)|\phi) = \frac{P_\phi(\psi \wedge \eta)}{P_\phi(\psi)}$  cannot be justified by a construction of the conditional probability from DBL.

**Logical independence and probabilistic independence.** This paragraph is dedicated to illustrating that the logical independence  $\times$  is not exactly the same as the probabilistic independence. In fact, the logical independence is a property *stronger* than the probabilistic independence. Logically, the independence  $\psi \times \phi$  is defined as  $(\psi|\phi) \equiv \psi$ . It is independent of the choice of a probability. In the probabilist paradigm,  $\phi$  and  $\psi$  are said to be independent when  $P(\psi|\phi) = P(\psi)$ . It is dependent of  $P$ . In particular, the condition  $\sigma \equiv \psi$  implies  $P(\sigma) = P(\psi)$  but the converse is false. The independence is a special case of this dissymmetry, where  $\sigma = (\psi|\phi)$ .

The regularity property is a particular illustration of this difference:

$$\vdash (\phi \wedge \eta) \rightarrow (\psi \wedge \eta) , \phi \times \eta \text{ and } \psi \times \eta \text{ imply } \vdash \neg\eta \text{ OR } \vdash \phi \rightarrow \psi .$$

In regards to the probabilistic independence, this property looks like a strange thing. Considering  $\phi, \psi, \eta$  as measurable sets, the probabilistic independences  $p(\phi \cap \eta) = p(\phi)p(\eta)$ ,  $p(\psi \cap \eta) = p(\psi)p(\eta)$  and the inclusion  $\phi \cap \eta \subset \psi \cap \eta$  do not yield the inclusion  $\phi \subset \psi$  in general, even when  $\eta \neq \emptyset$ .

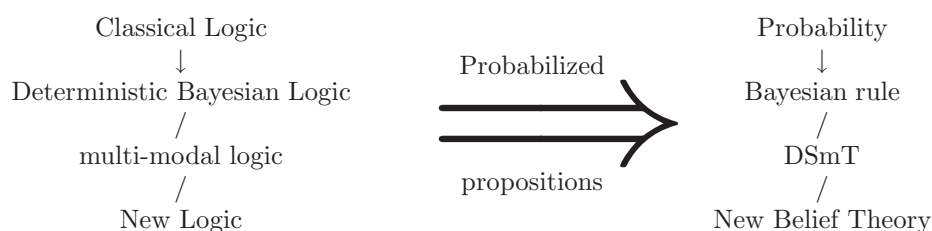
**Conclusion.** Finally, the probabilistic Bayesian inference has been recovered from DBL. At last, it has been shown that Probability and Bayesian inference could be defined separately. Somehow, the Bayesian inference appears here as a purely logical interpretation of the knowledge. It is not the only logical interpretation of the knowledge though; refer to [3].



## 6 Conclusion

In this contribution, a deterministic Bayesian logic has been defined. This logic has been proven to be coherent and probabilizable. It has been deduced several theorems of the logic and the probabilistic Bayesian rule has been recovered by probabilizing this logic. At last, it seems that the Bayesian reasoning may be considered separated from the probabilistic framework.

This is not our first result in such questions. In a previous work [3], it has also been proved that the DSmT, another non deterministic Theory, was also recovered by probabilizing a logic. At last, it seems possible to handle probabilized logics as a relatively general framework for manipulating non deterministic informations. The principle is first to construct a logic well adapted to the problem, second to probabilize this logic, and third to derive an implied new belief theory:



From a strictly logical viewpoint, the Deterministic Bayesian Logic has also some interesting properties. For example, the notion of independence in DBL establishes a link between the *proof of disjunction* and the *disjunction of proof*.

There are still many open questions. In particular, a general model for DBL has not been constructed yet (although a partial model of the free DBL has been constructed for the proof in appendix C). The author is also interested in combining DBL with modal logics, in order to obtain a common framework to the Bayesian and the DSm Theories.

## References

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## A Modal logics

This section is far from an introduction to Modal logics [19]. The axioms and rules of the “ $T$ -system”, a particular modal logic, are just recalled. The reader should notice some similarities with the modal-like axioms of DBL.

The  $T$ -system is obtained by adding the modal operators  $\Box$  and  $\Diamond$  to the classical logic. The proposition  $\Box\phi$  will mean that *the proposition  $\phi$  is true for sure*. The proposition  $\Diamond\phi$  will mean that *the proposition  $\phi$  is possibly true*. The modalities  $\Box$  and  $\Diamond$  could be seen as a *viewpoint*, a partial knowledge, about the world. The axioms and rules of the  $T$ -system are the classical axioms, the Modus Ponens and the following modal axioms and rule:

- i. From  $\vdash \phi$  is deduced  $\vdash \Box\phi$ ; *theorems are sure*,
- ii.  $\vdash \Box(\phi \rightarrow \psi) \rightarrow (\Box\phi \rightarrow \Box\psi)$ ; *the modus ponens works within the modality*,
- iii.  $\vdash \Box\phi \rightarrow \phi$ ; *sure propositions are true*,
- iv.  $\Diamond\phi \equiv \neg\Box\neg\phi$ ; *is unsure what cannot be false for sure*.

Axioms i and ii tell that the viewpoint is expressed by a “good logician”. Axiom iii tells that this “logician” manipulates true sentences. Notice the similarity of i, ii and iii with the DBL axioms/rule b1, b2 and b3.

## B The proof of coherence continued

Assume that the function  $f : DBL \rightarrow \{0, 1\}$  is constructed recursively by:

- $f(\perp) = 0$ ,  $f(\top) = 1$  and  $\forall i \in I, f(\theta_i) \in \{0, 1\}$ ,
- $f(\neg\phi) = 1 - f(\phi)$ ,
- $f(\phi \wedge \psi) = f(\phi)f(\psi)$  and  $f(\phi \vee \psi) = f(\phi) + f(\psi) - f(\phi)f(\psi)$ ,
- $f(\phi \rightarrow \psi) = 1 - f(\phi) + f(\phi)f(\psi)$ ,
- $f((\psi|\phi)) = f(\psi)$ .

The following properties are proven:

- $f(\top) = 1$ .

**Proof.** Definition.

□□□

- $f(\phi \rightarrow (\psi \rightarrow \phi)) = 1$ .

**Proof.**

$$\begin{aligned} f(\phi \rightarrow (\psi \rightarrow \phi)) &= 1 - f(\phi) + f(\phi)f(\psi \rightarrow \phi) = 1 - f(\phi) + f(\phi)(1 - f(\psi) + f(\psi)f(\phi)) \\ &= 1 - f(\phi) + f(\phi) - f(\phi)f(\psi) + f(\psi)f(\phi) = 1. \end{aligned}$$

□□□

- $f((\eta \rightarrow (\phi \rightarrow \psi)) \rightarrow ((\eta \rightarrow \phi) \rightarrow (\eta \rightarrow \psi))) = 1$ .

**Proof.**

$$\begin{aligned} f((\eta \rightarrow (\phi \rightarrow \psi)) \rightarrow ((\eta \rightarrow \phi) \rightarrow (\eta \rightarrow \psi))) &= 1 - f(\eta \rightarrow (\phi \rightarrow \psi)) \\ &\quad + f(\eta \rightarrow (\phi \rightarrow \psi))f((\eta \rightarrow \phi) \rightarrow (\eta \rightarrow \psi)) \\ &= f(\eta) - f(\eta)f(\phi \rightarrow \psi) \\ &\quad + (1 - f(\eta) + f(\eta)f(\phi \rightarrow \psi)) \times (1 - f(\eta \rightarrow \phi) + f(\eta \rightarrow \phi)f(\eta \rightarrow \psi)) \\ &= f(\eta) - f(\eta)(1 - f(\phi) + f(\phi)f(\psi)) \\ &\quad + (1 - f(\eta) + f(\eta)(1 - f(\phi) + f(\phi)f(\psi))) \\ &\quad \times (f(\eta) - f(\eta)f(\phi) + (1 - f(\eta) + f(\eta)f(\phi))(1 - f(\eta) + f(\eta)f(\psi))) \\ &= f(\eta)f(\phi) - f(\eta)f(\phi)f(\psi) + (1 - f(\eta)f(\phi) + f(\eta)f(\phi)f(\psi))^2 \\ &= f(\eta)f(\phi) - f(\eta)f(\phi)f(\psi) + 1 - f(\eta)f(\phi) + f(\eta)f(\phi)f(\psi) = 1. \end{aligned}$$

□□□

- $f((\neg\phi \rightarrow \neg\psi) \rightarrow ((\neg\phi \rightarrow \psi) \rightarrow \phi)) = 1$ .

**Proof.**

$$\begin{aligned} f((\neg\phi \rightarrow \neg\psi) \rightarrow ((\neg\phi \rightarrow \psi) \rightarrow \phi)) &= 1 - f(\neg\phi \rightarrow \neg\psi) \\ &\quad + f(\neg\phi \rightarrow \neg\psi)f((\neg\phi \rightarrow \psi) \rightarrow \phi) \\ &= f(\neg\phi) - f(\neg\phi)f(\neg\psi) \\ &\quad + (1 - f(\neg\phi) + f(\neg\phi)f(\neg\psi)) \times (1 - f(\neg\phi \rightarrow \psi) + f(\neg\phi \rightarrow \psi)f(\phi)) \\ &= f(\psi) - f(\phi)f(\psi) + (1 - f(\psi) + f(\phi)f(\psi)) \\ &\quad \times (f(\neg\phi) - f(\neg\phi)f(\psi) + f(\phi)(1 - f(\neg\phi) + f(\neg\phi)f(\psi))) \\ &= f(\psi) - f(\phi)f(\psi) + (1 - f(\psi) + f(\phi)f(\psi)) \\ &\quad \times (1 - f(\phi) - f(\psi) + f(\phi)f(\psi) + f(\phi)(f(\phi) + f(\psi) - f(\phi)f(\psi))) \\ &= f(\psi) - f(\phi)f(\psi) + (1 - f(\psi) + f(\phi)f(\psi)) \times (1 - f(\psi) + f(\phi)f(\psi)) \\ &= f(\psi) - f(\phi)f(\psi) + 1 - f(\psi) + f(\phi)f(\psi) = 1. \end{aligned}$$

□□□

- $f(\perp \rightarrow \neg\top) = 1$ .

**Proof.**

$$f(\perp \rightarrow \neg\top) = 1 - f(\perp) + f(\perp)f(\neg\top) = 1.$$

□□□

- $f(\neg\top \rightarrow \perp) = 1$ .

**Proof.**

$$f(\neg\top \rightarrow \perp) = 1 - f(\neg\top) + f(\neg\top)f(\perp) = f(\top) = 1$$

□□□

- $f((\phi \rightarrow \psi) \rightarrow (\neg\phi \vee \psi)) = 1$ .

**Proof.**

$$\begin{aligned} f((\phi \rightarrow \psi) \rightarrow (\neg\phi \vee \psi)) &= 1 - f(\phi \rightarrow \psi) + f(\phi \rightarrow \psi)f(\neg\phi \vee \psi) \\ &= f(\phi) - f(\phi)f(\psi) + (1 - f(\phi) + f(\phi)f(\psi)) \times (f(\neg\phi) + f(\psi) - f(\neg\phi)f(\psi)) \\ &= f(\phi) - f(\phi)f(\psi) + (1 - f(\phi) + f(\phi)f(\psi)) \times (1 - f(\phi) + f(\psi) - f(\psi)(1 - f(\phi))) \\ &= f(\phi) - f(\phi)f(\psi) + (1 - f(\phi) + f(\phi)f(\psi)) \times (1 - f(\phi) + f(\phi)f(\psi)) \\ &= f(\phi) - f(\phi)f(\psi) + 1 - f(\phi) + f(\phi)f(\psi) = 1. \end{aligned}$$

□□□

- $f((\neg\phi \vee \psi) \rightarrow (\phi \rightarrow \psi)) = 1$ .

**Proof.**

$$\begin{aligned} f((\neg\phi \vee \psi) \rightarrow (\phi \rightarrow \psi)) &= 1 - f(\neg\phi \vee \psi) + f(\neg\phi \vee \psi)f(\phi \rightarrow \psi) \\ &= 1 - f(\neg\phi) - f(\psi) + f(\neg\phi)f(\psi) \\ &\quad + (f(\neg\phi) + f(\psi) - f(\neg\phi)f(\psi)) \times (1 - f(\phi) + f(\phi)f(\psi)) \\ &= f(\phi) - f(\phi)f(\psi) + (1 - f(\phi) + f(\phi)f(\psi)) \times (1 - f(\phi) + f(\phi)f(\psi)) \\ &= f(\phi) - f(\phi)f(\psi) + 1 - f(\phi) + f(\phi)f(\psi) = 1. \end{aligned}$$

□□□

- $f((\phi \wedge \psi) \rightarrow \neg(\neg\phi \vee \neg\psi)) = 1$ .

**Proof.**

$$\begin{aligned} f((\phi \wedge \psi) \rightarrow \neg(\neg\phi \vee \neg\psi)) &= 1 - f(\phi \wedge \psi) + f(\phi \wedge \psi)f(\neg(\neg\phi \vee \neg\psi)) \\ &= 1 - f(\phi)f(\psi) + f(\phi)f(\psi) \times (1 - f(\neg\phi \vee \neg\psi)) = 1 - f(\phi)f(\psi)f(\neg\phi \vee \neg\psi) \\ &= 1 - f(\phi)f(\psi)(f(\neg\phi) + f(\neg\psi) - f(\neg\phi)f(\neg\psi)) \\ &= 1 - f(\phi)f(\psi)(2 - f(\phi) - f(\psi) - (1 - f(\phi))(1 - f(\psi))) \\ &= 1 - f(\phi)f(\psi)(1 - f(\phi)f(\psi)) = 1. \end{aligned}$$

□□□

- $f(\neg(\neg\phi \vee \neg\psi) \rightarrow (\phi \wedge \psi)) = 1$ .

**Proof.**

$$\begin{aligned} f(\neg(\neg\phi \vee \neg\psi) \rightarrow (\phi \wedge \psi)) &= 1 - f(\neg(\neg\phi \vee \neg\psi)) + f(\neg(\neg\phi \vee \neg\psi))f(\phi \wedge \psi) \\ &= f(\neg\phi \vee \neg\psi) + (1 - f(\neg\phi \vee \neg\psi))f(\phi)f(\psi) \\ &= f(\neg\phi) + f(\neg\psi) - f(\neg\phi)f(\neg\psi) + (1 - f(\neg\phi) - f(\neg\psi) + f(\neg\phi)f(\neg\psi))f(\phi)f(\psi) \\ &= 1 - f(\phi)f(\psi) + (f(\phi)f(\psi))^2 = 1. \end{aligned}$$

□□□

- $f(\phi) = 1$  and  $f(\phi \rightarrow \psi) = 1$  implies  $f(\psi) = 1$ .

**Proof.** Assume  $f(\phi) = 1$  and  $1 - f(\phi) + f(\phi)f(\psi) = 1$ .

Then  $1 - 1 + f(\psi) = 1$  and finally  $f(\psi) = 1$ .

□□□

- $f(\phi \rightarrow \psi) = 1$  implies  $f(\neg\phi) = 1$  OR  $f((\psi|\phi)) = 1$ .

**Proof.** Assume  $1 - f(\phi) + f(\phi)f(\psi) = 1$ .

Then  $f(\phi) = f(\phi)f(\psi)$ , so that  $f(\phi) = 0$  or  $f(\psi) = 1$ .

Now  $f((\psi|\phi)) = f(\psi)$  and  $f(\neg\phi) = 1 - f(\phi)$ .

□□□

- $f((\psi \rightarrow \eta|\phi) \rightarrow ((\psi|\phi) \rightarrow (\eta|\phi))) = 1$ .

**Proof.**

$$\begin{aligned} f((\psi \rightarrow \eta|\phi) \rightarrow ((\psi|\phi) \rightarrow (\eta|\phi))) &= 1 - f((\psi \rightarrow \eta|\phi)) + f((\psi \rightarrow \eta|\phi))f((\psi|\phi) \rightarrow (\eta|\phi)) \\ &= 1 - f(\psi \rightarrow \eta) + f(\psi \rightarrow \eta)(1 - f(\psi|\phi) + f(\psi|\phi)f(\eta|\phi)) \\ &= 1 - f(\psi \rightarrow \eta) + f(\psi \rightarrow \eta)(1 - f(\psi) + f(\psi)f(\eta)) \\ &= 1 - f(\psi \rightarrow \eta) + f(\psi \rightarrow \eta)f(\psi \rightarrow \eta) = 1. \end{aligned}$$

□□□

- $f((\psi|\phi) \rightarrow (\phi \rightarrow \psi)) = 1$ .

**Proof.**

$$\begin{aligned} f((\psi|\phi) \rightarrow (\phi \rightarrow \psi)) &= 1 - f((\psi|\phi)) + f((\psi|\phi))f(\phi \rightarrow \psi) \\ &= 1 - f(\psi) + f(\psi)f(\phi \rightarrow \psi) = 1 - f(\psi) + f(\psi)(1 - f(\phi) + f(\phi)f(\psi)) = 1. \end{aligned}$$

□□□

- $f(\neg(\neg\psi|\phi) \rightarrow (\psi|\phi)) = 1$ .

**Proof.**

$$\begin{aligned} f(\neg(\neg\psi|\phi) \rightarrow (\psi|\phi)) &= 1 - f(\neg(\neg\psi|\phi)) + f(\neg(\neg\psi|\phi))f((\psi|\phi)) \\ &= f((\neg\psi|\phi)) + (1 - f((\neg\psi|\phi)))f(\psi) = f(\neg\psi) + (1 - f(\neg\psi))f(\psi) \\ &= 1 - f(\psi) + f(\psi)f(\psi) = 1. \end{aligned}$$

□□□

- $f((\psi|\phi) \rightarrow \neg(\neg\psi|\phi)) = 1$ .

**Proof.**

$$\begin{aligned} f((\psi|\phi) \rightarrow \neg(\neg\psi|\phi)) &= 1 - f((\psi|\phi)) + f((\psi|\phi))f(\neg(\neg\psi|\phi)) \\ &= 1 - f(\psi) + f(\psi)(1 - f((\neg\psi|\phi))) = 1 - f(\psi) + f(\psi)(1 - f(\neg\psi)) \\ &= 1 - f(\psi) + f(\psi)f(\psi) = 1. \end{aligned}$$

□□□

- $f((\psi|\phi) \rightarrow \psi) = f(\psi \rightarrow (\psi|\phi)) = 1$  implies  $f((\phi|\psi) \rightarrow \phi) = f(\phi \rightarrow (\phi|\psi)) = 1$ .

**Proof.**

$$f((\phi|\psi) \rightarrow \phi) = 1 - f((\phi|\psi)) + f((\phi|\psi))f(\phi) = 1 - f(\phi) + f(\phi)^2 = 1$$

and

$$f(\phi \rightarrow (\phi|\psi)) = 1 - f(\phi) + f(\phi)f((\phi|\psi)) = 1 - f(\phi) + f(\phi)^2 = 1.$$

□□□

## C Free DBL is probabilizable (proof)

The proof is an increasing construction of the operator  $(\cdot|\cdot)$  and of the probability  $P$ , from a pre-existing Boolean algebra and its associated probability  $p$ . It results in the definition of a sequence of injective morphisms, which for simplicity will be described as a sequence of set inclusions.

It is recalled that  $\Theta$ , *ie.*  $I$ , is a finite set. Let  $\mathcal{B}$  be the free Boolean algebra generated by  $\Theta$ . It is now constructed a sequence  $(\Omega_n, L_n, (\cdot|\cdot)_n, P_n, \Lambda_n)_{n \in \mathbb{N}}$ , where:

- $L_n|_{n \in \mathbb{N}}$  is an increasing sequence (*ie.*  $L_n \subset L_{n+1}$ ) of boolean algebra such that  $L_0 = \mathcal{B}$ ,
- $\Omega_n$  is a set of disjoint propositions generating  $L_n$ ,
- $(\cdot|\cdot)_n$  is an increasing construction of the  $(\cdot|\cdot)$  operator (*ie.*  $(\psi|\phi)_n$  is defined implies  $(\psi|\phi)_{n+1}$  is defined by  $(\psi|\phi)_{n+1} = (\psi|\phi)_n$ ),
- $P_n$  is a probability over  $L_n$  increasing with  $n$  (*ie.*  $\forall \phi \in L_n, P_{n+1}(\phi) = P_n(\phi)$ ) and such that  $P_0 = p$ ,
- $\Lambda_n$  is a list of propositions describing where the operator  $(\cdot|\cdot)$  should be constructed next. This list is ordered so as to guarantee the construction of any  $(\psi|\phi)$ .

Moreover, it is verified at each step of the construction that the axioms of the free DBL (without the optional axiom) apply, when defined. It is also shown that  $P_n$  has the multiplicative property.

From this increasing construction will be deduced a probabilized model for the free DBL. Then it is deduced that the free DBL is probabilizable.

### C.1 Step 0

**Constructions.** Set  $L_0 = \mathcal{B}$  and define  $\Omega_0 = \{ \bigwedge_{i \in I} \tau_i / \forall i \in I, \tau_i \in \{\theta_i, -\theta_i\} \}$ . Then  $\Omega_0$  is a set of disjoint propositions generating  $L_0$ ; *ie.*  $\forall \omega, \omega' \in \Omega_0, \omega \neq \omega' \Rightarrow \omega \wedge \omega' = \perp$  and  $\bigvee_{\omega \in \Omega_0} \omega = \top$ . In particular:

$$L_0 = \left\{ \bigvee_{\omega \in \Sigma} \omega / \Sigma \subset \Omega_0 \right\}.$$

The probability  $P_0$  is defined by  $P_0 = p$ .

The operator  $(\cdot|\cdot)$  is partially constructed over  $L_0$  by setting:

$$\forall \phi \in L_0, (\phi|\perp)_0 = (\phi|\top)_0 = \phi.$$

The list  $\Lambda_0 = (s_0, f_0, \lambda_0)$  is defined by:

$$s_0 = 0; \quad f_0 = \text{card}(L_0) - 2; \quad \lambda_0 : \llbracket s_0, f_0 - 1 \rrbracket \longrightarrow L_0 \setminus \{\perp, \top\},$$

where  $\lambda_0$  is an onto mapping,  $s_0$  is the first index of the list and  $f_0 - 1$  is the last one. Since  $\{\lambda_0(t) / s_0 \leq t < f_0\} = L_0 \setminus \{\perp, \top\}$ , the list covers all the propositions for which  $(\cdot|\cdot)$  has not been constructed.

**Axioms work.** Define  $\phi \rightarrow \psi$  in the boolean algebra by  $\phi \rightarrow \psi = \neg\phi \vee \psi$ . Since  $L_0$  is a Boolean algebra, it is proven that:

- $\top = \top$ ,
- $\phi \rightarrow (\psi \rightarrow \phi) = \top$ ,

- $(\eta \rightarrow (\phi \rightarrow \psi)) \rightarrow ((\eta \rightarrow \phi) \rightarrow (\eta \rightarrow \psi)) = \top$ ,
- $(\neg\phi \rightarrow \neg\psi) \rightarrow ((\neg\phi \rightarrow \psi) \rightarrow \phi) = \top$ ,
- $\perp = \neg\top$ ,
- $\phi \rightarrow \psi = \neg\phi \vee \psi$  (definition),
- $\phi \wedge \psi = \neg(\neg\phi \vee \neg\psi)$ ,
- $\phi = \top$  and  $\phi \rightarrow \psi = \top$  implies  $\psi = \top$ .

These property are the exact analogues of the axiom c1-7 and modus ponens.

Now, similar results hold for the axiom b1-4, when  $(\cdot|\cdot)_0$  is defined:

- $\phi \rightarrow \psi = \top$  implies  $\neg\phi = \top$  or  $(\psi|\phi)_0 = \top$ ,

**Proof.** It is hypothesized that  $(\psi|\phi)_0$  is defined, which means  $\phi = \perp$  or  $\phi = \top$ .

Assume  $\phi \rightarrow \psi = \top$  and  $\neg\phi \neq \top$ .

Then  $\phi = \top$  and  $\psi = \phi \rightarrow \psi = \top$ .

At last  $(\psi|\phi)_0 = (\psi|\top)_0 = \psi = \top$ .

□□□

- $\neg(\neg\psi|\phi)_0 = (\psi|\phi)_0$ .

**Proof.** Hypothesizing  $\phi = \perp$  or  $\phi = \top$ , it comes  $(\psi|\phi)_0 = \psi$  and  $(\neg\psi|\phi)_0 = \neg\psi$ .

It is known that  $\neg\neg\psi = \psi$ .

□□□

- $(\psi \rightarrow \eta|\phi)_0 \rightarrow ((\psi|\phi)_0 \rightarrow (\eta|\phi)_0) = \top$ ,

**Proof.** It is still hypothesized  $\phi = \perp$  or  $\phi = \top$ .

Then  $(\psi \rightarrow \eta|\phi)_0 = \psi \rightarrow \eta$ ,  $(\psi|\phi)_0 = \psi$  and  $(\eta|\phi)_0 = \eta$ .

It is known that  $(\psi \rightarrow \eta) \rightarrow (\psi \rightarrow \eta) = \top$ .

□□□

- $(\psi|\phi)_0 \rightarrow (\phi \rightarrow \psi) = \top$ ,

**Proof.** Hypothesizing  $\phi = \perp$  or  $\phi = \top$ , it comes  $(\psi|\phi)_0 = \psi$ .

It is known that  $\psi \rightarrow (\phi \rightarrow \psi) = \top$ .

□□□

**Multiplicative property.** Assume that  $(\psi|\phi)_0$  is defined.

Then  $P_0((\psi|\phi)_0)P_0(\phi) = P_0(\phi \wedge \psi)$ .

proof. Since  $(\phi|\perp)_0 = (\phi|\top)_0 = \phi$ , it comes  $P_0((\phi|\perp)_0) = P_0((\phi|\top)_0) = P_0(\phi) = p(\phi)$ .

Then  $P_0((\phi|\perp)_0)P_0(\perp) = P_0(\perp \wedge \phi) = 0$  and  $P_0((\phi|\top)_0)P_0(\top) = P_0(\top \wedge \phi) = p(\phi)$ .

□□□

## C.2 From step $n$ to step $n + 1$

**Constructions.** Assume  $(\Omega_n, L_n, (\cdot|\cdot)_n, P_n, \Lambda_n)$  being defined. Let  $b_n = \lambda_n(s_n)$ , the next proposition on the list. If there is  $m < n$  such that  $b_m = b_n$  define  $\nu$  by:

$$\nu = \sup\{m \in \mathbb{N}/m < n \text{ and } b_m = b_n\}.$$

Then by construction,  $(\omega|b_n)_n$  is defined if and only if  $\omega \in \Omega_{\nu+1}$ .

Let  $B_n \subset \Omega_n$  be such that  $b_n = \bigvee_{\omega \in B_n} \omega$  and  $A_n = \Omega_n \setminus B_n$ .

The pair  $(L_{n+1}, (\cdot|\cdot)_{n+1})$  will be constructed so as to define  $(\cdot|b_n)_{n+1}$  over all currently

constructed propositions. Before constructing  $L_{n+1}$  and  $(\cdot|b_n)_{n+1}$  explicitly, the hypotheses necessary for this construction should be listed. In particular, it is necessary to take into account the previous construction of  $(\cdot|b_n)_n$ . The subsequent analysis shows how these hypotheses could be derived from DBL:

At step  $n + 1$ , the operator  $(\cdot|b_n)$  is applied over all propositions of  $L_n$ ; the new Boolean algebra  $L_{n+1}$  and the partially constructed operator  $(\cdot|b_n)_{n+1}$  should involve the resulting new propositions. Since we are interested in compatibility with DBL, it is just necessary to build  $(\omega|b_n)$  for  $\omega \in \Omega_n$  (the other propositions  $(\phi|b_n)$  should be obtained by applying the properties 3.2). Moreover, it is implied from DBL:

$$\forall \omega, \omega' \in \Omega_n, \omega \neq \omega' \implies (\omega|b_n) \wedge (\omega'|b_n) = \perp \quad \text{and} \quad \bigvee_{\omega \in B_n} (\omega|b_n) = \top .$$

In particular,  $(\omega|b_n) = \perp$  for any  $\omega \in A_n$ . Thus, *the only new ingredients are  $(\omega|b_n)$  where  $\omega \in B_n$ .*

At this stage, it is also necessary to check the intersections  $\omega \wedge (\omega'|b_n)$  where  $\omega \in \Omega_n$  and  $\omega' \in B_n$ . According to DBL,  $\omega \wedge (\omega|b_n) = \omega$  for any  $\omega \in B_n$  and  $\omega \wedge (\omega'|b_n) = \perp$  for any  $\omega, \omega' \in B_n$  such that  $\omega \neq \omega'$ .

The remaining unknown comes from the propositions  $\omega \wedge (\omega'|b_n)$  where  $\omega \in A_n$  and  $\omega' \in B_n$ . If it is the first construction of  $(\cdot|b_n)$  in the sequence, there is no way to reduce this unknown. But if there has been a previous partial construction of  $(\cdot|b_n)$ , one should derive from the step  $\nu + 1$  some additional constraints.

Let  $\omega \in B_\nu$ .

Then  $(\omega|b_n)_{\nu+1}$  and thus  $(\omega|b_n)_n$  are defined and  $(\omega|b_n)_n = (\omega|b_n)_{\nu+1} = (\omega|b_\nu)_{\nu+1}$ . Define the set  $C_\omega \subset \Omega_n$  such that  $(\omega|b_n)_n = \bigvee_{\omega' \in C_\omega} \omega'$ . Then  $(\omega|b_n)_n \wedge \omega' = \perp$  for any  $\omega' \in \Omega_n \setminus C_\omega = D_\omega$ . Now define the set  $E_\omega \subset B_n$  such that  $\omega = \bigvee_{\omega' \in E_\omega} \omega'$ . Then, it is implied from DBL  $\forall \omega' \in D_\omega, \forall \omega'' \in E_\omega, (\omega''|b_n) \wedge \omega' = \perp$ .

Notice that this constraint could contradict  $\forall \omega' \in B_n, \omega' \wedge (\omega'|b_n) = \omega'$ , previously derived. In order to avoid such contradiction, it is necessary to prove  $D_\omega \cap E_\omega = \emptyset$  or equivalently  $E_\omega \subset C_\omega$ .

Proof of  $E_\omega \subset C_\omega$ :

Since  $(\omega|b_n)_n \wedge \omega = (\omega|b_\nu)_{\nu+1} \wedge \omega = \omega$  (by construction), it is deduced  $(\bigvee_{\omega' \in C_\omega} \omega') \wedge (\bigvee_{\omega'' \in E_\omega} \omega'') = \bigvee_{\omega'' \in E_\omega} \omega''$ . Then  $E_\omega \subset C_\omega$ .

It is also proven  $C_\omega \subset A_n \cup E_\omega$  (this property will be useful):

Since  $(\omega|b_n)_n \wedge (\bigvee_{\omega' \in B_n \setminus E_\omega} \omega') = (\omega|b_\nu)_{\nu+1} \wedge (\bigvee_{\omega' \in B_\nu \setminus \{\omega\}} \omega') = \perp$ , it comes  $C_\omega \cap (B_n \setminus E_\omega) = \emptyset$  and  $C_\omega \subset A_n \cup E_\omega$ .

It is now possible to construct  $L_{n+1}$  and  $(\cdot|b_n)_{n+1}$ .

$L_{n+1}$  will be obtained by adding new propositions to  $L_n$ . More precisely, define a new proposition  $\langle \omega \rangle_{n+1}$  for any  $\omega \in \Omega_n$ . These propositions will be related to the construction of  $(\cdot|b_n)$ ; in particular, it will be implied  $(\omega|b_n)_{n+1} = \langle \omega \rangle_{n+1}$  for any  $\omega \in \Omega_n$ .

**First case.**  $\nu$  does not exist.

- $L_{n+1}$  is defined as the Boolean algebra generated by  $\Omega_n \cup \{\langle \omega \rangle_{n+1} / \omega \in \Omega_n\}$  such that:

$$\left\{ \begin{array}{l} \forall \omega, \omega' \in \Omega_n, \omega \neq \omega' \implies \omega \wedge \omega' = \perp \text{ and } \bigvee_{\omega \in \Omega_n} \omega = \top , \\ \forall \omega, \omega' \in \Omega_n, \omega \neq \omega' \implies \langle \omega \rangle_{n+1} \wedge \langle \omega' \rangle_{n+1} = \perp \text{ and } \bigvee_{\omega \in B_n} \langle \omega \rangle_{n+1} = \top , \\ \forall \omega \in B_n, \omega \wedge \langle \omega \rangle_{n+1} = \omega \text{ and } \forall \omega, \omega' \in B_n, \omega \neq \omega' \implies \omega \wedge \langle \omega' \rangle_{n+1} = \perp . \end{array} \right.$$



Then, the set  $\Omega_{n+1}$  defined by:

$$\begin{aligned}\Omega_{n+1} &= \{\omega \wedge \langle \omega' \rangle_{n+1} / \omega, \omega' \in \Omega_n\} \setminus \{\perp\} \\ &= B_n \cup \{\omega \wedge \langle \omega' \rangle_{n+1} / \omega \in A_n \text{ and } \omega' \in B_n\}\end{aligned}$$

is a set of disjoint propositions, generating  $L_{n+1}$ .

It is noticeable that  $\underline{L_n} \subset L_{n+1}$ .

- The probability  $P_{n+1}$  over  $L_{n+1}$  is characterized by its values over  $\Omega_{n+1}$  and is defined by:

$$\begin{cases} \forall \omega \in B_n, P_{n+1}(\omega) = P_n(\omega), \\ \forall \omega \in A_n, \forall \omega' \in B_n, P_{n+1}(\omega \wedge \langle \omega' \rangle_{n+1}) = P_n(\omega)P_n(\omega')/P_n(b_n) \quad \text{when } P_n(b_n) > 0, \\ \forall \omega \in A_n, \forall \omega' \in B_n, P_{n+1}(\omega \wedge \langle \omega' \rangle_{n+1}) = P_n(\omega)/\text{card}(B_n) \quad \text{when } P_n(b_n) = 0. \end{cases}$$

The whole probability is derived by  $\forall \Sigma \subset \Omega_{n+1}, P_{n+1}(\bigvee_{\omega \in \Sigma} \omega) = \sum_{\omega \in \Sigma} P_{n+1}(\omega)$ .

Since  $\forall \omega \in A_n, \omega = \omega \wedge \bigvee_{\omega' \in B_n} \langle \omega' \rangle_{n+1} = \bigvee_{\omega' \in B_n} (\omega \wedge \langle \omega' \rangle_{n+1})$ ,

it comes  $\forall \omega \in A_n, P_{n+1}(\omega) = \sum_{\omega' \in B_n} P_{n+1}(\omega \wedge \langle \omega' \rangle_{n+1})$ .

From the definition of  $P_{n+1}(\omega \wedge \langle \omega' \rangle_{n+1})$ , it comes  $\forall \omega \in A_n, P_{n+1}(\omega) = P_n(\omega)$ .

Then  $\forall \omega \in \Omega_n, P_{n+1}(\omega) = P_n(\omega)$  and finally  $\forall \phi \in L_n, P_{n+1}(\phi) = P_n(\phi)$ ; *ie.*  $\underline{P_n} \subset P_{n+1}$ .

- The operator  $(\cdot|\cdot)$  is updated as follows:

$$\begin{cases} \forall \phi \in L_n \setminus \{\perp, \top\}, \forall \psi \in L_n, (\psi|\phi)_{n+1} = (\psi|\phi)_n \quad \text{when } (\psi|\phi)_n \text{ is defined,} \\ \forall \psi \in L_{n+1}, (\psi|\perp)_{n+1} = (\psi|\top)_{n+1} = \psi, \\ \forall \Sigma \subset B_n, \forall \Gamma \subset A_n \times B_n, \left( \left( \bigvee_{\omega \in \Sigma} \omega \right) \vee \left( \bigvee_{(\omega, \omega') \in \Gamma} (\omega \wedge \langle \omega' \rangle_{n+1}) \right) \middle| b_n \right)_{n+1} \\ = \bigvee_{\omega \in \Sigma} \left( \omega \vee \bigvee_{\omega' \in A_n} (\omega' \wedge \langle \omega \rangle_{n+1}) \right). \end{cases}$$

It appears clearly that  $(\psi|\phi)_{n+1} = (\psi|\phi)_n$  when  $(\psi|\phi)_n$  is defined, *ie.*  $\underline{(\cdot|\cdot)}_n \subset \underline{(\cdot|\cdot)}_{n+1}$ .

- The list  $\Lambda_{n+1} = (s_{n+1}, f_{n+1}, \lambda_{n+1})$  is updated by:

$$s_{n+1} = s_n + 1; \quad f_{n+1} = f_n + \text{card}(L_{n+1} \setminus L_n) + 1; \quad \lambda_{n+1} : \llbracket 0, f_{n+1} - 1 \rrbracket \rightarrow L_{n+1} \setminus \{\perp, \top\},$$

where  $\forall t \in \llbracket 0, f_n - 1 \rrbracket, \lambda_{n+1}(t) = \lambda_n(t)$  and  $\lambda_{n+1}$  implies an onto mapping from  $\llbracket f_n, f_{n+1} - 1 \rrbracket$  to  $\{b_n\} \cup (L_{n+1} \setminus L_n)$ . Notice that  $\underline{\lambda_n} \subset \underline{\lambda_{n+1}}$ ; *ie.*  $\lambda_{n+1}$  is obtained by adding new “tasks” to the list  $\lambda_n$ . Since  $\{\lambda_{n+1}(t)/s_{n+1} \leq t < f_{n+1}\} = L_{n+1} \setminus \{\perp, \top\}$ , the task list covers all the possible constructions of  $(\cdot|\cdot)$  for this stage.

**Second case.**  $\nu$  exists.

For any  $\omega \in B_\nu$ , define  $C_\omega, \tilde{C}_\omega, D_\omega \subset \Omega_n$  and  $E_\omega \subset B_n$  by:

$$(\omega|b_n)_n = \bigvee_{\omega' \in C_\omega} \omega', \quad D_\omega = \Omega_n \setminus C_\omega, \quad \omega = \bigvee_{\omega' \in E_\omega} \omega' \quad \text{and} \quad \tilde{C}_\omega = C_\omega \setminus E_\omega.$$

- $L_{n+1}$  is defined as the Boolean algebra generated by  $\Omega_n \cup \{\langle \omega \rangle_{n+1} / \omega \in \Omega_n\}$  such that:

$$\begin{cases} \forall \omega, \omega' \in \Omega_n, \omega \neq \omega' \implies \omega \wedge \omega' = \perp \text{ and } \bigvee_{\omega \in \Omega_n} \omega = \top, \\ \forall \omega, \omega' \in \Omega_n, \omega \neq \omega' \implies \langle \omega \rangle_{n+1} \wedge \langle \omega' \rangle_{n+1} = \perp \text{ and } \bigvee_{\omega \in B_n} \langle \omega \rangle_{n+1} = \top, \\ \forall \omega \in B_n, \omega \wedge \langle \omega \rangle_{n+1} = \omega \text{ and } \forall \omega, \omega' \in B_n, \omega \neq \omega' \implies \omega \wedge \langle \omega' \rangle_{n+1} = \perp, \\ \forall \omega \in B_\nu, \forall \omega' \in D_\omega, \forall \omega'' \in E_\omega, \omega' \wedge \langle \omega'' \rangle_{n+1} = \perp. \end{cases}$$

It has been shown that  $E_\omega \subset C_\omega$ , which implies the compatibility of the above constraints. It has also been proven  $C_\omega \subset A_n \cup E_\omega$ , so that  $\tilde{C}_\omega \subset A_n$ . And from  $\bigcup_{\omega \in B_\nu} C_\omega = \Omega_n$ , it is deduced  $A_n = \bigcup_{\omega \in B_\nu} \tilde{C}_\omega$ . Then the set  $\Omega_{n+1}$  defined by:

$$\begin{aligned}\Omega_{n+1} &= \{\omega \wedge \langle \omega' \rangle_{n+1} / \omega, \omega' \in \Omega_n\} \setminus \{\perp\} \\ &= B_n \cup \bigcup_{\omega \in B_\nu} \{\omega' \wedge \langle \omega'' \rangle_{n+1} / \omega' \in \tilde{C}_\omega \text{ and } \omega'' \in E_\omega\}\end{aligned}$$

is a set of disjoint propositions, generating  $L_{n+1}$ .

It is noticeable that  $\underline{L_n} \subset \underline{L_{n+1}}$ .

- The probability  $P_{n+1}$  over  $L_{n+1}$  is characterized by its values over  $\Omega_{n+1}$  and is defined by:

$$\left\{ \begin{array}{l} \forall \omega' \in B_n, P_{n+1}(\omega') = P_n(\omega'), \\ \forall \omega \in B_\nu, \forall \omega' \in \tilde{C}_\omega, \forall \omega'' \in E_\omega, \\ \left\{ \begin{array}{l} P_{n+1}(\omega' \wedge \langle \omega'' \rangle_{n+1}) = P_n(\omega')P_n(\omega'')/P_n(\omega) \quad \text{when } P_n(\omega) > 0, \\ P_{n+1}(\omega' \wedge \langle \omega'' \rangle_{n+1}) = P_n(\omega')/\text{card}(E_\omega) \quad \text{when } P_n(\omega) = 0. \end{array} \right. \end{array} \right.$$

The whole probability is derived by  $\forall \Sigma \subset \Omega_{n+1}, P_{n+1}(\bigvee_{\omega' \in \Sigma} \omega') = \sum_{\omega' \in \Sigma} P_{n+1}(\omega')$ .

Assume  $\omega \in B_\nu$ . Since:

$$\forall \omega' \in \tilde{C}_\omega, \omega' = \omega' \wedge \bigvee_{\omega'' \in B_n} \langle \omega'' \rangle_{n+1} = \bigvee_{\omega'' \in B_n} (\omega' \wedge \langle \omega'' \rangle_{n+1}) = \bigvee_{\omega'' \in E_\omega} (\omega' \wedge \langle \omega'' \rangle_{n+1}),$$

it comes  $\forall \omega' \in \tilde{C}_\omega, P_{n+1}(\omega') = \sum_{\omega'' \in E_\omega} P_{n+1}(\omega' \wedge \langle \omega'' \rangle_{n+1})$ .

From the definition of  $P_{n+1}(\omega' \wedge \langle \omega'' \rangle_{n+1})$ , it comes  $\forall \omega' \in \tilde{C}_\omega, P_{n+1}(\omega') = P_n(\omega')$ .

Then  $\forall \omega' \in \Omega_n, P_{n+1}(\omega') = P_n(\omega')$  and finally  $\forall \phi \in L_n, P_{n+1}(\phi) = P_n(\phi)$ ; ie.  $\underline{P_n} \subset \underline{P_{n+1}}$ .

- The operator  $(\cdot|\cdot)$  is updated as follows:

$$\left\{ \begin{array}{l} \forall \phi \in L_n \setminus \{\perp, \top, b_n\}, \forall \psi \in L_n, (\psi|\phi)_{n+1} = (\psi|\phi)_n \quad \text{when } (\psi|\phi)_n \text{ is defined,} \\ \forall \psi \in L_{n+1}, (\psi|\perp)_{n+1} = (\psi|\top)_{n+1} = \psi, \\ \text{For any } (\Sigma_\omega)_{\omega \in B_\nu} \text{ such that } \forall \omega \in B_\nu, \Sigma_\omega \subset E_\omega \text{ and any } \Gamma \subset A_n \times B_n, \\ \left( \left( \bigvee_{\omega \in B_\nu} \bigvee_{\omega' \in \Sigma_\omega} \omega' \right) \vee \left( \bigvee_{(\omega', \omega'') \in \Gamma} \omega' \wedge \langle \omega'' \rangle_{n+1} \right) \middle| b_n \right)_{n+1} \\ = \bigvee_{\omega \in B_\nu} \bigvee_{\omega' \in \Sigma_\omega} \left( \omega' \vee \bigvee_{\omega'' \in \tilde{C}_\omega} (\omega'' \wedge \langle \omega' \rangle_{n+1}) \right). \end{array} \right.$$

It appears clearly that  $(\psi|\phi)_{n+1} = (\psi|\phi)_n$  when  $(\psi|\phi)_n$  is defined and  $\phi \neq b_n$ . Now, any proposition of  $L_{\nu+1}$  is of the form  $\bigvee_{\omega \in \Sigma_1 \cup \Sigma_2} \omega$ , where  $\Sigma_1 \subset B_\nu$  and  $\Sigma_2 \subset \Omega_{\nu+1} \setminus B_\nu$ . Moreover, there is a set  $\Sigma_3 \subset A_n$  such that  $\bigvee_{\omega \in \Sigma_2} \omega = \bigvee_{\omega' \in \Sigma_3} \omega'$ .

Then  $\bigvee_{\omega \in \Sigma_1 \cup \Sigma_2} \omega = (\bigvee_{\omega \in \Sigma_1} \bigvee_{\omega' \in E_\omega} \omega') \vee (\bigvee_{\omega' \in \Sigma_3} \bigvee_{\omega'' \in B_n} (\omega' \wedge \langle \omega'' \rangle_{n+1}))$  and:

$$\begin{aligned} \left( \bigvee_{\omega \in \Sigma_1 \cup \Sigma_2} \omega \middle| b_n \right)_{n+1} &= \bigvee_{\omega \in \Sigma_1} \bigvee_{\omega' \in E_\omega} \left( \omega' \vee \bigvee_{\omega'' \in \tilde{C}_\omega} (\omega'' \wedge \langle \omega' \rangle_{n+1}) \right) \\ &= \bigvee_{\omega \in \Sigma_1} \bigvee_{\omega' \in E_\omega} \bigvee_{\omega'' \in C_\omega} (\omega'' \wedge \langle \omega' \rangle_{n+1}) = \bigvee_{\omega \in \Sigma_1} \bigvee_{\omega' \in B_n} \bigvee_{\omega'' \in C_\omega} (\omega'' \wedge \langle \omega' \rangle_{n+1}) = \bigvee_{\omega \in \Sigma_1} \bigvee_{\omega'' \in C_\omega} \omega'' \\ &= \bigvee_{\omega \in \Sigma_1} (\omega|b_n)_{\nu+1} = \left( \bigvee_{\omega \in \Sigma_1} \omega \middle| b_n \right)_{\nu+1} = \left( \bigvee_{\omega \in \Sigma_1 \cup \Sigma_2} \omega \middle| b_n \right)_{\nu+1} = \left( \bigvee_{\omega \in \Sigma_1 \cup \Sigma_2} \omega \middle| b_n \right)_n. \end{aligned}$$

At last,  $(\psi|\phi)_{n+1} = (\psi|\phi)_n$  for any  $\phi, \psi$  such that  $(\psi|\phi)_n$  is defined, ie.  $(\cdot|\cdot)_n \subset (\cdot|\cdot)_{n+1}$ .

- The list  $\Lambda_{n+1} = (s_{n+1}, f_{n+1}, \lambda_{n+1})$  is again updated by:

$$s_{n+1} = s_n + 1; \quad f_{n+1} = f_n + \text{card}(L_{n+1} \setminus L_n) + 1; \quad \lambda_{n+1} : \llbracket 0, f_{n+1} - 1 \rrbracket \rightarrow L_{n+1} \setminus \{\perp, \top\},$$

where  $\forall t \in \llbracket 0, f_n - 1 \rrbracket$ ,  $\lambda_{n+1}(t) = \lambda_n(t)$  and  $\lambda_{n+1}$  implies an onto mapping from  $\llbracket f_n, f_{n+1} - 1 \rrbracket$  to  $\{b_n\} \cup (L_{n+1} \setminus L_n)$ . Notice that  $\lambda_n \subset \lambda_{n+1}$ . Moreover, the task list covers all the possible constructions of  $(\cdot|\cdot)$  for this stage, since  $\{\lambda_{n+1}(t)/s_{n+1} \leq t < f_{n+1}\} = L_{n+1} \setminus \{\perp, \top\}$ .

**Axioms work.** First, the analogues of the classical axioms (c1-7 and modus ponens) hold from the Boolean structure:

- $\top = \top$ ,
- $\phi \rightarrow (\psi \rightarrow \phi) = \top$ ,
- $(\eta \rightarrow (\phi \rightarrow \psi)) \rightarrow ((\eta \rightarrow \phi) \rightarrow (\eta \rightarrow \psi)) = \top$ ,
- $(\neg\phi \rightarrow \neg\psi) \rightarrow ((\neg\phi \rightarrow \psi) \rightarrow \phi) = \top$ ,
- $\perp = \neg\top$ ,
- $\phi \rightarrow \psi = \neg\phi \vee \psi$  (definition),
- $\phi \wedge \psi = \neg(\neg\phi \vee \neg\psi)$ ,
- $\phi = \top$  and  $\phi \rightarrow \psi = \top$  implies  $\psi = \top$ .

The Bayesian axioms (b1-4) also hold similarly, when  $(\cdot|\cdot)_{n+1}$  is defined (proof follows).

**Remark:** Since  $L_n \subset L_{n+1}$  and  $(\cdot|\cdot)_n \subset (\cdot|\cdot)_{n+1}$ , the axioms are partially inherited by step  $n+1$  from step  $n$ . It is just necessary to prove the axiom analogues involving  $(\cdot|b_n)_{n+1}$ .

- Assume  $(\psi|\phi)_{n+1}$  being defined. Then  $\phi \rightarrow \psi = \top$  implies  $\neg\phi = \top$  or  $(\psi|\phi)_{n+1} = \top$ ,

**Proof.** When  $\phi \neq b_n$ , the property is inherited from step  $n$ , by  $(\psi|\phi)_{n+1} = (\psi|\phi)_n$ .

It is now hypothesized  $\phi = b_n$ .

By construction, it is implied  $b_n \neq \perp$ , ie.  $\neg b_n \neq \top$ .

Then assume  $\phi \rightarrow \psi = \top$  and consequently  $\neg b_n \vee \psi = \top$ .

Then it is implied  $\psi = \bigvee_{\omega \in B_n \cup \Gamma} \omega$ , where  $\Gamma \subset \Omega_{n+1} \setminus B_n$ .

Two cases are considered.

*First case:  $\nu$  does not exist.* Then:

$$\begin{aligned} (\psi|\phi)_{n+1} &= (\psi|b_n)_{n+1} = \bigvee_{\omega \in B_n} \left( \omega \vee \bigvee_{\omega' \in A_n} (\omega' \wedge \langle \omega \rangle_{n+1}) \right) \\ &= \left( \bigvee_{\omega \in B_n} \omega \right) \vee \left( \left( \bigvee_{\omega' \in A_n} \omega' \right) \wedge \left( \bigvee_{\omega \in B_n} \langle \omega \rangle_{n+1} \right) \right) = b_n \vee (\neg b_n \wedge \top) = \top. \end{aligned}$$

*Second case:  $\nu$  exists.* Then:

$$\begin{aligned} (\psi|\phi)_{n+1} &= (\psi|b_n)_{n+1} = \bigvee_{\omega \in B_\nu} \bigvee_{\omega' \in E_\omega} \left( \omega' \vee \bigvee_{\omega'' \in \tilde{C}_\omega} (\omega'' \wedge \langle \omega' \rangle_{n+1}) \right) \\ &= \bigvee_{\omega \in B_\nu} \bigvee_{\omega' \in E_\omega} \bigvee_{\omega'' \in C_\omega} (\omega'' \wedge \langle \omega' \rangle_{n+1}) = \left( \bigvee_{\omega \in B_\nu} \bigvee_{\omega' \in E_\omega} \langle \omega' \rangle_{n+1} \right) \wedge \left( \bigvee_{\omega \in B_\nu} \bigvee_{\omega'' \in C_\omega} \omega'' \right) \\ &= \left( \bigvee_{\omega' \in B_n} \langle \omega' \rangle_{n+1} \right) \wedge \left( \bigvee_{\omega \in B_\nu} (\omega|b_\nu)_{\nu+1} \right) = \top \wedge \top = \top. \end{aligned}$$

Notice that there are many ways to derive  $\bigvee_{\omega \in B_\nu} (\omega|b_\nu)_{\nu+1} = \top$ .

One simple way is just to derive it from the axioms b1-4, which hold on  $(\omega|b_\nu)_{\nu+1}$  by recursion hypothesis.

□□□

- Assume  $(\psi|\phi)_{n+1}$  and  $(\neg\psi|\phi)_{n+1}$  being defined. Then  $\neg(\neg\psi|\phi)_{n+1} = (\psi|\phi)_{n+1}$ .

**Proof.** Assume  $\phi = b_n$ .

*First case:  $\nu$  does not exist.*

Define  $\Sigma \subset B_n$  and  $\Gamma \subset \Omega_{n+1} \setminus B_n$ , such that  $\psi = \bigvee_{\omega \in \Sigma \cup \Gamma} \omega$ .

Defining  $\bar{\Sigma} = B_n \setminus \Sigma$  and  $\bar{\Gamma} = (\Omega_{n+1} \setminus B_n) \setminus \Gamma$ , it comes  $\neg\psi = \bigvee_{\omega \in \bar{\Sigma} \cup \bar{\Gamma}} \omega$ .

The result is then a consequence of:

$$\text{Lemma 1. } \neg(\psi|b_n)_{n+1} = (\neg\psi|b_n)_{n+1} = \bigvee_{\omega \in \bar{\Sigma}} \left( \omega \vee \bigvee_{\omega' \in A_n} (\omega' \wedge \langle \omega \rangle_{n+1}) \right).$$

**Sub-proof.**

$$\begin{aligned} \neg(\psi|b_n)_{n+1} &= \neg \left( \left( \bigvee_{\omega \in \Sigma} \omega \right) \vee \left( \bigvee_{\omega \in \Sigma} \bigvee_{\omega' \in A_n} (\omega' \wedge \langle \omega \rangle_{n+1}) \right) \right) \\ &= \left( \left( \bigvee_{\omega \in \bar{\Sigma}} \omega \right) \vee \left( \bigvee_{\omega \in \bar{\Sigma}} \bigvee_{\omega' \in A_n} (\omega' \wedge \langle \omega \rangle_{n+1}) \right) \right) = (\neg\psi|b_n)_{n+1}. \end{aligned}$$

□□

*Second case:  $\nu$  exists.* Define  $\Gamma \subset \Omega_{n+1} \setminus B_n$  and  $\Sigma_\omega \subset E_\omega$  for any  $\omega \in B_\nu$ , such that  $\psi = \left( \bigvee_{\omega \in B_\nu} \bigvee_{\omega' \in \Sigma_\omega} \omega' \right) \vee \left( \bigvee_{\omega'' \in \Gamma} \omega'' \right)$ .

Defining  $\bar{\Gamma} = (\Omega_{n+1} \setminus B_n) \setminus \Gamma$  and  $\bar{\Sigma}_\omega = E_\omega \setminus \Sigma_\omega$  for any  $\omega \in B_\nu$ ,

it comes  $\neg\psi = \left( \bigvee_{\omega \in B_\nu} \bigvee_{\omega' \in \bar{\Sigma}_\omega} \omega' \right) \vee \left( \bigvee_{\omega'' \in \bar{\Gamma}} \omega'' \right)$ .

The result is then a consequence of:

$$\text{Lemma 2. } \neg(\psi|b_n)_{n+1} = (\neg\psi|b_n)_{n+1} = \bigvee_{\omega \in B_\nu} \bigvee_{\omega' \in \bar{\Sigma}_\omega} \left( \omega' \vee \bigvee_{\omega'' \in \bar{C}_\omega} (\omega'' \wedge \langle \omega' \rangle_{n+1}) \right).$$

**Sub-proof.**

$$\begin{aligned} \neg(\psi|b_n)_{n+1} &= \neg \left( \left( \bigvee_{\omega \in B_\nu} \bigvee_{\omega' \in \Sigma_\omega} \omega' \right) \vee \left( \bigvee_{\omega \in B_\nu} \bigvee_{\omega' \in \Sigma_\omega} \bigvee_{\omega'' \in \bar{C}_\omega} (\omega'' \wedge \langle \omega' \rangle_{n+1}) \right) \right) \\ &= \left( \left( \bigvee_{\omega \in B_\nu} \bigvee_{\omega' \in \bar{\Sigma}_\omega} \omega' \right) \vee \left( \bigvee_{\omega \in B_\nu} \bigvee_{\omega' \in \bar{\Sigma}_\omega} \bigvee_{\omega'' \in \bar{C}_\omega} (\omega'' \wedge \langle \omega' \rangle_{n+1}) \right) \right) = (\neg\psi|b_n)_{n+1}. \end{aligned}$$

□□

□□□

- Assume  $(\psi \rightarrow \eta|\phi)_{n+1}$ ,  $(\psi|\phi)_{n+1}$  and  $(\eta|\phi)_{n+1}$  being defined.

Then  $(\psi \rightarrow \eta|\phi)_{n+1} \rightarrow ((\psi|\phi)_{n+1} \rightarrow (\eta|\phi)_{n+1}) = \top$ ,

**Proof.** Assume  $\phi = b_n$ .

*First case:  $\nu$  does not exist.* Define  $\Sigma_1, \Sigma_2 \subset B_n$  and  $\Gamma_1, \Gamma_2 \subset \Omega_{n+1} \setminus B_n$ , such that  $\psi = \bigvee_{\omega \in \Sigma_1 \cup \Gamma_1} \omega$  and  $\eta = \bigvee_{\omega \in \Sigma_2 \cup \Gamma_2} \omega$ .

Defining  $\bar{\Sigma}_1 = B_n \setminus \Sigma_1$  and  $\bar{\Gamma}_1 = (\Omega_{n+1} \setminus B_n) \setminus \Gamma_1$ , it comes  $\neg\psi = \bigvee_{\omega \in \bar{\Sigma}_1 \cup \bar{\Gamma}_1} \omega$ .

Then  $(\psi \rightarrow \eta|b_n)_{n+1} = \bigvee_{\omega \in \bar{\Sigma}_1 \cup \Sigma_2} \left( \omega \vee \bigvee_{\omega' \in A_n} (\omega' \wedge \langle \omega \rangle_{n+1}) \right)$

and  $(\eta|b_n)_{n+1} = \bigvee_{\omega \in \Sigma_2} \left( \omega \vee \bigvee_{\omega' \in A_n} (\omega' \wedge \langle \omega \rangle_{n+1}) \right)$ .

By applying Lemma 1, it is deduced:

$$\begin{aligned} (\psi \rightarrow \eta|\phi)_{n+1} \rightarrow ((\psi|\phi)_{n+1} \rightarrow (\eta|\phi)_{n+1}) &= \neg \left( \bigvee_{\omega \in \overline{\Sigma}_1 \cup \Sigma_2} \left( \omega \vee \bigvee_{\omega' \in A_n} (\omega' \wedge \langle \omega \rangle_{n+1}) \right) \right) \\ &\vee \left( \bigvee_{\omega \in \overline{\Sigma}_1 \cup \Sigma_2} \left( \omega \vee \bigvee_{\omega' \in A_n} (\omega' \wedge \langle \omega \rangle_{n+1}) \right) \right) = \top. \end{aligned}$$

*Second case:  $\nu$  exists.*

Define  $\Gamma_1, \Gamma_2 \subset \Omega_{n+1} \setminus B_n$  and  $\Sigma_{1\omega}, \Sigma_{2\omega} \subset E_\omega$  for any  $\omega \in B_\nu$ , such that:

$$\psi = \left( \bigvee_{\omega \in B_\nu} \bigvee_{\omega' \in \Sigma_{1\omega}} \omega' \right) \vee \left( \bigvee_{\omega'' \in \Gamma_1} \omega'' \right) \quad \text{and} \quad \eta = \left( \bigvee_{\omega \in B_\nu} \bigvee_{\omega' \in \Sigma_{2\omega}} \omega' \right) \vee \left( \bigvee_{\omega'' \in \Gamma_2} \omega'' \right).$$

Defining  $\overline{\Gamma}_1 = (\Omega_{n+1} \setminus B_n) \setminus \Gamma_1$  and  $\overline{\Sigma}_{1\omega} = E_\omega \setminus \Sigma_{1\omega}$  for any  $\omega \in B_\nu$ , it comes  $\neg\psi = \left( \bigvee_{\omega \in B_\nu} \bigvee_{\omega' \in \overline{\Sigma}_{1\omega}} \omega' \right) \vee \left( \bigvee_{\omega'' \in \overline{\Gamma}_1} \omega'' \right)$ .

Then  $(\psi \rightarrow \eta|b_n)_{n+1} = \bigvee_{\omega \in B_\nu} \bigvee_{\omega' \in \overline{\Sigma}_{1\omega} \cup \Sigma_{2\omega}} \left( \omega' \vee \bigvee_{\omega'' \in \tilde{C}_\omega} (\omega'' \wedge \langle \omega' \rangle_{n+1}) \right)$

and  $(\eta|b_n)_{n+1} = \bigvee_{\omega \in B_\nu} \bigvee_{\omega' \in \Sigma_{2\omega}} \left( \omega' \vee \bigvee_{\omega'' \in \tilde{C}_\omega} (\omega'' \wedge \langle \omega' \rangle_{n+1}) \right)$ .

By applying Lemma 2, it is deduced:

$$\begin{aligned} (\psi \rightarrow \eta|\phi)_{n+1} \rightarrow ((\psi|\phi)_{n+1} \rightarrow (\eta|\phi)_{n+1}) &= \neg \left( \bigvee_{\omega \in B_\nu} \bigvee_{\omega' \in \overline{\Sigma}_{1\omega} \cup \Sigma_{2\omega}} \left( \omega' \vee \bigvee_{\omega'' \in \tilde{C}_\omega} (\omega'' \wedge \langle \omega' \rangle_{n+1}) \right) \right) \\ &\vee \left( \bigvee_{\omega \in B_\nu} \bigvee_{\omega' \in \overline{\Sigma}_{1\omega} \cup \Sigma_{2\omega}} \left( \omega' \vee \bigvee_{\omega'' \in \tilde{C}_\omega} (\omega'' \wedge \langle \omega' \rangle_{n+1}) \right) \right) = \top. \end{aligned}$$

□□□

- Assume  $(\psi|\phi)_{n+1}$  being defined. Then  $(\psi|\phi)_{n+1} \rightarrow (\phi \rightarrow \psi) = \top$ ,

**Proof.** Assume  $\phi = b_n$ .

Then  $(\psi|\phi)_{n+1} \rightarrow (\phi \rightarrow \psi) = \neg(\psi|b_n)_{n+1} \vee \neg b_n \vee \psi$ .

*First case:  $\nu$  does not exist.*

Define  $\Sigma \subset B_n$  and  $\Gamma \subset \Omega_{n+1} \setminus B_n$ , such that  $\psi = \bigvee_{\omega \in \Sigma \cup \Gamma} \omega$ .

Define also  $\overline{\Sigma} = B_n \setminus \Sigma$ . Then, by lemma 1:

$$\begin{aligned} (\psi|\phi)_{n+1} \rightarrow (\phi \rightarrow \psi) &= \left( \bigvee_{\omega \in \overline{\Sigma}} \left( \omega \vee \bigvee_{\omega' \in A_n} (\omega' \wedge \langle \omega \rangle_{n+1}) \right) \right) \vee \left( \bigvee_{\omega \in \Omega_{n+1} \setminus B_n} \omega \right) \\ &\vee \left( \bigvee_{\omega \in \Sigma \cup \Gamma} \omega \right) = \left( \bigvee_{\omega \in \Sigma \cup \overline{\Sigma}} \omega \right) \vee \left( \bigvee_{\omega \in \Omega_{n+1} \setminus B_n} \omega \right) = \bigvee_{\omega \in \Omega_{n+1}} \omega = \top. \end{aligned}$$

*Second case:  $\nu$  exists.* Define  $\Gamma \subset \Omega_{n+1} \setminus B_n$  and  $\Sigma_\omega \subset E_\omega$  for any  $\omega \in B_\nu$ , such that  $\psi = \left( \bigvee_{\omega \in B_\nu} \bigvee_{\omega' \in \Sigma_\omega} \omega' \right) \vee \left( \bigvee_{\omega'' \in \Gamma} \omega'' \right)$ .

Define also  $\bar{\Sigma}_\omega = E_\omega \setminus \Sigma_\omega$  for any  $\omega \in B_\nu$ . Then, by lemma 2:

$$\begin{aligned} (\psi|\phi)_{n+1} \rightarrow (\phi \rightarrow \psi) &= \bigvee_{\omega \in B_\nu} \bigvee_{\omega' \in \bar{\Sigma}_\omega} \left( \omega' \vee \bigvee_{\omega'' \in \tilde{C}_\omega} (\omega'' \wedge \langle \omega' \rangle_{n+1}) \right) \vee \left( \bigvee_{\omega' \in \Omega_{n+1} \setminus B_n} \omega' \right) \\ &\vee \left( \bigvee_{\omega \in B_\nu} \bigvee_{\omega' \in \Sigma_\omega} \omega' \right) \vee \left( \bigvee_{\omega' \in \Gamma} \omega' \right) = \left( \bigvee_{\omega \in B_\nu} \bigvee_{\omega' \in \Sigma_\omega \cup \bar{\Sigma}_\omega} \omega' \right) \vee \left( \bigvee_{\omega \in \Omega_{n+1} \setminus B_n} \omega' \right) = \bigvee_{\omega' \in \Omega_{n+1}} \omega' = \top. \end{aligned}$$

□□□

**Multiplicative property.** Assume that  $(\psi|\phi)_{n+1}$  is defined.

Then  $P_{n+1}((\psi|\phi)_{n+1})P_{n+1}(\phi) = P_{n+1}(\phi \wedge \psi)$ .

proof. Assume first that  $\phi \neq b_n$ .

Then  $(\psi|\phi)_n$  is defined and  $P_n((\psi|\phi)_n)P_n(\phi) = P_n(\phi \wedge \psi)$ .

Since  $P_n \subset P_{n+1}$  and  $(\cdot|\cdot)_n \subset (\cdot|\cdot)_{n+1}$ ,

it follows  $P_{n+1}((\psi|\phi)_{n+1})P_{n+1}(\phi) = P_{n+1}(\phi \wedge \psi)$ .

Now assume  $\phi = b_n$ .

When  $P_n(b_n) = 0$ , the property is obvious.

When  $P_n(b_n) > 0$ , two cases are considered.

*First case:  $\nu$  does not exist.* Assume  $\Sigma \subset B_n$  and  $\Gamma \subset \Omega_{n+1} \setminus B_n$ .

$$\begin{aligned} P_{n+1} \left( \left( \bigvee_{\omega \in \Sigma \cup \Gamma} \omega \middle| b_n \right)_{n+1} \right) P_{n+1}(b_n) &= \sum_{\omega \in \Sigma} \left( P_{n+1}(\omega) + \sum_{\omega' \in A_n} P_{n+1}(\omega' \wedge \langle \omega \rangle_{n+1}) \right) P_n(b_n) \\ &= \sum_{\omega \in \Sigma} P_n(\omega)P_n(b_n) + \sum_{\omega \in \Sigma} \sum_{\omega' \in A_n} P_n(\omega')P_n(\omega) = \sum_{\omega \in \Sigma} P_n(\omega)P_n(b_n) + \sum_{\omega \in \Sigma} P_n(\omega)(1 - P_n(b_n)) \\ &= \sum_{\omega \in \Sigma} P_n(\omega) = P_{n+1} \left( \bigvee_{\omega \in \Sigma} \omega \right) = P_{n+1} \left( \left( \bigvee_{\omega \in \Sigma \cup \Gamma} \omega \right) \wedge b_n \right). \end{aligned}$$

*Second case:  $\nu$  exists.* Let  $(\Sigma_\omega)_{\omega \in B_\nu}$  such that  $\forall \omega \in B_\nu$ ,  $\Sigma_\omega \subset E_\omega$  and  $\Gamma \subset \Omega_{n+1} \setminus B_n$ .

$$\begin{aligned} P_{n+1} \left( \left( \left( \bigvee_{\omega \in B_\nu} \bigvee_{\omega' \in \Sigma_\omega} \omega' \right) \vee \left( \bigvee_{\omega' \in \Gamma} \omega' \right) \middle| b_n \right)_{n+1} \right) P_{n+1}(b_n) \\ = \sum_{\omega \in B_\nu} \sum_{\omega' \in \Sigma_\omega} \left( P_{n+1}(\omega') + \sum_{\omega'' \in \tilde{C}_\omega} P_{n+1}(\omega'' \wedge \langle \omega' \rangle_{n+1}) \right) P_n(b_n) \end{aligned}$$

In the case  $P_n(\omega) = 0$ , and since  $P_n(b_n) > 0$ , it happens  $P_n((\omega|b_n)_n) = 0$ . Then  $P_{n+1}(\omega') = P_{n+1}(\omega'') = P_{n+1}(\omega'' \wedge \langle \omega' \rangle_{n+1}) = 0$  for any  $\omega' \in E_\omega$  and  $\omega'' \in C_\omega$ . Then  $P_{n+1}(\omega') + \sum_{\omega'' \in \tilde{C}_\omega} P_{n+1}(\omega'' \wedge \langle \omega' \rangle_{n+1}) = 0 = P_n(\omega')/P_n(b_n)$  for any  $\omega' \in \Sigma_\omega$ . Otherwise when  $P_n(\omega) > 0$ , it also happens for any  $\omega' \in \Sigma_\omega$ :

$$\begin{aligned} P_{n+1}(\omega') + \sum_{\omega'' \in \tilde{C}_\omega} P_{n+1}(\omega'' \wedge \langle \omega' \rangle_{n+1}) &= P_n(\omega') + \sum_{\omega'' \in \tilde{C}_\omega} P_n(\omega'')P_n(\omega')/P_n(\omega) \\ &= P_n(\omega') + (P_n((\omega|b_n)_n) - P_n(\omega)) \frac{P_n(\omega')}{P_n(\omega)} = P_n((\omega|b_n)_n) \frac{P_n(\omega')}{P_n(\omega)} = \frac{P_n(\omega')}{P_n(b_n)}. \end{aligned}$$

Followingly:

$$\begin{aligned}
P_{n+1} \left( \left( \left( \bigvee_{\omega \in B_\nu} \bigvee_{\omega' \in \Sigma_\omega} \omega' \right) \vee \left( \bigvee_{\omega' \in \Gamma} \omega' \right) \middle| b_n \right)_{n+1} \right) P_{n+1}(b_n) &= \sum_{\omega \in B_\nu} \sum_{\omega' \in \Sigma_\omega} P_{n+1}(\omega') \\
= P_{n+1} \left( \bigvee_{\omega \in B_\nu} \bigvee_{\omega' \in \Sigma_\omega} \omega' \right) &= P_{n+1} \left( \left( \left( \bigvee_{\omega \in B_\nu} \bigvee_{\omega' \in \Sigma_\omega} \omega' \right) \vee \left( \bigvee_{\omega' \in \Gamma} \omega' \right) \right) \wedge b_n \right) .
\end{aligned}$$

□□□

### C.3 Conclusion

Define  $\mathcal{L} = \bigcup_{n \in \mathbb{N}} L_n$ ,  $P = \bigcup_{n \in \mathbb{N}} P_n$  and  $(\cdot|\cdot) = \bigcup_{n \in \mathbb{N}} (\cdot|\cdot)_n$ .

Since  $(L_n)_{n \in \mathbb{N}}$ ,  $(P_n)_{n \in \mathbb{N}}$  and  $((\cdot|\cdot)_n)_{n \in \mathbb{N}}$  are increasing sequences, it is implied:

- $\mathcal{L}$  is a Boolean algebra (with an infinity of propositions), containing the free Boolean algebra  $\mathcal{B}$  generated by  $\Theta$ ,
- $\mathcal{L}$  verifies the classical axioms, *ie.* c1 – 7 and Modus ponens,
- $P$  is a probability over  $\mathcal{L}$ , which extends the probability  $p$  over  $\mathcal{B}$ ,
- $(\cdot|\cdot)$  is an operator defined over  $\mathcal{L}$ , which verify the Bayesian axioms b1 – 4,
- $P$  is multiplicative.

Moreover, the definition of the list  $(\Lambda_n)_{n \in \mathbb{N}}$  ensures that for any  $p, q \in \mathbb{N}$  and  $\phi \in L_p \setminus \{\perp, \top\}$ , there is  $r > q$  such that  $b_r = \phi$ . Consequently, for any  $\phi, \psi \in \mathcal{L}$ , the operation  $(\psi|\phi)$  is defined (just take  $p$  and  $q$  such that  $\phi \in L_p$  and  $\psi \in L_q$ ):

*The operator  $(\cdot|\cdot)$  is defined for any pair  $(\phi, \psi) \in \mathcal{L}^2$ .*

It is notheworthy that the pair  $(\mathcal{L}, (\cdot|\cdot))$  has been constructed independently of  $P$ . The pair  $(\mathcal{L}, (\cdot|\cdot))$  is not far from a model of the free DBL. It is likely that the method presented here could be adapted for the construction of a true model.

Now, it is a mere formality to derive a construction of a probability over the free DBL. In order to distinguish between the logical propositions of DBL and the propositions of  $\mathcal{L}$ , the  $\mathcal{L}$ -counterpart of any logical proposition  $\phi$  is now denoted  $\phi_{\mathcal{L}}$ . Since  $(\mathcal{L}, (\cdot|\cdot))$  verifies the axioms of DBL, it is implied:

$$\phi \equiv \psi \quad \text{implies} \quad \phi_{\mathcal{L}} = \psi_{\mathcal{L}} .$$

The probability  $P$  is thus compatible with the logic, and it is possible to set:

$$\forall \phi \in DBL, P(\phi) \triangleq P(\phi_{\mathcal{L}}) .$$

The additivity, coherence and finiteness of  $P$  over the free DBL is of course derived from  $\mathcal{L}$ . It is the same for the multiplicativity. More precisely,  $(\psi|\phi) \equiv \psi$  implies  $(\psi_{\mathcal{L}}|\phi_{\mathcal{L}}) = \psi_{\mathcal{L}}$  and:

$$P(\psi)P(\phi) = P(\psi_{\mathcal{L}})P(\phi_{\mathcal{L}}) = P\left((\psi_{\mathcal{L}}|\phi_{\mathcal{L}})\right)P(\phi_{\mathcal{L}}) = P(\psi_{\mathcal{L}} \wedge \phi_{\mathcal{L}}) = P((\psi \wedge \phi)_{\mathcal{L}}) = P(\psi \wedge \phi) .$$

At last, it has been shown that *the free DBL is probabilizable*.

□□□

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