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**GLOBAL MINIMALITY OF GENERIC MANIFOLDS AND  
HOLOMORPHIC EXTENDIBILITY OF CR FUNCTIONS**

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**Introduction.**

Let  $M$  be a smooth generic submanifold of  $\mathbf{C}^n$ . Several authors have studied the property of CR functions on  $M$  to extend locally to manifolds with boundary attached to  $M$  and holomorphically to generic wedges with edge  $M$  (*cf.* [14], [67], [68]). In a recent work ([69]), Tumanov has showed that CR-extendibility of CR functions on  $M$  propagates along curves that run in complex tangential directions to  $M$ . His main result appears as a natural generalization of results by Trépreau on propagation of singularities of CR functions ([61]). Indeed, Theorem 5.1 in [69] states that the direction of CR-extendibility moves parallelly with respect to a certain differential geometric partial connection in a quotient bundle of the normal bundle to  $M$ , and this variation is dual to the one introduced by Trépreau, according to Proposition 7.3 in [69].

In this paper we give a new and simplified presentation of the connection introduced in Tumanov's work. Let  $M$  be a real manifold and  $N$  a submanifold of  $M$ ,  $K$  a subbundle of  $TM$  with the property that  $K|_N \subset TN$ . Then by means of the Lie bracket, we can define a  $K$ -partial connection on the normal bundle of  $N$  in  $M$  (Proposition 1.1). In general, the parallel translation associated with that partial connection will be induced by the flow of  $K$ -tangent sections of  $TM$  (Proposition 1.2). When  $M$  is a generic submanifold of  $\mathbf{C}^n$  containing a CR submanifold  $S$  with the same CR dimension we recover in section 2 the  $T^cS$ -partial connection constructed by Tumanov in [69].

Recall that the *CR-orbit* of a point  $z \in M$  is the set of points that can be reached by piecewise smooth integral curves of complex tangent vector fields. We then say that  $M$  is *globally minimal* at a point  $z \in M$  if the CR-orbit of  $z$  contains a neighborhood of  $z$  in  $M$ . Using previous results, we show that vector space generated by the directions of CR-extendibility of CR functions on  $M$  exchanges by the induced composed flow between two points in a same CR-orbit (Lemma 3.5). As an application, we prove the main result of this paper, conjectured by Trépreau in [61] : *for wedge extendibility of CR functions to hold at every point in the CR-orbit of  $z \in M$  it is sufficient that  $M$  be globally minimal at  $z$*  (Theorem 3.4). Up till now we can only conjecture the converse (for a local result, see [6]).

I wish to thank J.-M. Trépreau for helpful critical and simplifying remarks.

**Remark :** After this paper was completed, we have received a preprint by B. Jöricke *Deformation of CR-manifolds, minimal points and CR-manifolds with the microlocal analytic extension property*, which contains also a proof of Theorem 3.4 and Theorem 3.6. Our proof seems quite different since we obtain these results relying on Tumanov's propagation theorems, the generic manifold  $M$  being fixed, whereas B. Jöricke works with conic perturbations of the base manifold so as to produce minimal points.

### §1. Partial connections associated with a system of vector fields.

Let  $M$  be a real differentiable manifold of class  $C^2$  of dimension  $n$  and  $H \rightarrow M$  a  $r$ -dimensional vector bundle over  $M$ . Recall that a connection  $\nabla$  on the bundle  $H \rightarrow M$  is a bilinear mapping which assigns to each pair of a vector field  $X$  with domain  $U$  and a section  $\eta$  of  $H$  over  $U$  a section  $\nabla_X \eta$  of  $H$  over  $U$  and satisfy

$$\nabla_{\phi X} = \phi \nabla_X, \quad \nabla_X(\phi \eta) = \phi \nabla_X \eta + (X\phi)\eta, \quad \phi \in C^1(M, \mathbf{R}).$$

When the covariant derivative  $\nabla_X \eta$  can only be defined for vectors  $X$  that belong to a subbundle  $K$  of  $TM$ , we call the connection  $\nabla$  a  $K$ -partial connection (cf. [69]).

If  $N$  is a submanifold of  $M$ , let  $T_N M$  be the *normal bundle of  $N$  in  $M$* , i.e.

$$T_N M = TM|_N / TN.$$

PROPOSITION 1.1. *Let  $M$  be a real manifold of class  $C^2$ ,  $N \subset M$  a submanifold of class  $C^2$  too and let  $K$  be a  $C^1$  subbundle of  $TM$  with the property that  $K|_N \subset TN$ . Then there exists a natural  $K$ -partial connection  $\nabla$  on the bundle  $T_N M$  which is defined as follows. If  $x \in N, X \in K[x]$  and  $\eta$  is a local section of  $T_N M$  over a neighborhood of  $x$ , then take*

$$\nabla_X \eta = [\tilde{X}, \tilde{Y}](x) \text{ mod } T_x N$$

where  $\tilde{X}$  is a  $C^1$  local section of  $K$  extending  $X$  and  $\tilde{Y}$  is a lifting of  $\eta$  in  $TM$  in a neighborhood of  $x$ .

PROOF. We first check that the definition is independent of the lifting  $\tilde{Y}$ . In fact, when  $\tilde{Y}$  is tangent to  $N$ , as  $\tilde{X}$  is tangent to  $N$  too, the Lie bracket  $[\tilde{X}, \tilde{Y}]$  remains tangent to  $N$  hence is zero in the quotient bundle.

Next we have to check that the definition of  $\nabla$  is independent of the chosen section  $\tilde{X}$  or, to rephrase, that if  $\tilde{X}(x) = 0$  then  $[\tilde{X}, \tilde{Y}](x)$  belongs to  $T_x N$ . Since  $K$  is a fiber bundle we can write

$$\tilde{X} = \sum_{j=1}^r f_j \tilde{X}_j \quad f_j(0) = 0 \quad j = 1, \dots, r$$

where  $r = \text{rank } K$ ,  $(\tilde{X}_j)_{j=1, \dots, r}$  is a frame for  $K$  near  $x$  and the  $f_j$  are  $C^1$  real valued functions defined near  $x$ . Noting that

$$[f \tilde{X}, \tilde{Y}] \equiv f[\tilde{X}, \tilde{Y}] - (\tilde{Y}f)\tilde{X} \equiv f[\tilde{X}, \tilde{Y}] \text{ mod } TN$$

the result follows and the mapping  $\nabla$  is well-defined. Moreover the preceding implies that if  $\phi \in C^1(M, \mathbf{R})$

$$\nabla_{\phi X} \eta \equiv \phi \nabla_X \eta.$$

Last, we check that  $\nabla_X$  is a derivation. Indeed

$$\nabla_X(\phi \eta) \equiv [\tilde{X}, \phi \tilde{Y}](x) \equiv (\tilde{X} \cdot \phi) \tilde{Y} + \phi [\tilde{X}, \tilde{Y}] \equiv (X\phi)\eta + \phi \nabla_X \eta$$

and the proof is complete.

With the connection  $\nabla$  it is associated the *parallel translation* of fibers of  $T_N M$  along smooth curves on the base  $N$  that run in directions tangent to  $K$ . Let  $I \ni t$  be a subinterval of  $\mathbf{R}$  and  $\gamma : I \rightarrow N$  be a smooth curve with the property that  $\dot{\gamma}(t) \in K[\gamma(t)]$ , where  $\dot{\gamma} = \frac{d}{dt}\gamma(t)$ . A curve  $\eta(t) \in T_N M[\gamma(t)]$  is a *horizontal lift* of  $\gamma$  if  $\nabla_{\dot{\gamma}}\eta = 0$ . Existence and uniqueness of horizontal lifts provide linear isomorphisms

$$\Phi_{t_0,t} : T_N M[\gamma(t_0)] \rightarrow T_N M[\gamma(t)]$$

obtained by moving elements of  $K$  along horizontal lifts of  $\gamma$ .

Recall (cf. [58]) that the Lie bracket  $[\tilde{X}, \tilde{Y}]$  is defined as the Lie derivative  $L_{\tilde{X}}\tilde{Y}$  of  $\tilde{Y}$  with respect to  $\tilde{X}$

$$[\tilde{X}, \tilde{Y}](x) = L_{\tilde{X}}\tilde{Y} = \lim_{h \rightarrow 0} [\tilde{Y}(x) - d\tilde{X}_{-h}(\tilde{Y}(\tilde{X}_h(x)))]$$

where  $\tilde{X}_t$  is the local flow on  $M$  generated in a neighborhood of  $x$  by the vector field  $\tilde{X}$ , and  $d\tilde{X}_t$  denotes its differential. In the assumptions of Proposition 1.1,  $\tilde{X}$  is of class  $C^1$  so the mapping  $x \rightarrow \tilde{X}_t(x)$  is of class  $C^1$  and the differential is a well-defined continuous mapping. When  $\tilde{X}$  is  $K$ -tangent its flow (and more generally any piecewise smooth composition of such flows) stabilizes the tangent bundle  $TN$  of the manifold  $N$ , hence its differential induces isomorphisms of fibers of  $T_N M$ , which we denote by  $dX_t$ . Assume moreover that the curve  $\gamma$  is an integral curve of a  $C^1$   $K$ -tangent vector field  $\tilde{X}$ , (which cannot be true for most general smooth curves  $\gamma$  but is sufficient enough for the applications) :  $\gamma(0) = x$  and  $\gamma(t) = \tilde{X}_t(x)$ . Then we claim that the mapping

$$dX_t : T_N M[x] \rightarrow T_N M[\gamma(t)]$$

provides the parallel translation  $\Phi_{0,t}$ . Indeed let  $\eta_0 \in T_N M[x]$  and take  $\eta(t) = dX_t(\eta_0)$ . Then by the definition of the partial connection  $\nabla$  and the definition of the Lie bracket we have

$$\nabla_{\dot{\gamma}}\eta(t) \equiv 0.$$

By uniqueness of solutions of linear differential equations of order one it must be that

$$\eta(t) = \Phi_{0,t}(\eta_0).$$

**PROPOSITION 1.2.** *Under the hypotheses of Proposition 1.1, let  $\gamma(t) = X_t(x_1)$  be a smooth (piecewise smooth) integral curve of a  $K$ -tangent vector field  $X$  (a finite number of  $K$ -tangent vector fields) running from  $x_1 \in N$  to  $x_2 \in N$ . Then the parallel translation along  $\gamma$  associated with the  $K$ -partial connection  $\nabla$  is induced by the differential of the flow of  $X$  (composed flow).*

In order to give an expression of the covariant derivatives induced by the partial connection  $\nabla$ , we choose coordinates on  $M$ ,  $x = (x', x'') \in \mathbf{R}^l \times \mathbf{R}^m$  such that the base point corresponds to  $x = 0$  and the submanifold  $N$  is defined by the equation  $x'' = 0$ . Let  $(x, \eta) = (x', x'', \eta', \eta'')$  be the canonical coordinates on  $TM$ , and  $(x', \eta'') \in \mathbf{R}^l \times \mathbf{R}^m$  the associated coordinates on  $T_N M$ .

If  $X = \sum_{j=1}^{l+m} a_j(x) \frac{\partial}{\partial x_j}$  is a  $C^1$  section of  $K$ , it must be tangent to  $N$ , so  $a_j(x', 0) = 0$ ,  $j = l+1, \dots, l+m$ . We choose a local section  $\eta$  of  $T_N M$  over a neighborhood of 0 in  $N$ , in fact a section  $\tilde{Y}$  of  $TM$  of the form

$$\tilde{Y} = \sum_{j=l+1}^{l+m} \eta_j(x') \frac{\partial}{\partial x_j},$$

Recalling Proposition 1.1 we have the following expression for the covariant derivative of  $\eta$  in the direction of  $X$

$$\nabla_X \eta = \sum_{j=l+1}^{l+m} \sum_{k=1}^l a_k(x', 0) \frac{\partial \eta_j(x')}{\partial x_k} \frac{\partial}{\partial x_j} - \sum_{j,k=l+1}^{l+m} \eta_k(x') \frac{\partial a_j(x', 0)}{\partial x_k} \frac{\partial}{\partial x_j}.$$

Given an integral curve  $\gamma(t) = (\gamma'(t), 0)$  of the field  $X$ , the equations for the horizontal lifts look like

$$X.\eta_j = \dot{\eta}_j(t) = \sum_{k=l+1}^{k=l+m} \frac{\partial a_j}{\partial x_k}(\gamma'(t), 0) \eta_k(t) \quad j = l+1, \dots, l+m$$

so the curve  $(\gamma'(t), \eta''(t))$  is the integral curve of the following vector field  $\check{X}$  on  $T_N M$

$$\check{X}(x', \eta'') = \sum_{j=1}^l a_j(x', 0) \frac{\partial}{\partial x_j} + \sum_{j,k=l+1}^{l+m} \frac{\partial a_j}{\partial x_k}(x', 0) \eta_k(x') \frac{\partial}{\partial \eta_j}.$$

Alternately, the partial connection  $\nabla$  can be defined by the family of horizontal subspaces  $H(\eta) \subset T_\eta(T_N M)$  generated by vectors of the form  $\check{X}$ .

Let us consider the *dual connection*  $\nabla^*$  to the connection  $\nabla$  on the dual bundle  $T_N^* M$ . Recall that the conormal bundle of  $N$  in  $M$ ,  $T_N^* M$ , consists of forms in  $T^* M$  that vanish on  $TN$ . It has fiber over a point  $x \in N$

$$T_N^* M[x] = \{\phi \in T_x^* M; \phi|_{T_x N} = 0\}.$$

The dual connection  $\nabla^*$  is defined by the following relation : if  $X$  is a  $K$ -tangent vector to  $N$  at  $x$ ,  $\eta$  is any section of  $T_N M$  near  $x$  and  $\phi$  is any section of  $T_N^* M$

$$X \langle \phi, \eta \rangle = \langle \nabla_X^* \phi, \eta \rangle + \langle \phi, \nabla_X \eta \rangle.$$

It is easily checked that such a relation defines a  $K$ -partial connection on  $T_N^* M$ .

Along with the coordinates on  $T_N M$  we introduced before we can introduce the canonical coordinates  $(x', \xi'')$  on the conormal bundle  $T_N^* M$ . These are dual to the coordinates  $(x', \eta'')$  for the canonical duality  $\langle, \rangle$  between  $T_N M$  and  $T_N^* M$  :

$$\left\langle \sum_{j=l+1}^{l+m} \xi_j dx_j, \sum_{j=l+1}^{l+m} \eta_j \frac{\partial}{\partial x_j} \right\rangle = \sum_{j=l+1}^{l+m} \xi_j \eta_j.$$

Using the previous definition of the dual connection we can then compute the covariant derivative of a section  $\sum \xi_j dx_j = \phi$  of  $T_N^* M$ . One easily shows

$$\nabla_X^* \phi = \sum_{j=l+1}^{l+m} (X.\xi_j + \sum_{k=l+1}^{l+m} \xi_k \frac{\partial a_k}{\partial x_j}) dx_j.$$

Hence, under the assumption of Proposition 1.2, the parallel translation associated with the connection  $\nabla^*$  is given by means of the integral curves of the following vector field on  $T_N^* M$

$$\hat{X} = \sum_{j=1}^l a_j(x', 0) \frac{\partial}{\partial x_j} - \sum_{j,k=l+1}^{l+m} \frac{\partial a_j}{\partial x_k}(x', 0) \xi_j \frac{\partial}{\partial \xi_k}.$$

There is another way of thinking the connection  $\nabla^*$  dual to the partial connection  $\nabla$  which has been considered by Trépreau in [61].

To a general vector field  $X$  on  $M$  it is associated its symbol  $\sigma(X)$  which is an invariantly defined function on the cotangent bundle  $T^*M$  of  $M$ . To a function  $f$  of class  $C^1$  on  $T^*M$  it is associated its hamiltonian field  $H_f$ .

Let  $X_j, j = 1, \dots, r$  be a local basis of  $K$ -tangent sections of  $TM$ . Let  $\Sigma_K$  be the orthogonal complement of  $K$  in  $T^*M$ . If  $X = \sum_{j=1}^r \phi_j X_j$  is a  $C^1$  section of  $K$  we have

$$H_{\sigma(X)}|_{\Sigma_K} = \sum_{j=1}^r \phi_j H_{\sigma(X_j)}|_{\Sigma_K} + \sum_{j=1}^r \sigma(X_j) H_{\phi_j}|_{\Sigma_K}.$$

Since  $\sigma(X_j), j = 1, \dots, r$  is zero on  $\Sigma_K$ , we deduce that the restricted hamiltonian field

$$H_{\sigma(X)}|_{\Sigma_K} = \sum_{j=1}^r \phi_j H_{\sigma(X_j)}|_{\Sigma_K}$$

depends only on the value of  $X$  at the base point and not on the chosen section. If  $X$  is tangent to  $N$ ,  $H_{\sigma(X)}$  when restricted to  $T_N^*M$  is tangent to  $T_N^*M$ . Hence we have constructed another vector field on  $T_N^*M$  which is in fact the same as the one associated with the connection dual to the partial connection  $\nabla$ .

Indeed, let as before  $(x', \xi'')$  be the canonical coordinates on the conormal bundle  $T_N^*M$ . Recall that the hamiltonian field of a function  $f = f(x, \xi)$  just looks like

$$H_f = \sum_{j,k=1}^{j,k=l+1} \frac{\partial f}{\partial \xi_k} \frac{\partial}{\partial x_j} - \frac{\partial f}{\partial x_j} \frac{\partial}{\partial \xi_k}.$$

The symbol of the section

$$X = \sum_{j=1}^{l+m} a_j \frac{\partial}{\partial x_j} \quad a_j(x', 0) = 0, \quad j = l+1, \dots, l+m$$

of  $K$  being  $\sigma(X) = \sum a_j \xi_j$  we can compute

$$H_{\sigma(X)}|_{T_N^*M} = \sum_{j=1}^l a_j(x', 0) \frac{\partial}{\partial x_j} - \sum_{j,k=l+1}^{l+m} \frac{\partial a_j}{\partial x_k}(x', 0) \xi_j \frac{\partial}{\partial \xi_k}$$

and the last expression proves that  $H_{\sigma(X)}$  is the same vector field on  $T_N^*M$  as  $\hat{X}$  computed previously, so the set of restricted hamiltonian fields  $H_{\sigma(X)}|_{T_N^*M}$  defines the same family of horizontal subspaces for the partial connection  $\nabla^*$ .

The next section is devoted to the application of the preceding results to the geometry of CR submanifolds of  $\mathbf{C}^n$ .

## §2. Application to generic submanifolds of $\mathbf{C}^n$

In this section we apply results of section 1 in the context of differential geometry in the complex euclidean space  $\mathbf{C}^n$ . Afterwards we check that our definitions recover those of Trépreau [61] and Tumanov [62].

Let  $T\mathbf{C}^n$  be the real tangent bundle of  $\mathbf{C}^n$  and  $J$  be the standard complex structure operator on  $T\mathbf{C}^n$ . Let  $T^*\mathbf{C}^n$  be the bundle of *holomorphic* ( $\mathbf{C}$ -linear) 1-forms on  $\mathbf{C}^n$ . In the canonical coordinates  $z = (z_1, \dots, z_n)$  its fiber over a point  $z$  consists of  $(1,0)$ -forms  $\omega = \sum_{j=1}^n \zeta_j dz_j, \zeta_j \in \mathbf{C}, j = 1, \dots, n$ . Then

$T^*\mathbf{C}^n$  is a *complex* manifold. It can be (and it is usually) identified with the real dual bundle of  $T\mathbf{C}^n$  introducing the real duality defined by

$$(\omega, X) \in T^*\mathbf{C}^n \times T\mathbf{C}^n \quad (\omega, X) \longmapsto \text{Im} \langle \omega, X \rangle .$$

In other words we identify real and holomorphic forms by  $\text{Im} \omega \leftrightarrow \omega$ .

Now, let  $M$  be a real submanifold of  $\mathbf{C}^n$ . In this identification, the conormal bundle  $T_M^*\mathbf{C}^n$  is a subbundle of  $T^*\mathbf{C}^n$  and it has fiber spaces

$$T_M^*\mathbf{C}^n[z] = \{ \omega \in T^*\mathbf{C}^n; \text{Im} \omega |_{T_z M} = 0 \} .$$

Hence the bundles  $T_M\mathbf{C}^n = T\mathbf{C}^n|_M/TM$  and  $T_M^*\mathbf{C}^n$  are in duality by

$$(\omega, X \text{ mod } TM) \longmapsto \text{Im} \langle \omega, X \rangle .$$

Assume moreover that  $M$  is generic (that is  $TM + JTM = T\mathbf{C}^n|_M$ ) and let  $\Sigma_M$  be the orthogonal complement of the complex tangent bundle  $T^cM$  in the cotangent bundle  $T^*M$ . In the terminology of linear partial differential equations it is the *characteristic set* (and since  $T^cM$  is a fiber bundle, the characteristic *manifold*) of the system of CR vector fields. It is easily checked that  $\Sigma_M$  and  $TM/T^cM$  are in duality in the same way.

Since  $M$  is CR,  $\Sigma_M$  is a fiber bundle and there is a canonical bundle epimorphism

$$\theta : T_M^*\mathbf{C}^n \rightarrow \Sigma_M,$$

defined by  $\theta(\omega) = \iota_M^* \omega$  where  $\iota_M : M \rightarrow \mathbf{C}^n$  is the natural injection. Since  $M$  is generic,  $\theta$  is an isomorphism.

On the other hand the complex structure  $J$  induces an isomorphism, still denoted by  $J$

$$J : TM/T^cM \rightarrow T_M\mathbf{C}^n .$$

LEMMA 2.1.  $\theta$  is the transposed of  $J$ , i.e.

$$(\omega, JX) = (\theta(\omega), X)$$

for every  $\omega, X$ .

(Indeed  $\langle \omega, JX \rangle = i \langle \omega, X \rangle$ ).

From now on we let  $S \subset M$  be a CR submanifold of  $M$  with the property that  $CRdim S = CRdim M$ . Equivalently it is required that  $T^cS = T^cM|_S$ . By restriction analogous pairs of bundles remain isomorphic when  $TM/T^cM$  is replaced by  $T_S M$ ,  $\Sigma_M$  is replaced by  $T_S^* M$ ,  $T_M\mathbf{C}^n$  is replaced by  $T\mathbf{C}^n|_S / (TM|_S + JTS) = E$ , and  $T_M^*\mathbf{C}^n$  is replaced by  $T_M^*\mathbf{C}^n \cap iT_N^*\mathbf{C}^n = E^*$ , but now  $T^cS$ -partial connections can be defined by means of the isomorphisms  $J$  and  $\theta$  on the two new bundles  $E$  and  $E^*$ . Note that the duplication essentially deals with complex differential geometry.

First, the results of the previous section apply with  $K = T^cM = TM \cap JTM$  and  $N = S$  and produce a  $T^cS$ -partial connection  $\nabla$  on  $T_S M$  together with the dual connection  $\nabla^*$  on  $T_S^* M$ . On the other hand, the push forward by  $J$  of  $\nabla$  defines a  $T^cS$ -partial connection  $\Theta$  on  $E$ ; its action on a section  $\vartheta$  of  $E$  in the direction of a complex tangent vector  $X$  is simply

$$\Theta_X \vartheta := J \nabla_X (J^{-1} \vartheta).$$

Similarly, the pull-back of the  $T^cS$ -partial connection  $\nabla^*$  by  $\theta$  defines a  $T^cS$ -partial connection  $\Theta^*$  on  $E^*$ , and  $\Theta^*$  is the connection dual to  $\Theta$  since  $\theta$  is the transposed of  $J$  (lemma 2.1).

Recall from section 1 that if  $X$  is a section of  $T^cM$  then  $\hat{X} = H_{\sigma(X)}|_{T_S^*M}$  is tangent to  $T_S^*M$ . In [61], Trépreau showed that  $E^*$  is a CR manifold, using a lemma which states that given such a vector field  $\hat{X}$  tangent to  $T_S^*M$  with horizontal part  $X$  complex tangent to  $M$ , there exists a unique vector field  $\tilde{X}$  complex tangent to  $E^*$  with the same horizontal part  $X$ . Moreover Trépreau states that

$$\hat{X} = d\theta(\tilde{X})$$

Hence we deduce that the  $T^cS$ -partial connection  $\Theta^* = \theta^*\nabla^*$  can alternately be given, as is originally done in [61], by means of the vector fields of the form  $\tilde{X}$ , i.e. horizontal subspaces of  $\Theta^*$  are spanned by tangent vectors to integral curves of  $\tilde{X}$ . We then have checked that the parallel translation in  $E^*$  introduced by Trépreau with the assumption of Proposition 1.2 is the same as the one associated with the  $T^cS$ -partial connection  $\Theta^*$  previously defined starting, as in section 1, with the partial connection associated with the bundle of complex tangents to  $M$ ,  $K = T^cM$ . Moreover, since  $\tilde{X}$  is complex tangent to  $E^*$ , we see that  $T^cE^*$  is the set of horizontal subspaces for the  $T^cS$ -partial connection  $\Theta^*$ . This has been noticed in [69] and will be useful in the next section when proving Theorem 3.4.

### §3. Orbits and the extension of CR functions.

In this section, it is assumed that  $M$  is a generic submanifold of  $\mathbf{C}^n$  of smoothness class  $C^2$ , and we let  $\mathbf{X}$  be the set of  $C^1$  sections over open subsets of  $M$  of  $T^cM$ . If  $z \in M$ , the subset of  $M$  consisting of points of  $M$  which can be reached by piecewise  $C^1$ -smooth integral curves of elements of  $\mathbf{X}$ , starting at  $z$ , is called the *CR-orbit* of  $z$ , and is denoted by  $\mathcal{O}[z]$ .

If  $U$  is an open subset of  $M$ ,  $\mathbf{X}|_U$  denotes the set of elements of  $\mathbf{X}$  restricted to  $U$ . It is well-known (cf. [59], [6], [61]) that

$$\lim_{\substack{\longrightarrow \\ U}} \mathcal{O}(\mathbf{X}|_U, \ddagger)$$

where  $U$  runs over the open neighborhoods of  $z$  in  $M$  defines the germ at  $z$  of the unique CR-submanifold of  $M$  with the same CR dimension as  $M$  of minimal dimension passing through  $z$ , which is called the *local CR-orbit* of  $z$  and is denoted by  $\mathcal{O}^{loc}[z]$ . When considering  $\mathcal{O}^{loc}[z]$  in the following we shall mean such a submanifold of a neighborhood of  $z$  in  $M$ , i.e. an actual representative of the germ. It plays the crucial role in the study of automatic extendibility of CR functions (cf. Theorem 3.1 below).

Recall that a smooth complex-valued function on  $M$  is called a *CR function* if it is annihilated by every antiholomorphic tangent vector field on  $M$ . A continuous function can be thought CR in the sense of distribution theory. We denote by  $CR(M)$  the set of all continuous CR functions on  $M$ .

For completeness we recall definitions from [61] and [69]. We say that a manifold  $\tilde{M}$  with boundary is *attached to  $M$  at  $(m, u)$* ,  $m \in M$ ,  $u \neq 0$ ,  $u \in T_M\mathbf{C}^n[m]$  if  $b\tilde{M} \cap U = M \cap U$  for some neighborhood  $U$  of  $m$ , and  $u$  is represented by a vector  $u_1 \in T_m\tilde{M}$  directed inside  $\tilde{M}$ .

Let  $f$  be a CR function on  $M$ ; we say that  $f$  is *CR-extendible* at  $(m, u)$  if it extends continuously to be CR on some  $\tilde{M}$  attached to  $M$  at  $(m, u)$ . When there is a CR submanifold  $S$  of  $M$  through  $m$  and a manifold  $\tilde{M}$  attached to  $M$  at  $(m, u)$ ,  $u \in T_M\mathbf{C}^n[m]$ , we also say that  $\tilde{M}$  is attached to  $M$  at  $(m, \eta)$ , if  $u$  represents  $\eta \in E_m$ ,  $\eta \neq 0$  ( $E$  is the bundle defined in section 2). Similarly it makes sense to consider CR-extendibility at  $(m, \eta)$ ,  $m \in S$ ,  $\eta \in E_m$ . But it should be noted that given  $\eta \neq 0$  in  $E_m$  does not determine  $\tilde{M}$  unambiguously unless  $S$  is complex



From now on we will require that  $M$  belong to the class  $C^{(k,\alpha)}$ ,  $k \geq 2, 0 < \alpha < 1$ . This regularity assumption can be justified since it behaves well when proving the strongest local results on CR-extendibility (In fact, it behaves well through the so-called *Bishop equation*, [68], Theorem 1.), and constructing wedges with ribs and an edge having such a regularity (cf. [4]). Moreover, we need manifolds of class at least  $C^2$  in order to apply Proposition 1.1. Since it will be of use in the proof of Theorem 3.4 we recall the following theorem due to Tumanov ([68])

**THEOREM 3.1.** (A. E. TUMANOV) *Let  $M$  be a generic submanifold of  $\mathbf{C}^n, n = p + q$ , with  $\dim M = 2p + q$ ,  $CRdim M = p$ , and of smoothness class  $C^{k,\alpha}$  ( $k \geq 2$ ),  $0 < \alpha < 1$ . For every point  $z \in M$  there exist  $r = r(z) = \dim \mathcal{O}_{[z]}^{loc} - 2$   $CRdim M$  manifolds with boundary  $\tilde{M}_1, \dots, \tilde{M}_r$  attached to  $M$  at  $z$ , of class  $C^{(k,\beta)}$  whenever  $0 < \beta < \alpha$  such that*

- (a) *Every CR function on  $M$  is CR-extendible to  $\tilde{M}_1, \dots, \tilde{M}_r$*
- (b)  $\sum_{j=1}^r T_{z'} \tilde{M}_j = T_{z'} M + JT_{z'} \mathcal{O}_{[z]}^{loc} \quad z' \text{ close to } z \text{ in } \mathcal{O}_{[z]}^{loc}$ .

*Moreover the manifold germ  $\mathcal{O}_{[z]}^{loc}$  is of class  $C^{(k,\beta)}$  whenever  $0 < \beta < \alpha$ .*

Note that  $\mathcal{O}_{[z]}^{loc}$  is at least of class  $C^2$  so it can play the role of  $N$  in Propositions 1 and 2. Using the connections constructed in section 2 we can reinterpret the main result on propagation of analyticity for CR functions recently proved by Tumanov.

According to Tumanov ([69], Proposition 7.3), the connection dual to the one that is constructed during the paper has the property that its horizontal subspaces are exactly fibers of the complex tangent bundle  $T^c E^*$ , hence, concludes Tumanov, the induced parallel translation need be the same as the one introduced on  $E^*$  by Trépreau. We have shown in section 2 that our connection  $\Theta$  has as a dual connection a connection  $\Theta^*$  with the same property; so  $\Theta = J_* \nabla$  coincides with the connection constructed by Tumanov.

Proposition 1.2 together with Theorem 5.1 in [69] leads to

**THEOREM 3.2.** *Let  $M \subset \mathbf{C}^n$  be a generic manifold and  $S \subset M$  a CR submanifold of  $M$  with the property that  $CRdim S = CRdim M$ . Let  $\gamma$  be a piecewise smooth integral curve of  $T^c M$  running from  $z' \in S$  to  $z'' \in S$  and let  $\Phi_\gamma$  be the associated composed flow. Then for every  $\epsilon > 0$ , every  $\eta' \in E_{z'}$  and every manifold  $\tilde{M}'$  attached to  $M$  at  $(z', \eta')$ , there exists another manifold  $\tilde{M}''$  attached to  $M$  at  $(z'', \eta'')$ ,  $\eta'' \in E_{z''}$  such that*

- (a)  $|\eta'' - Jd\Phi_\gamma(z).J^{-1}\eta'| < \epsilon$
- (b) *if a CR function on  $M$  extends to be CR on  $\tilde{M}'$  it extends to be CR on  $\tilde{M}''$*
- (c) *if  $M, \tilde{M}'$  belong to  $C^{k,\alpha}$  ( $k \geq 2$ ),  $0 < \gamma < \alpha < 1$  then there exists such a  $\tilde{M}'' \in C^{(k,\gamma)}$ .*

Theorem 3.2 shows that the so-called propagation of analyticity for CR functions is intrinsically related to the geometry of the base manifold  $M$ . Moreover, it fundamentally means that the study of extendibility for CR functions is closely related to the study of sections of the complex tangent space to  $M$ .

Following Sussmann ([59]), we begin with some adapted terminology and recalls. Let  $X \in \mathbf{X}$  be a local section of  $T^c M$ . The  $C^1$  integral curves  $t \rightarrow \gamma(t)$  of  $X$  generate local diffeomorphisms of  $M$  where they are defined (the so-called *flow* of  $X$ ) which we will denote by  $z \rightarrow X_t z$ . Composites of several maps of the form  $X_t$  can produce local diffeomorphisms of neighborhoods of points that are *far* from each other in a same CR-orbit. If  $X = (X_1, \dots, X_m)$  is an element of  $\mathbf{X}^m$  such that for  $t = (t_1, \dots, t_m) \in \mathbf{R}^m$ , the map  $z \rightarrow X_{m,t_m} \cdots X_{1,t_1} z$  is well defined in a neighborhood of  $z$ , we will still denote it for convenience by  $X_t$  or  $\Phi$  (cf. Proposition 1.2).

Let  $\Delta_{\mathbf{X}}$  be the *distribution spanned by  $\mathbf{X}$* , i.e. the mapping which to  $z \in M$  assigns the linear hull of vectors  $X(z)$  where  $X$  belongs to  $\mathbf{X}$ : it is just the distribution associated with the complex tangent bundle of  $M$ . We let  $P_{\mathbf{X}}$  denote the smallest distribution which contains  $\Delta_{\mathbf{X}}$  and is invariant under complex-flow

diffeomorphisms, or for short the smallest  $\mathbf{X}$ -invariant distribution which contains  $\Delta_{\mathbf{X}}$ . Precisely,  $P_{\mathbf{X}}(z)$  is the linear hull of vectors of the form  $dX_t(v)$  where  $v \in \Delta_{\mathbf{X}}(z')$  and  $z = X_t z'$ . A  $C^1$  distribution  $P$  on  $M$  has the *maximal integral manifold property* if for every  $z \in M$  there exists a submanifold  $S$  of  $M$  such that  $z \in S$  and for every  $z' \in S$ ,  $T_{z'} S = P(z')$ . Moreover,  $S$  is said to be a *maximal integral manifold* of  $P$  if  $S$  is an integral manifold of  $P$  such that every connected integral manifold of  $P$  which intersects  $S$  is an open submanifold of  $S$ .

Then the results of Sussmann, which extend to the  $C^2$  case tell us that  $\mathcal{O}[z]$  is a (connected) maximal integral submanifold of  $P_{\mathbf{X}}$  (perhaps with a finer topology) and admits a unique differentiable structure making the injection  $i : \mathcal{O}[z] \rightarrow M$  an immersion of class  $C^1$ .

We now introduce the following definitions.

DEFINITION 3.3. Let  $M$  be a generic submanifold of  $\mathbf{C}^n$  and  $z \in M$ .  $M$  is called *minimal at  $z$*  if  $\mathcal{O}^{loc}[z]$  contains a neighborhood of  $z$  in  $M$ . It is called *globally minimal at  $z$*  if  $\mathcal{O}[z]$  contains a neighborhood of  $z$ .

In view of the global results of Sussmann definition 3.3 means that the generic manifold  $M$  is globally minimal at a point  $z$  if and only if there exist a finite number of points  $z'_l$ ,  $l = 1, \dots, d$  in the CR-orbit of  $z$  and composed flow diffeomorphisms  $\Phi'_l$ ,  $l = 1, \dots, d$  of a neighborhood of  $z'_l$  in  $M$  on a neighborhood of  $z$  in  $M$  respectively such that

$$T_z M = \sum_{l=1}^d d\Phi'_l(z'_l) \cdot (T_{z'_l}^c M).$$

We are now able to prove the theorem conjectured by Trépreau in [61] which is the natural generalization of a celebrated theorem of Tumanov ([67]). Here is the substance of this paper.

THEOREM 3.4. Let  $M$  be a generic submanifold of  $\mathbf{C}^n$  of smoothness class  $C^{(k,\alpha)}$ ,  $k \geq 2, 0 < \alpha < 1$  which is globally minimal at a point  $z \in M$ . Then for every  $z'$  in the CR-orbit of  $z$  there exists a wedge  $\mathcal{W}$  of edge  $M$  at  $z'$  such that

(\*) every CR function on  $M$  extends holomorphically into  $\mathcal{W}$ .

PROOF. We shall make use of the following abuse of language : we will say that a CR-function  $u$  is CR-extendible in the direction  $v \in TM/T^c M[z]$  if it is in fact CR-extendible in the direction of  $Jv$ . Let us consider the set

$$H_z = Vect \{v \in T_z M/T_z^c M; u \text{ is CR-extendible at } (z, v)\}$$

and its preimage under the natural surjection  $\pi : TM \rightarrow TM/T^c M$

$$\hat{H}_z = \pi^{-1}(H_z) \subset T_z M.$$

LEMMA 3.5. Let  $X$  be a  $C^1$  section of  $T^c M$  over a neighborhood of  $z \in M$  and let  $\Phi_t$  be the flow of  $X$  and  $\Phi = \Phi_t$  for some  $t$ . Then, if  $v \in T_z M$ ,

$$v \in \hat{H}_z \iff d\Phi(z).v \in \hat{H}_{\Phi(z)}.$$

PROOF. Since the statement is a symmetric and a transitive one we can assume that  $z$  and  $z'$  are so close that  $z' := \Phi(z)$  is contained in a CR submanifold  $S$  of  $M$  with  $CRdim S = CRdim M$  which is minimal at  $z$  (for instance take for  $S$  the local CR-orbit of  $z$ ) and such that  $z'$  belongs to the boundary of the manifolds whose existence comes from Theorem 3.1. Hence

$$(*) \quad \hat{H}_{z'} \supset T_{z'} S.$$

So if  $v$  belongs to  $T_z S$  there is nothing to add. On the other hand, if  $\xi = pr_{T_S M} v \neq 0$  we apply the propagation result Theorem 3.2 and obtain that for every  $\epsilon > 0$   $u$  is CR-extendible at  $(z', \xi'')$ , where  $\xi''$  is  $\epsilon$ -close in euclidean norm to  $\xi' = d\Phi(x).\xi$ ; so letting  $\epsilon$  decrease to zero, since every finite-dimensional vector space is closed, we have  $\xi' \in pr_{T_S M}(\hat{H}_{z'})$ . Because of (\*) the indetermination on the specific representative of  $\xi'$  is removed whence

$$d\Phi(z).v \in \hat{H}_{z'}$$

and the lemma is proved.

END OF PROOF OF THEOREM 3.4. The global lemma 3.5 and the condition of global minimality imply immediately that

$$\hat{H}_{z'} = T_{z'} M$$

for every  $z'$  in the (global) CR-orbit of  $z$ . The conclusion follows by the edge-of-the-wedge theorem and the proof is complete.

Theorem 4.1 admits an obvious generalization which involves the concept of  $\mathcal{W}_r$ -wedges. Recall that a  $\mathcal{W}_r$ -wedge at  $z$  with edge  $M$  is locally the general intersection of a wedge of edge  $M$  at  $z$  and a generic manifold containing  $M$  as a submanifold of codimension  $r$ .

**THEOREM 3.6.** *Let  $M$  be a generic submanifold of  $\mathbf{C}^n$  of smoothness class  $C^{(k,\alpha)}$ ,  $k \geq 2, 0 < \alpha < 1$ , and let  $r = \dim \mathcal{O}[z] - 2CR \dim M$ . Then for every  $z'$  in the CR-orbit of  $z$ , every  $\gamma$  with  $0 < \gamma < \alpha$ , there exists a  $\mathcal{W}_r$ -wedge  $\mathcal{W}$  of edge  $M$  at  $z'$  and of smoothness class  $C^{k,\gamma}$  such that*

(\*) every CR function on  $M$  extends to be CR on  $\mathcal{W}$ .

Moreover, the tangent space to  $\mathcal{W}$  at  $z'$  spans  $T_{z'} M + JT_{z'} \mathcal{O}[z]$ .

**PROOF.** The same argument runs in proving that  $\hat{H}_{z'}$  contains  $T_{z'} M + JT_{z'} \mathcal{O}[z]$  and the conclusion then follows by the edge-of-the wedge theorem of Ayrapetyan ([4]), in the classes  $C^{(k,\alpha)}$ ,  $k \geq 2, 0 < \alpha < 1$ .

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