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# Resonance modes in a 1D medium with purely absorbing boundaries

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Studying the problem of wave propagation in dissipative media can be made by searching for "resonance modes" or free oscillations regimes. In the present article, a simple case is investigated, which allows to enlighten the respective interest of different, classical methods, without any approximation, some of them being rather delicate. This case is the 1D propagation in a homogeneous medium having two purely resistive terminations, the calculation of the Green function being done by using three methods. The first one is the straightforward use of the "compact" solution in the frequency domain, then the rules of residues. The second one is the expansion in orthogonal modes in the frequency domain, the modes having eigenfrequencies depending on the frequency. Calculating the derivative of the eigenfrequencies with respect to frequency allows to get the same result than the first method. Finally the method of separation of variables (space and time) leads to a solution depending on the initial conditions, and can be applied to the Green function. The question of the orthogonality of the (complex) resonance modes is investigated with several points of view. The results of the three methods generalize or/and correct some results already existing in the literature. Finally the extension of the problem to a medium where dissipation is coupled to propagation is investigated.

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## I. INTRODUCTION

Studying the problem of wave propagation in dissipative media can be made by searching for "resonance modes" [1], or free oscillations regimes. These modes can be non orthogonal

for a certain choice of their product, entailing some difficulties depending on the mathematical treatment, made either in the time or frequency domain. Two classical methods exist for such a problem:

i) in the frequency domain, the complex dependence of dissipation with respect to frequency does not make impossible the use of orthogonal decomposition, but leads to eigenmodes and eigenfrequencies *depending on frequency*. It is the case for the classical theory of room acoustics (see e.g. Morse and Ingard [2]). To return to time domain in order to deduce the resonance modes is a rather delicate task, especially because of the calculation of the derivation of eigenfrequencies with respect to frequency.

ii) in the time domain, the use of time and space variable as separate variables leads directly to the basis of modes, but they are non orthogonal for the most common product, entailing difficulties when searching the coefficients depending for instance on the initial conditions. Nevertheless Guyader [3] has solved the problem for a particular case.

The present article is devoted to the study of a special, simple 1D case for which a straightforward solution exists, by applying the rule of residues to the compact form or the Fourier domain solution: as discussed by Levine[4], this compact solution, avoiding the sum of a series, is “relatively poorly, if not entirely, unknown to the general acoustics community”. For the studied case all calculations can be carried out analytically without any approximations, exhibiting the properties of the different methods. This case corresponds to the onedimensional propagation in a homogeneous medium bounded by two other semi-infinite media with different characteristic impedances, dissipation being therefore due to radiation at infinity. This case is especially interesting because of its physical significance (it is probably the simplest radiation problem), and also because it realizes a possible transition between Neumann and Dirichlet boundary conditions.

In section II, the equations to be solved are stated, with some possible physical interpretations. As a first step, the classical, compact solution in the frequency domain is established (section III), with its inverse Fourier Transform, corresponding to the successive reflections (section IV). The second step is the use of the rule of residues in order to determine the resonance modes (section V, the basic result being given by equation 37). Then the results are compared to those of the two aforementioned methods, i.e. the method of the eigenmodes in the frequency domain (section VI), and the method of separation of variables (section VII), which gives the result for given initial conditions (the corresponding results being equations

66, 77 and 79). Finally an extension to the case where absorption is coupled to propagation is treated, with some approximated formulas and numerical results, which are compared to the time domain solutions with successive reflections (section VIII).

## II. STATEMENT OF THE PROBLEM, PHYSICAL INTERPRETATION

The Green function  $g(x, t | x_0, t_0)$  for the wave equation is solution of the following equation:

$$\left[ \partial_{xx}^2 - \frac{1}{c^2} \partial_{tt}^2 \right] g(x, t | x_0, t_0) = -\delta(x - x_0)\delta(t - t_0), \quad (1)$$

where  $x$  and  $x_0$  are the spatial coordinates of the receiver and source, respectively (or vice-versa),  $t$  and  $t_0$  the times of observation and excitation, respectively,  $c$  the speed of sound.  $\delta(x)$  is the Dirac function.

For sake of simplicity,  $x_0$  and  $t_0$  are considered to be fixed, and the following notation is used :  $g(x, t) = g(x, t | x_0, t_0)$ . Moreover in the whole paper, the choice of  $t_0 = 0$  is made, except for the final results. For negative  $t$ , the function is zero, as well as its first derivative. The Green function satisfies the following boundary conditions:

$$\partial_x g(x, t) = \frac{1}{c\zeta} \partial_t g(x, t) \quad \text{at } x = 0, \quad (2)$$

and

$$\partial_x g(x, t) = -\frac{1}{c\zeta_\ell} \partial_t g(x, t) \quad \text{at } x = \ell. \quad (3)$$

where  $\zeta = Z/\rho c$ ,  $\rho$  is the density of the fluid, and  $Z$  the impedance at  $x = 0$ , which is assumed to be a real quantity, independent of the frequency. Possible corresponding physical problems are discussed hereafter. Similarly,  $\zeta_\ell = Z_\ell/\rho c$ , where  $Z_\ell$  is the impedance at  $x = \ell$  ( $\ell$  being positive).

An obvious physical interpretation for the quantities  $\zeta$  and  $\zeta_\ell$  is the following: consider for  $x < 0$  and  $x > \ell$  (see figure 1) two media with characteristic impedances  $\rho_- c_-$  and  $\rho_+ c_+$ , respectively. If the media are non dissipative, the impedances are real, and can be larger or smaller than the impedance of the bounded medium,  $\rho c$ . Moreover, they are positive, because they correspond to waves outgoing from the bounded medium. Therefore this is the problem of planar pressure waves in a stratified medium, the direction of propagation

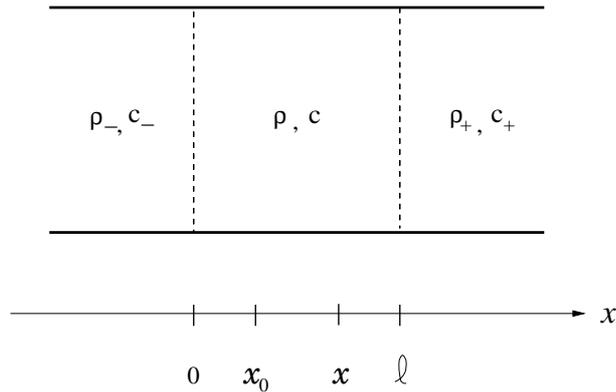


FIG. 1: 1D medium bounded with two other media

being normal to the interfaces. A generalization to more complex stratified media would be possible, at least numerically. In this problem, the Green function corresponds to the acoustic pressure: of course, it has not the dimension of a pressure, but the solution for a “concrete” problem with source can be easily solved, as explained in standard textbooks, and discussed in a recent paper by Levine[4].

Other problems can correspond to the previous equations :

i) in an approximate way, ignoring higher order duct modes, the problem of planar waves in a duct terminating in two semi-infinite ducts with different cross sections areas, the quantities  $\zeta$  and  $\zeta_\ell$  being the ratios of the areas. The approximation is good at low frequencies.

ii) the problem of absorbing termination : the terminal impedances  $Z$  and  $Z_\ell$  can be the impedances of absorbing media (at low frequencies, a porous medium open to a large space can be an approximation of a pure resistance, due to viscous effects).

In all the previous problems, the quantities  $\zeta$  and  $\zeta_\ell$  are real and positive, the terminations being passive. If the termination is active, they can be negative. An example is the beginning of self-sustained oscillations in musical instruments: a nonlinear excitor, like a reed for a clarinet, can be linearized as a pure resistance. When the main control parameter, i.e. the pressure in the mouth of the musician, increases, the resistance becomes negative and the static regime becomes unstable, and an oscillation starts as an increasing exponential (see e.g. [5]).

### III. COMPACT SOLUTION FOR THE FOURIER TRANSFORM

The Fourier Transform of  $g(x, t)$  is denoted  $G(x, \omega)$ . It is equal to :

$$G(x, \omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} g(x, t) e^{-i\omega t} dt, \quad (4)$$

where

$$g(x, t) = \int_{-\infty}^{+\infty} G(x, \omega) e^{i\omega t} d\omega. \quad (5)$$

The Fourier Transform of equation (1), is found to be:

$$\left( \partial_{xx}^2 + \frac{\omega^2}{c^2} \right) G(x, \omega) = -\frac{1}{2\pi} \delta(x - x_0), \quad (6)$$

and similarly for the boundary conditions (2) and (3) :

$$\partial_x G(x, \omega) = \frac{i\omega}{c\zeta} G(x, \omega) \text{ at } x = 0 \quad (7)$$

$$\partial_x G(x, \omega) = -\frac{i\omega}{c\zeta_\ell} G(x, \omega) \text{ at } x = \ell. \quad (8)$$

We notice the difficulty of the problem treated by the present article: while the terminal impedances are independent of frequency, the boundary conditions are frequency dependent. Nevertheless a classical, compact, solution is well known, as explained hereafter. If  $x \neq x_0$  the solutions of equation (6) can be written as

$$G(x, \omega) = A^- \cosh \left[ \frac{i\omega x}{c} + \eta \right] \text{ if } x < x_0 \quad (9)$$

and

$$G(x, \omega) = A^+ \cosh \left[ \frac{i\omega(\ell - x)}{c} + \eta_\ell \right] \text{ if } x > x_0. \quad (10)$$

In order to satisfy the boundary conditions, the following definition is used:

$$\zeta = \coth \eta ; \zeta_\ell = \coth \eta_\ell \quad (11)$$

(as we will see later, the quantities  $\eta$  and  $\eta_\ell$  can be either real or complex, depending on the values of  $\zeta$  and  $\zeta_\ell$ ).

At  $x = x_0$ , writing the continuity of the function and the jump of its first derivative, the following result is obtained :

$$G(x, \omega) = \frac{c}{2\pi i \omega} \frac{\cosh[\eta + i\omega x_0/c] \cosh[\eta_\ell + i\omega(\ell - x)/c]}{\sinh(i\omega\ell/c + \eta + \eta_\ell)} \text{ if } x \geq x_0 \quad (12)$$

and a similar result if  $x \leq x_0$ , by interchanging  $x$  and  $x_0$ .

The case  $\zeta = 1$  (semi-infinite tube or medium) corresponds to  $\eta = \infty$  : it is discussed in the next subsection.

#### IV. SOLUTION IN THE TIME DOMAIN (SUCCESSIVE REFLECTIONS)

Equation (12) can be transformed in the time domain, leading to a solution corresponding to the successive reflections of the Green function in infinite space at the two boundaries. It will be the reference solution for the check of the validity of the modal expansion. The second factor of the denominator can be written as

$$\frac{1}{\sinh(i\omega\ell/c + \eta + \eta_\ell)} = \frac{2e^{-\eta - \eta_\ell - i\omega\ell/c}}{1 - e^{-2\eta - 2\eta_\ell - 2i\omega\ell/c}}, \quad (13)$$

and, if the modulus of the exponential at the denominator is less than unity (this is discussed hereafter), as:

$$\frac{1}{\sinh(i\omega\ell/c + \eta + \eta_\ell)} = 2e^{-\eta - \eta_\ell - i\omega\ell/c} [1 + F(\omega) + F^2(\omega) + F^3(\omega) + \dots], \quad (14)$$

where

$$F(\omega) = e^{-2\eta - 2\eta_\ell - 2i\omega\ell/c} \quad (15)$$

is the function corresponding to a complete round trip of a wave in the tube, of duration  $2\ell/c$ . The quantities  $r = e^{-2\eta}$  and  $r_\ell = e^{-2\eta_\ell}$  are the reflection coefficients at the boundaries and are equal to:

$$r = e^{-2\eta} = (\zeta - 1)/(\zeta + 1); \quad r_\ell = e^{-2\eta_\ell} = (\zeta_\ell - 1)/(\zeta_\ell + 1). \quad (16)$$

Concerning the numerator of (12), it can be written as follows :

$$\cosh\left[\eta + \frac{i\omega x}{c}\right] \cosh\left[\eta_\ell + i\omega \frac{\ell - x_0}{c}\right] = \frac{1}{4} e^{+\eta + \eta_\ell + i\omega\ell/c} G_p(x, \omega), \quad (17)$$

where

$$G_p(x, \omega) = e^{-i\omega(x-x_0)/c} + re^{-i\omega(x+x_0)/c} + r_\ell e^{-i\omega(2\ell-x-x_0)/c} + rr_\ell e^{-i\omega(2\ell-x+x_0)/c}. \quad (18)$$

Therefore the Green function in the frequency domain is:

$$G(x, \omega) = \frac{c}{4\pi i\omega} G_p(x, \omega) [1 + F(\omega) + F^2(\omega) + F^3(\omega) + \dots]. \quad (19)$$

The factor  $G_p(x, \omega)/i\omega$  corresponds to the four “primary” waves arriving during the first cycle of duration  $2\ell/c$ , and this packet is simply reproduced at times  $2\ell/c, 4\ell/c, 6\ell/c$ , etc... (see for a detailed explanation e.g. Kergomard [5]). The inverse Fourier Transform of the function  $G(x, \omega)/i\omega$  is now easily obtained as

$$\partial_t g(x, t) = \frac{1}{2} c g_p(x, t) * [\delta(t) + f(t) + f(t) * f(t) + f(t) * f(t) * f(t) + \dots], \quad (20)$$

where

$$g_p(x, t) = \delta(t - (x - x_0)/c) + r(t) * \delta(t - (x + x_0)/c) + r_\ell(t) * \delta(t - (2\ell - x - x_0)/c) + r(t) * r_\ell(t) * \delta(t - (2\ell - x + x_0)/c), \quad (21)$$

with

$$r(t) = r\delta(t), \quad r_\ell(t) = r_\ell\delta(t) \quad (22)$$

and

$$f(t) = r(t) * r_\ell(t) * \delta(t - 2\ell/c). \quad (23)$$

Finally, calculating the integral  $h_p(x, t)$  of the function  $g_p(x, t)$ , with the condition for negative times, the final result is found to be, whatever the sign of  $(x - x_0)$ :

$$g(x, t) = \frac{c}{2} h_p(x, t) * [\delta(t) + f(t) + f(t) * f(t) + f(t) * f(t) * f(t) + \dots], \quad (24)$$

where

$$h_p(x, t) = H[t - |x - x_0|/c] + rH[t - (x + x_0)/c] + rH[t - (2\ell - x - x_0)/c] + rr_\ell H[t - (2\ell - |x - x_0|)/c] \quad (25)$$

and

$$f(t) = rr_\ell \delta(t - 2\ell/c), \quad (26)$$

$H(t)$  being the step function. The condition of validity of the expansion (14) is  $|rr_\ell| < 1$ . We notice that if  $\zeta$  is real and positive,  $|r| < 1$ , and similarly for  $\zeta_\ell$ . Therefore the condition is satisfied when the two boundaries are absorbing, or, more precisely, if the combination of the two reflections is absorbing. What happens in the case  $|rr_\ell| > 1$ ? It is possible to replace equation (13) by an equivalent expression with  $1 - e^{2\eta+2\eta_\ell+2i\omega\ell/c}$  at the denominator, but this would correspond to an infinite series of non causal functions in the time domain: the question will be discussed in section V C.

Otherwise if  $r$  is positive ( $|\zeta| > 1$ ),  $\eta$  is real, and if  $r$  is negative ( $|\zeta| < 1$ ),  $\eta = \mu + i\pi/2$ , where  $\mu$  is real; a similar remark can be made for the end  $\ell$ . The most of the following calculations are valid for all cases,  $\eta$  and  $\eta_\ell$  being real or complex.

Finally, if  $\zeta$  (respectively  $\zeta_\ell$ ) is unity, the reflection coefficient  $r$  (respectively  $r_\ell$ ) vanishes, as well as  $f(t)$ : the first term of the Green function is the Green function of an infinite medium, the first two terms correspond to a semi-infinite medium, etc... As it will be seen in the next section, no modes can be found for these cases, because either  $\eta$  or  $\eta_\ell$  tends to infinity.

## V. EXPANSION IN RESONANCE MODES USING THE INVERSE FOURIER TRANSFORM

Putting the expression in the frequency domain (12) in the equation (5) leads to the modal expansion of the time domain expression. The tool is the rule of residues. If all the poles of the expression (12) are simple and located on or above the real axis, the following equation can be used:

$$g(x,t) = 2\pi i \Sigma \quad \text{if } t > 0 \quad \text{and } 0 \quad \text{if } t < 0, \quad (27)$$

where  $\Sigma$  is the sum of the residues (see e.g. Morse and Ingard[2] p 17, changing  $i$  in  $-i$ ).

### A. Calculation of the poles

The poles corresponding to the zeros of function  $\sinh$  satisfy:

$$\sinh \left[ \frac{i\omega_n \ell}{c} + \eta + \eta_\ell \right] = 0, \quad (28)$$

or

$$\frac{i\omega_n \ell}{c} + \eta + \eta_\ell = in\pi, \quad (29)$$

where  $n$  is an integer. In order for the poles to be above the real axis, the condition is  $\text{Re}(\eta + \eta_\ell) > 0$ . It is equivalent to the condition previously obtained for the successive reflections expansion :  $|rr_\ell| < 1$ .

Depending on the values of  $\zeta$  and  $\zeta_\ell$ , different cases must be distinguished :

i) if  $|\zeta| > 1$  and  $|\zeta_\ell| > 1$  (real  $\eta$  and  $\eta_\ell$ ):

$$\omega_n = \frac{n\pi c}{\ell} + i \frac{(\eta + \eta_\ell)c}{\ell}; \quad n = -\infty, \dots, -2, -1, 0, 1, 2, \dots + \infty. \quad (30)$$

As already remarked by Filippi[1] or Guyader [3], the imaginary part of the complex frequency is independent of  $n$ , and the real part is independent of the absorption, corresponding to the values for pure Neumann conditions (infinite  $\zeta$  and  $\zeta_\ell$ ).

ii) if  $|\zeta| < 1$  and  $|\zeta_\ell| < 1$  (complex  $\eta$  and  $\eta_\ell$ :  $\eta = \mu + i\pi/2$ ;  $\eta_\ell = \mu_\ell + i\pi/2$ ):

$$\omega_n = \frac{n\pi c}{\ell} + i \frac{(\mu + \mu_\ell)c}{\ell}; \quad n = -\infty, \dots, -2, -1, 0, 1, 2, \dots + \infty. \quad (31)$$

The imaginary part is still independent of  $n$ , and the real part independent of the absorption, corresponding to the values for pure Dirichlet conditions (zero  $\zeta$  and  $\zeta_\ell$ ).

iii) if  $|\zeta| > 1$  and  $|\zeta_\ell| < 1$  (real  $\eta$  and complex  $\eta_\ell$ :  $\eta_\ell = \mu_\ell + i\pi/2$ )

For this mixed case

$$\omega_n = \frac{m\pi c}{2\ell} + i \frac{(\eta + \mu_\ell)c}{\ell}; \quad m = -\infty, \dots, -3, -1, 1, 3, \dots + \infty, \quad (32)$$

where  $m = 2n - 1$  is an odd integer. The real part corresponds to a problem with Neumann and Dirichlet conditions at  $x = 0$  at  $x = \ell$ , respectively. The last (mixed) case ( $|\zeta| < 1$  and  $|\zeta_\ell| > 1$ ) is obvious, and is not treated here.

## B. Calculation of the residues

In all cases, the Taylor expansion of the function  $\sinh$  in equation (12) at the first order of the quantity  $(\omega - \omega_n)$  can be determined. The result is:

$$\sinh \left[ i \frac{\omega \ell}{c} + \eta + \eta_\ell \right] = i(-1)^n (\omega - \omega_n) \frac{\ell}{c}. \quad (33)$$

Thus, for  $\omega$  close to the pole  $\omega_n$ :

$$G(x, \omega) = -\frac{c^2}{2\pi\omega_n \ell} \frac{\cosh [\eta + i\omega_n x_0/c] \cosh [\eta_\ell + i\omega_n(\ell - x)/c]}{(-1)^n (\omega - \omega_n)}, \quad (34)$$

or, using equation (29)

$$G(x, \omega) = -\frac{c^2}{2\pi\omega_n \ell} \frac{\cosh [\eta + i\omega_n x_0/c] \cosh [\eta + i\omega_n x/c]}{(\omega - \omega_n)}. \quad (35)$$

The residue corresponding to the pole  $\omega = 0$ , remains to be calculated. For small  $\omega$ ,

$$G(x, \omega) = \frac{c}{2\pi i \omega} \frac{\cosh \eta \cosh \eta_\ell}{\sinh(\eta + \eta_\ell)} = \frac{c}{2\pi i \omega} \frac{1}{\tanh \eta + \tanh \eta_\ell} = \frac{c}{2\pi i \omega} \frac{1}{\zeta^{-1} + \zeta_\ell^{-1}}. \quad (36)$$

Finally, using equation (27), the inverse FT of  $G(x, \omega)$  is obtained, as follows:

$$g(x, t) = H(t) \frac{c^2}{\ell} \sum_n \frac{f_n(x) f_n(x_0)}{i\omega_n} e^{i\omega_n t} + H(t) \frac{c}{\zeta^{-1} + \zeta_\ell^{-1}}, \quad (37)$$

where

$$f_n(x) = \cosh(\eta + i\omega_n x/c) = (-1)^n \cosh(i\omega_n(\ell - x)/c + \eta_\ell). \quad (38)$$

This formula is valid for all aforementioned cases, which are detailed in Appendix A. Some comments can be made:

- the mode shapes  $f_n(x)$  are complex functions of the space variable, and the question of their orthogonality will be discussed in section VII;
- the decreasing of the modes is identical for all non constant modes;
- there are two non oscillating modes, corresponding to the poles  $\omega = \omega_0$  and  $\omega = 0$ ;
- if one of the impedances  $\zeta$  or  $\zeta_\ell$  is zero, the constant mode disappears, as it is intuitive, in order to satisfy a Dirichlet condition;
- for the three considered cases in Appendix A, we notice that

$$\omega_{-n} = -\omega_n^* \text{ and } f_n(x) = f_{-n}^*(x). \quad (39)$$

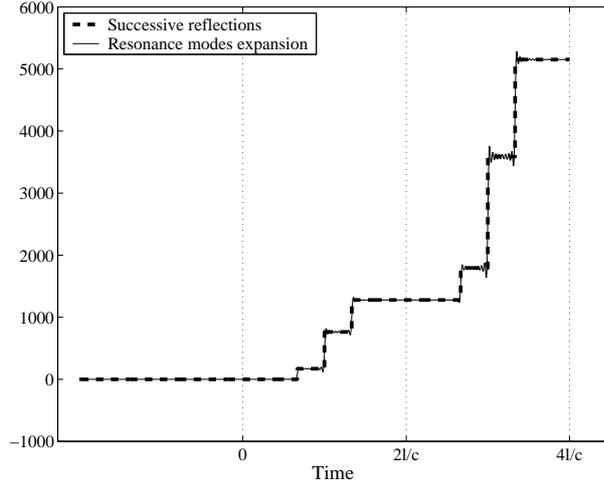


FIG. 2: Green function as a function of time with active boundary conditions  $\zeta_i = \zeta_\ell = -3.6975$ . Comparison between the successive reflections method and modal expansion (102 modes, i.e. maximum  $n = 50$ ). Locations of the source and receiver are  $x_0 = 0.1m$  and  $x = 0.5m$ , respectively. The scale for the  $y$ -axis corresponds to  $c = 340 \text{ m.s}^{-1}$ . Notice that constant mode is equal to  $-628.5 \text{ m.s}^{-1}$ .

As a consequence, the solution  $g(x, t)$  is real. It could be possible to transform the sum by adding the two terms corresponding to  $n$  and  $-n$ , as it is usually done for non absorbing boundaries. Nevertheless it appears that the formulas become intricate.

- when the boundaries tend to non absorbing boundaries, the result tends to the classical result. A particular case occurs when both  $\zeta$  and  $\zeta_\ell$  tend to infinity ( $\eta$  and  $\eta_\ell$  tend to zero) : the combination of the mode of frequency  $\omega_0$  and the constant mode results in a mode increasing linearly with time, equal to  $H(t)tc^2/\ell$ .

- equations (35) and (36) lead directly to another form of the Fourier Transform of the result given by (37), written as a series.

An example of comparison of the successive reflections method and modal expansion is shown in figure 2. We notice that it is satisfactory and that the Gibbs phenomenon appears, because of the truncated series of modes.

### C. The case of active boundaries

What happens when the combination of boundaries is active, i.e. when  $|rr_\ell| > 1$ , or  $\text{Re}(\eta + \eta_\ell) < 0$  (at least one of the impedances  $\zeta$  or  $\zeta_\ell$  is negative)? An equation similar to equation (27) can be used, leading to a non causal solution, as well as the solution with successive reflections, as discussed in section IV. This solution is easy to find, replacing  $(t - t_0)$  by  $-(t - t_0)$ ,  $\zeta$  by  $-\zeta$ ,  $\zeta_\ell$  by  $-\zeta_\ell$  in the equations (1 to 3), but it is not the Green function. On the contrary, it is possible to prove that equations (24) and (37) **remain valid for active boundary conditions**. The detailed solution given in appendix A exhibits that if a new function  $\tilde{g}(x, t) = g(x, t) \exp(-\tilde{\eta}t)$ , where  $\tilde{\eta} > -\text{Re}(\eta + \eta_\ell) > 0$  can be substituted in the initial problem, equation (1) becomes

$$\partial_{xx}^2 \tilde{g}(x, t) - \frac{1}{c^2} [\partial_t + \tilde{\eta}]^2 \tilde{g}(x, t) = -\delta(x - x_0) e^{-\tilde{\eta}t} \delta(t)$$

and similarly for equations (2) and (3). Going in the frequency domain leads to equations (6 to 8), where  $G(x, \omega)$  is replaced by  $\tilde{G}(x, \omega)$  and  $i\omega$  by  $(i\omega + \tilde{\eta})$ , and a similar result for equation (12). The analysis of both successive reflections and poles and residues leads to the result  $\tilde{g}(x, t) = g(x, t) \exp(-\tilde{\eta}t)$ , where  $g(x, t)$  is given by equations (24) and (37), respectively, and the proof is achieved. We do not repeat here the complete procedure.

## VI. EIGENMODES EXPANSION IN THE FREQUENCY DOMAIN, AND ITS INVERSE FOURIER TRANSFORM

### A. Eigenmodes expansion in the frequency domain

In order to calculate the inverse Fourier Transform of  $G(x, \omega)$ , another solution is possible: the expansion of  $G(x, \omega)$  in (orthogonal) eigenmodes. This is done for a particular case by Filippi ([1]) (this author considers another type of excitation instead of the Dirac function, thus uses the Laplace Transform instead of the Fourier Transform). We will see how this method leads to the same poles and residues that the direct method using the compact expression (12). We are searching for the following expansion :

$$G(x, \omega) = \sum_n G_n(x, \omega), \quad (40)$$

where the eigenmodes  $G_n(x, \omega)$  are solutions of the equation

$$\left[ \partial_{xx}^2 + \frac{\theta_n^2(\omega)}{c^2} \right] G_n(x, \omega) = 0, \quad (41)$$

and satisfy the boundary conditions (7) and (8).  $\theta_n(\omega)$  are the eigenvalues. The key point is that the eigenmodes  $G_n(x, \omega)$  and eigenvalues  $\theta_n(\omega)$  depend on the frequency  $\omega$  because of the boundary conditions. The solutions of equation (41) can be written as follows :

$$G_n(x, \omega) = \cosh(i\theta_n(\omega)x/c + \varphi_n(\omega)) \quad (42)$$

where  $\theta_n(\omega)$  and  $\varphi_n(\omega)$  are given by the boundary conditions. Thus they satisfy:

$$\theta_n(\omega) \tanh \varphi_n(\omega) = \frac{\omega}{\zeta} \quad (43)$$

and

$$\theta_n(\omega) \tanh(i\theta_n(\omega)\ell/c + \varphi_n(\omega)) = -\frac{\omega}{\zeta_\ell}. \quad (44)$$

Eliminating quantity  $\varphi_n(\omega)$ , the eigenvalues are found to satisfy the following equation:

$$\tanh(i\theta_n(\omega)\ell/c) \left[ \theta_n(\omega) + \frac{\omega^2}{\theta_n(\omega)\zeta\zeta_\ell} \right] = -\omega \left[ \frac{1}{\zeta} + \frac{1}{\zeta_\ell} \right]. \quad (45)$$

When  $\theta_n(\omega)$  and  $\omega$  are not simultaneously zero, this equation can be rewritten as:

$$e^{2i\theta_n\ell/c} = \left[ \frac{\theta_n\zeta - \omega}{\theta_n\zeta + \omega} \right] \left[ \frac{\theta_n\zeta_\ell - \omega}{\theta_n\zeta_\ell + \omega} \right]. \quad (46)$$

The calculation of all solutions of this equation is not necessary, only a few number of them being useful in the following. We notice that orthogonality of the eigenmodes is ensured, and the norm of the modes is equal to:

$$\Lambda_n = \int_0^\ell G_n^2(x, \omega) dx = \frac{\ell}{2} + \frac{c}{4i\theta_n} [\sinh 2(i\theta_n\ell/c + \varphi_n) - \sinh 2\varphi_n]. \quad (47)$$

Therefore the solution of equation (6) can written as follows:

$$G(x, \omega) = \frac{c^2}{2\pi} \sum_n \frac{G_n(x, \omega)G_n(x_0, \omega)}{\Lambda_n(\theta_n^2(\omega) - \omega^2)}. \quad (48)$$

## B. Calculation of poles and residues

In order to calculate the inverse Fourier Transform, the rule of residues will be used again. The only terms of the series contributing to poles verify:

$$\theta_n(\omega) = \pm\omega. \quad (49)$$

Looking at equation (46), it can be seen that these two solutions lead to the same equation for  $\omega$ . Rewriting equation (46) by using equations (43) and (44), the resonance modes frequencies are found to be solution of equation (28):

$$\sinh \left[ \frac{i\omega\ell}{c} + \eta + \eta_\ell \right] = 0.$$

The solutions  $\omega_p$  of this equation are the non zero poles of the integral in the inverse FT. Nevertheless the pole  $\omega = 0$  exists again, because the zero value satisfies equation (49), the eigenvalue  $\theta_n(\omega) = 0$  satisfying equation (45)!

It remains to calculate the residues. Starting with the poles  $\omega_p \neq 0$ , we need to select in the series (48) the terms involving poles. For a given  $\omega_p$ , there are two terms. However it appears that the modes corresponding to  $\theta_n$  and  $-\theta_n$  are identical. As a consequence, only one term of the series contributes to the inverse FT: it will be denoted  $\theta_p(\omega)$ . The corresponding residue is found by expanding equation (48) for  $\omega$  close to  $\omega_p$ , as follows:

$$G(x, \omega) = \frac{c^2}{2\pi} \frac{G_p(x, \omega_p)G_p(x_0, \omega_p)}{\Lambda_p(2\omega_p)(\omega_p - \omega)(1 - \left[ \frac{d}{d\omega}\theta_p(\omega) \right]_{\omega=\omega_p})}. \quad (50)$$

Similar expression can be found in Filippi [1], which points out that Morse and Ingard forgot the derivative (reference [2], p. 559). An interesting remark is that the derivative of  $\theta_p(\omega)$ , denoted  $\theta'_p(\omega)$  can be calculated analytically, as follows. Taking the derivative of equation (46) with respect to  $\omega$ , or more conveniently, taking the logarithmic derivative of equations (43) and (44), the following results are obtained:

$$\frac{\theta'_p}{\theta_p} + \frac{2\varphi'_p}{\sinh 2\varphi_p} = \frac{1}{i\omega_p} \quad (51)$$

and

$$\frac{\theta'_p}{\theta_p} + \frac{2(i\theta'_p\ell/c + \varphi'_p)}{\sinh 2(i\theta_p\ell/c + \varphi_p)} = \frac{1}{i\omega_p}. \quad (52)$$

Thus, eliminating the derivative  $\varphi'_p$ ,

$$D_p \left[ \frac{\theta'_p}{\theta_p} - \frac{1}{i\omega_p} \right] = \frac{2i\theta'_p \ell}{c}, \quad \text{where } D_p = [\sinh 2\varphi_p - \sinh 2(i\theta_p \ell/c + \varphi_p)]. \quad (53)$$

Therefore, since  $\theta_p = \omega_p$ ,

$$1 - \theta'_p = \frac{1}{1 - cD_p/2li\theta_p}. \quad (54)$$

Using equation (47), the norm is deduced :

$$\Lambda_p = \frac{\ell}{2} \left[ 1 - \frac{c}{2li\theta_p} D_p \right]. \quad (55)$$

For  $\omega$  close to  $\omega_p$ , the following equation is obtained :

$$G(x, \omega) = -\frac{c^2}{2\pi\ell} \frac{G_p(x, \omega_p)G_p(x_0, \omega_p)}{\omega_p(\omega - \omega_p)}, \quad (56)$$

which is in accordance with equation (35).

Finally, for  $\omega$  close to 0, the solution  $\theta(\omega)$  which is close to 0, solution of equation (49), satisfies the following equation, deduced from (45) :

$$\left[ 1 - \frac{1}{3}\Theta^2 + O(\Theta^4) \right] \left[ \Theta^2 + \frac{\Omega^2}{\zeta\zeta_\ell} \right] = i\Omega \left[ \frac{1}{\zeta} + \frac{1}{\zeta_\ell} \right]$$

where  $\Theta = \theta\ell/c$  and  $\Omega = \omega\ell/c$ . Therefore  $\Theta^2$  is of order  $\Omega$ , and

$$\theta^2 = i\omega \frac{c}{\ell} \left[ \frac{1}{\zeta} + \frac{1}{\zeta_\ell} \right] + O(\omega^2). \quad (57)$$

Using equation (48), the residue for the pole  $\omega = 0$  is obtained, and equation (36) is confirmed. We conclude that the method of the expansion in orthogonal modes in the Fourier domain leads to the same result (equation (37)) than the ‘‘direct’’ method. Nevertheless the derivation is more delicate.

## VII. METHOD OF SEPARATION OF VARIABLES

Guyader[3] has treated a particular case of the problem (zero  $\zeta$ , large  $\zeta_\ell$ ) using the method of separation of variables, as well as Cox and Zuazua [6], which studied the case of a non homogenous medium. He get non orthogonal modes for the inner product  $\int_0^\ell f_n(x)f_p(x)dx$ . We will see that the method is valid whatever the values of the boundary conditions, and that the derivation can be largely simplified. Finally the expression of the Green function is checked and orthogonality of the modes is discussed.

### A. Solutions with separated variables

We are searching for the solutions  $p(x, t)$  of homogeneous equation (1) (without second member), with the boundary conditions (2) and (3), and with given initial conditions. Assuming that the general solution of the problem is a superposition of solutions with separate variables (this will be proven, at least for the case of the Green function, if the results are identical to the previous ones), the solutions with separate variables are written in the following form:

$$p(x, t) = f(x)h(t), \quad (58)$$

where

$$h(t) = B^+ e^{i\omega t} + B^- e^{-i\omega t} \quad (59)$$

and

$$f(x) = \cosh(i\omega x/c + \varphi). \quad (60)$$

The decomposition (58) differs from the ordinary Fourier transform, because a priori  $\omega$  is a complex quantity, depending on the boundary conditions. Considering first the solution  $B^+ e^{i\omega t}$ , this leads to :

$$\omega \sinh \varphi = \frac{\omega}{\zeta} \cosh \varphi \quad (61)$$

and

$$\omega \sinh(i\omega \ell/c + \varphi) = -\frac{\omega}{\zeta_\ell} \cosh(i\omega \ell/c + \varphi). \quad (62)$$

$\omega = 0$  is a solution, corresponding to the constant mode. The other modes are given by equation (61):

$$\sinh(\varphi - \eta) = 0 \quad (63)$$

thus

$$f(x) = \cosh(i\omega x/c + \eta). \quad (64)$$

Actually there is a sign  $\pm$  in the right-hand side member of equation (64), but it is without importance, because it can be included in the coefficient  $B^+$  of the solution. The eigenvalues equation is deduced from equations (63) and (62), as follows:

$$\sinh(i\omega \ell/c + \eta + \eta_\ell) = 0, \quad (65)$$

which is identical to (28), the solutions of which being already studied. The solution in time  $B^- e^{-i\omega t}$  does not lead to new solutions for  $f(x)$ , therefore the general solution of the problem can be written as:

$$p(x, t) = \sum_n A_n \phi_n(x, t) + A = \sum_n A_n f_n(x) e^{i\omega_n t} + A, \quad (66)$$

where  $\omega_n$  and  $f_n(x)$  are given by equations (29) and (38), respectively, and the coefficients  $A_n$  and  $A$  depend on the initial conditions, and can be determined using the orthogonality relation of the modes.

### B. Orthogonality relation of the modes

In order to derive an orthogonality relation of the modes the following bilinear form is calculated:

$$A_{np} = \int_0^\ell f_n(x) f_p(x) dx. \quad (67)$$

Because  $f_n(x) = f_{-n}^*(x)$ , the calculation of the quantities defined in equation (67) for all values of the index  $n$  is equivalent to the calculation of the quantities defined when replacing  $f_p(x)$  by its conjugate.

Writing

$$\int_0^\ell \left[ f_n(x) \frac{d^2 f_p(x)}{dx^2} - f_p(x) \frac{d^2 f_n(x)}{dx^2} \right] dx = \left[ f_n(x) \frac{df_p(x)}{dx} - f_p(x) \frac{df_n(x)}{dx} \right]_0^\ell$$

and using equation (38), the following result is obtained:

$$\frac{(\omega_p^2 - \omega_n^2)}{c^2} A_{np} = i(\omega_p - \omega_n) \left[ \frac{f_n(0) f_p(0)}{\zeta} + \frac{f_n(\ell) f_p(\ell)}{\zeta_\ell} \right].$$

For  $\omega_p \neq \omega_n$ , because  $\omega_p + \omega_n \neq 0$ , the result is deduced. For  $\omega_p = \omega_n$ , the calculation is straightforward. The general formula is found to be:

$$A_{np} = \frac{ci}{\omega_p + \omega_n} \left[ \frac{f_n(0) f_p(0)}{\zeta} + \frac{f_n(\ell) f_p(\ell)}{\zeta_\ell} \right] + \frac{1}{2} \ell \delta_{np} \quad (68)$$

where  $\delta_{np}$  is the Kronecker symbol. An equivalent formula is:

$$A_{np} = -\frac{c \sinh 2\eta + (-1)^{n+p} \sinh 2\eta_\ell}{2 i(\omega_p + \omega_n)} + \frac{1}{2} \ell \delta_{np}. \quad (69)$$

Modes are found to be non orthogonal for the product defined by (67), but, as shown by Guyader[3], it is possible to solve the problem from the knowledge of initial conditions. A discussion on orthogonality is given in the subsection VII E. When absorption tends to zero ( $\eta$  and  $\eta_\ell$  tend to zero), the first term does not vanish, tending to  $\frac{1}{2}\ell\delta_{n(-p)}$ . It is due to the choice of considering separately the modes  $\omega_n$  and  $\omega_{-n}$ .

Finally we notice that formula (68) remains valid when one of the modes is the constant mode  $f(x) = 1$ , and the other one a non constant mode:

$$A_n = \int_0^\ell f_n(x)dx = -\frac{c}{i\omega_n} \left[ \frac{f_n(0)}{\zeta} + \frac{f_n(\ell)}{\zeta_\ell} \right] = -\frac{c}{i\omega_n} (\sinh \eta + (-1)^n \sinh \eta_\ell). \quad (70)$$

Obviously the norm of the constant mode is  $\ell$ .

### C. Solution with respect to initial conditions

According to equation (66), the initial conditions can be written as:

$$p(x, 0) = \sum_n A_n \cosh(i\omega_n x/c + \eta) + A \quad (71)$$

and

$$\partial_t p(x, 0) = \sum_n A_n i\omega_n \cosh(i\omega_n x/c + \eta). \quad (72)$$

Using equation (69) for a non constant mode  $p$ , the following results are obtained:

$$\int_0^\ell p(x, 0) f_p(x) dx = \sum_n A_n A_{np} + A A_p \quad (73)$$

and

$$\int_0^\ell \partial_t p(x, 0) f_p(x) dx = \sum_n A_n i\omega_n A_{np}. \quad (74)$$

Multiplying equation (73) by  $i\omega_p$ , then adding equation (74), leads to:

$$\begin{aligned} & \int_0^\ell [i\omega_p p(x, 0) + \partial_t p(x, 0)] f_p(x) dx \\ &= i \sum_n A_n (\omega_p + \omega_n) A_{np} + i A \omega_p A_p \\ &= -c \widehat{\sum_n} A_n \left[ \frac{f_n(0) f_p(0)}{\zeta} + \frac{f_n(\ell) f_p(\ell)}{\zeta_\ell} \right] + i A_p \ell \omega_p \end{aligned} \quad (75)$$

$$= -c \left[ \frac{f_p(0) p(0, 0)}{\zeta} + \frac{f_p(\ell) p(\ell, 0)}{\zeta_\ell} \right] + i A_p \ell \omega_p \quad (76)$$

The notation  $\widehat{\Sigma}$  for the series in equation (75) indicates that it involves the constant mode. As noticed by Guyader[3], this series is related to the initial conditions at the two ends  $x = 0$  and  $x = \ell$ . Thus

$$A_p \ell i \omega_p = \int_0^\ell [i \omega_p p(x, 0) + \partial_t p(x, 0)] f_p(x) dx + c p(0, 0) \sinh \eta + c p(\ell, 0) \varepsilon_p \sinh \eta_\ell \quad (77)$$

is the result for a non constant mode. Using (39), the following property is deduced :

$$A_{-p} = A_p^*. \quad (78)$$

The coefficient  $A$  remains to be determined. Using a similar calculation starting from  $\int_0^\ell \partial_t p(x, 0) dx$  the following result is obtained:

$$A = \frac{c^{-1} \int_0^\ell \partial_t p(x, 0) dx + p(0, 0) \tanh \eta + p(\ell, 0) \tanh \eta_\ell}{\tanh \eta + \tanh \eta_\ell}. \quad (79)$$

What is the condition for which this coefficient vanishes? If for instance at  $x = 0$ ,  $\zeta$  is zero,  $\eta$  is infinite, and, according to the boundary condition,  $p(0, 0)$  vanishes, thus  $A$  vanishes too. This confirms the remark concerning result (37).

#### D. Deduction of the Green function

How is it possible to check the accordance between formula (37) and the previous results? Using an integral formulation (in the time domain) in order to deduce the solution of the problem with initial conditions from the knowledge of the Green function leads to heavy calculations. We prefer to check that, for the particular case of the Green function, the results of the present section are valid. For this purpose, we use the knowledge of the time domain solution for the Green function (equation (24)).

First we will determine the integrals involved in equation (77). On one hand, for  $t \rightarrow 0^+$ ,  $g(x, t)$  is equivalent to the Green function in infinite space,

$$\frac{c}{2} H \left( t - \frac{|x - x_0|}{c} \right)$$

therefore the integral  $\int_0^\ell g(x, t) f_p(x) dx$  tends to 0. On the other hand, the integral  $\int_0^\ell \partial_t g(x, t) f_p(x) dx$  is equivalent to

$$\frac{c}{2} \int_0^\ell \left[ \delta \left[ t - \frac{x - x_0}{c} \right] + \delta \left[ t + \frac{x - x_0}{c} \right] \right] f_p(x) dx$$

$$= \frac{c^2}{2} [f_p(ct + x_0) + f_p(-ct + x_0)].$$

Thus it tends to  $c^2 f_p(x_0)$ . The initial quantities  $p(0, 0)$  et  $p(\ell, 0)$  being zero, result (37) is verified for positive times. Similarly, concerning the constant mode (equation 79), the integral  $\int_0^\ell \partial_t g(x, t) dx$  tends to  $c^2$ , and the result is also verified.

### E. Discussion on the orthogonality

Two remarks can be made concerning the non orthogonality found in equations (69) and (70) for the product defined in (67): i) it can be interpreted by taking into account that the studied system produces energy outside of the interval  $[0, \ell]$ , using the first interpretation of the problem to be solved (see figure 1). ii) another product can be defined, leading to a true orthogonality condition. These remarks are examined hereafter.

#### 1. Influence of the energy outside the studied medium

We consider the problem extended to the two outside media, between  $-\infty$  and  $0$ , and between  $\ell$  and  $+\infty$ . The complete medium is viewed as an inhomogeneous medium, with two jumps in density  $\rho$  or/and sound velocity  $c$ . In order to solve this new problem, all the previous methods can be applied, and we choose the separation of variables. When using this method, neither the time dependence  $\omega_n$  of the solutions, nor the functions  $f_n(x)$  are changed in the interval  $[0, \ell]$ . In order to deduce the solutions outside this interval, the continuity at  $x = 0$  (a similar reasoning is done at  $x = \ell$ , but not given here) of pressure  $p$  and velocity, i.e. the spatial derivative of the pressure divided by the density,  $\partial_x p / \rho$  is used:

$$p_- = p \text{ and } \partial_x p_- / \rho^- = \partial_x p / \rho. \quad (80)$$

In order to obtain orthogonal modes, it is necessary to ensure the continuity of the wronskian  $f_n(x) \partial_x f_p(x) - f_p(x) \partial_x f_n(x)$  at these abscissae. It means that it is necessary to change the function solution of the homogeneous equation (1). Replacing  $p(x, t)$  by  $q(x, t) = p(x, t) [\rho / \rho(x)]^{1/2}$ , the following continuity conditions at  $x = 0$  are obtained:

$$q^-(x, t) (\rho_- / \rho)^{1/2} = q(x, t) \text{ and } \partial_x q^-(x, t) / (\rho^- / \rho)^{1/2} = \partial_x q(x, t). \quad (81)$$

This change of function, keeping the function unchanged in the interval  $[0, \ell]$ , ensures the nullity of the wronskian, and is equivalent, for a continuous inhomogeneity, to the reduction of the Sturm-Liouville equation  $\left[\partial_{xx}^2 - \frac{\partial_x \rho}{\rho} \partial_x - \frac{1}{c^2} \partial_{tt}^2\right] p(x, t) = 0$  in its canonical form, without term involving derivative of first order, by changing  $p(x, t)$  in  $q(x, t) = p(x, t)/\rho^{1/2}(x)$ .

It remains to calculate the solution of the wave equation for negative  $x$ : using, for the absorbing case ( $\text{Re}(\eta + \eta_\ell) > 0$ ), the condition of vanishing field when  $x$  tends to  $-\infty$ , and the continuity condition, the result for the modes of  $q(x, t)$  is

$$f_n(x) = (\rho/\rho_-)^{1/2} \cosh(\eta) \exp(-i\omega_n x/c_-) \text{ for } x < 0 \quad (82)$$

and for the “constant” mode

$$f(x) = (\rho/\rho^-)^{1/2}.$$

The orthogonality integral is given by:

$$\int_{-\infty}^0 f_n(x) f_p(x) dx = \frac{\rho}{\rho_-} \cosh^2(\eta) \int_{-\infty}^0 \exp(-i(\omega_n + \omega_p)x/c_-) dx$$

or

$$\int_{-\infty}^0 f_n(x) f_p(x) dx = -\frac{1}{2} \frac{\rho}{\rho^-} \zeta \sinh(2\eta) \frac{c_-}{i(\omega_n + \omega_p)} = -\frac{c_-^2}{2c} \frac{\sinh 2\eta}{i(\omega_n + \omega_p)} \quad (83)$$

and for the constant mode

$$\int_{-\infty}^0 f_n(x) f(x) dx = -\frac{c_-^2}{c} \frac{\sinh \eta}{i\omega_n}. \quad (84)$$

If  $\eta_\ell$  is zero, the integrals given by the last two equations correspond to the terms responsible for the non orthogonality in equations (69) and (70), respectively, provided that  $c_- = c$ . If  $\eta_\ell$  is not zero, a similar calculation can be done for  $\ell < x < +\infty$ . We conclude that an interpretation of the non orthogonality can be done by interpreting the jump in characteristic impedances in 0 and  $\ell$  as a jump in densities  $\rho_-$  and  $\rho_+$ , respectively, in the two outside media: considering the entire space, orthogonality is ensured for the product  $\int_{-\infty}^{+\infty} f_n(x) f(x) dx$ .

## 2. Change in product of modes

For vibrating systems, the product defined by (67) corresponds to the product with respect to the mass. It is interesting to calculate the product related to the stiffness (see

e.g. Meirovitch [7]):

$$\Lambda'_{np} = \int_0^\ell \frac{d}{dx} f_n(x) \frac{d}{dx} f_p(x) dx \quad (85)$$

Integrating by parts, and using equation (68) it is found, for  $n \neq p$ , to be equal to:

$$\Lambda'_{np} = \left[ f_n(x) \frac{d}{dx} f_p(x) \right]_0^\ell + \frac{\omega_p^2}{c^2} \Lambda_{np} = -\frac{\omega_n \omega_p}{c^2} \Lambda_{np}. \quad (86)$$

Therefore the modes become orthogonal if we define a new product for the modes  $\phi_n(x, t) = f_n(x)h_n(t)$ , as follows:

$$\Lambda'_{np} = \int_0^\ell \left[ \frac{\partial}{\partial x} \phi_n(x, t) \frac{\partial}{\partial x} \phi_p(x, t) - \frac{1}{c^2} \frac{\partial}{\partial t} \phi_n(x, t) \frac{\partial}{\partial t} \phi_p(x, t) \right]_{t=0} dx = \frac{\omega_n^2}{c^2} \ell \delta_{np}. \quad (87)$$

We remark that the modes  $\phi_n(x, t)$  and  $\phi_{-n}(x, t) = \phi_n^*(x, t)$  are orthogonal. For the calculation of the solution from the initial conditions, using equation (66) at  $t = 0$ , the following result is obtained:

$$\int_0^\ell \left[ \frac{d}{dx} f_p(x) \frac{\partial}{\partial x} p(x, 0) - \frac{i\omega_p}{c^2} f_p(x) \frac{\partial}{\partial t} p(x, 0) \right] dx = A_p \frac{\omega_p^2}{c^2} \ell. \quad (88)$$

As a consequence, the initial conditions need to be written by using the derivatives of the function  $p(x, t)$  with respect to the abscissa and time, respectively. Result (77) can be checked by integrating the first term of the integral by parts. Nevertheless, the scalar product (87) is not useful for the constant mode, and the first method needs to be used (see subsection VII C).

## VIII. GENERALIZATION WHEN ADDING ABSORPTION TO PROPAGATION

### A. Statement of the problem

Returning to the initial problem in a bounded medium, we will now consider absorption in the medium. In general absorption is frequency-dependent (see e.g. Pierce[8]), but for simplicity we assume a coefficient,  $\alpha$ , independent of both the space coordinate and the frequency, and we assume arbitrarily the following form for a traveling wave in the frequency domain:  $\exp(-\alpha x) \exp(i\omega x/c)$ . For frequency-dependent absorption, it can be assumed that the obtained formulas can be regarded to be valid by replacing  $\alpha$  by its value for the frequency corresponding to a given mode. This is in general satisfactory for small losses, but

is not discussed further hereafter. For structural vibration, our model is called “proportional damping” (see e.g. Woodhouse[9], or Adhikari[10] and an application in vibroacoustics see Mattei [11]). Our interest in this section is limited to the study of the resonance modes and their frequencies.

In the frequency domain, equation (6) is replaced by the following :

$$\left[ \partial_{xx}^2 - \left( i\frac{\omega}{c} + \alpha \right)^2 \right] G(x, \omega) = -\frac{1}{2\pi} \delta(x - x_0), \quad (89)$$

the boundary conditions being unchanged. After elementary algebra, equation (12) is replaced by the following :

$$G(x, \omega) = \frac{1}{2\pi z} \frac{\cosh[\chi + zx_0] \cosh[\chi_\ell + z(\ell - x)]}{\sinh(z\ell + \chi + \chi_\ell)} \text{ if } x \geq x_0 \quad (90)$$

where

$$z = \frac{i\omega}{c} + \alpha ; \coth \chi = \zeta \frac{z}{z - \alpha} ; \coth \chi_\ell = \zeta_\ell \frac{z}{z - \alpha}. \quad (91)$$

## B. Calculation of the frequencies

The poles corresponding to the function  $\sinh$  are solutions of

$$z\ell + \chi(z) + \chi_\ell(z) = in\pi. \quad (92)$$

For  $n \neq 0$ , the modulus of the solutions  $z$  are expected to be large, thus this of  $\alpha/z$  is small, and a perturbation method can be used. To the zeroth order of  $\alpha/z$ , the solution is given by equation (28) :

$$z^{(0)}\ell = -\eta - \eta_\ell + in\pi. \quad (93)$$

Writing the unknown  $\chi^{(2)} = \eta + \varepsilon$ , and calculating the quantity  $\tanh(\chi^{(2)})$  by using equations (91), the following (exact) result is obtained

$$\tanh \varepsilon = -\frac{\alpha}{z} \frac{\zeta}{\zeta^2 - 1} \left[ 1 - \frac{\alpha}{z} \frac{1}{\zeta^2 - 1} \right]. \quad (94)$$

Therefore the quantity  $\varepsilon$  is of order  $\alpha/z$ , and to the second order in  $\varepsilon$ , we get:

$$\chi^{(2)} = \eta - \frac{\alpha}{z^{(0)}} \frac{\zeta}{\zeta^2 - 1} \left[ 1 - \frac{\alpha}{z^{(0)}} \frac{1}{\zeta^2 - 1} \right] + O\left(\left(\alpha/z^{(0)}\right)^2\right), \quad (95)$$

which, with a similar result for  $\chi_\ell^{(2)}$ , leads to the value of  $z^{(2)}$ , deduced from equation (92). This method can be iterated, in order to converge to the exact result. Some comments can be made :

- because of the term  $(\zeta^2 - 1)$  at the denominator, the perturbation method is not appropriate when one of the boundary coefficient is close to this of an anechoic termination, therefore we have tried to determine the limit of the method. Numerical calculations have been done and presented in section VIII E;

- such a perturbation method does not allow to know the exact number of solutions of equation (92) ; we assume that if  $(\zeta^2 - 1)$  is not too small, the number of solutions remains the same as for the case of zero  $\alpha$  ;

- we notice that if  $(\zeta^2 - 1)$  is not small, the real part of  $z^{(0)}$  is small, whatever the considered case ( $\eta$  and  $\eta_\ell$  being real or complex), compared to the imaginary part, because  $n \neq 0$  : as a consequence, the perturbation is significant for the real part of the frequencies  $\omega_n$ , and negligible for its imaginary part, i.e. the attenuation rate. Moreover the effect of losses decreases when  $n$  increases, which is obvious when looking at equation (89). The result at the first order of the perturbation can be written as follows :

$$i\frac{\omega_n \ell}{c} = -\alpha \ell - \eta - \eta_\ell + in\pi - \varepsilon - \varepsilon_\ell$$

or

$$i\frac{\omega_n \ell}{c} = -\alpha \ell - \eta - \eta_\ell + in\pi + \frac{\alpha \ell}{in\pi - \eta - \eta_\ell} \left[ \frac{\zeta}{\zeta^2 - 1} + \frac{\zeta_\ell}{\zeta_\ell^2 - 1} \right]. \quad (96)$$

The question of the poles corresponding to the case  $n = 0$ , for which a real solution can be expected is not treated in general, but for an example in section VIII E. Obviously, it remains also the pole  $z = 0$ .

### C. Calculation of the residues

As done in section V B, the Taylor expansion of the function  $\sinh(z\ell + \chi + \chi_\ell)$  to the first order in  $(\omega - \omega_n)$  has to be determined to calculate the residues. We get

$$\sinh(z\ell + \chi + \chi_\ell) = (\omega - \omega_n)(-1)^n \Gamma_n, \quad (97)$$

having denoted  $\Gamma_n$  the term

$$\Gamma_n = \left[ \frac{i\ell}{c} - \frac{1}{\omega_n} \left( \frac{\zeta - \coth \chi_n}{\coth^2 \chi_n - 1} + \frac{\zeta_\ell - \coth \chi_{\ell n}}{\coth^2 \chi_{\ell n} - 1} \right) \right], \quad (98)$$

where  $\chi_n = \chi(z_n)$  and  $\chi_{\ell n} = \chi_\ell(z_n)$ , Therefore the residue of  $G(x, \omega)$  at  $\omega = \omega_n$ , for  $n \neq 0$ , is given by

$$G(x, \omega) = \frac{1}{2\pi z_n} \frac{\cosh[\chi_n + z_n x_0] \cosh[\chi_{\ell n} + z_n(\ell - x)]}{(\omega - \omega_n)(-1)^n \Gamma_n}, \quad (99)$$

and with the use of (92), results in

$$G(x, \omega) = \frac{1}{2\pi z_n} \frac{\cosh[\chi_n + z_n x_0] \cosh[\chi_n + z_n x]}{(\omega - \omega_n) \Gamma_n}. \quad (100)$$

#### D. Successive reflections

Using the same procedure as for the lossless case (see section IV), the transformation of equation (90) leads to the solution in the time domain. The addition of losses has implication in the expression of the Green function of the primary waves (see eq.(18)) with the appearing of an exponentially damped factor  $e^{-\alpha ct}$ . It also modifies the characterization of the reflections at the boundaries which are no longer described by constant values but by functions depending on time. As a consequence, products in equations (25) and (26) have to be replaced by convolution products. Finally, the Green function in the time domain becomes

$$g(x, t) = \frac{c}{2} g_p(x, t) * [\delta(t) + f(t) + f(t) * f(t) + f(t) * f(t) * f(t) + \dots] \quad (101)$$

where the factor corresponding to the primary waves  $g_p$  is given by

$$g_p(x, t) = e^{-\alpha ct} c H(t - |x - x_0|/c) + r(t) * [e^{-\alpha ct} H[t - (x + x_0)/c] + r_\ell(t) * [e^{-\alpha ct} H(t - (2\ell - x - x_0)/c)] + r(t) * r_\ell(t) * [e^{-\alpha ct} H(t - (2\ell - |x - x_0|)/c)]] \quad (102)$$

the function  $f(t)$  is

$$f(t) = e^{-2\alpha \ell} r(t) * r_\ell(t) * \delta(t - 2\ell/c)$$

and, from equations (91), the reflections functions are found to be

$$r(t) = e^{-2\eta} \delta(t) + H(t) \frac{4\pi \zeta \alpha c}{(\zeta + 1)^2} e^{-\frac{\alpha c \zeta}{\zeta + 1} t}, \quad (103)$$

and

$$r_\ell(t) = e^{-2\eta_\ell} \delta(t) + H(t) \frac{4\pi \zeta_\ell \alpha c}{(\zeta_\ell + 1)^2} e^{-\frac{\alpha c \zeta_\ell}{\zeta_\ell + 1} t}. \quad (104)$$

### E. An example for $\zeta_\ell = 0$ (Dirichlet condition at $x=\ell$ )

We consider the case of a Dirichlet condition at  $x = \ell$  ( $\zeta_\ell = 0$ ,  $\chi_\ell = i\pi/2$ ). Poles are given by the equation

$$z\ell + \chi(z) = im\pi/2, \quad (105)$$

where  $m = 2n - 1$  is a (non zero) odd integer. It is easy to check that  $z = 0$  is not a pole of  $G(x, \omega)$ , because both the numerator and the denominator vanish. If  $|\zeta| > 1$ , it can be shown that no real solution exists, so that the perturbation method can be applied for all poles, which are given to first order by

$$i\frac{\omega_n \ell}{c} = -\alpha\ell - \eta + im\frac{\pi}{2} + \frac{2\alpha\ell}{im\pi - 2\eta} \frac{\zeta}{\zeta^2 - 1}. \quad (106)$$

When  $|\zeta| < 1$ ,  $\zeta = \tanh \mu$ , where  $\mu = \eta - i\pi/2$ , and equation (105) can be replaced by

$$z\ell + \kappa(z) = in\pi,$$

where  $n$  is an integer, and  $\kappa(z) = \chi(z) - i\pi/2$ . The first order solution is given by the formula:

$$i\frac{\omega_n \ell}{c} = -\alpha\ell - \mu + in\pi - \frac{\alpha\ell}{in\pi - \mu} \frac{\zeta}{1 - \zeta^2}. \quad (107)$$

For  $n = 0$ , we recognize a particular case, because the perturbation method cannot be applied if  $\mu$  is small. Nevertheless, using equations (91) and (92), the unknown  $z$  is found to be solution of

$$\coth z\ell = -\left(1 - \frac{\alpha}{z}\right) \coth \mu. \quad (108)$$

If  $\mu$  is small,  $z\ell$  is expected to be small as well, and a series expansion of the coth function leads to the following result:

$$i\frac{\omega_0 \ell}{c} = -\tanh(\mu) \left[1 + \frac{(\alpha\ell - \tanh \mu)^2}{3}\right] = -\mu - \mu \frac{\alpha^2 \ell^2}{3} + O(\mu^2). \quad (109)$$

We notice that to the first order of  $\mu$ , the two expressions (107) and (109) are identical. It means that for small  $\zeta$ , the first order approximation is good, while the second order one is not: in equation (95), there is a factor  $\zeta$  at the numerator of the first order term, but not in the bracket involving the second order term.

Figure 3 shows the variation of the real part of the frequencies with respect to parameter  $\zeta$ , chosen to be positive. When  $\zeta$  is not close to unity, the variation of frequency is small,

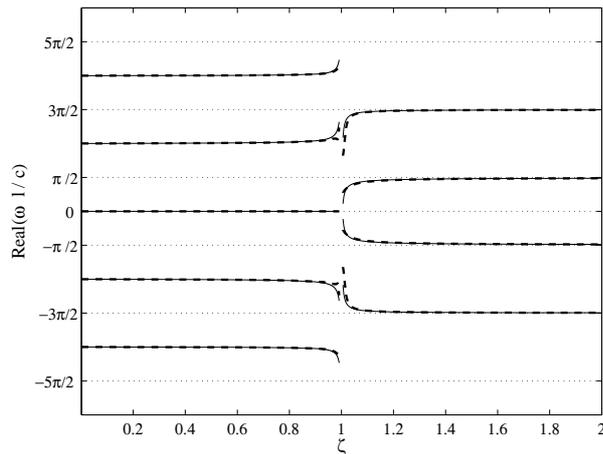


FIG. 3: Real part of the frequencies as a function of  $\zeta$  obtained with the perturbation method when losses are taken into account  $\alpha\ell = 0.1$ ,  $\zeta_\ell = 0$ . Comparison between numerical, iterative solution (dotted) and first order approximation solution (dashed-dot) (equations 106 and 107 for  $\zeta > 1$  and

and the first order of the iteration computation (equations 106 and 107) is satisfactory. For  $\zeta$  approaching unity, the iterative computation converges, but it is not true for  $\zeta$  very close to unity.

For large  $\zeta$  the real part of the frequencies are close to the frequencies of the Neumann-Dirichlet problem, and for small  $\zeta$ , the frequencies are close to the frequencies of the Dirichlet-Dirichlet problem. According to equations (106) and (107), because  $\zeta$  is positive, the frequencies are smaller than the frequencies for zero  $\alpha$  if  $\zeta$  is larger than unity, and larger if  $\zeta$  is smaller than unity. For  $\zeta$  smaller than 0.5 the relative effect of the losses on the real part of the first frequency is less than 0.03%, while when it is larger than 2.0, this effect is smaller than 0.25%.

$\zeta < 1$ , respectively.)

Figure 4 shows the result of the same calculation for the imaginary part of the first frequency ( $m = 1$  for  $\zeta > 1$ ;  $n = 1$  for  $\zeta < 1$ ), corresponding to the attenuation. The first order approximation is excellent. Obviously it is a fortiori true for higher order frequencies. Figure 5 shows the result for  $n = 0$  when  $\zeta < 1$ . Equation (109) is satisfactory for small  $\zeta$ , while the numerical, iterative method fails.

$\zeta < 1$ , respectively.)

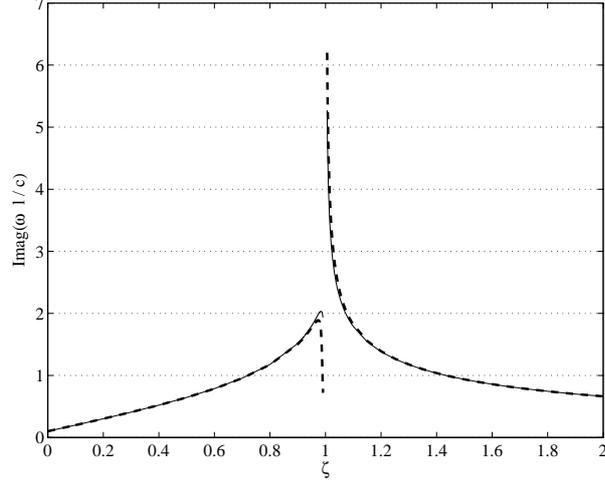


FIG. 4: Imaginary part of the first frequency as a function of  $\zeta$ , obtained with the perturbation method when losses are taken into account  $\alpha\ell = 0.1$ ,  $\zeta_\ell = 0$ . Comparison between numerical, iterative solution (dotted) and first order approximation solution (dashed-dot) (equation 106 and 107 for  $\zeta > 1$  and

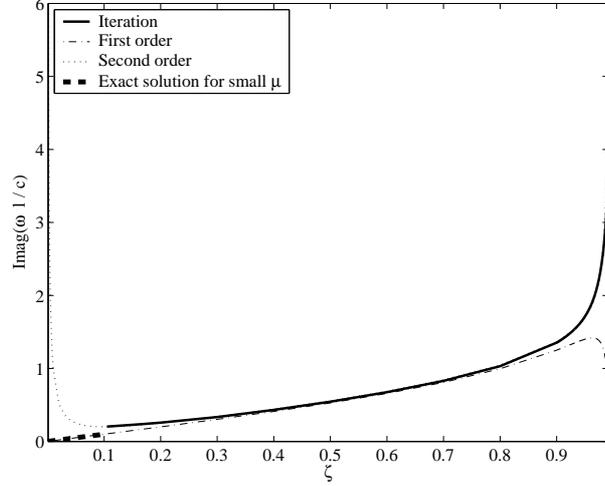


FIG. 5: Imaginary part of the frequency  $\omega_0$  as a function of  $\zeta$ , when  $\zeta < 1$ , obtained with the perturbation method when losses are taken into account.  $\alpha\ell = 0.1$ ,  $\zeta_\ell = 0$ . Comparison between several results : first order approximation (107) is satisfactory till  $\zeta = 0.8$ ; second order approximation (from equation 95) as well as numerical, iterative method are satisfactory from  $\zeta = 0.4$ .

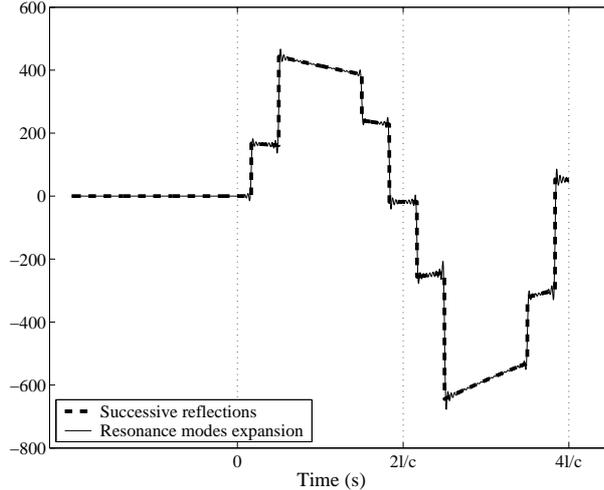


FIG. 6: Green function calculated using modal expansion and successive reflections method. Case of active boundary at  $x = 0$ ,  $\zeta = -3.6975$ , and Dirichlet boundary condition  $\zeta_\ell = 0$ .  $\alpha\ell = 0.1$ , 100 modes Location of the source and receiver are  $x_0 = 0.1m$  and  $x = 0.2m$ , respectively.

Examples of numerical calculation of the Green function can now be computed. Figures 6 and 7 show the results for the modal expansion method and for the successive reflections method, for two cases with negative  $\zeta$  (active termination) and with positive  $\zeta$  (passive termination), respectively. For both cases  $|\zeta| > 1$ . The agreement is satisfactory for a number of 100 modes ( $m = -101$  to  $m = 101$ ).

## IX. CONCLUSION

The studied simple problem we have studied, which can be regarded in particular as a radiation problem, exhibits interesting properties for the resonance modes: they are complex, and non orthogonal for the simple product (67) because of the bounded character of the considered medium, but except the constant mode, they are orthogonal for a product modified in a proper way. When interpreting the absorbing boundaries as interfaces with semi-infinite media with different densities, they are orthogonal for the simple product. When losses are ignored in the propagation, their frequencies are independent of the parameters for boundary conditions on the same type, i.e. for terminating impedances involved in either interval  $[0, 1[$  or interval  $]1, +\infty[$ .

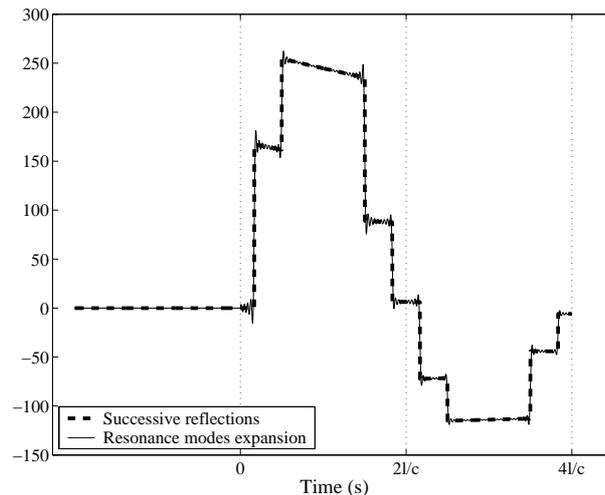


FIG. 7: Green function calculated using modal expansion and successive reflections method. Case of passive boundary at  $x = 0$ ,  $\zeta = 3.6975$ , and Dirichlet boundary condition  $\zeta_\ell = 0$ .  $\alpha\ell = 0.1$ , 100 modes. Locations of the source and receiver are  $x_0 = 0.1m$  and  $x = 0.2m$ , respectively.

When losses are taken into account, the variation of the imaginary part of the frequencies (i.e. attenuation), and, to a smaller extent, the real part of them, is small when the boundary impedances vary, except near unity, when the medium tends to be (semi-)infinite. Approximation formulas are excellent.

Thanks to the simplicity of the problem, the analytical treatment is possible with several methods, enlightening the relationship between them, which can be useful for more intricate problems. Active boundaries can also be considered, thanks to a change in functions.

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## APPENDIX A: MAIN RESULT FOR THE DIFFERENT CASES

i)  $|\zeta| > 1$  and  $|\zeta_\ell| > 1$  (real  $\eta$  and  $\eta_\ell$ ):

The modes and Green function are:

$$f_n(x) = \cosh [\eta(1 - x/\ell) - \eta_\ell x/\ell + in\pi x/\ell];$$

$$g(x, t) = H(t)c e^{-(\eta+\eta_\ell)ct/\ell} \sum_{n=-\infty}^{+\infty} \frac{f_n(x)f_n(x_0)}{in\pi - \eta - \eta_\ell} e^{in\pi ct/\ell} + H(t) \frac{c}{\zeta^{-1} + \zeta_\ell^{-1}}.$$

ii) if  $|\zeta| < 1$  and  $|\zeta_\ell| < 1$  (complex  $\eta$  and  $\eta_\ell$ :  $\eta = \mu + i\pi/2$ ;  $\eta_\ell = \mu_\ell + i\pi/2$ ):

The modes and Green function are :

$$f_n(x) = \sinh [\mu(1 - x/\ell) - \mu_\ell x/\ell + in\pi x/\ell],$$

$$g(x, t) = -H(t)c e^{-(\mu+\mu_\ell)ct/\ell} \sum_{n=-\infty}^{+\infty} \frac{f_n(x)f_n(x_0)}{in\pi - \mu - \mu_\ell} e^{in\pi ct/\ell} + H(t) \frac{c}{\zeta^{-1} + \zeta_\ell^{-1}}.$$

iii) if  $|\zeta| > 1$  and  $|\zeta_\ell| < 1$  (real  $\eta$  and complex  $\eta_\ell$ :  $\eta_\ell = \mu_\ell + i\pi/2$ )

For this mixed case, the modes and Green function are,

$$f_n(x) = \cosh [\eta(1 - x/\ell) - \mu_\ell x/\ell + im\pi x/2\ell],$$

$$g(x, t) = H(t)c e^{-(\eta+\mu_\ell)ct/\ell} \sum_{m \text{ odd}=-\infty}^{+\infty} \frac{f_n(x)f_n(x_0)}{im\pi/2 - \eta - \mu_\ell} e^{im\pi ct/2\ell} + H(t) \frac{c}{\zeta^{-1} + \zeta_\ell^{-1}}.$$

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