Volume entropy, systole and stable norm on graphs

Florent N. Balacheff

Volume entropy, systole and stable norm on graphs

Florent Balacheff *

November 25, 2004

Abstract

We study some new isoperimetric inequalities on graphs. We etablish a relation between the volume entropy (or asymptotic volume), the systole and the first Betti number of weighted graphs. We also find bounds for the volume, associated to some special measure, of the unit ball for the stable norm of graphs.

Mathematics Subject Classification (2000) : 05C35, 37B10 .

1 Introduction

Some isoperimetric inequalities on manifolds can sometimes be generalized to graphs. For example, the asymptotic behaviour of the systolic volume of a surface in terms of its genus, investigated in [6] and [8], has a 1-dimensional analog valid on weighted graphs [5]. Namely, let (Γ, w) be a weighted graph (see subsection 2.1 for a precise definition). The *volume* of (Γ, w) (or *size*), denoted by $\mathrm{Vol}(\Gamma, w)$, is the sum of the weight of its edges

$$\mathrm{Vol}(\Gamma, w) = \sum_{e \in E} w(e).$$

The *systole* of (Γ, w) (or *weighted girth*) is defined as

$$\mathrm{sys}(\Gamma, w) = \inf\{l_w(\gamma) \mid \gamma \text{ non trivial cycle of } \Gamma\},$$

where the length of a cycle γ, noted $l_w(\gamma)$, is the sum of the weights of its edges.

With this two quantities, we can define the *systolic volume* of Γ as

$$\sigma(\Gamma) = \inf_w \frac{\mathrm{Vol}(\Gamma, w)}{\mathrm{sys}(\Gamma, w)}.$$

where the infimum is taken over all the weight functions on the graph Γ.

Theorem (Bollobás & Szemerédi [5]) *Let Γ be a finite graph with first Betti number $b \geq 3$. We have :*

$$\sigma(\Gamma) \geq \frac{3 \ln 2}{2} \frac{b-1}{\ln(b-1) + \ln\ln(b-1) + 4\ln 2 - \ln\ln 2}.$$

*F. Balacheff UMR 5030, Département des Sciences Mathématiques Université Montpellier II Case Courrier 051 - place Eugène Bataillon 34095 Montpellier Cedex 5, France (email : balachef@math.univ-montp2.fr)

This lower bound gives an asymptotic lower estimate in terms of the first Betti number b:

$$\sigma(\Gamma) \geq \frac{3\ln 2}{2}\frac{b}{\ln b} + o\left(\frac{b}{\ln b}\right). \tag{1.1}$$

On the other hand, for each $b \geq 2$, we can construct a weighted graph (Γ, w) (see [2]) of first Betti number b such that

$$\sigma(\Gamma, w) \leq 8\ln 2\frac{b}{\ln b},$$

where $\sigma(\Gamma, w) = \mathrm{Vol}(\Gamma, w)/\mathrm{sys}(\Gamma, w)$.

Thus, the estimate (1.1) gives a good asymptotic behaviour of the systolic volume in terms of the first Betti number. This is the analog of an estimate on surfaces, which states that the systolic volume of a surface S of genus g, denoted by $\sigma(S)$ (see [8] for a precise definition), has the following lower bound

$$\sigma(S) \geq C\frac{g}{(\ln g)^2}, \tag{1.2}$$

where C is some positive constant. In [6], the authors construct a metric on surfaces of genus g for which the systolic volume is asymptotically close to (1.2).

In this article, we are interested in the normalization of the volume entropy by the systole. Let $(\tilde{\Gamma}, \tilde{w})$ be the universal (weighted) covering of (Γ, w). Fix $x_0 \in \Gamma$ and $\tilde{x}_0 \in \tilde{\Gamma}$ a lift of x_0. The *volume entropy* (or *asymptotic volume*) of (Γ, w) is defined as

$$h_{vol}(\Gamma, w) = \lim_{R\to+\infty} \frac{\ln(\mathrm{Vol}_{\tilde{w}}B(\tilde{x}_0, R))}{R} \tag{1.3}$$

where $\mathrm{Vol}_{\tilde{w}}B(\tilde{x}_0, R)$ is the volume of the ball centered at $\tilde{x}_0$ with radius R in $(\tilde{\Gamma}, \tilde{w})$. Since the weighted graph (Γ, w) is compact, the limit in (1.3) exists and does not depend on the point $x_0 \in \Gamma$ and its lift (see [10]).

The product $h_{vol}(\Gamma, w).\mathrm{sys}(\Gamma, w)$ is invariant under scaling and has been studied in [12] for surfaces. The author has proved that this quantity is bounded from above for each surface. We will prove an upper bound on this product for weighted graphs, which is asymptotically optimal.

Theorem 1 *Let (Γ, w) be a weighted graph with first Betti number equal to b. Then*

$$h_{vol}(\Gamma, w).sys(\Gamma, w) \leq 2\ln(8b^3 - 1).$$

We state an analogue result for topological Markov chains associated to a $n \times n$ matrix A, relating the topological entropy h_{top} and the smallest period of a periodic orbit T_{min}.

Proposition *For each topological Markov chain (Σ_A, ϕ_A),*

$$h_{top}(\Sigma_A, \phi_A).T_{min}(\Sigma_A, \phi_A) \leq \ln b_A,$$

where $b_A = \sum_{i,j=1}^{n} A_{ij} - n + 1$.

In an another direction, we can associate to a weighted graph an object called the *stable norm* on the real first homology space (see subsection 3.1 for a definition). We can define a natural measure μ_w on $H_1(\Gamma, \mathbb{R})$, which allows us to define the volume of the *stable ball* $\mathcal{B}_{st}(\Gamma, w)$, defined as the unit ball of the stable norm. For usual combinatorial graphs, which can of course be identified with a weighted graph in which all the edges have weight 1, we obtain the following inequalities.

Theorem 2 *Let Γ be a graph with first Betti number b. Then*

$$\frac{2^b}{b!}\left(\frac{b}{Vol(\Gamma)}\right)^b \leq \mu_w(\mathcal{B}_{st}(\Gamma)) \leq \frac{2^b}{b!},$$

The two cases of equality are attained by the bouquet of b circles $\bigvee_{i=1}^b S_i^1$.

We also obtain an inequality on weighted graphs relating the volume of the graph to the volume of the stable ball.

Theorem 3 *For every weighted graph (Γ, w) with first Betti number b,*

$$\mu_w(\mathcal{B}_{st}(\Gamma, w)).Vol(\Gamma, w)^{b/2} \geq \omega_b,$$

where ω_b is the volume of the Euclidean unit ball of $\mathbb{R}^b$.

The first part of this paper is dedicated to the proof of theorem **1** and the proposition above. Some related problems are also considered. In the second part, we recall some definitions and show theorems **2** and **3**.

The author would like to thank his thesis advisor I. Babenko and express his gratitude to S. Sabourau for several helpful remarks.

2 Volume entropy, systole and scale of graphs

2.1 Preliminaries

By a finite graph, we mean a finite non-oriented multigraph (we allow multiple edges and loops). For a finite graph Γ, we denote by V the set of vertices and by E the set of edges. Each element of E is an element of $V \times V$. A *weighted graph* is a pair (Γ, w), where Γ is a finite graph and $w : E \to \mathbb{R}^+$ is a weight function. For $e \in E$, we call $w(e)$ the weight of e. From now on, all graphs considered are finite.

We recall the definition of the exponential growth rate of a group (see [9]), that we will use in the proof of theorem **1**, and in the proof of proposition **3** and **5**.

Let G be a group of finite presentation and Σ be a finite generating set. We define the *algebraic length* of an element α of G with respect to Σ as the smallest integer k such that $\alpha = \alpha_1 \dots \alpha_k$, where $\alpha_i \in \Sigma \cup \Sigma^{-1}$. It is denoted by $|\alpha|_\Sigma$.

The *exponential growth rate* of G with respect to the system Σ is defined as

$$\omega(G, \Sigma) = \lim_{R \to +\infty} \sqrt[k]{N_\Sigma(R)}$$

where $N_\Sigma(R) = \text{card}\{\alpha \in G \mid |\alpha|_\Sigma \leq R\}$ is the cardinal of the ball of radius R of $(G, |.|_\Sigma)$ centered at its origin. For a group of finite presentation G and a finite generating set Σ of G,

$$\omega(G, \Sigma) \leq 2.\text{card}(\Sigma) - 1. \tag{1.3}$$

2.2 Volume entropy and systole of regular graphs

In this subsection, we etablish an upper bound on the normalization of the volume entropy by the systole valid on regular graphs which is better than the bound of theorem **1**. Recall that the *valence* of a vertex of a graph is the number of incident edges at this vertex. Let $v \geq 2$ be an integer. We say that a graph Γ is *regular* of valence v if the valence of each vertex is constant equal to v. Recall that a (usual combinatorial) graph is naturally identified with a weighted graph in which all the edges have weight 1. Remark that in this case $\mathrm{Vol}(\Gamma) = \mathrm{card}(E)$.

We will show the following result.

Proposition 1 *Let Γ be a regular graph with first Betti number b. Then*

$$h_{vol}(\Gamma).sys(\Gamma) \leq 3 \ln b. \tag{1.4}$$

Proof of the proposition. Let v be the valence of Γ. We have

$$h_{vol}(\Gamma) = \ln(v-1). \tag{1.5}$$

To see this, fix some vertex $\tilde{x}_0$ in the universal covering $\tilde{\Gamma}$. As $\tilde{\Gamma}$ is an infinite regular tree of valence v,

$$\begin{aligned} \mathrm{Vol}(B(\tilde{x}_0, R)) &= v(1 + (v-1) + \ldots + (v-1)^{R-1}) \\ &= v\frac{(v-1)^R - 1}{v-2} \end{aligned}$$

for each positive integer R. We deduce (1.5).

In the case $b = 1$, we obtain $v = 2$ and so $h_{vol}(\Gamma) = 0$. The inequality (1.4) is then trivial.

In the case $\mathrm{sys}(\Gamma) = 1$, we have $b = \mathrm{card}(E) - \mathrm{card}(V) + 1$, and as, by elementary considerations, $2.\mathrm{card}(E) = v.\mathrm{card}(V)$, and $\mathrm{card}(V) \geq 1$, we get $b \geq v/2$. Therefore,

$$\begin{aligned} h_{vol}(\Gamma) &= \ln(v-1) \\ &\leq \ln(2b-1) \\ &\leq 3 \ln b, \end{aligned}$$

and (1.4) follows in this case.

Now suppose that $b > 1$ and $\mathrm{sys}(\Gamma) > 1$. We will show the following lemma.

Lemma 1

$$sys(\Gamma) \leq \frac{3 \ln b}{\ln(v-1)}. \tag{1.6}$$

Proof. For all $R < \mathrm{sys}/2$, the ball centered at any point x in Γ of radius R is a tree. Thus, the calcul of the volume of a ball centered in a vertex of radius $[\mathrm{sys}(\Gamma)/2]$ gives the following estimate (compare with [4], p.14)

$$\mathrm{card}(E) \geq v\frac{(v-1)^{[\mathrm{sys}(\Gamma)/2]} - 1}{v-2}.$$

With $\mathrm{card}(E) = v(b-1)/(v-2)$, we deduce

$$[\mathrm{sys}(\Gamma)/2] \leq \frac{\ln b}{\ln(v-1)}.$$

Then

$$\mathrm{sys}(\Gamma) \leq 1 + 2\frac{\ln b}{\ln(v-1)} \leq 3\frac{\ln b}{\ln(v-1)},$$

and we are done.

Now we combine inequalities (1.5) and (1.6) to get the inequality (1.4). □

The asymptotic behaviour of (1.4) is optimal when b goes to infinity. It is realized by the bouquet of b circles viewed as the graph with one vertex and b loops of weight 1.

In the case of a graph provided with some control on the valence of its vertices, we easily obtain a lower and an upper bound on the volume entropy. We denote by $v(s)$ the valence of a vertex $s \in V$.

Proposition 2 *Let $\Gamma = (V, E)$ be a graph. Suppose that there exists two integers $2 \leq \delta \leq \Delta$ such that for all $s \in V$, $\delta \leq v(s) \leq \Delta$. Then*

$$\ln(\delta - 1) \leq h_{vol}(\Gamma) \leq \ln(\Delta - 1).$$

2.3 Volume entropy and systole of weighted graphs

We now generalize (1.4) to weighted graphs.

Theorem 1 *Let (Γ, w) be a weighted graph with first Betti number equal to b. Then*

$$h_{vol}(\Gamma, w).sys(\Gamma, w) \leq 2\ln(8b^3 - 1). \tag{1.7}$$

Proof of the theorem. The proof involves techniques coming from [15]. Fix a fundamental domain D of $\tilde{\Gamma}$ and a point $\tilde{x}_0$ in D such that points of the boundary are not vertices of $\tilde{\Gamma}$. D is a tree and denote by $\{\tilde{y}_1, \ldots, \tilde{y}_m\}$ the boundary. We have $m \leq 2b$. Denote by $p : \tilde{\Gamma} \to \Gamma$ the universal covering projection and by $\{z_1, \ldots, z_k\}$ the image of $\{\tilde{y}_1, \ldots, \tilde{y}_m\}$ under p. It is obvious that $k \leq b$. Denote by x_0 the projection of $\tilde{x}_0$.

Let $s = \mathrm{sys}(\Gamma, w)/2$. Denote by $p(s)$ the minimal number of translated domains $\gamma.D$ under the action of $\pi_1(\Gamma, x_0)$ needed to form a neighbourhood of D such that each point not belonging to this neighbourhood is at a distance more than s from D. We will find an upper bound of $p(s)$. Fix a vertex $\tilde{y}_i$ in the boundary and enumerate the paths starting at $\tilde{y}_i$ of length s going outside D. The number of these paths is less than the number of paths starting at $p(\tilde{y}_i)$ of length s, which is less than $2b$. It is clear that each of these paths passes at most once through each element of $\{z_1, \ldots, z_k\}$ (as $s = \mathrm{sys}(\Gamma, w)/2$). If we consider a path $\tilde{c}$ of $\tilde{\Gamma}$ starting at $\tilde{y}_i$ of length s, the number of translated domain that it passes through is exactly the number of points in $\{z_1, \ldots, z_k\}$ that belongs to $p(\tilde{c})$. Thus, this number is bounded from above by $k \leq b$. The number of translated domains of non null

intersection with a path starting from $\tilde{y}_i$ with length s is then bounded from above by $2b^2$.

As there are m elements in the boundary of D, and $m \leq 2b$, we get $p(s) \leq 4b^3$.

Denote by $\Sigma(s) = \{\gamma_i\}_{i=1}^{p(s)}$ a generating set of $\pi_1(\Gamma, x_0)$ such that for each $x \notin \cup_{i=0}^{p(s)} \gamma_i.D$ we have $d_{\tilde{w}}(x, D) \geq s$ (here γ_0 is by convention the neutral element).

We will estimate the volume of the ball centered at $\tilde{x}_0 \in D$ with radius ns. We can easily show that

$$B^{\tilde{\Gamma}}(x_0, ns) \subset B(\Sigma(s), n),$$

where $B(\Sigma(s), n) := \cup\{\gamma.D \mid |\gamma|_{\Sigma(s)} \leq n\}$.

We deduce

$$\mathrm{Vol}_{\tilde{w}}(B^{\tilde{\Gamma}}(\tilde{x}_0, ns)) \leq \mathrm{Vol}_{\tilde{w}}(\Gamma).N_{\Sigma(s)}(n),$$

and so, by (1.3),

$$s.h_{vol}(\Gamma, w) \leq \ln(2.p(s) - 1).$$

Inequality (1.7) follows. □

Inequality (1.7) yields the asymptotic behaviour

$$h_{vol}(\Gamma, w).\mathrm{sys}(\Gamma, w) \lesssim 6 \ln b.$$

We say that a weighted graph (Γ, w) has a *systolic basis* if there exist $x_0 \in V(\Gamma)$ and a generating set $\Sigma_0 = \{\gamma_i\}_{i=1}^{b}$ of $\pi_1(\Gamma, x_0)$ such that $l_w(\gamma_i) = \mathrm{sys}(\Gamma, w)$ for $i = 1, \ldots, b$. For weighted graph with systolic basis, we have the following lower bound.

Proposition 3 *Let (Γ, w) be a weighted graph with first Betti number b. Suppose that (Γ, w) has a systolic basis. Then*

$$h_{vol}(\Gamma, w).sys(\Gamma, w) \geq \ln(2b - 1), \tag{1.8}$$

and the equality case is attained by the bouquet of b circles $\bigvee_{i=1}^{b} S_i^1$.

Proof. Let $x_0 \in \Gamma$ be the vertex of Γ which is the base-point of the systolic basis. We fix $\tilde{x}_0 \in \tilde{\Gamma}$ a lift of x_0. The geometric length of an element $\alpha \in \pi_1(\Gamma, x_0)$ is defined by

$$\begin{aligned} |\alpha|_w &= \min\{l_w(\gamma) \mid \gamma \text{ loop based at } x_0 \text{ representing } \alpha\} \\ &= d_{\tilde{w}}(\tilde{x}_0, \alpha.\tilde{x}_0). \end{aligned}$$

This geometric length does not depend on the lift $\tilde{x}_0$.

We then have (see [12])

$$h_{vol}(\Gamma, w) = \lim_{R \to +\infty} \frac{\ln(N_w(R))}{R}$$

where $N_w(R) = \mathrm{card}\{\alpha \in \pi_1(\Gamma, x_0) \mid |\alpha|_w \leq R\}$ is the cardinal of the ball of $(\pi_1(\Gamma, x_0), |.|_w)$ with radius R centered at the origin.

We have easily

Lemma 2 *Let (Γ, w) be a weighted graph and Σ be a generating set of $G = \pi_1(\Gamma, x_0)$. If there exist $\lambda, \mu > 0$ such that*

$$\lambda.|.|_\Sigma \leq |.|_w \leq \mu.|.|_\Sigma,$$

then

$$\frac{1}{\mu} \ln \omega(G, \Sigma) \leq h_{vol}(\Gamma, w) \leq \frac{1}{\lambda} \ln \omega(G, \Sigma). \tag{1.9}$$

Now, as (Γ, w) has a systolic basis Σ_0 at x_0, for every $\alpha \in \pi_1(\Gamma, x_0)$,

$$|\alpha|_w \leq |\alpha|_{\Sigma_0}.\mathrm{sys}(\Gamma, w).$$

We immediately deduce (1.8).

The case of equality is attained by the bouquet $\vee_{i=1}^b S_i^1$. □

2.4 Volume entropy and scale of graphs

Let (Γ, w) be a weighted graph. A *chain* of Γ is a path such that the valence of each intermediate vertex is 2.

We define the *microscopic scale* of (Γ, w) as

$$C_{min}(\Gamma, w) = \min\{l_w(C) \mid C \text{ chain of } \Gamma\},$$

and the *macroscopic scale* of (Γ, w) as

$$C_{max}(\Gamma, w) = \max\{l_w(C) \mid C \text{ chain of } \Gamma\}.$$

The aim of this subection is to prove isoperimetric inequalities involving the volume entropy and the scale of weighted graphs.

Proposition 4 *Let (Γ, w) be a weighted graph such that $v(s) \leq 3$ for every $s \in V$. Then*

$$\frac{\ln 2}{C_{max}(\Gamma, w)} \leq h_{vol}(\Gamma, w) \leq \frac{\ln 2}{C_{min}(\Gamma, w)}. \tag{1.10}$$

Proof. Let T_3 be the infinite regular tree of valence 3. Fix a vertex v in T_3 and v' in Γ. We denote by w_{min} (respectively w_{max}) the constant weight function defined on T_3 equal to $C_{min}(\Gamma, w)$ (respectively $C_{max}(\Gamma, w)$). Then, for every $R > 0$,

$$\mathrm{Vol}_{w_{max}}(B^{T_3}_{w_{max}}(v, R)) \leq \mathrm{Vol}_w(B^{\tilde{\Gamma}}(v', R)) \leq \mathrm{Vol}_{w_{min}}(B^{T_3}_{w_{min}}(v, R))$$

and so

$$3(2^{[R/C_{max}]} - 1).C_{max} \leq \mathrm{Vol}_w(B^{\tilde{\Gamma}}(v', R)) \leq 3(2^{[R/C_{min}]+1} - 1).C_{min}.$$

We deduce (1.10). □

We can prove a stronger result for the normalization of the volume entropy by the minimum scale.

Proposition 5 *Let (Γ, w) be a weighted graph of first Betti number b. Then*

$$h_{vol}(\Gamma, w).C_{min}(\Gamma, w) \leq \ln(2b-1). \tag{1.11}$$

The equality case is attained by the bouquet of b circles $\bigvee_{i=1}^{b} S_i^1$.

Proof. Fix a vertex x_0 and let $\Sigma = \{\gamma_1, \ldots, \gamma_b\}$ be a minimal generating set of $\pi_1(\Gamma, x_0)$. Choose a maximal tree T of Γ containing x_0, and denote by e_i the edge of $\Gamma \setminus T$ such that $e_i \in \gamma_i$ for all $i = 1, \ldots, b$. Also denote by $p : \Sigma \cup \Sigma^{-1} \to \{e_1, \ldots, e_b\}$ the map defined by $p(\gamma_i) = p(\gamma_i^{-1}) = e_i$. For every $\alpha \in \pi_1(\Gamma, x_0)$, we choose a reduced form $\alpha = \alpha_1 \ldots \alpha_k$ where $\alpha_j \in \Sigma \cup \Sigma^{-1}$ and $k = |\gamma|_\Sigma$. For every β in α, β may be written as the concatenation of paths

$$[\beta_1, p(\alpha_1), \beta_2, p(\alpha_2), \ldots, p(\alpha_k), \beta_{k+1}]$$

where β_i are paths of Γ. As $l_w(e_i) \geq C_{min}(\Gamma, w)$ for $i = 1, \ldots, k$, we get

$$l_w(\beta) \geq |\alpha|_\Sigma.C_{min}(\Gamma, w),$$

so

$$|\alpha|_w \geq |\alpha|_\Sigma.C_{min}(\Gamma, w).$$

From inequality (1.9), we deduce (1.11). □

2.5 Entropy and topological Markov chain

We show here the proposition **6** stated in the introduction. We recall first some definitions (see [13]). Let n be a positive integer. We denote by $\mathbb{N}_n$ the set $\{1, \ldots, n\}$ and $\Sigma(n)$ the product space $\mathbb{N}_n^{\mathbb{Z}}$. The product topology is then induced by the following metric on $\Sigma(n)$

$$d(a, b) = \sum_{i=-\infty}^{+\infty} \frac{d_i(a, b)}{2^{2|i|+1}},$$

where $d_i(a, b)$ is equal to 0 if $a_i = b_i$, or to 1 otherwise. Note that the sequence $(a^j)_{j\in\mathbb{N}}$ converges if and only if for all $i \in \mathbb{Z}$ the sequence $(a_i^j)_{j\in\mathbb{N}}$ converges. Let ϕ be the homeomorphism of $\Sigma(n)$ defined as $(\phi(a))_i = a_{i+1}$. The homeomorphism ϕ is called the *shift*.

Denote by M_n the space of the $n \times n$ matrices all of whose entries are 0 or 1. If $A \in M_n$, we define

$$\Sigma_A = \{a \in \Sigma(n) \mid A_{a_i, a_{i+1}} = 1\}.$$

The set Σ_A is a closed ϕ-invariant subspace of $\Sigma(n)$. We denote by $\phi_A : \Sigma_A \to \Sigma_A$ the restriction of ϕ to Σ_A. The pair (Σ_A, ϕ_A) is called a *subshift of finite type* or *topological Markov chain*.

The *topological entropy* of a dynamical system (see [1] for a definition) is denoted by h_{top}. In the case of a topological Markov chain (Σ_A, ϕ_A), we deduce from [1] that

$$h_{top}(\Sigma_A, \phi_A) = \lim_{k \to +\infty} \frac{\ln(N(\Sigma_A, k))}{k},$$

where $N(\Sigma_A, k)$ is the number of open sets in the following cover

$$\mathcal{A}_k = \{\{a \mid \forall j = 0, \ldots, k-1, a_j = i_j\} \mid \{i_j\} \in \mathbb{N}_n^k\}.$$

We introduce also $N_i(\Sigma_A, k)$ the cardinal of open sets in $\mathcal{A}_k \cap \{a \mid a_0 = i\}$. We have

$$N(\Sigma_A, k) = \sum_{i=0}^{k} N_i(\Sigma_A, k).$$

It is a classic result that the topological entropy of (Σ_A, ϕ_A) is exactly the natural logarithm of the spectral radius of A.

From a different point of view, Σ_A can be identified with the space of bi-infinite paths of an oriented graph Γ_A whose vertices lie in $\mathbb{N}_n$ and edges are the pairs (i, j) of vertices such that $a_{i,j} = 1$. We define the *minimal period* of (Σ_A, ϕ_A), denoted by $T_{min}(\Sigma_A, \phi_A)$, as the smallest period of a periodic point of the dynamical system (Σ_A, ϕ_A). It coincides with the shortest length of an oriented cycle of Γ_A. We denote by b_A the first Betti number of Γ_A which can be expressed as $b_A = \sum_{i,j=1}^{n} A_{ij} - n + 1$.

Proposition 6 *For each topological Markov chain (Σ_A, ϕ_A),*

$$h_{top}(\Sigma_A, \phi_A).T_{min}(\Sigma_A, \phi_A) \leq \ln b_A. \tag{1.12}$$

The equality case is attained for each $b \geq 1$ by the following Markov chain defined as the matrix of size $b + 1$

$$\begin{cases} A_{1,j} = A_{j,1} = 1 \text{ for } j \neq 1, \\ A_{i,j} = 0 \text{ otherwise.} \end{cases} \tag{1.13}$$

Proof. Fix $t = T_{min}(\Sigma_A, \phi_A)$. For each vertex $v \in \mathbb{N}_n$ of γ_A, the number of oriented paths starting at v of length t is at most b_A.

We obtain

$$N_i(\Sigma_A, nt) \leq b_A^n,$$

and so

$$N(\Sigma_A, nt) \leq n.b_A^n.$$

Then

$$t.h(\Sigma_A, \phi_A) \leq \ln b_A.$$

and we get the claim (1.12). It is easy to compute that the topological Markov chain defined by (1.13) realize the equality case. □

3 Stable norm and volume of graphs

3.1 Unit ball of the stable norm

Let $\Gamma = (V, E)$ be a finite graph with first Betti number b. For the following classical notions, we refer to [14]. The graph Γ is a simplicial complex of dimension 1, whose 1-simplices are the edges and 0-simplices are the vertices. We denote by $C(\Gamma, \mathbb{R})$ the *oriented chain complex* of the simplicial complex Γ which is, in this 1-dimensional context and after a choice of orientation for each edge, the real vectorspace $\oplus_{e \in E} \mathbb{R}e$ of dimension $|E|$. Since the homology groups of dimension at least two are null, the 1-dimensional homology group with real coefficients $H_1(\Gamma, \mathbb{R})$ is embedded in $C(\Gamma, \mathbb{R})$ as a subspace of dimension b. Finally, the 1-homology group with integer

coefficients $H_1(\Gamma,\mathbb{Z})$ is identified with a lattice of the subspace $H_1(\Gamma,\mathbb{R})$ (compare with [3]).

Let w be a weight function on Γ. Denote by the sequence $\{e_i\}_{i=1}^{|E|}$ the edges of E and by $w_i = w(e_i)$ the weight of each edge. We define the following scalar product denoted by $< .,. >_w$

$$< e_i, e_j >_w = w_i \delta_{ij},$$

for $i, j = 1, \ldots, |E|$, where δ_{ij} is the Kronecker symbol.

Recall now the definition of the stable norm defined on $H_1(\Gamma,\mathbb{R})$ (see [7]). For an element $u \in H_1(\Gamma,\mathbb{R})$,

$$||u||_{st} = \lim_{n\to+\infty} \frac{1}{n} |\overline{nu}|_w, \tag{2.1}$$

where $\overline{nu}$ is the point of $H_1(\Gamma,\mathbb{Z})$ closest (for an arbitrary fixed Euclidean metric) to nu. The limit in (2.1) always exists and is independent on x_0.

We define the *unit ball of the stable norm* of (Γ, w) by

$$\mathcal{B}_{st}(\Gamma, w) = \{u \in H_1(\Gamma,\mathbb{R}) \mid ||u||_{st} \leq 1\}.$$

Consider the Haar measure on $\mathbb{R}^{|E|}$ for which the volume of the parallelepiped generated by $(e_1, \ldots, e_{|E|})$ is the square root of the determinant of the scalar product $< .,. >_w$. The restriction of this measure to $H_1(\Gamma,\mathbb{R})$ is noted μ_w. When we speak of a combinatorial graph, we forget the indice w in the notations $< .,. >_w$ and μ_w. For a combinatorial graph Γ, the measure μ coincides with the restriction to $H_1(\Gamma,\mathbb{R})$ of the Lebesgue measure of $\mathbb{R}^{|E|}$.

3.2 Estimates for combinatorial graphs

Theorem 2 *Let Γ be a graph with first Betti number b. We have the following inequalities*

$$\frac{2^b}{b!} \left(\frac{b}{Vol(\Gamma)} \right)^b \leq \mu_w(\mathcal{B}_{st}(\Gamma)) \leq \frac{2^b}{b!}. \tag{2.2}$$

The two cases of equality are attained by the bouquet of b circles $\bigvee_{i=1}^{b} S_i^1$.

Proof of the theorem. We have

$$\mathcal{B}_{st}(\Gamma) = B_1^k \cap H_1(\Gamma,\mathbb{R}), \tag{2.3}$$

where $k = |E|$ and $B_1^k = \{(x_i) \in \mathbb{R}^k \mid \sum_{i=1}^k |x_i| \leq 1\}$.

Indeed, we can show that the stable norm agrees with the norm $|.|_1$ defined by $|u|_1 = \sum_{i=1}^k |u_i|$ for all $u = \sum_{i=1}^k u_i e_i \in H_1(\Gamma,\mathbb{R})$. Namely,

$$\begin{aligned} |u|_{st} &= \lim_{n\to+\infty} \frac{1}{n} |\overline{nu}|_w \\ &= \lim_{n\to+\infty} \frac{1}{n} |\overline{nu}|_1 \\ &= |u|_1. \end{aligned}$$

We can find in [11] the following estimate. For every b-plane P^b in $\mathbb{R}^k$

$$\mu_k(B_1^k \cap P^b) \geq \mu_k(B_1^k)^{b/k},$$

where μ_n is the canonical volume of $\mathbb{R}^n$. We deduce with (2.3) that

$$\mu(B_{st}(\Gamma)) \geq \mu_k(B_1^k)^{b/k}.$$

Since $\mu_n(B_1^n) = 2^n/n!$ for every positive integer n, we have

$$\mu(B_{st}(\Gamma)).k^b \geq \frac{2^b.k^b}{(k!)^{b/k}}.$$

With $k \geq b$, we get

$$\mu(B_{st}(\Gamma')).k^b \geq \frac{(2b)^b}{b!}.$$

As $k = |E| = \mathrm{Vol}(\Gamma)$, the left inequality of (2.2) is then proved.

For the upper bound, we start with an other estimate obtained in [11]. For every b-plane P^b in $\mathbb{R}^k$,

$$\mu_k(B_1^k \cap P^b) \leq \mu_b(B_1^b).$$

So

$$\mu(B_{st}(\Gamma)) \leq \mu_b(B_1^b),$$

which gives the right inequality of (2.2).

For the bouquet of b circles (viewed as the combinatorial graph with one vertex and b loops)

$$\mu(B_{st}(\bigvee_{i=1}^{b} S_i^1)) = \frac{2^b}{b!}$$

and $\mathrm{Vol}(\bigvee_{i=1}^b S_i^1) = b$, so

$$\mu(B_{st}(\bigvee_{i=1}^{b} S_i^1)).(\mathrm{Vol}(\bigvee_{i=1}^{b} S_i^1))^b = \frac{(2b)^b}{b!}.$$

□

Remark For a regular graph Γ of valence $v \geq 3$, we have

$$\left(\frac{v-2}{v}\right)^b \frac{2^b}{b!} \leq \mu_w(\mathcal{B}_{st}(\Gamma)) \leq \frac{2^b}{b!}.$$

3.3 Estimate for weighted graphs

Theorem 3 *For every weighted graph (Γ, w) with first Betti number b,*

$$\mu_w(\mathcal{B}_{st}(\Gamma, w)).Vol(\Gamma, w)^{b/2} \geq \omega_b, \tag{2.4}$$

where ω_b is the volume of the Euclidean unit ball of $\mathbb{R}^b$.

Proof. Suppose that the result holds for graphs. Let (Γ, w) be a weighted graph with first Betti number b. For each $\epsilon > 0$, we can find a weight function w_ϵ close enough to w in the sense of the C^0 topology such that

$$|\mu_{w_\epsilon}(\mathcal{B}_{st}(\Gamma, w_\epsilon)) - \mu_w(\mathcal{B}_{st}(\Gamma, w))| < \epsilon,$$

$$|\mathrm{Vol}(\Gamma, w_\epsilon) - \mathrm{Vol}(\Gamma, w)| < \epsilon,$$

and such that $w_\epsilon(e)$ is rational for every $e \in E$. Fix an integer λ such that $\lambda w_\epsilon(e) \in \mathbb{N}^*$ for all $e \in E$. We have

$$\mu_{\lambda w_\epsilon}(\mathcal{B}_{st}(\Gamma, \lambda w_\epsilon)).\mathrm{Vol}(\Gamma, \lambda w_\epsilon)^{b/2} = \mu_{w_\epsilon}(\mathcal{B}_{st}(\Gamma, w_\epsilon)).\mathrm{Vol}(\Gamma, w_\epsilon)^{b/2}.$$

Choose an enumeration $\{e_i\}_{i=1}^{|E|}$ of the edges of Γ. If we subdivise each edge e_i of Γ in $k_\epsilon(i) = \lambda w_\epsilon(e_i)$ edges denoted by $e'_{i,j}$ for $j = 1, \ldots, k_\epsilon(i)$, we get a graph Γ'_ϵ with first Betti number b, which is isometric to $(\Gamma, \lambda w_\epsilon)$ when it is endowed with the trivial weight function $w = 1$. Denote by f this isometry and by E' the set of the edges of Γ'_ϵ. We have

$$\mu(\mathcal{B}_{st}(\Gamma'_\epsilon)) = \mu_{\lambda w_\epsilon}(\mathcal{B}_{st}(\Gamma, \lambda w_\epsilon)). \tag{2.5}$$

To see this, observe that the isometry f induces a linear homomorphism F from $C(\Gamma, \mathbb{R})$ to $C(\Gamma'_\epsilon, \mathbb{R})$ (and an isomorphism between $H_1(\Gamma, \mathbb{R})$ and $H_1(\Gamma'_\epsilon, \mathbb{R})$) which satisfies

$$F(e_i) = \sum_{j=1}^{k_\epsilon(i)} e'_{i,j},$$

for $i = 1, \ldots, |E|$. The familly

$$\left\{ \frac{1}{\sqrt{k_\epsilon(i)}} e_i \right\}_{i=1}^{|E|}$$

is an orthonormal basis for the scalar product $<.,.>_{\lambda w_\epsilon}$ of $\mathbb{R}^{|E|}$ and

$$\left\{ \frac{1}{\sqrt{k_\epsilon(i)}} \sum_{j=1}^{k_\epsilon(i)} e'_{i,j} \right\}_{i=1}^{|E|}.$$

an orthonormal basis of $I(C(\Gamma, \mathbb{R}))$ for $<.,.>$ in $\mathbb{R}^{|E'|}$. We then find that the map $F_{|H_1(\Gamma,\mathbb{R})}$ expressed in these orthonormal basis is the identity map, so we get the claim (2.5).

This construction can be realized for every $\epsilon > 0$. So if the result holds for graphs, it holds for every weighted graph.

Let $\Gamma' = (E', V')$ be a graph with first Betti number b and denote $k = |E'|$. For every $u \in \mathbb{R}^k$,

$$|u|_1 \leq k.|u|_2$$

where $|u|_2 = \sqrt{\sum_{i=1}^k u_i^2}$ is the Euclidean norm. So

$$\mu(\mathcal{B}_{st}(\Gamma')) \geq \mu(B_2(\frac{1}{\sqrt{k}})),$$

where $B_2(R)$ is the ball with radius R for the norm $|.|_2$ in $H_1(\Gamma', \mathbb{R})$. We deduce the inequality (2.4) for graphs. □

References

[1] Adler, R., Konheim, A. & McAndrew, M. : *Topological entropy*, Trans. Amer. Math. Soc., Vol 114, 1965, 309-319.

[2] Babenko, I. & Balacheff, F.: *Géométrie systolique des sommes connexes et des revêtements cycliques*, preprint (www.math.univ-montp2.fr/prepublications/04-05.pdf).

[3] Bacher, R., de la Harpe, P. & Nagnibeda T. : *Lattices defined by finite graphs*, Bull. Soc. Math. France, Vol 125, 1997, 167-198.

[4] Bollobás, B. : *Extremal graph theory with emphasis on probabilistic methods*, CBMS Regional Conference Series in Mathematics, Vol 62, (1986), 1-64.

[5] Bollobás, B. & Szemerédi, E. : *Girth of sparse graphs*, J. Graph Theory, Vol 39, 2002, 194-200.

[6] Buser, P., Sarnak, P. : *On the period matrix of a Riemann surface of large genus (with an appendix by J.H. Conway and N.J.A. Sloane)*, Invent. Math., Vol 117, 1994, 27-56.

[7] Gromov, M. : *Structures métriques pour les variétés riemanniennes*, Textes Mathématiques, Vol 1, CEDIC, Paris, 1981.

[8] Gromov, M. : *Filling Riemannian Manifolds*, J. Differential Geom., Vol 18, 1983, 1-147.

[9] De la Harpe, P. : *Uniform growth in groups of exponential growth*, Proceedings of the conference on the geometric and combinatorial group theory , Part II(Haifia 2000), Geom. Dedicata, Vol 95, 2002, 1-17.

[10] Manning, A. : *Topological entropy for geodesic flows*, Ann. of Math., Vol 110, no 2, 1979, 567-573.

[11] Meyer, M. & Pajor, A. : *Sections of the unit ball of l_p^n*, J. Funct. Anal., Vol 80, 1988, 109-123.

[12] Sabourau, S. : *Entropy and systoles on surfaces*, preprint.

[13] Shub, M. : *Stabilité globale des systèmes dynamiques*, Astérisque 56, SMF, Paris, 1978.

[14] Spanier, E. : *Algebraic topology*, Springer-Verlag, Berlin, 1995.

[15] Sv́arc, A. : *A volume invariant of coverings*, Dokl. Ak. Nauk. SSSR, Vol 1058, 1955, 32-34.