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SEMIGROUP ACTIONS ON TORI AND STATIONARY MEASURES ON PROJECTIVE SPACES

YVES GUIVARC'H AND ROMAN URBAN

Dedicated with admiration to Hillel Furstenberg on the occasion of his 70th birthday

Abstract. Let Γ be a sub-semigroup of $G = GL(d, \mathbb{R})$, $d > 1$. We assume that the action of Γ on $\mathbb{R}^d$ is strongly irreducible and that Γ contains a proximal and expanding element. We describe contraction properties of the dynamics of Γ on $\mathbb{R}^d$ at infinity. This amounts to the consideration of the action of Γ on some compact homogeneous spaces of G, which are extensions of the projective space $\mathbb{P}^{d-1}$. In the case where Γ is a sub-semigroup of $GL(d, \mathbb{R}) \cap M(d, \mathbb{Z})$ and Γ has the above properties, we deduce that the Γ-orbits on $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ are finite or dense.

1. Introduction and main results

Let Γ be a multiplicative semigroup of integers. The semigroup Γ is said to be lacunary if the members $\{\gamma \in \Gamma : \gamma > 0\}$ are of the form γ_0^k, $k \in \mathbb{N}$, $\gamma_0 \in \mathbb{N}^*$. Otherwise Γ is non-lacunary. In 1967 in [12] Furstenberg proved that if Γ is a non-lacunary semigroup of integers and α is an irrational number, then the orbit $\Gamma\alpha$ is dense modulo 1. The problem of approximating a number θ modulo 1 by numbers of the form $q\alpha$, where α is a fixed irrational and q varies in a specified subset $Q \subset \mathbb{N}$ was considered by Hardy and Littlewood in [20] for various subsets Q of $\mathbb{N}$. In particular, the result of Furstenberg above can be considered as a generalization of a theorem of [20], which asserts that if r is a positive integer and α is an irrational number, the set $\{q^r\alpha : q \in \mathbb{N}\}$ is dense modulo 1; furthermore, this result draws attention to the role of the multiplicative structure of Q in Diophantine approximation, hence of the role of the corresponding dynamical properties of endomorphisms of $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Hence, one is led, more generally, to consider separately the properties depending on the multiplicative structure of Γ and the properties depending on the additive structure implied in reduction modulo 1. In

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this direction, a generalization of Furstenberg's result to a commutative semigroup $\Gamma \subset M_{inv}(d,\mathbb{Z}) := GL(d,\mathbb{R}) \cap M(d,\mathbb{Z})$, where $M(d,\mathbb{Z})$ is the set of $d \times d$ matrices with integer entries, acting by endomorphisms on the torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ was given by Berend in [4].

Following [4], we say that the semigroup of endomorphisms of a d-dimensional torus $\mathbb{T}^d$ has ID-property (cf. [4, 25, 26]) if the only infinite closed Γ-invariant subset of $\mathbb{T}^d$ is $\mathbb{T}^d$ itself. (ID-property stands for *infinite invariant is dense.*) Berend in [4] gave necessary and sufficient conditions in arithmetical terms for a commutative semigroup to have ID-property.

On the other hand, starting from [4] and [12], Margulis in [24] asked for necessary and sufficient conditions on sub-semigroup $\Gamma \subset M_{inv}(d,\mathbb{Z})$ in order that the Γ-orbit closures on $\mathbb{T}^d$ are finite union of manifolds. We observe that it follows from general results of Dani and Raghavan on linear actions [9] that the orbits of $\Gamma = SL(d,\mathbb{Z})$ acting on $\mathbb{T}^d$ are finite or dense. In this direction a detailed study of Γ-orbits in $\mathbb{R}^d$ of a general subgroup $\Gamma \subset SL(d,\mathbb{R})$ was developed by Conze and Guivarc'h in [7]. The homogeneity at infinity of Γ-orbits was pointed out there as well as the role of "Γ-irrational" vectors in the construction of limit points of Γ-orbits, if Γ is a general subgroup of $SL(d,\mathbb{R})$.

Some results in direction of general question of Margulis has been obtained recently. Muchnik in [25] proved that if the semigroup Γ of $SL(d,\mathbb{Z})$ is Zariski dense in $SL(d,\mathbb{R})$, then Γ acting on $\mathbb{T}^d$ has ID-property. Starkov in [29] proved the same result in case Γ is a strongly irreducible subgroup of $SL(d,\mathbb{Z})$. In the next paper [26] Muchnik generalized the results of Berend to semigroups of $M_{inv}(d,\mathbb{Z})$. In the same time Guivarc'h and Starkov in [19] derived an important part of Muchnik's result using different methods, based on [6, 7]. We observe that in [19], the property $\Gamma \subset SL(d,\mathbb{Z})$ is used only when additive aspects connected with reduction modulo one comes into the play. It turned out that the property of Γ-orbits in $\mathbb{R}^d$ which is responsible for density in $\mathbb{T}^d$ is "thickness" at infinity of Γ-orbits (see Theorem 5.22 and the comment). Hence, this property can be studied separately in full generality; Γ is then a general sub-semigroup of $GL(d,\mathbb{R})$ and the use of boundaries and random walks is natural in this context.

In this paper we consider this problem in a simplified setting, we give a self-contained exposition of some of the methods developed in [6, 7, 19] in the more general context of random walks and linear actions, and we use the results to prove ID-property in our setting. We prove also some new results for actions on tori and on certain compact G-factor spaces of $\mathbb{R}^d$.

The general idea is to lift the automorphisms of the torus $\mathbb{T}^d$ to its universal cover $\mathbb{R}^d$ and to study the action of the lifts at infinity. The action of Γ at infinity can be expressed in terms of some compact homogeneous spaces of $GL(d,\mathbb{R})$ which are closely related to the projective spaces $\mathbb{P}^{d-1}$.

The random walk framework allows to take into account the global semigroup asymptotic behavior in terms of stationary measures and convergence to them. As in [13] and [15] the results can be used to obtain topological properties of the Γ-action. Furthermore, this general framework allows us to obtain a series of informations on linear actions which are of interest in other problems.

Before we state the results we need to introduce some notions. A matrix $\gamma \in GL(d, \mathbb{R})$ is said to be *proximal* if it has an eigenvalue λ_γ such that $|\lambda_\gamma| > |\lambda|$ for all other eigenvalues λ of γ. A matrix γ is said to be *expanding* if it has an eigenvalue λ such that $|\lambda| > 1$.

Let Γ be a sub-semigroup of $GL(d, \mathbb{R})$. The Γ-action on $\mathbb{R}^d$ (or simply Γ) is said to be *strongly irreducible* if there do not exist any finite union of proper subspaces which is Γ-invariant.

The first main theorem of this paper is as follows:

Theorem 1.1. *Let Γ be a sub-semigroup of $M_{inv}(d, \mathbb{Z})$, $d > 1$, such that Γ contains a proximal and expanding element and the Γ-action on $\mathbb{R}^d$ is strongly irreducible. Then the semigroup Γ acting on $\mathbb{T}^d$ has the ID-property, that is, every infinite Γ-invariant subset of $\mathbb{T}^d$ is dense.*

If $d = 1$, one has $M_{inv}(1, \mathbb{Z}) = \mathbb{Z}^* \subset \mathbb{R}^*$. As said above, the conclusion of Theorem 1.1 is valid in this case too, if and only if Γ is non-lacunary, i.e., not contained in a cyclic subgroup of $\mathbb{R}^*$. For $d > 1$, the condition in Theorem 1.1 imply that Γ is not contained in a finite extension of an abelian subgroup of $M_{inv}(d, \mathbb{Z})$; in particular, here Γ is non-abelian, hence the situation of [4] is excluded from our setting.

The first step in order to get Theorem 1.1 is to study infinite Γ-invariant subsets Σ of $\mathbb{T}^d$ such that 0 is a limit point of Σ. Then we notice that the inverse image in $\mathbb{R}^d$ of such an infinite Γ-invariant subset contains some asymptotic set which consists of lines. Moreover, there are some rays with good properties, that is which are not contained in a subspace having a basis which consists of integer vectors. This allows us to project them using canonical projection $p : \mathbb{R}^d \to \mathbb{R}^d/\mathbb{Z}^d$, $p(x) = x + \mathbb{Z}^d$ on $\mathbb{T}^d$ and obtain the result in case when 0 is a limit point of the subset Σ. Furthermore, using arguments close to [4] and [12] and reduction to a finitely generated semigroup of Γ, we show that the opposite situation does not exist.

Let $L_\Gamma \subset \mathbb{P}^{d-1}$ be the closure of the set of directions corresponding to dominant eigenvectors of the proximal elements in Γ. We denote by $\tilde{L}_\Gamma$ the set of corresponding nonzero vectors in $V = \mathbb{R}^d$, by σ the symmetry $\sigma : v \mapsto -v$ in V, and by $\tilde{V}$ the factor space $\tilde{V} = V/\{\sigma, \mathrm{Id}\}$.

The following is the basic tool in the proof of Theorem 1.1.

Theorem 1.2. *Suppose Γ is a sub-semigroup of $GL(d, \mathbb{R})$, $d > 1$ which is strongly irreducible and contains a proximal and expanding element. Let Σ*

be a Γ*-invariant subset of* $\tilde{V} \setminus \{0\}$ *and suppose* $0 \in \overline{\Sigma}$*. Then*

$$\overline{\Sigma} \supset \tilde{L}_\Gamma / \{\sigma, Id\}.$$

To have a simple example in mind illustrating Theorem 1.1 and Theorem 1.2, consider the torus $\mathbb{T}^2$. One of the simplest example of a sub-semigroup of $SL(2,\mathbb{Z})$ satisfying properties of Theorem 1.1 is the semigroup $\Gamma = \langle a, b\rangle$ generated by two matrices $a = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ and $b = \left(\begin{smallmatrix} 3 & 2 \\ 1 & 1 \end{smallmatrix}\right)$ from $SL(2,\mathbb{Z})$.

¿From Theorem 1.1 we obtain that the Γ-orbits in $\mathbb{T}^2$ are finite or dense. Furthermore we observe that, in the context of Theorem 1.2, the dynamics of Γ on $\mathbb{R}^2$ is easy to visualize. The closure of the eigen-directions in the positive quadrant $\mathbb{R}^2_+$ form a Cantor set and the corresponding lines form an "attractor set" $\tilde{L}^1_\Gamma$ for the action of Γ in $\mathbb{R}^2_+$. There exist in $\mathbb{R}^2$ vectors whose orbit closures contain 0, for example dominant eigenvectors of elements of Γ^{-1}. The Γ-orbit for such a vector tends to fill $\tilde{L}^1_\Gamma \cup -\tilde{L}^1_\Gamma$ since the dynamics of its Γ-orbit consists of attraction towards 0 and expansion along the eigenvectors sitting in $\tilde{L}^1_\Gamma \cup -\tilde{L}^1_\Gamma$.

For a general vector, for example a vector $v \in \mathbb{R}^2_+$, there is attraction towards $\tilde{L}^1_\Gamma$ and expansion along $\tilde{L}^1_\Gamma$ and the Γ-orbit of v is "thick at infinity" due to the irrationality properties of eigenvalues of elements in Γ. In general the situation is similar, in particular the projections of general Γ-orbits in $\mathbb{T}^d$ are large, hence one can expect the closed Γ-orbits in $\mathbb{T}^d$ to be finite unions of special manifolds, as conjectured in [24].

Let us consider now for $c > 1$, the factor space $\mathbb{P}^{d-1}_c$ of $V \setminus \{0\}$ by the subgroup of homotheties with ratio $\pm c^k$ $(k \in \mathbb{Z})$. The action of $g \in G = GL(d,\mathbb{R})$ on $v \in \mathbb{P}^{d-1}_c$ will be denoted $v \mapsto g.v$. Let μ be a probability measure on $\Gamma \subset GL(d,\mathbb{R})$ whose support generates Γ. Then we can define an associated Markov operator P_μ on $\mathbb{P}^{d-1}_c$ by the formula:

$$P_\mu(v, \cdot) = \int \delta_{g.v} d\mu(g).$$

The iterates P^n_μ of P_μ define a random walk on $\mathbb{P}^{d-1}_c$.

The following describes the asymptotic behavior of the iterates P^n_μ; it is an essential tool in the proof of Theorem 1.2, hence of Theorem 1.1.

Theorem 1.3. *Assume that* $\Gamma \subset GL(d,\mathbb{R})$ *is a sub-semigroup which is strongly irreducible and contains a proximal element. Then with the above notations, the Markov operator* P_μ *on* $\mathbb{P}^{d-1}_c$ *has a unique stationary measure* ρ*, the support* S_ρ *of* ρ *is the unique closed* Γ*-invariant minimal subset of* $\mathbb{P}^{d-1}_c$ *and for any* $v \in \mathbb{P}^{d-1}_c$ *the sequence of measures* $P^n_\mu(v, \cdot)$ *converges to* ρ*. Moreover, the trajectories of* P_μ*, starting from* v *converge a.e. to* S_ρ*.*

Along the way, we get some new results and informations. For example, we show a-priori that *weak* ID-property (that is the closures of the orbits Γx,

$x \in \mathbb{T}^d$ are either finite or equal $\mathbb{T}^d$ itself) and ID-property are equivalent, a fact implicitly used in previous papers, but apparently unproved in the literature.

We also clarify the relations between a fundamental cocycle equation on $\Gamma \times \mathbb{P}^{d-1}$ and an aperiodicity condition for the dominant eigenvalues of proximal elements in Γ which occur in [22] and which has also a geometric interpretation in terms of lengths of closed geodesics (see [11]).

Furthermore, the result in Theorem 1.3 extends results of [17] but is new in this generality.

Also the result of Theorem 1.1 is not covered by [19] since, in our setting, Γ is allowed to be a sub-semigroup of $M_{inv}(d, \mathbb{Z})$, $(d > 1)$. Then we need to prove that Γ can be supposed to be finitely generated (see Proposition 2.6).

The structure of the paper is as follows. In Sect. 2 we set the notation and give all necessary definitions. In particular, we define a dominant vector, a proximal element and state our two hypothesis (H_1) - strong irreducibility and (H_2) - proximality, under which we prove Theorem 1.3. We introduce hypothesis (H_0), i.e. the unboundedness of Γ-orbits in $V \setminus \{0\}$. Under (H_1) and (H_2), this condition is equivalent (see Proposition 2.4) to the existence of a proximal and expanding element in Γ which allows to prove Theorem 1.1 and Theorem 1.2. It is clear, that this condition is necessary for validity of ID-property.

We observe that conditions (H_0), (H_1) and (H_2) are analogous to those used in [16, Theorem 2.5] in order to get a homogeneous behavior at infinity of the potential measure in $\tilde{V}$ associated with μ, hence also of the Γ-orbits at infinity in $\tilde{V}$. (See also [10] for the case of affine actions.)

In Sect. 3 we prove the equivalence of the weak ID-property and ID-property (Proposition 3.1).

In Sect. 4 we study the Γ-actions on various spaces, namely on the projective space $\mathbb{P}(V)$, the compact homogeneous space $\mathbb{P}_c(V)$ and V itself. We define the asymptotic sets for Γ-actions and we study their properties. We also clarify the role of aperiodicity hypotheses of Γ considered by Kesten in [22] (see Corollary 4.8).

Sect. 5 develops the random walks techniques which are used in the proof of the main result of this section which is Theorem 5.18. This theorem together with the method presented in [12] allow us to prove in Sect. 6 Theorem 1.1. Theorem 5.18 follows from a detailed study of random walks on V and various Γ-spaces, governed by a measure μ sitting on Γ and such that the convolution iterations μ^{*k} fill Γ. Some of these results are well known but we have included the proofs in order to be self-contained. Some others are new.

Finally, in Sect. 6 we give the proof of Theorem 1.1.

2. Proximality, irreducibility, expansivity

In what follows Γ will denote a sub-semigroup of $GL(d,\mathbb{R})$. We consider the actions of Γ on the vector space $V=\mathbb{R}^d$, on the associated projective space $\mathbb{P}^{d-1}=\mathbb{P}(V)$, and on $\tilde{V}=V/\{\mathrm{Id},\sigma\}=V/\{\pm\mathrm{Id}\}$. We denote by π the projection of $V\setminus\{0\}$ on $\mathbb{P}^{d-1}=\mathbb{P}(V)$ and we identify $\mathbb{P}(V)$ with the unit sphere $\mathbb{S}^{d-1}$ divided by the symmetry $\sigma: x\mapsto -x$. Also $K=SO(d,\mathbb{R})$ will denote the special orthogonal group and m the unique K-invariant probability measure on $\mathbb{P}(V)$.

The action of the matrix g on the vector $x\in V$ we denote by $(g,x)\mapsto gx$, whereas for the action of g on the projective space $\mathbb{P}(V)$ we write: $g.\pi(x)=\pi(gx)$.

A matrix $\gamma\in GL(d,\mathbb{R})$ is said to be *proximal* if it has an eigenvalue λ_γ such that $|\lambda_\gamma|>|\lambda|$ for all other eigenvalues λ of γ. Thus $\lambda_\gamma\in\mathbb{R}$. For such a γ an eigenvector $v_\gamma\in V$ corresponding to the eigenvalue λ_γ is called a *dominant eigenvector* or simply *dominant vector* of γ. By Δ_Γ we denote the set of all proximal elements in Γ. An element $\gamma\in GL(d,\mathbb{R})$ is said to be *expanding* if it has an eigenvalue λ such that $|\lambda|>1$.

More generally, for $u\in\mathrm{End}(V)$, we denote $|\lambda_u|$ the spectral radius of u.

If $\gamma\in\Delta_\Gamma$ then we define $\gamma^+\in\mathbb{P}(V)$ as a point corresponding to the line in V generated by v_γ. By $V_\gamma^<$ we denote the unique γ-invariant hyperplane complementary to $V_\gamma^{\max}=\mathbb{R}v_\gamma$.

We consider the following assumptions.

- (H_0) For every $v\in V\setminus\{0\}$, the orbit Γv is unbounded.
- (H_1) The Γ-action is *strongly irreducible* (in short, Γ is *strongly irreducible*), that is that there does not exist any finite union of proper subspaces which is Γ-invariant.
- (H_2) Γ contains a matrix γ which is proximal.

Remark 2.1. (i) The condition (H_1) can be equivalently formulated as follows. A sub-semigroup Γ of $GL(V)$ acts strongly irreducibly on V if every finite index subgroup H of the group $\langle\Gamma,\Gamma^{-1}\rangle$ acts irreducibly on V, that is every H-invariant subspace of V is either 0 or V.

(ii) If Γ is a sub-semigroup of $SL(d,\mathbb{R})$, then conditions (H_1) and (H_2) imply (H_0), since otherwise the determinant of the proximal element γ would be strictly less than 1 (see Proposition 2.4 b) below).

(iii) The condition (H_1) (resp. (H_2)) if valid for Γ, is also valid for Γ^t, the transposed semigroup acting on the dual space V^*.

(iv) Conditions (H_0), (H_1) for Γ imply condition (H_0) for Γ^t. This will be used in the proof of Theorem 1.1 and can be seen as follows. Let $W\subset V^*$ be the subspace of vectors with bounded Γ^t-orbits. Then W is Γ^t-invariant, hence (iii) implies $W=\{0\}$ or $W=V^*$. In case $W=V^*$, Γ^t

is relatively compact in $\mathrm{End}(V^*)$, hence Γ is relatively compact in $\mathrm{End}(V)$. This contradicts condition (H_0) for Γ.

The concept of Zariski closure, defined below, will be freely used in dealing with the above conditions (see [27]).

Let Γ be a subset of $GL(d,\mathbb{R})$. We recall that the *Zariski closure* $\mathrm{Zc}(\Gamma)$ of Γ is the set of zeros of all real polynomials with variables in the coefficients of $g \in GL(d,\mathbb{R})$ and $(\det g)^{-1}$, which vanish on Γ.

If Γ is a sub-semigroup of $GL(d,\mathbb{R})$ then $\mathrm{Zc}(\Gamma)$ is a group which contains Γ, is closed and has a finite number of connected components in the real topology (see [27]). The connected component of the identity in the Zariski topology is a subgroup of finite index which will be denoted by $\mathrm{Zc}_0(\Gamma)$.

We have the following generalization of Lemma 2.8 in [6] to the case of semigroups.

Lemma 2.2. *Let $\Gamma \subset GL(V)$ be a sub-semigroup. The Γ-action satisfies condition (H_1), if and only if there do not exist a non-zero vector v such that the orbit Γv is contained in a finite union of proper vector subspaces of V.*

Proof. Suppose (H_1) to be valid and $v \in V$ be such that $\Gamma v \subset \bigcup_{j=1}^n V_j$ where V_j are proper subspaces of V. Let W be a finite union of subspaces of V such that $\Gamma v \subset W$ and $\mathcal{W}$ the set of such W. We observe that $\Gamma v \subset \bigcap_{W\in\mathcal{W}} W$. Since a strictly decreasing family of elements of $\mathcal{W}$ is finite we get that $\bigcap_{W\in\mathcal{W}} W$ belongs also to $\mathcal{W}$, in other words $W_0 := \bigcap_{W\in\mathcal{W}} W$ is the minimum element in $\mathcal{W}$. We write $W_0 = \bigcup_{j=1}^m V_j$ and we are going to show that W_0 is preserved by Γ. Since $W \in \mathcal{W}$ is algebraic we have: $\mathrm{Zc}(\Gamma)v \subset W$, in particular $\langle \Gamma, \Gamma^{-1}\rangle v \subset W$. It follows that, for any $\gamma \in \Gamma$:

$$\gamma W \supset \gamma\langle \Gamma, \Gamma^{-1}\rangle v \supset \Gamma v.$$

Hence, $\gamma W \in \mathcal{W}$. Since W_0 is the minimum element of $\mathcal{W}$, we have $\gamma W_0 \supset W_0$, $\gamma W_0 = W_0$; hence, $\Gamma W_0 = W_0$. Condition (H_1) says that it is impossible.

Conversely, suppose V_j $(1 \le j \le n)$ is a family of proper subspaces which is preserved by Γ. Let $v \in V_1$, then $\Gamma v \in \bigcup_{i=1}^n V_i$. From the hypothesis this is impossible, hence condition (H_1) is satisfied. □

Let X be a compact metric space with distance function δ. We say that the action of a semigroup Γ of continuous transformations of X is *proximal* if, given $x, y \in X$, there exists a sequence $\{\gamma_n\} \subset \Gamma$ such that $\delta(\gamma_n.x, \gamma_n y) \to 0$ as $n \to \infty$.

Define the distance function δ on $\mathbb{P}(V)$ as follows:

$$\delta(\bar{u}, \bar{v}) = \|u \wedge v\|/\|u\|\|v\|,\ \bar{u}, \bar{v} \in \mathbb{P}(V),$$

where u and v are the corresponding vectors in the vector space V.

Proposition 2.3 (Theorem 2.9 in [14]). *Let Γ be a sub-semigroup of the group $GL(V)$.*

Then we have the following equivalence:

(a) Γ *satisfies* (H_1) *and* (H_2).
(b) Γ *acts proximally on* $\mathbb{P}(V)$ *and is strongly irreducible.*

Proof. ((a) $\Rightarrow$ (b)) We consider a proximal element $\gamma \in \Gamma$ and denote

$$\delta = \lim_n \|\gamma^{2n}\|^{-1}\gamma^{2n}, \qquad \mathfrak{z} = \operatorname{Ker}\delta \subset \mathbb{P}(V^*).$$

Then if $x, y \in \mathbb{P}(V)$ do not belong to $\operatorname{Ker}\delta$, we have $\lim_n \gamma^n.x = \gamma^+$, $\lim_n \gamma^n.y = \gamma^+$. Hence, $\lim_n \delta(\gamma^n.x, \gamma^n.y) = 0$.

In general, if $x, y \in \mathbb{P}(V)$ are given we can find $h \in \Gamma$ such that $h.x \notin \operatorname{Ker}\delta$ and $h.y \notin \operatorname{Ker}\delta$, otherwise, passing to the dual space V^*, transposing maps, and using the hyperplanes $x^\perp$ and $y^\perp$ of V^* defined by x and y, one would have

$$\forall\, h \in \Gamma:\ h^t.\mathfrak{z} \subset x^\perp \text{ or } h^t.\mathfrak{z} \subset y^\perp.$$

But Remark 2.1 iii) and Lemma 2.2 say that this is impossible under condition (H_2).

((b) $\Rightarrow$ (a)) It follows from proximality of Γ on $\mathbb{P}(V)$ (see [13]) that, given a finite subset $E \subset \mathbb{P}(V)$, there exist a sequence $\{g_n\} \subset \Gamma$ and $x \in \mathbb{P}(V)$ such that

$$\forall\, y \in E:\ \lim_n g_n.y = x.$$

We consider a finite system $E = \{x_1, x_2, \dots, x_{2d-1}\}$ of $2d-1$ points in $\mathbb{P}(V)$ such that any $d-$subsystem consists of independent points.

We consider the linear maps $u_n = \frac{g_n}{\|g_n\|}$ and using a convergent subsequence, we can assume suppose than u_n converges towards $u \in \operatorname{End}(V)$, $\|u\| = 1$. We show that u has rank one.

¿From the definition of E it follows that, at least d points of E do not belong to $\operatorname{Ker} u$. We replace these points, as well as x, by the corresponding unit vectors in V, say $\tilde{x}_1, \dots, \tilde{x}_d, \tilde{x}$. Then we obtain

$$u\tilde{x}_i = \lambda_i \tilde{x},\ 1 \le i \le d,$$

where $\lambda_i \neq 0$; the points $\{\tilde{x}_i\}$ form a basis of V, hence the rank of u is one, i.e., $\dim \operatorname{Ker} u = d-1$. We can moreover suppose that $\operatorname{Im} u \not\subset \operatorname{Ker} u$, since otherwise we could replace g_n by gg_n, where $g \in \Gamma$ satisfies $\operatorname{Im} gu = g(\operatorname{Im} u) \not\subset \operatorname{Ker} u$ and $\operatorname{Ker} gu = \operatorname{Ker} u$. The existence of $g \in \Gamma$ such that $g(\operatorname{Im} u) \not\subset \operatorname{Ker} u = \operatorname{Ker} gu$ follows from Lemma 2.2.

Under this condition, u is proportional to the projection on $\operatorname{Im} u$, along the hyperplane $\operatorname{Ker} u$. In particular, u has a unique non-zero eigenvalue λ. Since the sequence $\frac{g_n}{\|g_n\|} - u$ converges to zero, we obtain that for n large,

$\frac{g_n}{\|g_n\|}$ has also a unique dominant eigenvalue close to λ. The same is true for g_n, hence Γ satisfies (H_2). $\square$

Proposition 2.4. *Let Γ be a sub-semigroup of $GL(V)$. Then we have the following equivalence:*

(a) *Γ satisfies (H_1), (H_2) and the element γ in the condition (H_2) satisfies $|\lambda_\gamma| > 1$.*
(b) *Γ is unbounded and satisfies (H_1) and (H_2).*
(c) *Γ satisfies (H_0), (H_1) and (H_2).*
(d) *Γ satisfies (H_1), (H_2) and there exists $\gamma \in \Gamma$ such that $|\lambda_\gamma| > 1$.*

Proof. ((d) $\Rightarrow$ (b)) Let $\gamma \in \Gamma$ be an expanding element in Γ, hence $|\lambda_\gamma| > 1$. Then: $\|\gamma^n\| \geq |\lambda_\gamma|^n$. Hence $\lim_n \|\gamma^n\| = \infty$, i.e., Γ is unbounded.

((b) $\Rightarrow$ (a)) We will use the basic [1, Theorem 4.1], which allows to construct new proximal maps and, which implies the following. If $\Gamma \subset GL(V)$ satisfies (H_1) and (H_2) there exists $\varepsilon > 0$, $r > 1$, and a finite subset $M \subset \Gamma$, such that, for any $g \in GL(V)$, there exists $a \in M$ such that ag is proximal, the distance in $\mathbb{P}(V)$ of $(ag)^+$ to $V_{ag}^<$ is at least ε, and:

$$|\lambda_{ag}| \geq r\|(ag)|_{V_{ag}^<}\|.$$

Since Γ is unbounded, there exists a sequence $\{\gamma_n\} \subset \Gamma$ such that

$$\lim_n \|\gamma_n\| = \infty.$$

Using a subsequence of γ_n we can suppose that, for some $a \in M$, $a\gamma_n$ is proximal, $(a\gamma_n)^+$, ($V_{a\gamma_n}^<$ resp.) converges to $x \in \mathbb{P}(V)$ ($W_n = V_{a\gamma_n}^<$ converges to the hyperplane W of $\mathbb{P}(V)$, resp.). We have $x \notin W$, since the distance of $(a\gamma_n)^+$ to W_n is at least ε. We can suppose also that $\frac{a\gamma_n}{\|a\gamma_n\|}$ converges to $u \in \mathrm{End}(V)$ with $\|u\| = 1$. Clearly, V is the direct sum of the hyperplane W and of the line generated by x. Furthermore, $|\lambda_u| = \lim_n \frac{|\lambda_{a\gamma_n}|}{\|a\gamma_n\|}$, $\|u|_W\| = \lim_n \frac{1}{\|a\gamma_n\|}\|(a\gamma_n)|_{W_n}\|$. Since $u \neq 0$ preserves the above direct sum we have $|\lambda_u| > 0$. Then the condition $|\lambda_{a\gamma_n}| \geq r\|(a\gamma_n)|_{W_n}\|$ implies:

$$|\lambda_u| \geq r\|u|_W\|, \ |\lambda_u| > \|u|_W\|.$$

In particular, u has a dominant eigenvalue which is simple. Since, $\|u - \frac{a\gamma_n}{\|a\gamma_n\|}\|$ converges to zero, we have for n large:

$$|\lambda_{a\gamma_n}| \geq \|a\gamma_n\|\frac{|\lambda_u|}{2}.$$

Moreover, the condition $\lim_n \|\gamma_n\| = \infty$ implies

$$\lim_n \|a\gamma_n\| \geq \lim_n \|a^{-1}\|^{-1}\|\gamma_n\| = \infty.$$

In particular, for n large, $|\lambda_{a\gamma_n}| > 1$, hence $a\gamma_n$ is proximal and expanding, i.e., (a) is valid.

((b) $\Rightarrow$ (c)) We consider the subspace $W \subset V$ of vectors in V having a bounded Γ-orbit. Clearly, this subspace is Γ-invariant. Then condition (H_1) implies $W = V$ or $W = \{0\}$. In the second case (H_0) has been proved. The first case do not occur since it contradicts the hypothesis that Γ is unbounded.

((c) $\Rightarrow$ (b)) and ((a) $\Rightarrow$ (d)) are trivial. □

The following is a useful characterization of strong irreducibility in terms of Zariski closure.

Proposition 2.5. *Let Γ be a sub-semigroup of $GL(V)$. Then Γ satisfies (H_1) if and only if $Zc_0(\Gamma)$ acts irreducibly on V.*

Proof. Assume that Γ satisfies (H_1) and let $W \subset V$ be a non-zero and $\mathrm{Zc}_0(\Gamma)$-invariant subspace. For some finite set $F \subset \Gamma$ we have $\Gamma \subset \mathrm{Zc}(\Gamma) = \bigcup_{\gamma \in F} \gamma \mathrm{Zc}_0(\Gamma)$, hence $\Gamma W = \bigcup_{\gamma \in F} \gamma W$. Since Γ satisfies (H_1) we get $W = V$, hence $\mathrm{Zc}_0(\Gamma)$ acts irreducibly on V.

Assume that $\mathrm{Zc}_0(\Gamma)$ acts irreducibly on V and let W be a non-zero subspace of V, F a finite subset of Γ such that $\Gamma W = \bigcup_{\gamma \in F} \gamma W$. Since ΓW is an algebraic manifold, $\mathrm{Zc}(\Gamma)$ leaves ΓW invariant, hence permutes the subspaces γW $(\gamma \in \Gamma)$. Since $\mathrm{Zc}_0(\Gamma)$ is connected, we have for any $\gamma \in F$:

$$\mathrm{Zc}_0(\Gamma)\gamma W = \gamma W.$$

¿From the irreducibility of the action of $\mathrm{Zc}_0(\Gamma)$ on V, we get $W = V$. □

The following will be essential in the proof of Theorem 1.1

Proposition 2.6. *Assume that the semigroup $\Gamma \subset GL(V)$ satisfies (H_1) and (H_2). Then Γ contains a finitely generated sub-semigroup which satisfies (H_1) and (H_2).*

The proof of the above proposition depends on the following

Lemma 2.7. *Assume that Γ satisfies (H_1), (H_2). Denote by D (resp. C) the commutator subgroup (resp. connected center) of $Zc_0(\Gamma)$. Then $Zc_0(\Gamma)$ is the almost direct product of D and C. Furthermore, D is semisimple without compact factors and C consists of homotheties.*

Proof. Since Γ acts irreducibly on V, $\mathrm{Zc}(\Gamma)$ is a $\mathbb{R}$-reductive group (see [27]), hence D is semisimple and $\mathrm{Zc}_0(\Gamma)$ is the almost direct product of C and D. We can write D as as the almost direct product $D = D_1 D_2$, where D_1 is compact and D_2 is semisimple without compact factor. Since Γ contains a proximal element and $\mathrm{Zc}(\Gamma)/\mathrm{Zc}_0(\Gamma)$ is finite, $\mathrm{Zc}_0(\Gamma)$ contains also a proximal element. We denote by γ this element and write $\gamma v_\gamma = \lambda_\gamma v_\gamma$, $\gamma = c d_1 d_2$ with $c \in C$, $d_1 \in D_1$, $d_2 \in D_2$. Since d_1 and γ commute, and the direction of v_γ is uniquely defined by γ, $d_1 v_\gamma$ is proportional to v_γ. Since D_1 is compact

we have $d_1 v_\gamma = \pm v_\gamma$, hence cd_2 is also proximal with dominant eigenvector v_γ. Since C commutes with cd_2, and v_γ is cd_2-dominant, there exists a $\mathbb{R}$-character χ of C such that for every $g \in C : gv_\gamma = \chi(g)v_\gamma$. Since the subspace $W = \{v : cv = \chi(c)v, \forall c \in C\}$ is Γ-invariant, contains v_γ and the action of Γ is irreducible, we have that for every $v \in V$ and for every $c \in C$, $cv = \chi(c)v$. Thus, C consists of homotheties, $D_1 D_2$ acts also irreducibly on V and v_γ is d_2-dominant. Since D_1 commutes with d_2 we get, as above, that D_1 preserves the direction of v_γ. Since D_1 is compact and connected, we obtain that v_γ is D_1-invariant. Since D_1 commutes with CD_2, the subspace of D_1-invariant vectors is preserved by the action of CD_1D_2. From the irreducibility of $\mathrm{Zc}_0(\Gamma)$, we get that $D_1 = \mathrm{Id}$, hence $\mathrm{Zc}_0(\Gamma) = CD_2$. □

Proof of Proposition 2.6. We consider the semigroup $\Gamma(S)$ generated by the finite set $S \subset \Gamma$. Clearly, if $S \subset S'$, then $\Gamma(S) \subset \Gamma(S')$. We take a totally ordered family S_i $(i \in I)$ such that $\Gamma = \bigcup_{i \in I} \Gamma(S_i)$; we denote by G_0^i the connected component of the identity in $\mathrm{Zc}(\Gamma(S_i))$. Then, since $G_0^i \subset G_0^j$ if $S_i \subset S_j$, we get that for some $\iota \in I$:

$$H_0 := G_0^\iota = \bigcup_{i \in I} G_0^i = G_0^k \text{ if } S_k \supset S_\iota.$$

We can suppose that for any $i \in I$, $G_0^i = G_0^\iota$. It follows that H_0 is normal in $\mathrm{Zc}(\Gamma(S_i))$ for any $i \in I$, hence H_0 is normal in $\mathrm{Zc}(\Gamma)$. In particular, H_0 is contained in $\mathrm{Zc}_0(\Gamma)$. We observe that H_0 has finite index in $\mathrm{Zc}(\Gamma(S_i))$, hence $L = \mathrm{Zc}_0(\Gamma)/H_0$ is an algebraic group which is the Zariski closure of the union of the finite subgroups Φ_i corresponding to $\mathrm{Zc}_0(\Gamma(S_i))$. In view of Lemma 2.7 we know that the algebraic group L has the same structure as $\mathrm{Zc}_0(\Gamma)$, in particular is reductive. We write it as the almost direct product of its connected center $C' \subset \mathbb{R}^*$ and its commutator subgroup D'. Passing to the factor group L/D', using the finite subgroups Φ_i, we get $C' = \{\mathrm{Id}\}$, $L = D'$. We consider a faithful, irreducible representation of the adjoint group of L into a real vector space V'. Then each finite subgroup Φ_i leaves a positive definite quadratic form q_i invariant. We can suppose that the forms q_i are normalized and we denote by q a cluster value of the $(q_i)_{i \in I}$. Then q is invariant under the action of topological closure Φ of $\bigcup_{i \in I} \Phi_i$. Since $\mathrm{Zc}(\bigcup_{i \in I} \Phi_i) = L = \mathrm{Zc}(\Phi)$, we get that Φ acts irreducibly on V', as L. Since the kernel of q is Φ-invariant, we get that this kernel is trivial, hence q is positive definite. It follows that Φ is compact. Since $\mathrm{Zc}(\Phi) = L$, we conclude that $L = \Phi$ is compact, hence from Lemma 2.7, $L = \{\mathrm{Id}\}$. It follows: $H_0 = \mathrm{Zc}_0(\Gamma) = \mathrm{Zc}_0(\Gamma(S_\iota))$. We can suppose that $\Gamma(S_\iota)$ contains a proximal element from $\Gamma \cap \mathrm{Zc}_0(\Gamma)$. Then $\Gamma(S_\iota)$ is finitely generated, and satisfies (H_2). The condition (H_1) is also satisfied by $\Gamma(S_\iota)$, since $\mathrm{Zc}_0(\Gamma(S_\iota)) = \mathrm{Zc}_0(\Gamma)$ acts irreducibly on V. □

Remark 2.8. We will see in Lemma 3.3 below that condition (H_0) remains also valid while passing to a convenient finitely generated sub-semigroup. However, this property cannot be achieved with condition (H_1) alone. A simple counterexample is the following: suppose Γ is the semigroup of rational rotations of the Euclidean plane, centered at the origin. Then any finitely generated sub-semigroup Γ' preserves a regular polygon inscribed in the unit circle. Hence, condition (H_1) is not satisfied by Γ'.

This explains why we consider (H_1) and (H_2) simultaneously in Proposition 2.6.

3. Equivalence of the weak ID-property and ID-property

Let us recall the definitions of the weak ID-property and ID-property once again in the context of sub-semigroups of $M_{inv}(d,\mathbb{Z})$ acting in the usual way on d-dimensional tori. We say that a sub-semigroup Γ of $M_{inv}(d,\mathbb{Z})$ has *ID-property* if every infinite Γ-invariant subset of $\mathbb{T}^d$ is dense in $\mathbb{T}^d$. This is of course equivalent to the fact that every infinite closed Γ-invariant subset of $\mathbb{T}^d$ is $\mathbb{T}^d$ itself.

Let us now recall the definition of the weak ID-property which is defined in terms of orbits. We say that a sub-semigroup Γ of $M_{inv}(d,\mathbb{Z})$ acting on d-dimensional torus has the *weak ID-property* if for every $x \in \mathbb{T}^d$, the closure of the orbit $\overline{\Gamma x}$ is either finite or the whole $\mathbb{T}^d$.

Here it is convenient to use condition (H_0) which is weaker than the hypothesis in Theorem 1.1.

Proposition 3.1. *Let Γ be a sub-semigroup of $M_{inv}(d,\mathbb{Z})$ acting on $\mathbb{T}^d$ and satisfying assumption (H_0). Then Γ has weak ID-property if and only if Γ has ID-property.*

Before we go to the proof of the above equivalence we need the following three lemmas.

Lemma 3.2. *Suppose S is a finite subset of $GL(d,\mathbb{R})$, which generates a semigroup Γ which satisfies (H_0). Denote*

$$C = \sup\{\|s\| : s \in S\}.$$

Then for every $x \in V$, $\|x\| \leq 1$ there exists an element $g \in \Gamma$ such that

$$1 < \|gx\| \leq C.$$

Proof. Note that $C > 1$, since Γ is unbounded. We consider a sequence $s_k \in S$ such that the sequence $s_n \dots s_1 x$, $\|x\| \leq 1$, is unbounded and denote

$$k = \sup\{n \in \mathbb{N} : \|s_n \dots s_1 x\| \leq 1\}.$$

Then we have

$$\|s_{k+1}\| \leq C,\ \|s_k \dots s_1 x\| \leq 1,\ \|s_{k+1} \dots s_1 x\| > 1.$$

It follows

$$1 < \|s_{k+1}s_k \dots s_1 x\| \leq C\|s_k \dots s_1 x\| \leq C.$$

Then the conclusion follows with $g = s_{k+1} \dots s_1$. □

Lemma 3.3. *Let Γ be a sub-semigroup of $GL(d, \mathbb{R})$ which satisfies condition (H_0). Then Γ contains a finitely generated sub-semigroup which satisfies (H_0).*

Proof. For any finite subset $S \subset \Gamma$, we denote by $\Gamma(S)$ the semigroup generated by S, by $V(S)$ the subspace of vectors $v \in V$ such that $\Gamma(S)v$ is bounded. We observe that the inclusion $S \subset S'$ implies $V(S') \subset V(S)$ We consider a totally ordered family of finite subsets S_ι,$(\iota \in I)$ such that $\Gamma = \bigcup_{\iota \in I} \Gamma(S_\iota)$. Then $W = \bigcap_{\iota \in I} V(S_\iota)$ is of the form $V(S_j)$ for some $j \in I$ and we have $V(S_\iota) = V(S_j)$ if $S_\iota \supset S_j$. It follows that W is Γ-invariant. Furthermore, for any $v \in W$, $\iota \in I$, $\Gamma(S_\iota)v$ is bounded.

We show that, if $W \neq \{0\}$, then Γv is bounded, for some $v \in W \setminus \{0\}$. Hence using condition $(H_0) : W = \{0\}$. This implies that condition (H_0) is satisfied by $\Gamma(S_j)$.

We consider the complexified vector space $W^{\mathbb{C}} \subset V^{\mathbb{C}}$, a Γ-irreducible subspace $U \subset W^{\mathbb{C}}$, and the action of $\Gamma(S_\iota)$ on U. Since every $\Gamma(S_\iota)$-orbit in W is bounded for any $\gamma \in \Gamma(S_\iota)$ we have $|\lambda_\gamma| \leq 1$, hence: for any $\gamma \in \Gamma$, $|\operatorname{Tr} \gamma|_U| \leq \dim U$.

Since the action of Γ on U is irreducible, Burnside's theorem implies that the algebra $\operatorname{End}(U)$ is generated by Γ, i.e. there exist $\gamma_1, \dots, \gamma_r$ in Γ such that the linear forms $f_1, \dots, f_r$ on $\operatorname{End}(U)$ defined by

$$f_k(w) = \operatorname{Tr}(\gamma_k w) \ (1 \leq k \leq r)$$

form a basis of $(\operatorname{End}(U))^*$. It follows that for every $\gamma \in \Gamma$, $|f_k(\gamma)| \leq \dim U$. Since the $\{f_k\}$ form a basis of $(\operatorname{End}(U))^*$, we obtain that $\Gamma|_U$ is bounded. Then any Γ-orbit in U is bounded. Then the same is true for the conjugated space $\bar{U} \subset V^{\mathbb{C}}$, and for the sum $U + \bar{U} \subset V^{\mathbb{C}}$. In particular, any $v \in (U + \bar{U}) \cap V$ has a bounded Γ-orbit. Hence from condition (H_0), $(U + \bar{U}) \cap V = \{0\}$, $U = \{0\}$, $W = \{0\}$. □

Let $B_\varepsilon \subset \mathbb{R}^d$ denote the ball with the radius ε and the center 0. If we consider $\mathbb{T}^d$ and $\varepsilon < 1/2$, then we denote also by B_ε the homeomorphic image of the ball $B_\varepsilon \subset \mathbb{R}^d$ by the canonical quotient map $p : \mathbb{R}^d \to \mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$.

Lemma 3.4. *Let Γ be a sub-semigroup of $M_{inv}(d, \mathbb{Z})$ which satisfies (H_0). Then, there exists $\varepsilon = \varepsilon_\Gamma > 0$ such that for every $0 \neq x \in \mathbb{T}^d$:*

$$\Gamma x \cap \mathbb{T}^d \setminus B_\varepsilon \neq \{0\}.$$

Proof. ¿From Lemmas 3.3 and 3.2 above we can find $C > 1$ such that for any $x \in B_\varepsilon \subset \mathbb{R}^d$, $\varepsilon < 1/2$, there exists $g \in \Gamma$ such that

$$\varepsilon \leq \|gx\| \leq C\varepsilon.$$

If $\varepsilon_\Gamma = 1/2C < 1/2$ we get that, for every $x \in B_\varepsilon \subset \mathbb{T}^d$, we have $\Gamma x \not\subset B_\varepsilon$. If, for some $y \notin B_\varepsilon$ we had $\Gamma y \subset B_\varepsilon$, then, for some $\gamma \in \Gamma$, $x = \gamma y \in B_\varepsilon$; hence, from the above observation $\Gamma x \not\subset B_\varepsilon$; in particular, since $\Gamma x \subset \Gamma y$, we have $\Gamma y \not\subset B_\varepsilon$ and this is a contradiction. □

Now we are ready to give the proof of Proposition 3.1.

Proof of Proposition 3.1. It is obvious that ID-property implies weak ID-property. Therefore we have to prove the converse.

If for some $x \in \Sigma$, Γx is infinite then the hypothesis implies $\overline{\Gamma x} = \mathbb{T}^d = \Sigma$. Hence we can suppose that Σ is infinite, $\Sigma = \bigcup_{x\in\Sigma} \Gamma x$ and each Γx, $x \in \Sigma$ is finite. It follows that $\Sigma \subset p(\mathbb{Q}^d)$, hence Σ is countable.

Now consider the sequence of derived sets,

$$\Sigma^0 = \Sigma \supset \Sigma^1 \supset \ldots \supset \Sigma^n \supset \ldots, \tag{3.5}$$

that is Σ^{n+1} is the set of limit points of Σ^n. Actually, the sequence (3.5) terminates, i.e. there is an index n such that $\Sigma^n = \emptyset$. If not we consider $\Sigma^\infty := \bigcap_{n=0}^\infty \Sigma^n$. Clearly, Σ^∞ is a closed and countable set such that the set $(\Sigma^\infty)^{\mathrm{ac}}$ of limit points of Σ^∞ is equal to Σ^∞. This means that we have so called *perfect set* Σ^∞ which is non-void and countable. The Baire theorem says that, since every point is closed and has empty interior in Σ^∞, the set Σ^∞ has the same property, i.e. has empty interior in Σ^∞, which is impossible. Therefore, there is $n \in \mathbb{N}$ such that

$$\Sigma^0 = \Sigma \supset \Sigma^1 \supset \ldots \supset \Sigma^n = \emptyset.$$

Without loss of generality we may assume $n = 2$. It follows that Σ^1 is finite. In fact, otherwise Σ^2 would not be an empty set. Let $\{x_1, \ldots x_n\} = \Sigma^1 \subset p(\mathbb{Q}^d)$ be the set of limit points of Σ and let q be a common denominator of x_i, $1 \leq i \leq n$. Then $0 \in q\Sigma^1$ is the unique limit point of $q\Sigma$. Let us consider a ball B_ε around 0 with $\varepsilon < \varepsilon_\Gamma$ given by Lemma 3.4. Then the points of $q\Sigma$ outside B_ε have no limit point, hence form a finite set F. Now we can consider the Γ-orbits of these points, i.e. Γx, $x \in F$. They form a Γ-invariant finite set $F' = \bigcup_{x\in F} \Gamma x$ that we can exclude from $q\Sigma$ without changing its properties. Therefore, now we have the new set $\Sigma' = q\Sigma \setminus F'$ which is closed, infinite, Γ-invariant and fully included in B_ε, and this contradicts Lemma 3.4. □

4. Asymptotic sets for Γ-actions and a cohomological equation

As in Sect. 2, Γ is a sub-semigroup of $GL(V) = GL(d, \mathbb{R})$ and we consider its action on V and $\mathbb{P}(V) = \pi(V \setminus \{0\})$.

We define the *asymptotic sets*:

$$L_\Gamma = \overline{\{\pi(v_0) : v_0 \text{ is a dominant vector for } \Gamma\}}$$
$$\tilde{L}_\Gamma = \{v \neq 0 : \pi(v) \in L_\Gamma\} = \pi^{-1}(L_\Gamma).$$

We start with the following proposition which is a semigroup version of a result of Guivarc'h and Conze (cf. [6] Proposition 2.2).

Proposition 4.1. *Let Σ be a Γ-invariant subset of $V \setminus \{0\}$ such that $0 \in \overline{\Sigma}$. Then, under assumptions (H_0), (H_1) and (H_2), for any proximal and expanding element $\gamma \in \Gamma$ there exists a γ-dominant vector u_0 such that*

$$\gamma^{\mathbb{Z}} u_0 := \{\gamma^k u_0 : k \in \mathbb{Z}\} \subset \overline{\Sigma}. \tag{4.2}$$

Proof. Let $V = V_\gamma^{\max} \oplus V_\gamma^{<}$ be the decomposition of the space V relative to a proximal and expanding element $\gamma \in \Gamma$.

Let $x_i \in \Sigma$ and $x_i \to 0$ as $i \to \infty$. Then there exists a sequence $\{\alpha_i\}$ of reals and $w \in V$ such that $\alpha_i x_i \to w$ as $i \to \infty$. We will show that without loss of generality we can assume that $w \notin V_\gamma^{<}$. In fact, since Γ acts strongly irreducibly on V, by Lemma 2.2, one can find an element $h \in \Gamma$ such that $hw \notin V_\gamma^{<}$, i.e. $w \notin h^{-1}V_\gamma^{<}$. Define

$$\Gamma_1 = h^{-1}\Gamma h,\ \gamma_1 = h^{-1}\gamma h \in \Gamma_1,\ \Sigma_1 = h^{-1}\Sigma.$$

Then γ_1 is proximal in Γ_1, $w \notin h^{-1}V_\gamma^{<} = V_{\gamma_1}^{<}$, and Σ_1 is a Γ_1-invariant subset that contains 0 as a limit point. Assume that we have found a non-zero vector $u_0 \in V_{\gamma_1}^{\max}$, such that $\gamma_1^{\mathbb{Z}} u_0 \subset \overline{\Sigma}_1$. Then $h^{-1}\gamma^{\mathbb{Z}} h u_0 \subset h^{-1}\overline{\Sigma}$, i.e., $\gamma^{\mathbb{Z}} h u_0 \subset \overline{\Sigma}$. But $hu_0 \in V_\gamma^{\max}$ and we are done.

Thus from the very beginning we can assume that $w \notin V_\gamma^{<}$.

Let $e_1, \ldots, e_n$ be a basis of V such that $e_1 \in V_\gamma^{\max}$, $\|e_1\| = 1$ and $e_2, \ldots, e_n \in V_\gamma^{<}$. Let $\phi_j : V \to \mathbb{R}$ be linear forms such that

$$x = \sum_{j=1}^{n} \phi_j(x) e_j,\ x \in V.$$

Let $\Phi : V = V_\gamma^{\max} \oplus V_\gamma^{<} \to V_\gamma^{\max}$ be the projection along $V_\gamma^{<}$, i.e., $\Phi(x) = \phi_1(x)e_1$. Since $w \notin V_\gamma^{<}$, it follows without loss of generality that $\Phi(x_i) \neq 0$ (since $\alpha_i x_i \to w \notin V_\gamma^{<}$). Since $|\lambda_\gamma| = \lambda > 1$, there exists a sequence of integers $\{p_i\}$ such that $p_i \to \infty$ and

$$1 \leq \lambda^{p_i} |\phi_1(x_i)| \leq \lambda.$$

Now passing to a subsequence if necessary one can find a γ-dominant vector $u_0 \in V_\gamma^{\max}$ such that $\gamma^{p_i}(\Phi(x_i)) \to u_0$ as $i \to \infty$. We will prove that $\gamma^{p_i}(x_i) \to u_0$ as $i \to \infty$.

Clearly, it is enough to show that

$$\phi_j(x_i)\gamma^{p_i}(e_j) \to 0 \text{ as } i \to \infty, \text{ for any } j > 1. \tag{4.3}$$

In fact, $\gamma^{p_i}(x_i) \to u_0$ if $\gamma^{p_i}(\sum_{j=1}^n \phi_j(x_i)e_j) = \sum_{j=1}^n \phi_j(x_i)\gamma^{p_i}(e_j) \to u_0$ and so we are led to (4.3).

Therefore, we are going to prove (4.3). One has

$$\phi_j(x_i)\gamma^{p_i}(e_j) = \frac{\phi_j(x_i)}{|\phi_1(x_i)|}\frac{\gamma^{p_i}(e_j)}{\lambda^{p_i}}\|\gamma^{p_i}(\Phi(x_i))\|,$$

where the first fraction tends to $\phi_j(w)/|\phi_1(w)|$, the second one tends to zero, and the third term tends to $\|u_0\|$ as $i \to \infty$.

Since $x_i \in \Sigma$ and $\gamma^{p_i} \in \Gamma$ for every $i \in \mathbb{N}$ it follows that $u_0 \in \overline{\Sigma}$. Since $\lim_i \gamma^{p_i}(x_i) = u_0$, we get that $\gamma^{-m}(u_0) = \lim_i \gamma^{p_i - m}(x_i)$. We see that for almost all i, $\gamma^{p_i-m}(x_i) \in \Sigma$, thus $\gamma^{-m}u_0 \in \overline{\Sigma}$. □

Remark 4.4. *Notice that the condition* (4.2) *implies that* $0 \in \overline{\Sigma}$. *In fact, simply take a sequence* $\mathbb{Z} \ni k_n$ *such that* $k_n \to -\infty$.

Proposition 4.5. *Under conditions* (H_1) *and* (H_2), *the set* L_Γ *is the unique minimal* Γ*-invariant closed subset of* $\mathbb{P}(V)$.

Proof. We first show that L_Γ is Γ-invariant. Consider $g \in \Delta_\Gamma$, and $u = \lim_n \frac{g^{2n}}{\|g^{2n}\|}$, where the limit exists in $\mathrm{End}(V)$ by proximality of g. Consider a decomposition $V = V_g^< \oplus V_g^{\max}$. Then u is a multiple of the projection of V onto $V_g^{\max}$, along $V_g^<$.

On the other hand, we consider $\gamma \in \Gamma$ and want to show that $\gamma.g^+ \in L_\Gamma$. We observe that for any $\delta \in \Gamma$, we have

$$\lim_n \left\| \gamma \frac{g^{2n}}{\|g^{2n}\|}\delta - \gamma u \delta \right\| = 0.$$

We have $\mathrm{Im}(\gamma u \delta) = \gamma(\mathrm{Im}\, u)$ and $\mathrm{Ker}(\gamma u \delta) = \delta^{-1}(\mathrm{Ker}\, u)$. We note that $\gamma u \delta$ has rank one, like u, hence $\gamma u \delta$ will be a multiple of a one-dimensional projection if $\gamma(\mathrm{Im}\, u) \not\subset \delta^{-1}(\mathrm{Ker}\, u)$, i.e. $\delta(\mathrm{Im}(\gamma u)) \not\subset \mathrm{Ker}\, u$.

Since Γ is strongly irreducible, Lemma 2.2 shows that such a $\delta \in \Gamma$ exists. Then, having δ chosen as above, we know from perturbation theory, as in the proof of Proposition 2.3, that for n large $\gamma \frac{g^{2n}}{\|g^{2n}\|}\delta$ has a simple dominant eigenvalue and the corresponding eigenvector is close to $\gamma(v_g)$. In other words:

$$\gamma g^{2m}\delta \in \Delta_\Gamma, \ \gamma.g^+ = \lim_m (\gamma g^{2m}\delta)^+.$$

Hence $\gamma L_\Gamma \subset L_\Gamma$.

Now let Λ be a closed Γ-invariant subset of $\mathbb{P}(V)$ and let us show that $\Lambda \supset L_\Gamma$. Since Γ is strongly irreducible, Λ is not contained in a proper subspace. In particular $\Lambda \not\subset V_\gamma^<$, hence there exists $x \in \Lambda$ with $x \notin V_\gamma^<$. Then $\gamma^+ = \lim_n \gamma^n.x \in \Lambda$. Since Λ is closed, we have $L_\Gamma \subset \Lambda$. This shows that L_Γ is minimal and is the unique minimal subset of $\mathbb{P}(V)$. $\square$

Proposition 4.6 (Proposition 2.2 in [7])**.** *Let Γ be a sub-semigroup of the group $GL(d,\mathbb{R})$, $d > 1$, satisfying hypothesis (H_1) and (H_2). Let S be a subset of Γ generating it. If φ is a non-zero, continuous function on $\mathbb{P}^{d-1}$, t is real and $\theta \in [0, 2\pi)$, then the following equation:*

$$\forall\, \gamma \in S,\, \forall\, x \in L_\Gamma :\, \varphi(\gamma.x)\|\gamma x\|^{it} = e^{i\theta}\varphi(x)$$

has no solution, unless $\theta = 0$, $t = 0$, $\varphi(x) \equiv$ const on L_Γ.

Proof. Clearly we can suppose that $|\det\gamma| = 1$ for $\gamma \in \Gamma$. Consider the function $\psi(v)$ defined on $\tilde{L}_\Gamma$ by formula $\psi(v) = \varphi(\pi(v))\|v\|^{-it}$. Then the relation for φ is

$$\forall\, \gamma \in S,\, \psi(\gamma v) = e^{i\theta}\psi(v).$$

Suppose that $t \neq 0$ and put $\log\rho = 2\pi/|t|$. Then additionally we have $\psi(\pm\rho^k v) = \psi(v)$ and the condition $\psi(\lambda v) = \psi(v)$ for some $v \in \tilde{L}_\Gamma$ and some $\lambda \in \mathbb{R}_*^+$ implies that $\lambda = \pm\rho^k$, where $k \in \mathbb{Z}$.

Let c be any of the values of ψ and put $L_c = \psi^{-1}(\{c\}) \subset \tilde{L}_\Gamma$. Then, since ψ is continuous, L_c is a nonempty closed subset of $\tilde{L}_\Gamma$, which satisfies

$$\forall\, \gamma \in S,\, \gamma(L_c) \subset L_{ce^{i\theta}}.$$

We also have for every $\lambda \in H_\rho$, which is the group of homotheties of the form $\pm\rho^k$, $k \in \mathbb{Z}$,

$$\lambda L_c = L_c.$$

If now $u \in \mathrm{End}(\mathbb{R}^d)$ satisfies $u = \lim_k \rho^{-n_k}\gamma_k$ with $\gamma_k \in \Gamma$, $\rho^{-1} \leq \|u\| < 1$, then we have $|\det u| = \lim_k \rho^{-n_k d} = 0$ and $u(L_c) \subset L_{ce^{i\alpha}} \cup \{0\}$ with $\alpha \in \mathbb{R}$. From condition (H_2) we can choose $\gamma_k = \gamma^k \in \Gamma$ with $W = \mathrm{Ker}\, u \neq 0$ having codimension 1. Since $\mathrm{Im}\, u = \mathbb{R}v_\gamma = \mathbb{R}a$, $a \neq 0$, we get that $u(L_c) \subset H_\rho a \cup \{0\}$. Since $u^{-1}(a) = b + W$ with $b \in \mathbb{R}^d \setminus \{0\}$ we get that $L_c \subset W \cup H_\rho(b+W)$. It follows that, in the quotient space $\mathbb{R}^d/W$, L_c is projected on a set which is countable and H_ρ-invariant. If W_i, $1 \leq i \leq r$ is a family of such subspaces then the subspace $\bigcap_{i=1}^r W_i$ has the same property. In fact $V/\bigcap_{i=1}^r W_i$ can be identified with the diagonal subspace of $V/W_1 \times \ldots \times V/W_r$, and so the projection of L_c into $V/\bigcap_{i=1}^r W_i$ can be identified with a subset of the product of the projections of L_c. Hence, such a projection is countable and H_ρ-invariant. Since the intersection of any family of subspaces with the above properties is a finite intersection, there exists a minimum subspace W_0, which has the considered properties. This subspace is unchanged when

c is replaced by $ce^{i\alpha}$. As a consequence the condition $\gamma(L_c) \subset L_{ce^{i\theta}}$ for $\gamma \in \Gamma$ implies that $\gamma(W_0) = W_0$. Since $\pi(L_c) = L_\Gamma$ and L_Γ is uncountable (see Lemma 5.1) W_0 is proper. This contradicts the irreducibility of Γ, and so $t = 0$, $e^{-i\theta}\varphi(\gamma.x) = \varphi(x)$ for every $x \in L_\Gamma$ and for every $\gamma \in S$. Therefore, for every n, and for all $\gamma_i \in S$, and for every $x \in L_\Gamma$ we have $e^{-in\theta}\varphi(\gamma_1 \dots \gamma_n.x) = \varphi(x)$. Since $\Gamma = \bigcup_0^\infty S^n$ satisfies (H_2), we can deduce that $\varphi \equiv$ const on L_Γ, and $e^{i\theta} = 1$. In fact, suppose that there are two different points x and y in L_Γ such that $\varphi(x) \neq \varphi(y)$. Then

$$\varphi(x) = e^{-in\theta}\varphi(\gamma_1 \dots \gamma_n.x) \neq e^{-in\theta}\varphi(\gamma_1 \dots \gamma_n y) = \varphi(y)$$

and so,

$$0 < |\varphi(x) - \varphi(y)| = |\varphi(\gamma_1 \dots \gamma_n.x) - \varphi(\gamma_1 \dots \gamma_n y)| \tag{4.7}$$

for every $n \in \mathbb{N}$ and all $\gamma_i \in S$. By proximality of Γ on $\mathbb{P}(V)$ (see Proposition 2.3) we have that there exists a sequence $\{\gamma_i\}_1^\infty$ such that

$$\lim_n \delta(\gamma_1 \dots \gamma_n.x, \gamma_1 \dots \gamma_n.y) = 0.$$

By continuity of φ we a get a contradiction to (4.7). □

The following corollary which is a complement to Proposition 3 p. 45 in [17] clarifies the role of the aperiodicity hypotheses of Γ considered by Kesten in [22], Guivarc'h and Raugi in [17] (Proposition 3 and Lemma p. 45), Lalley in [23] (Corollaries 11.3, 11.4), Eberlein in [11] and Dal'bo in [8] in the context of lengths of closed geodesics in negative curvature. For an extension of these results and their use in the more general setting of semisimple groups see [3] and [19]. It explains why aperiodicity conditions are not explicitly stated in Theorem 1.1, as in [4] and [26].

Corollary 4.8. *Suppose $\Gamma \subset GL(d, \mathbb{R})$ is a sub-semigroup which satisfies (H_1), (H_2) and denote $S_\Gamma = \{\log|\lambda_g| : g \in \Delta_\Gamma\}$. Then S_Γ generates a dense subgroup of $\mathbb{R}$.*

For the proof, we need three lemmas, the first of them being well known (see [5], pp. 90–94).

Lemma 4.9. *Let A be a finite set, Ω the compact metric space $A^\mathbb{N}$ and θ the shift transformation on Ω given by $(\theta\omega)_k = \omega_{k+1}$, $k \in \mathbb{N}$. For a function φ on Ω we denote:*

$$S_n\varphi(\omega) = \sum_0^{n-1} \varphi \circ \theta^k(\omega).$$

Suppose φ is Hölder continuous, and for any periodic point ω of period p, the sum $S_p\varphi(\omega)$ belongs to $\mathbb{Z}$. Then there exists a Hölder $\mathbb{Z}$-valued function φ' on Ω and a Hölder function ψ such that $\varphi = \varphi' + \psi - \psi \circ \theta$.

Lemma 4.10. *Suppose $g, h \in GL(d, \mathbb{R})$ are such that h is proximal and $g.h^+ \notin V_h^<$. Then, for $n = 2p$ large, gh^n is proximal and*

$$\lim_n (gh^n)^+ = g.h^+,$$
$$\lim_n V_{gh^n}^< = V_h^<.$$

Proof. We consider the sequence of linear maps $u_n = \frac{h^n}{\|h^n\|}$ and observe that u_n converges towards a map π_h which is proportional to the projection on $\mathbb{R}v_h$ along the subspace $V_h^<$. Hence gu_n converges towards $g\pi_h$. We have

$$\mathrm{Im}(g\pi_h) = \mathbb{R}g.h^+, \ \mathrm{Ker}(g\pi_h) = V_h^<.$$

Hence, if $g.h^+ \notin V_h^<$, then $g\pi_h$ is collinear to a projection onto a one-dimensional subspace. Since $g\pi_h$ has a simple dominant eigenvalue, the same is true for gu_n for n large. Therefore, for n large, gu_n is proximal and moreover, we have the required convergence. □

Lemma 4.11. *Suppose $\Gamma \subset GL(V)$ is a sub-semigroup and satisfies conditions (H_1), (H_2). Then, there exists $a, b \in \Delta_\Gamma$ such that $a^+ \neq b^+$, $V_a^< \neq V_b^<$ and $a^+ \notin V_b^<$, $b^+ \notin V_a^<$.*

Proof. We consider the transposed semigroup Γ^t acting on the dual space V^* and the projective space $\mathbb{P}(V^*)$. From Remark 2.1 iii) conditions (H_1) and (H_2) remain valid for Γ^t and we can consider the corresponding limit set $L_{\Gamma^t} = L_\Gamma^*$. We fix $a \in \Delta_\Gamma$ and we observe that we can find $b \in \Delta_\Gamma$ such that $V_b^< \neq V_a^<$, $a^+ \notin V_b^<$. Otherwise there will be a dense subset of L_Γ^* contained in a union of the two projective subspaces defined by $V_a^<$ and a^+ in $\mathbb{P}(V^*)$. Hence L_Γ^* itself will be contained in such an union. But, from Lemma 5.1 below this is impossible. If $b^+ \notin a^+ \cup V_a^<$, we have found the required pair (a, b). If not, we consider $g \in \Gamma$ and the sequence gb^n, $n \in 2\mathbb{N}$. In view of Lemma 2.2 we can choose $g \in \Gamma$ such that:

$$g.b^+ \notin V_b^< \cup V_a^< \cup a^+.$$

Then we can apply Lemma 4.10 and replace b by $gb^n = b'$ for n large. Under this condition, $V_{b'}^<$ is close to $V_b^<$ and the relations are still satisfied. Since $(b')^+$ is close to $g.b^+$ and $g.b^+ \notin a^+ \cup V_a^<$, the condition $(b')^+ \notin a^+ \cup V_a^<$ is also satisfied. Hence, we can take (a, b') as the required pair. □

Proof of Corollary 4.8. Since Γ satisfies (H_1) and (H_2) we can choose a_1, a_2 in Γ according to the Lemma 4.11. Let C_1, C_2 be two closed and disjoint neighborhoods of a_1^+, a_2^+ in $\mathbb{P}(V)$ such that $(C_1 \cup C_2) \cap (V_{a_1}^< \cup V_{a_2}^<) = \emptyset$ and let o be a point outside $V_{a_1} \cup V_{a_2}^<$. Then, for $i = 1, 2$:

$$\lim_n a_i^n.(C_1 \cup C_2) = a_i^+$$
$$\lim_n a_i^n.o = a_i^+.$$

If we take n large and set $a = a_1^n$, $b = a_2^n$, we have

$$a.o \in C_1,\ b.o \in C_2,\ a.(C_1 \cup C_2) \subset \operatorname{Int} C_1,\ b.(C_1 \cup C_2) \subset \operatorname{Int} C_2. \tag{4.12}$$

It follows from (4.12) that the semigroup $\Gamma(a,b)$ generated by a, b is free. In order to prove Corollary 4.8 we can suppose $\Gamma = \Gamma(a,b)$. We consider the trivial metric δ on $\{a,b\}$ and endow $\Omega = \{a,b\}^{\mathbb{N}}$ with the metric $\delta(\omega,\omega') = \sum_1^\infty 2^{-k}\delta(\omega_k,\omega'_k)$. We define a homeomorphism $\mathfrak{z}$ between Ω and L_Γ as follows. We observe that if $\omega = (a_k)_{k\in\mathbb{N}}$ and $a_k \in \{a,b\}$, then it follows from (4.12) that the sequence $a_1 \dots a_n.o$ converges to $\mathfrak{z}(\omega) \in C_1 \cup C_2$. It is easy to verify that $\mathfrak{z}$ is a bi-Hölder homeomorphism, hence we can transfer properties of (Ω,θ) to the action of Γ on L_Γ. We consider $\mathfrak{z}(\omega)$ as a unit vector in V and we observe that, by definition of $\mathfrak{z}$:

$$a_1(\omega)\mathfrak{z}(\theta\omega) = \|a_1(\omega)\mathfrak{z}(\theta\omega)\|\mathfrak{z}(\omega).$$

It follows that, if we denote

$$\varphi(\omega) = \log\|a_1(\omega)\mathfrak{z}(\theta\omega)\|,$$
$$S_n\varphi(\omega) = \sum_0^{n-1}\varphi(\theta^k\omega)$$

we have, with $\gamma = a_1 \dots a_{n-1} \in \Gamma$ and $x = \mathfrak{z}(\theta^n\omega) \in L_\Gamma$:

$$S_n\varphi(\omega) = \log\|\gamma x\|.$$

Given a Hölder function ψ on Ω we define a Hölder function $\bar\psi$ on L_Γ by $\bar\psi[\mathfrak{z}(\omega)] = \psi(\omega)$ and we have also $\psi(\omega) - \psi(\theta^n\omega) = \bar\psi(\gamma.x) - \bar\psi(x)$. In particular, if $\omega \in \Omega$ is periodic with period p, ($\theta^p\omega = \omega$), then $\mathfrak{z}(\omega)$ is a dominant eigenvector of $\gamma = a_1 \dots a_{p-1}$ and the corresponding eigenvalue λ_γ satisfies:

$$\log|\lambda_\gamma| = S_p\varphi(\omega).$$

If S_Γ do not generate a dense subgroup of $\mathbb{R}$, then, for some positive c we have $S_\Gamma \subset c\mathbb{Z}$, hence $S_p\varphi(\omega) \in c\mathbb{Z}$ for any periodic point ω and we can apply Lemma 4.9 to the function $\frac{1}{c}\varphi$. In particular, the function $e^{2i\pi\varphi/c}$ can be written in the form $e^{2i\pi(\psi-\psi\circ\theta)}$, where ψ is a Hölder function on Ω. We can define $\bar\psi$ on L_Γ as above and write $u(x) = e^{2i\pi\bar\psi}$. Then u is continuous and we obtain with $\gamma = a_1 \dots a_n$, $x = \mathfrak{z}(\theta^n\omega)$:

$$\|\gamma x\|^{2i\pi/c} = \frac{u(\gamma.x)}{u(x)}.$$

We extend u to $\mathbb{P}(V)$ as a continuous function, again denoted by u. Then we have

$$\forall\, x \in L_\Gamma,\ \forall\, \gamma \in \Gamma,\ \|\gamma x\|^{2i\pi/c} = \frac{u(\gamma.x)}{u(x)}.$$

In view of the Proposition 4.6, this implies $\frac{2i\pi}{c} = 0$, $u = 1$ and this is impossible. $\square$

5. Random walks on a vector space and its factor spaces

In this section, relying strongly on [13], [17] (see also [18]), we develop the random walk approach to the study of Γ-orbits on $V \setminus \{0\}$ and other related Γ-spaces. The main new results are Theorems 5.9 and 5.18 and their corollaries. They give Theorem 1.3 and Theorem 1.2 of the Introduction. In particular, Corollary 5.21 is one of the main tools for the study of Γ-orbits on the torus $\mathbb{T}^d$ with $\Gamma \subset M_{inv}(d, \mathbb{Z})$, i.e., for Theorem 1.1.

Let μ be a probability measure on $G = GL(V)$, Γ_μ the closed sub-semigroup generated by the support S_μ of μ. We denote by $M^1(X)$ the set of probability measures on a given Polish space X. We denote $\Omega = S_\mu^{\mathbb{N}}$ and we consider the probability measure $\mathbf{P}_\mu = \mu^{\otimes \mathbb{N}}$ on Ω; the shift θ on Ω given by $(\theta\omega)_k = \omega_{k+1}$, $(k \in \mathbb{N})$ preserves $\mathbf{P}_\mu$ and the components $\omega_k = g_k(\omega)$ of ω are i.i.d. G-valued random variables of law μ. From Markov-Kakutani theorem, there exists a probability measure ν on $\mathbb{P}(V)$ which is μ-stationary, i.e.,

$$\mu * \nu = \int g.\nu d\mu(g) = \nu.$$

We are going to establish that $[\mathbb{P}(V), \nu]$ is a μ-boundary (see [13]), i.e.,

$$\lim_n g_1 g_2 \dots g_n.\nu = \delta_{z_\omega},$$

where $z_\omega \in \mathbb{P}(V)$. This will allow us to derive some properties of the typical sequences:

$$S_n = g_n g_{n-1} \dots g_1 \text{ and } X_n = g_1 g_2 \dots g_n,$$

and transposed maps S_n^t and X_n^t.

Lemma 5.1. *Assume that $\Gamma = \Gamma_\mu$ satisfies condition (H_1), and let ν be a μ-stationary measure. Then ν gives zero mass to every projective subspace. Furthermore, if Γ satisfies also (H_2), then L_Γ is not contained in a countable union of subspaces.*

Proof. Let W be a projective subspace of minimal dimension such that $\nu(W) > 0$.

Define,

$$\eta = \eta_\mu = \sum_{k \geq 1} (1/2^k) \mu^{*k} \tag{5.2}$$

and consider the function $f(g) = g.\nu(W) = \nu(g^{-1}.W)$. This function is μ-*harmonic*, i.e. satisfies

$$\int f(gh)d\mu(h) = \int f(gh)d\eta(h) = f(g)$$

and reaches its maximum. In fact, the hypothesis on W gives $\nu(g.W \cap g'.W) = 0$ if $g.W \neq g'.W$, so the set of $g.W$ such that $\nu(g.W) > \delta$ is finite for every δ. Then if $f(g_0) = \sup_{g\in G} f(g)$, the equation $f(g_0) = \int f(g_0h)d\eta(h)$ gives $f(g_0h) = f(g_0)$, η-a.e. Consequently, there exists a finite set E of projective subspaces such that $h^{-1}g_0^{-1}.W$ belongs to E for a.e. h, and therefore for every $h \in \Gamma^{-1}$. Then Γ permutes a finite set of projective subspaces and the strong irreducibility of Γ gives the contradiction.

If (H_2) is also satisfied by Γ, then L_Γ is well defined. From Markov-Kakutani theorem, we know that there exists a μ-stationary measure λ such that $\lambda(L_\Gamma) = 1$, hence $S_\lambda \subset L_\Gamma$. Since λ gives zero measure to every subspace, the same is true for a countable union of subspaces, hence L_Γ cannot be contained in such an union. □

Proposition 5.3. *Let ν be a μ-stationary measure on $\mathbb{P}(V)$ and η be as in* (5.2)*. Then the sequence $g_1 \dots g_n.\nu$ converges $\mathbf{P}_\mu$-a.e. and for $\mathbf{P}_\mu \otimes \eta$-a.e. $(\omega, g) \in \Omega \times G$ we have*

$$\lim_n g_1(\omega) \dots g_n(\omega).\nu = \lim_n g_1(\omega) \dots g_n(\omega)g.\nu.$$

Proof. For a continuous function φ, we set $F_\varphi(g) = g.\nu(\phi)$ and we observe that the relation $\int g.\nu d\mu(g) = \nu$ gives $\int F_\varphi(gh)d\mu(h) = F_\varphi(g)$ and consequently that $F_\varphi(X_n) = g_1 \dots g_n.\nu(\varphi)$ is a bounded martingale. This martingale converges and, letting φ vary in a dense countable part of $\mathbb{P}(V)$, we obtain the convergence of $g_1 \dots g_n.\nu$. In order to obtain the second claim, it suffices to show that $F_\varphi(X_n g) - F_\varphi(X_n)$ converges to zero, $\mathbf{P}_\mu \otimes \mu^{*r}$-a.e. for every $r \geq 1$. But:

$$\int |F_\varphi(X_n g) - F_\varphi(X_n)|^2 d\mu^{*r}(g)d\mathbf{P}(\omega) = \mu^{*n+r}(F_\varphi^2) - \mu^{*n}(F_\varphi^2)$$

because of the relation $\int F_\varphi(X_n g)d\mu^{*r}(g) = F_\varphi(X_n)$. One deduces that:

$$\int \int \sum_{n=0}^{p} |F_\varphi(X_n g) - F_\varphi(X_n)|^2 d\mathbf{P}(\omega)d\mu^{*r}(g) \leq 2r\|\varphi\|_\infty$$

for every $r \geq 1$; this proves the convergence $\mathbf{P} \otimes \mu^{*r}$-a.e. of the series $\sum |F_\varphi(X_n g) - F_\varphi(X_n)|^2$ and consequently the convergence of $F_\varphi(X_n g) - F_\varphi(X_n)$ to zero □

In the sequel we are going to use concepts introduced in [13]. Therefore, we recall them briefly.

To every linear transformation of $\mathbb{R}^d$ is associated a *quasi-projective* transformation acting on the lines of $\mathbb{R}^d$ not contained in the kernel of the transformation. So we have maps of $\mathbb{P}^{d-1}$ defined outside a projective subspace: these maps are continuous outside the exceptional subspace and are limits, outside this subspace, of a sequence of projective transformations. Furthermore, from every sequence of projective transformations, we can extract a subsequence converging to a quasi-projective one, outside a projective subspace.

Theorem 5.4. *Let ν be a μ-stationary measure on $\mathbb{P}(V)$. Assume that $\Gamma = \Gamma_\mu$ satisfies conditions (H_1) and (H_2). Then we have $\mathbf{P}_\mu$-a.e.*

$$\lim_n g_1 g_2 \dots g_n.\nu = \delta_{z_\omega}.$$

In particular ν is unique and its support is L_Γ.

Proof. The proof goes like in [17]. For a fixed ω, we consider the relation given by Proposition 5.3,

$$\theta(\omega) = \lim_n g_1(\omega) \dots g_n(\omega).\nu = \lim_n g_1(\omega) \dots g_n(\omega) g.\nu,$$

which is true for $\mathbf{P}_\mu \otimes \eta$-a.e. (ω, g). One can extract from $g_1(\omega) \dots g_n(\omega)$ a subsequence converging outside a projective subspaces to a quasi-projective map $\tau(\omega)$. As ν gives zero-measure to any projective subspace (Lemma 5.1), one has from Proposition 5.3 above: $\tau(\omega).\nu = \tau(\omega)g.\nu = \theta(\omega)$, for η-a.e. g, and therefore for all $g \in \Gamma$. As Γ satisfies (H_1) and (H_2), one can find a sequence $g_n \in \Gamma$ such that $g_n.\nu$ converges to a Dirac measure δ_z with z belonging to the open set of continuity of $\tau(\omega)$. Then, in the limit $\theta(\omega) = \tau(\omega).\delta_z$. This proves that $\theta(\omega)$ is a Dirac measure δ_{z_ω}. The law of random variable z is necessarily ν by the martingale convergence theorem. Since z is independent of the choice of the μ-stationary measure ν we get the uniqueness of ν.

Clearly, S_ν is closed and Γ-invariant. Hence, Proposition 4.5 gives $S_\nu \supset L_\Gamma$. The Markov-Kakutani theorem and uniqueness of ν give, as in the proof of Lemma 5.1, $\nu(L_\Gamma) = 1$, hence $S_\nu = L_\Gamma$. □

Corollary 5.5. *Let ρ (ρ^* resp.) be a probability measure on $\mathbb{P}(V)$ ($\mathbb{P}(V^*)$ resp.) which gives zero mass to every projective subspace. Then we have $\mathbf{P}_\mu$-a.e.*

$$\lim_n g_1 \dots g_n.\rho = \delta_{z_\omega} \ \ (\lim_n g_1^t \dots g_n^t.\rho^* = \delta_{z_\omega^*} \ \text{resp.})$$

In particular:

$$\lim_n g_1 \dots g_n.m = \delta_{z_\omega} \ \ (\lim_n g_1^t \dots g_n^t.m^* = \delta_{z_\omega^*} \ \text{resp.}),$$

where m (m^* resp.) is the K-invariant probability measure on $\mathbb{P}(V)$ ($\mathbb{P}(V^*)$ resp.)

Proof. We observe that $\frac{g_1 \dots g_n}{\|g_1 \dots g_n\|} \in \mathrm{End}(V)$ has norm one. We consider an arbitrary convergent subsequence:

$$u = \lim_k \frac{g_1 \dots g_{n_k}}{\|g_1 \dots g_{n_k}\|}.$$

Clearly $u \neq 0$, since $\|u\| = 1$. We note that u defines a continuous map from $\mathbb{P}(V) \setminus \mathrm{Ker}\, u$ onto $\mathbb{P}(V)$. We will denote it again by u, and observe that, since $\rho(\mathrm{Ker}\, u) = 0$, $u.\rho$ is well defined, and from dominated convergence:

$$u.\rho = \lim_k g_1 \dots g_{n_k}.\rho.$$

In particular from Theorem 5.4 and Lemma 5.1: $u.\nu = \delta_{z_\omega}$. This means that the linear map u has rank one and satisfies $u(\mathbb{P}(V) \setminus \mathrm{Ker}\, u) = \delta_{z_\omega}$. Hence $u.\rho = \delta_{z_\omega}$. The convergent subsequence chosen above was arbitrary, hence:

$$\lim_n g_1 \dots g_n.\rho = \delta_{z_\omega}.$$

In particular, we have the above convergence for $\rho = m$.

The results for $\mathbb{P}(V^*)$, ρ^*, m^*, z^*_ω follow from $\Gamma_{\mu^*} = (\Gamma_\mu)^t$ and Remark 2.1 iii). □

Recall m is the K-invariant probability measure on $\mathbb{P}(V)$, where $K = SO(d, \mathbb{R})$. One says that a sequence $f_n \in GL(d, \mathbb{R})$ has the *contraction property on* $\mathbb{P}(V)$ *towards* z if the sequence of measures $f_n.m$ on $\mathbb{P}(V)$ converges weakly towards δ_z. A point $z \in \mathbb{P}(V)$ will be identified with a vector of norm one, defined up to a sign.

We will use, as in [17] and [18], the $K\overline{A^+}K$ decomposition of $g \in GL(d, \mathbb{R})$, $g = kak'$, k and k' are orthogonal matrices and $\overline{A^+} \ni a = \mathrm{diag}(a^1, \dots, a^d)$ with $a^1 \geq a^2 \geq \dots \geq a^d$ and $(e_1, \dots, e_d)$ denotes the canonical basis of $\mathbb{R}^d$. In particular, if $g \in SL(d, \mathbb{R})$ then $k, k' \in K = SO(d, \mathbb{R})$ and $a \in \overline{A^+} = \{\mathrm{diag}(a^1, \dots, a^d) : a^1 \geq a^2 \geq \dots \geq a^d > 0 \text{ and } \prod_1^d a^i = 1\}$.

If one writes the polar decomposition of f_n as $f_n = k_n a_n k'_n$, where $k_n, k'_n \in K$, $a \in \overline{A^+}$, one sees that the contraction property is equivalent to $a^i_n = o(a^1_n)$, $1 < i \leq d$, $\lim_n k_n.e_1 = z$.

In proposition below and its corollary, the point $z \in \mathbb{P}(V)$ is considered as a unit vector, hence $|z(x)|$ is well defined for $x \in V$.

Proposition 5.6. *Assume that $f_n \in GL(V)$ is a sequence such that f^t_n has the contraction property on $\mathbb{P}(V^*)$ towards $z \in \mathbb{P}(V^*)$. Then for any $x, y \in \mathbb{P}(V)$, one has the following convergence*

$$\lim_n \frac{\|f_n(x)\|}{\|f_n\|} = |z(x)|,$$

$$\lim_n z(x)z(y)\frac{\delta(f_n.x, f_n.y)}{\delta(x, y)} = 0.$$

The second convergence is uniform when x, y belong to a compact subset of $\mathbb{P}(V) \setminus Ker\ z$.

If $f_n \in SL(V)$, then one has for every $x \notin Ker\ z$, $\lim_n \|f_n(x)\| = +\infty$.

Proof. Recall that the distance between elements $\bar{u} = \pi(u)$ and $\bar{v} = \pi(v)$ in $\mathbb{P}(V)$ is equal to $\delta(\bar{u}, \bar{v}) = \|u \wedge v\|/\|u\|\|v\|$.

One writes $f_n = k_n a_n k'_n$ as above, with $k_n, k'_n \in K$ and $a_n \in \overline{A^+}$. From the hypotheses we get:

$$\lim_n k_n'^{-1}.\bar{e_1} = z,\ a_n^i = o(a_n^1),\ (i > 1).$$

Writing $x = \sum_1^d x_i e_i$, we get:

$$\|f_n x\|^2 = \|a_n k'_n x\|^2 = \sum_1^d (a_n^i)^2 |\langle k'_n x, e_i\rangle|^2 \geq (a_n^1)^2 |\langle k'_n x, e_1\rangle|^2.$$

Since the norm of f_n is a_n^1, we get

$$\begin{aligned}\lim_n \frac{\|f_n x\|^2}{\|f_n\|^2} &= \lim_n |\langle x, k_n'^{-1} e_1\rangle|^2 + \lim_n \sum_{i>1} \left(\frac{a_n^i}{a_n^1}\right)^2 |\langle k'_n x, e_1\rangle|^2 \\ &= \lim_n |\langle x, k_n'^{-1} e_1\rangle|^2 = |z(x)|^2.\end{aligned}$$

Also

$$\begin{aligned}\|f_n(x) \wedge f_n(y)\|^2 &= \sum_{i \neq j} (a_m^i a_n^j)^2 |\langle k'_n(x \wedge y), e_i \wedge e_j\rangle|^2 \\ \|f_n(x) \wedge f_n(y)\| &\leq d^2 a_n^1 a_n^2 \|x \wedge y\|.\end{aligned}$$

Therefore,

$$\frac{\delta(f_n.x, f_n.y)}{\delta(x, y)} = \frac{\|f_n x \wedge f_n y\|}{\|f_n x\|\|f_n y\|\|x \wedge y\|} \leq d^2 \frac{a_n^2}{a_n^1} \frac{1}{|\langle k'_n x, e_1\rangle\langle k'_n y, e_1\rangle|}$$

and

$$\lim_n |z(x)||z(y)| \frac{\delta(f_n.x, f_n.y)}{\delta(x, y)} = 0.$$

The uniformity of the required convergence is clear from the previous formula, if $z(x)z(y) \not\equiv 0$.

In order to obtain the last assertion, it suffices to show, in view of the first statement, that $\|f_n\|$ converges to $+\infty$. The relation $a_n^2 = o(a_n^1)$ implies $\det f_n = \prod_{i=1}^d a_n^i = o(a_n^1)$. Since $\det f_n = 1$, we get $\lim_n \|f_n\| = \lim_n a_n^1 = +\infty$. □

Corollary 5.7 (see [17, 18])**.** *If μ, z_ω^* are as in Theorem 5.4 and Corollary 5.5, then, as n tends to infinity, we have uniformly in $x, y \in \mathbb{P}(V)$:*

$$\lim_n \mathbf{E}_\mu \delta(S_n.x, S_n.y) = 0$$

and $\mathbf{P}_\mu$*-a.e.*

$$\lim_n \frac{\|S_n x\|}{\|S_n\|} = |z^*_\omega(x)|.$$

If $\mu \in M^1(SL(V))$*, one has, for every* $x \in V$ *and* $\mathbf{P}_\mu$*-a.e.* $\lim_n \|S_n x\| = +\infty$.

Proof. Note that Corollary 5.5 implies that $S^t_n(\omega)$ has the contraction property towards z^*_ω. Hence Proposition 5.6 implies:

$$\lim_n \frac{\|S_n x\|}{\|S_n\|} = |z^*_\omega(x)|.$$

For the first convergence it suffices to show that for any sequence $x_n, y_n \in \mathbb{P}(V)$ we have $\mathbf{P}_\mu$-a.e.

$$\lim_n \delta(S_n.x_n, S_n.y_n) = 0. \tag{5.8}$$

One can suppose that $\lim_n x_n = x$ and $\lim_n y_n = y$. From Corollary 5.5 and Lemma 5.1, one knows that the law of $z^*_\omega = \lim_n S^t_n.m^*$ gives zero measure to every subspace. Hence, for almost every $\omega \in \Omega$, x and y are not in $\operatorname{Ker} z^*_\omega$, and the same is true for x_n, y_n for large n. Then (5.8) follows from dominated convergence.

The last assertion is proved as follows. If x is fixed, then $\mathbf{P}_\mu$-a.e. as above $|z^*_\omega(x)| \neq 0$, hence from Proposition 5.6, $\lim_n \|S_n x\| = +\infty$. □

Now we are going to study stationary measures on factor spaces of $V \setminus \{0\}$.

Let $c > 1$ be fixed and let us denote by $\mathbb{P}_c(V) = \mathbb{P}^{d-1}_c$ the factor space of $V \setminus \{0\}$ by the multiplicative subgroup of $\mathbb{R}^*$:

$$\pm c^{\mathbb{Z}} := \{\pm c^n : n \in \mathbb{Z}\},$$

and by $\mathbb{T}_c$ the 1-torus $\mathbb{T}_c = \mathbb{R}^*/\pm c^{\mathbb{Z}}$.

We can consider the projection from $V \setminus \{0\}$ to $\mathbb{P}(V) \times \mathbb{T}_c$ given by

$$v \mapsto (\bar{v}, \|v\|^{i\alpha}),$$

where $\alpha = 2\pi/\log c$ and we observe that $\mathbb{P}_c(V)$ is then naturally identified with $\mathbb{P}(V) \times \mathbb{T}_c$. Hence a point of $\mathbb{P}_c(V)$ will be written as $v = (\bar{v}, z)$, where $\bar{v} \in \mathbb{P}(V)$ is the projection of v and $z = \|v\|^{i\alpha}$. The action of $g \in G = GL(V)$ on $\mathbb{P}_c(V)$ can then be written as

$$g.v = g.(\bar{v}, z) = (g.\bar{v}, z\|gv\|^{i\alpha}).$$

$\mathbb{R}^*$ acts also on this space and the two actions commute. The corresponding formula is

$$t.(\bar{v}, z) = (\bar{v}, z|t|^{i\alpha}),\ t \in \mathbb{R}^*.$$

We denote by $\lambda_c = dz$ the normalized Lebesgue measure on $\mathbb{T}_c$ and observe that every measure of the form $\nu \otimes \lambda_c$, where $\nu \in M^1(\mathbb{P}(V))$ is invariant

under the action of $\mathbb{R}^*$ on $\mathbb{P}_c(V)$. Furthermore, if $\mu \in M^1(G)$ and $\nu \in M^1(\mathbb{P}(V))$ is μ-stationary, then $\nu \otimes \lambda_c$ is also μ-stationary. If $L_\Gamma \subset \mathbb{P}(V)$ is the limit set of Γ ($\Gamma = \Gamma_\mu$ satisfies (H_1) and (H_2) conditions), then $L_\Gamma(c) = L_\Gamma \times \mathbb{T}_c$ is a closed and Γ-invariant subset of $\mathbb{P}(V) \times \mathbb{T}_c$.

Theorem 5.9. *Assume that $\mu \in M^1(G)$ is such that $\Gamma = \Gamma_\mu$ satisfies conditions (H_1) and (H_2). Then, with the above notations, for every $\psi \in C[\mathbb{P}_c(V)]$ the sequence $\check{\mu}^{*n} * \psi$ converges uniformly to $(\nu \otimes \lambda_c)(\psi)$, where ν is μ-stationary measure on $\mathbb{P}(V)$. Furthermore, for any $v \in \mathbb{P}_c(V)$ we have the following a.e. convergence:*

$$\lim_n \delta^c(S_n(\omega).v, L_\Gamma(c)) = 0,$$

where δ^c is the distance on $\mathbb{P}(V) \times \mathbb{T}_c$ given by

$$\delta^c(v, v') = \delta(\bar{v}, \bar{v'}) + |z - z'|, \tag{5.10}$$

and $v = (\bar{v}, z)$, $v' = (\bar{v'}, z')$.

Corollary 5.11. *Assume $\Gamma \subset G$ is a sub-semigroup of G which satisfies property (H_1) and (H_2) and $c > 1$ is fixed. Then the closed Γ-invariant subset $L_\Gamma(c) = L_\Gamma \times \mathbb{T}_c$ of $\mathbb{P}_c(V)$ is the unique minimal set. Furthermore, any $\mu \in M^1(G)$ such that $\Gamma = \Gamma_\mu$ satisfies conditions (H_1) and (H_2) has a unique stationary measure on $\mathbb{P}_c(V)$.*

Clearly, Theorem 5.9 and its corollary imply Theorem 1.3 of the Introduction.

For the proof of Theorem 5.9 we need three lemmas.

Lemma 5.12. *If μ is as in Theorem 5.9 then for $x, y \in V$,*

$$\lim_{y \to x} \limsup_n \mathbf{E}_\mu \left| \left(\frac{\|S_n x\|}{\|S_n y\|} \right)^{i\alpha} - 1 \right| = 0.$$

Proof. ¿From Corollary 5.7 we know that if $x_n \to x$ and $y_n \to y$, then:

$$\lim_n \frac{\|S_n x_n\|}{\|S_n y_n\|} = \frac{|z_\omega(x)|}{|z_\omega(y)|}.$$

Hence, from dominated convergence:

$$\limsup_n \mathbf{E}_\mu \left| \left(\frac{\|S_n x_n\|}{\|S_n y_n\|} \right)^{i\alpha} - 1 \right| = \mathbf{E}_\mu \left| \left| \frac{z_\omega(x)}{z_\omega(y)} \right|^{i\alpha} - 1 \right|.$$

The formula in the lemma corresponds to the special case $x_n = x$, $y = x$. □

Lemma 5.13. *If μ is as in Theorem 5.9, for every $\psi \in C(\mathbb{P}_c(V))$ the sequence of functions $\check{\mu}^{*k} * \psi$ is uniformly equicontinuous.*

Proof. One considers the distance δ^c on $\mathbb{P}_c(V)$ given by (5.10). Then, in view of the form of the action of G on $\mathbb{P}_c(V)$:

$$\delta^c(S_n.v, S_n.v') = \delta(S_n.\bar{v}, S_n.\bar{v'}) + \left| \left(\frac{\|S_n\bar{v}\|}{\|S_n\bar{v'}\|} \right)^{i\alpha} - 1 \right| + |z - z'|.$$

¿From Lemma 5.12, we get:

$$\lim_n \delta^c(S_n.v, S_n.v') = \left| \left| \frac{\langle \bar{v}, z^*_\omega \rangle}{\langle \bar{v'}, z^*_\omega \rangle} \right|^{i\alpha} - 1 \right| + |z - z'|.$$

Hence, using dominated convergence

$$\limsup_n \mathbf{E}_\mu \delta^c(S_n.v, S_n.v') = |z - z'| + \mathbf{E}_\mu \left(\left| \frac{\langle \bar{v}, z^*_\omega \rangle}{\langle \bar{v'}, z^*_\omega \rangle} \right|^{i\alpha} - 1 \right). \tag{5.14}$$

The right hand side of this formula is uniformly small when $\delta^c(v, v')$ is small.

Now, if $\psi \in C(\mathbb{P}_c(V))$ is Lipschitz, with coefficient $[\psi]$:

$$|\check{\mu}^{*n} * \psi(v) - \check{\mu}^{*n} * \psi(v')| \leq \mathbf{E}_\mu[\delta^c[(S_n.v, S_n.v')][\psi].$$

Since Lipschitz functions are dense in $C(\mathbb{P}_c(V))$ the above inequality and (5.14) imply equicontinuity of the sequence $\check{\mu}^{*n} * \psi(v)$ for $\psi \in C(\mathbb{P}_c(V))$. □

Lemma 5.15. *Suppose $\theta \in \mathbb{R}$, $\eta \in C(\mathbb{P}(V))$ and $\eta \not\equiv 0$ and satisfies the equation*

$$\int \eta(g.\bar{v}) \|g\bar{v}\|^{i\alpha} d\mu(g) = e^{i\theta} \eta(\bar{v}). \tag{5.16}$$

Then $\alpha = 0$, $\theta = 0$ and $\eta = const$ on $\mathbb{P}(V)$.

Proof. Passing to absolute values in (5.16) we get

$$|\eta(\bar{v})| \leq \int |\eta(g.\bar{v})| d\mu(g). \tag{5.17}$$

Let $M = \{\bar{v} \in \mathbb{P}(V) : |\eta(\bar{v})| = \|\eta\|_\infty\}$. Then from (5.17) the condition $\bar{v} \in M$ implies $g.\bar{v} \in M$ μ a.e. Hence from continuity of $|\eta|$, we have $gM \subset M$ for every $g \in S_\mu$ and $\Gamma_\mu M \subset M$. Since L_Γ is minimal we get $L_\Gamma \subset M$. In particular for every $\bar{v} \in L_\Gamma$, $|\eta(\bar{v})| = \|\eta\|_\infty$. From strong convexity of the unit disc in $\mathbb{C}$ we get from (5.16) that

$$\forall\, \bar{v} \in L_\Gamma, \forall g \in S_\mu,\ \eta(g.\bar{v}) \|gv\|^{i\alpha} = e^{i\theta} \eta(\bar{v}).$$

¿From Proposition (4.6) it follows that $\alpha = 0$, $\theta = 0$ and $\eta = \text{const}$ on L_Γ.

Now we have on $\mathbb{P}(V)$:

$$\int \eta(g.\bar{v}) d\mu(g) = \eta(\bar{v}).$$

We can suppose η to be real and consider the set M' (M'' resp.) of points where η attains its maximum (minimum resp.). As above we obtain $M' \supset L_\Gamma$. Replacing η by $-\eta$, we obtain also that $M'' \supset L_\Gamma$, hence $M' = M''$. We conclude that

$$\forall\, \bar{v} \in \mathbb{P}(V),\ \eta(\bar{v}) = \text{const}.$$

□

Proof of Theorem 5.9. We use the following result of [28]. Let P be a Markov operator on the compact metric space X, which preserves $C(X)$ and is equicontinuous, i.e., for any $\psi \in C(X)$, the sequence $P^k\psi$, $k \in \mathbb{N}$ is equicontinuous. Then if 1 is the only eigenvalue of modulus one in $C(X)$, the sequence $P^k\psi$ converges uniformly. Here we have $P(x,\cdot) = \mu * \delta_x$, and $X = \mathbb{P}_c(V)$. From Lemma 5.13 we know that P is equicontinuous. Suppose that $\eta \in C(X)$, $\eta \not\equiv 0$ satisfies $P\eta = e^{i\theta}\eta$, i.e.,

$$\int \eta(g.v)d\mu(g) = e^{i\theta}\eta(v)$$

for any v in $\mathbb{P}_c(V)$. Now we can consider the Fourier coefficients ($k \in \mathbb{Z}$),

$$\eta_k(\bar{v}) = \int \eta(\bar{v}, z)z^k d\lambda_c(z)$$

and we obtain

$$\int \eta_k(g.\bar{v})\|g\bar{v}\|^{ik\alpha}d\mu(g) = e^{i\theta}\eta_k(\bar{v}).$$

¿From Lemma 5.15 we get $e^{i\theta} = 1$, $\eta_k(\bar{v}) = 0$ for $k \neq 0$, $\eta_0(\bar{v}) \equiv \text{const}$. Hence $\eta = \text{const}$ on $\mathbb{P}_c(V)$. Now the result of [28] give the uniform convergence of the sequence $\psi_n = \check{\mu}^{*n} * \psi$.

Clearly, if $\lim_n \psi_n = \eta$, one has $P\eta = \check{\mu} * \eta = \eta$ and η is continuous. From the above result, we deduce $\eta \equiv \text{const}$. Furthermore,

$$\eta = (\nu \otimes \lambda_c)(\eta) = \lim_n (\nu \otimes \lambda_c)(\psi_n) = (\nu \otimes \lambda_c)(\psi).$$

Hence the formula $\eta = (\nu \otimes \lambda_c)(\psi)$ and the required convergence.

In order to prove the second statement of the theorem, notice that since $L_\Gamma(c)$ is the inverse image of L_Γ in $\mathbb{P}_c(V)$ we have:

$$\delta^c(S_n(\omega).v, L_\Gamma(c)) = \delta(S_n(\omega).\bar{v}, L_\Gamma).$$

Proposition 5.6 implies that, given $\bar{v}$ and $\bar{w}$ in $\mathbb{P}(V)$, we have the a.e. convergence of the sequence $\delta(S_n(\omega).\bar{v}, S_n(\omega).\bar{w})$ to zero. If we choose $\bar{w}$ in L_Γ, then $S_n(\omega).\bar{w} \in L_\Gamma$, hence:

$$\delta(S_n(\omega).\bar{v}, L_\Gamma) \leq \delta(S_n(\omega).\bar{v}, S_n(\omega).\bar{w}).$$

It follows: $\lim_n \delta(S_n(\omega).\bar{v}, L_\Gamma) = \delta(S_n(\omega).\bar{v}, S_n(\omega).\bar{w}) = 0$. □

Proof of Corollary 5.11. Suppose $\xi \in M^1(\mathbb{P}_c(V))$ is another μ-stationary measure. Since $\psi_n = \check{\mu}^{*n} * \psi$ converges uniformly to $(\nu \otimes \lambda_c)(\psi)$, we get

$$\xi(\lim_n \psi_n) = \lim_n \xi(\check{\mu}^{*n} * \psi) = \xi(\psi).$$

Hence, $(\nu \otimes \lambda_c)(\psi) = \xi(\psi)$, $\nu \otimes \lambda_c = \xi$ and the uniqueness follows.

Suppose Δ is a closed Γ_μ-invariant subset of $\mathbb{P}_c(V)$. Then from the Markov-Kakutani theorem, there is a μ-stationary measure carried by Δ. From the uniqueness of the stationary measure we get

$$\Delta \supset \text{supp } \nu \otimes \lambda_c = L_\Gamma(c).$$

□

Theorem 5.18. *Suppose that Γ is a sub-semigroup of $GL(d, \mathbb{R})$ satisfying conditions (H_0), (H_1) and (H_2) and let Σ be Γ-invariant subset of $\tilde{V} \setminus \{0\}$ such that 0 is a limit point of Σ. Then*

$$\overline{\Sigma} \supset \tilde{L}_\Gamma / \{Id\,, \sigma\}. \tag{5.19}$$

Proof. We denote by Σ' the inverse image of Σ in $V \setminus \{0\}$. Let u_0 be a γ-dominant vector as in Proposition 4.1, that is satisfying

$$\gamma^{\mathbb{Z}} u_0 := \{\gamma^k u_0 : k \in \mathbb{Z}\} \subset \overline{\Sigma'}. \tag{5.20}$$

Applying Corollary 5.11 with $c = \lambda$, where λ is the unique eigenvalue of γ of maximum modulus, greater than 1 since γ is expanding, we get that if $\bar{u_0}$ denotes the projection of u_0 on $\mathbb{P}_c(V)$ then $\overline{\Gamma \bar{u_0}} \supset L_\Gamma(c)$. It follows that if $\bar{y} \in L_\Gamma(c)$ is given, then there is a sequence $\{\gamma_n\} \subset \Gamma$, such that $\gamma_n.\bar{u_0}$ converges to $\bar{y}$. This implies that there is a sequence of integers $\{p_n\}$ such that $\lambda^{p_n}\gamma_n u_0 \to y \in V \setminus \{0\}$ but this implies

$$\gamma_n \lambda^{p_n} u_0 = \gamma_n \gamma^{p_n} u_0 \to y.$$

But $\gamma^{p_n} u_0 \in \overline{\Sigma'}$ by (5.20). Thus $y \in \overline{\Sigma'}$. Since $\bar{y}$ was an arbitrary point from $L_\Gamma(c)$ we get that $\tilde{L}_\Gamma \subset \overline{\Sigma'}$ and (5.19) is proved. □

Clearly, Theorem 5.18 gives Theorem 1.2 of the Introduction.

Theorem 5.18 will be used below in the special case $\Gamma \subset M_{inv}(d, \mathbb{Z})$.

Corollary 5.21. *Let Γ be a sub-semigroup of $M_{inv}(d, \mathbb{Z})$ satisfying (H_0), (H_1) and (H_2). Let Σ be a Γ-invariant subset of $\tilde{V} \setminus \{0\}$ such that 0 is a limit point of Σ. Then $\overline{\Sigma} \supset \tilde{L}_\Gamma / \{Id, \sigma\}$.*

Theorem 5.18 does not give information on a general Γ-orbit closure $\overline{\Gamma v}$, $v \in \tilde{V} \setminus \{0\}$ if 0 is not a limit point. On the other hand Theorem 5.9 and its corollary describe the behavior of a general Γ-orbit in $\mathbb{P}_c(V)$. Using more precise informations on products of random matrices, i.e the renewal

theorem as in [19] (see also [22]), one can go further and describe the behavior at infinity of a general orbit $\Gamma v \subset \tilde{V} \setminus \{0\}$ as follows. For any c, d $(1 \leq c < d)$ we denote by $\tilde{V}_{[c,d]} \subset \tilde{V} \setminus \{0\}$ the "c-shell" $\mathbb{P}^{d-1} \times [c, d]$, by $\tilde{L}_{\Gamma,c} \subset \tilde{V}_c := \mathbb{P}^{d-1} \times [1, c]$ the closed subset $L_\Gamma \times [1, c]$. Then by the methods of [7] we can obtain the following

Theorem 5.22. *Assume that the semigroup $\Gamma \subset GL(d, \mathbb{R})$, $d > 1$, satisfies (H_0), (H_1) and (H_2). Then, with the above notations, for any $c > 1$, $v \in \tilde{V} \setminus \{0\}$ we have the following convergence*

$$\lim_{t\to\infty} c^{-t}(\overline{\Gamma v} \cap \tilde{V}_{[c^t, c^{t+1}]}) = \tilde{L}_{\Gamma,c}.$$

This can be interpreted as "thickness" at infinity in the direction of $\tilde{L}_\Gamma$ of the orbit closure $\overline{\Gamma v} \subset \tilde{V}$.

Theorems 5.18 and Corollary 5.21 can also be deduced from Theorem 5.22.

Remark 5.23. The conclusions in statements 5.18 to 5.22 are valid also if $d = 1$, if one supposes the semigroup Γ of $\mathbb{R}^*$ to be non-lacunary. The corresponding aperiodicity condition in the statements above is automatically satisfied if $d > 1$, because of Corollary 4.8.

6. Proof of Theorem 1.1

In order to prove the theorem, we use ideas of [12] and [4]. The first step is to prove that if $\Sigma \subset \mathbb{T}^d$ is a closed Γ-invariant subset that contains $0 \in \mathbb{T}^d$ as a limit point, then $\Sigma = \mathbb{T}^d$. Here we apply Corollary 5.21 to the inverse image $p^{-1}(\Sigma)$ of Σ in $\mathbb{R}^d$.

In the general case, we suppose Σ to be infinite and we construct other closely related closed Γ-invariant subsets of $\mathbb{T}^d$ which contains 0. Then we use the special case to get information on Σ and we conclude that $\Sigma = \mathbb{T}^d$.

6.1. **The case when 0 is a limit point of Σ.** The statement $\Sigma = \mathbb{T}^d$ will hold by Corollary 5.21 applied to $p^{-1}(\Sigma)$ if we are able to see that $\tilde{L}_\Gamma$ contains at least one ray which is not contained in a rational subspace. But the set of rational subspaces is countable and, by Lemma 5.1 L_Γ is not contained in a countable union of subspaces. The result follows.

We can observe that the set $\tilde{L}_\Gamma$ is very large, since it was proved in [6] that L_Γ has strictly positive Hausdorff dimension.

6.2. **The general case.** In order to show that the above case is the only one, we make use of previous ideas from [12] and [4].

If $\gamma \in M_{inv}(d, \mathbb{Z})$ and $m \in \mathbb{N}$ is fixed we write

$$\gamma \equiv \mathrm{Id} \pmod{m} \iff \gamma - \mathrm{Id} = mA,$$

with $A \in M(d, \mathbb{Z}) := \{d \times d$ matrices with integer entries$\}$.

For a fixed $m \in \mathbb{N}$ define

$$\Gamma^{(m)} = \{\gamma \in \Gamma : \gamma \equiv \mathrm{Id} \pmod{m}\}.$$

We observe that Γ acts on on the finite set $(\mathbb{Z}/m\mathbb{Z})^d$. We denote by $\gamma \mapsto \bar{\gamma}$ the corresponding homomorphism of Γ into the semigroup $\Lambda_{m,d}$ of maps of $(\mathbb{Z}/m\mathbb{Z})^d$ into itself and we write:

$$\Gamma_m = \{\bar{\gamma} \in \Lambda_{m,d} : \gamma \in \Gamma\}.$$

The proof depends on the following

Lemma 6.1. *Assume that Γ is finitely generated and satisfies (H_0), (H_1) and (H_2). Let m be a prime number not dividing the elements of the multiplicative semigroup $\{\det \gamma : \gamma \in \Gamma\}$. Then Γ_m is a group of permutations of $(\mathbb{Z}/m\mathbb{Z})^d$ and the semigroup $\Gamma^{(m)}$ satisfies (H_0), (H_1) and (H_2).*

Proof. Here $\mathbb{Z}/m\mathbb{Z}$ is a finite field and for $\gamma \in \Gamma$, $\bar{\gamma}$ is an endomorphism of the vector space $(\mathbb{Z}/m\mathbb{Z})^d$. Then $\det \bar{\gamma}$ is the congruence class of $\det \gamma$ in $\mathbb{Z}/m\mathbb{Z}$. Since m is a prime number not dividing $\det \gamma$, we conclude that $\det \bar{\gamma} \neq \bar{0}$, hence $\bar{\gamma}$ belongs to the group $GL(d, \mathbb{Z}/m\mathbb{Z})$. Then Γ_m is a semigroup contained in the finite group $GL(d, \mathbb{Z}/m\mathbb{Z})$; it follows that Γ_m is a group. We write $\Gamma_m = \{\bar{a}_i : a_i \in \Gamma : i = 1, \ldots, q\}$, and we observe that the inverse of $\bar{a}_i$ is of the form $\bar{a}_{i'}$ with $a_{i'} \in \Gamma$ and $1 \leq i' \leq q$. Since for every $\gamma \in \Gamma$, we have $\bar{\gamma} = \bar{a}_i$ for some i, we get $\bar{a}_{i'}\bar{\gamma} = \mathrm{Id}$, $a_{i'}\gamma \in \Gamma^{(m)}$.

Assume condition (H_1) is not satisfied by $\Gamma^{(m)}$; then for some subspace $W \subset V$, the orbit $\Gamma^{(m)}W$ is finite, hence the set $\{a_{i'}\gamma W : \bar{a}_{i'} \in \Gamma_m, \gamma \in \Gamma, \bar{a}_{i'}\bar{\gamma} = \mathrm{Id}\}$ is also finite. It follows that the set $\{\gamma W : \gamma \in \Gamma\}$ is finite and this contradicts condition (H_1) for Γ. Hence $\Gamma^{(m)}$ satisfies condition (H_1).

Let $\gamma \in \Gamma$ be a proximal and expanding element element of Γ. Since the group Γ_m is finite, there exists $k \leq |\Gamma_m|$ such that $\bar{\gamma}^k = \mathrm{Id}$, hence $\gamma^k \in \Gamma^{(m)}$. Clearly, γ^k is proximal and expanding. Then the result follows from the equivalence of a) and c) in Proposition 2.4. □

The following lemma will be used also. Its proof is analogous to the classical case of one endomorphism of $\mathbb{T}^d$ (see for example [2]). In this lemma, the torus $\mathbb{T}^d$ is endowed with its normalized Haar measure, which is Γ-invariant.

Lemma 6.2. *Assume $\Gamma \subset M_{inv}(d, \mathbb{Z})$ and $\Sigma \subset \mathbb{T}^d$ is measurable, has positive measure and satisfies $\Gamma\Sigma \subset \Sigma$. Then, if any character $\chi \neq Id$ has unbounded Γ^t-orbit, then Σ has measure 1, in particular Γ is ergodic on $\mathbb{T}^d$.*

In order to prove Theorem 1.1, we can suppose Γ to be finitely generated. In fact, Proposition 2.6 implies that Γ contains a finitely generated semigroup Γ_1 which satisfies (H_1) and (H_2), and we can add to Γ_1 an expanding

element γ from Γ. Then the semigroup generated by Γ_1 and γ satisfies (H_0), (H_1) and (H_2) in view of equivalence (c), (d) in Proposition 2.4.

Since Σ is infinite and closed, it contains limit points. We have two cases.

Case 1. Some limit point of Σ is rational. So, let p/q be a limit point of Σ. Then the set $q\Sigma$ is Γ-invariant and has 0 as its limit point. Therefore, by considerations in subsection 6.1 we get that $q\Sigma = \mathbb{T}^d$. Hence, Σ has positive Haar measure (greater than $(1/q)^d$). Since Γ satisfies (H_0), (H_1) we get, from Remark 2.1 (iv) that Γ^t satisfies (H_0), hence Lemma 6.2 allows us to conclude that Σ has measure 1. Since Σ is closed, we have $\Sigma = \mathbb{T}^d$.

Case 2. Every limit point of Σ is irrational. Let Σ^{ac} be the set of limit points of Σ. For m fixed and prime not dividing the elements of the finitely generated semigroup $\{\det\gamma : \gamma \in \Gamma\}$, let $\Sigma^{(m)} \subset \Sigma^{\mathrm{ac}}$ be a minimal $\Gamma^{(m)}$-invariant set. Since Σ^{ac} consists of irrational points, $\Sigma^{(m)}$ is infinite, hence 0 is a limit point of the closed $\Gamma^{(m)}$-invariant subset $\Sigma^{(m)} - \Sigma^{(m)}$. From Lemma 6.1 above and consideration in subsection 6.1 we get that $\Sigma^{(m)} - \Sigma^{(m)} = \mathbb{T}^d$. Therefore, for every $r = (r_1, \ldots, r_d) \in \mathbb{Z}^d$ there are x and y in $\Sigma^{(m)}$ such that

$$x - y = (r_1/m, \ldots, r_d/m) = r/m.$$

Let $\Sigma_r^{(m)}$ be defined as follows:

$$\Sigma_r^{(m)} = \{x \in \Sigma^{(m)} : \exists\, y \in \Sigma^{(m)} : x - y = r/m)\}.$$

Clearly, $\Sigma_r^{(m)}$ is closed and nonempty. Since r/m is fixed by $\Gamma^{(m)}$ it follows that $\Sigma_r^{(m)}$ is $\Gamma^{(m)}$-invariant. Thus, by minimality of $\Sigma^{(m)}$ we get that $\Sigma^{(m)} = \Sigma_r^{(m)}$. Therefore, for every $m \in \mathbb{N}$, $x \in \Sigma^{(m)}$, $r \in \mathbb{Z}^d$ we have

$$x + r/m = y \in \Sigma^{(m)}.$$

Hence $\Sigma^{(m)}$ is invariant under translations in $\mathbb{T}^d$ by r/m, $r \in \mathbb{Z}^d$. It follows that $\Sigma^{(m)}$ is $1/m$-dense, hence Σ^{ac} is $1/m$-dense for every prime m as above. We observe that the set of such primes is infinite, thus $1/m$ can be chosen arbitrary small. Since Σ^{ac} is closed we have $\Sigma^{\mathrm{ac}} = \mathbb{T}^d$, which contradicts the hypothesis. Thus, only case 1 is possible, hence $\Sigma = \mathbb{T}^d$. □

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