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PURE-INJECTIVE HULLS OF MODULES OVER VALUATION RINGS

FRANÇOIS COUCHOT

ABSTRACT. If \widehat{R} is the pure-injective hull of a valuation ring R , it is proved that $\widehat{R} \otimes_R M$ is the pure-injective hull of M , for every finitely generated R -module M . Moreover $\widehat{R} \otimes_R M \cong \bigoplus_{1 \leq k \leq n} \widehat{R}/A_k \widehat{R}$, where $(A_k)_{1 \leq k \leq n}$ is the annihilator sequence of M . The pure-injective hulls of uniserial or polyserial modules are also investigated. Any two pure-composition series of a countably generated polyserial module are isomorphic.

The aim of this paper is to study pure-injective hulls of modules over valuation rings. If R is a valuation domain and S a maximal immediate extension of R , then, in [10], Warfield proved that S is a pure-injective hull of R . Moreover, for each finitely generated R -module M , he showed that $S \otimes_R M$ is a pure-injective hull of M and a direct sum of $\text{gen } M$ indecomposable pure-injective modules. We extend this last result to every valuation ring R by replacing S with the pure-injective hull \widehat{R} of R . As in the domain case \widehat{R} is a faithfully flat module. Moreover, for each $x \in \widehat{R}$ there exist $r \in R$ and $y \in 1 + P\widehat{R}$ such that $x = ry$. This property allows us to prove most of the main results of this paper. We extend results obtained by Fuchs and Salce on pure-injective hulls of uniserial modules over valuation domains ([5, chapter XIII, section 5]). We show that the length of any pure-composition series of a polyserial module M is its Malcev rank $\text{Mr } M$ and its pure-injective hull \widehat{M} is a direct sum of p indecomposable pure-injective modules, where $p \leq \text{Mr } M$. But it is possible to have $p < \text{Mr } M$ and we prove that the equality holds for all M if and only if R is maximal (Theorem 4.5). This result is a consequence of the fact that R is maximal if and only if R/N and R_N are maximal, where N is the nilradical of R (Theorem 4.4). If U_1, \dots, U_n are the factors of a pure-composition series of a polyserial module M then the collection $(\widehat{R} \otimes_R U_k)_{1 \leq k \leq n}$ is uniquely determined by M . To prove this, we use the fact that $\widehat{R} \otimes_R U$ is an unshrinkable uniserial T -module for each uniserial R -module U , where $T = \text{End}_R(\widehat{R})$. When R satisfies a countable condition, the collection of uniserial factors of a polyserial module M is uniquely determined by M (Proposition 3.7).

In this paper all rings are associative and commutative with unity and all modules are unital. As in [3] we say that an R -module E is **divisible** if, for every $r \in R$ and $x \in E$, $(0 : r) \subseteq (0 : x)$ implies that $x \in rE$, and that E is **fp-injective** (or **absolutely pure**) if $\text{Ext}_R^1(F, E) = 0$, for every finitely presented R -module F . A ring R is called **self fp-injective** if it is fp-injective as R -module. An exact sequence $0 \rightarrow F \rightarrow E \rightarrow G \rightarrow 0$ is **pure** if it remains exact when tensoring it with any R -module. In this case we say that F is a **pure** submodule of E . Recall that a module E is fp-injective if and only if it is a pure submodule of every overmodule. A module is said to be **uniserial** if its submodules are linearly ordered

by inclusion and a ring R is a **valuation ring** if it is uniserial as R -module. Recall that every finitely presented module over a valuation ring is a finite direct sum of cyclic modules [9, Theorem 1]. Consequently a module E over a valuation ring R is fp-injective if and only if it is divisible.

An R -module F is **pure-injective** if for every pure exact sequence

$$0 \rightarrow N \rightarrow M \rightarrow L \rightarrow 0$$

of R -modules, the following sequence

$$0 \rightarrow \text{Hom}_R(L, F) \rightarrow \text{Hom}_R(M, F) \rightarrow \text{Hom}_R(N, F) \rightarrow 0$$

is exact. An R -module B is a **pure-essential extension** of a submodule A if A is a pure submodule of B and, if for each submodule K of B , either $K \cap A \neq 0$ or $(A + K)/K$ is not a pure submodule of B/K . We say that B is a **pure-injective hull** of A if B is pure-injective and a pure-essential extension of A . By [10] or [5, chapter XIII] each R -module M has a pure-injective hull and any two pure-injective hulls of M are isomorphic.

In the sequel R is a valuation ring, P its maximal ideal, Z its subset of zerodivisors and \widehat{M} the pure-injective hull of M , for each R -module M . As in [5, p.69], for every proper ideal A , we put $A^\# = \{s \in R \mid (A : s) \neq A\}$. Then $A^\# / A$ is the set of zerodivisors of R/A whence $A^\#$ is a prime ideal. In particular $\{0\}^\# = Z$. When $A^\# = P$, we say that A is an **archimedean** ideal. Then A is archimedean if and only if R/A is self fp-injective.

1. PROPERTIES OF \widehat{R}

The first assertion of the following proposition will play a crucial role to prove the main results of this paper.

Proposition 1.1. *The following assertions hold:*

- (1) *For each $x \in \widehat{R}$ there exist $a \in R$, $p \in P$ and $y \in \widehat{R}$ such that $x = a + pay$.*
- (2) *For each archimedean ideal A of R , $\widehat{R}/A\widehat{R}$ is an essential extension of R/A .*
- (3) *$\widehat{R}/P\widehat{R} \cong R/P$.*

Proof. The third assertion is an immediate consequence of the first.

We also deduce the second assertion from the first. Since R is a pure submodule of \widehat{R} , the natural map $R/A \rightarrow \widehat{R}/A\widehat{R}$ is monic. Let $x \in \widehat{R} \setminus R + A\widehat{R}$. We have $x = a + pay$ for $a \in R$, $p \in P$ and $y \in \widehat{R}$. Hence $pa \notin A$. Since A is archimedean, there exists $r \in (A : pa) \setminus (A : a)$. So $rx \in R + A\widehat{R} \setminus A\widehat{R}$.

We proceed by steps to prove the first assertion.

Step 1. Suppose that R is self fp-injective. In this case, $\widehat{R} \cong E_R(R)$ by [5, Lemma XIII.2.7]. We may assume that $x \notin R$. Then there exists $d \in R$ such that $dx \in R$ and $dx \neq 0$. Since R is a pure submodule of \widehat{R} we have $dx = db$ for some $b \in R$. By [1, Lemma 2] $(0 : x) = (0 : b)$, whence $x = bz$ for some $z \in \widehat{R}$ since \widehat{R} is divisible. In the same way, there exists $c, u \in R$ such that $cz = cu \neq 0$. We get that $(0 : u) = (0 : z) = b(0 : b) = 0$. So u is a unit of R . Since $z - u \notin R$, there exists $s, q \in R$ and $y \in \widehat{R}$ such that $0 \neq sq = s(z - u) \in R$ and $z - u = qy$. We have $c \in (0 : z - u) = (0 : q)$. So $q \in P$. Now we put $a = bu$ and $p = qu^{-1}$ and we get $x = a + pay$.

Step 2. Now we prove that $\widehat{R}/r\widehat{R} \cong E_{R/rR}(R/rR)$ for each $0 \neq r \in P$. If $\bigcap_{a \neq 0} aR = 0$ then it is an immediate consequence of [3, Theorem 5.6]. Else P is

not faithful, R is self fp-injective and $\widehat{R} \cong E_R(R)$. By Step 1 and the implication $1 \Rightarrow 2$ the second assertion holds. So it remains to show that $\widehat{R}/r\widehat{R}$ is injective over R/rR . Let J be an ideal of R such that $Rr \subset J$ and $g : J/Rr \rightarrow \widehat{R}/r\widehat{R}$ be a nonzero homomorphism. For each $x \in \widehat{R}$ we denote by \bar{x} the image of x in $\widehat{R}/r\widehat{R}$. Let $a \in J \setminus Rr$ such that $\bar{y} = g(\bar{a}) \neq 0$. Then $(Rr : a) \subseteq (r\widehat{R} : y)$. Let $t \in R$ such that $r = at$. Thus $ty = rz$ for some $z \in \widehat{R}$. It follows that $t(y - az) = 0$. So, since $at = r \neq 0$, we have $(0 : a) \subset Rt \subseteq (0 : y - az)$. The injectivity of \widehat{R} implies that there exists $x \in \widehat{R}$ such that $y = a(x + z)$. We put $x_a = x + z$. If $b \in J \setminus Ra$ then $a(x_a - x_b) \in r\widehat{R}$. Hence $x_b \in x_a + (r\widehat{R} :_{\widehat{R}} a)$. Since \widehat{R} is pure-injective, by [10, Theorem 4] there exists $x \in \bigcap_{a \in J} x_a + (r\widehat{R} :_{\widehat{R}} a)$. It follows that $g(\bar{a}) = a\bar{x}$ for each $a \in J$.

Step 3. Now we prove the first assertion in the general case. If $\bigcap_{r \neq 0} rR \neq 0$, then R is self fp-injective. So the result holds by Step 1. If $\bigcap_{r \neq 0} rR = 0$, we put $F = \bigcap_{r \neq 0} r\widehat{R}$. We will show that $F = 0$. Let $x \in F \cap R$. Then $x \in R \cap r\widehat{R} = rR$ for each $r \in R$, $r \neq 0$. Therefore $x = 0$ and $F \cap R = 0$. Let $x \in \widehat{R}$, $r, a \in R$ and $z \in F$ such that $rx = a + z$. There exists $y \in \widehat{R}$ such that $z = ry$. So $r(x - y) = a$, whence there exists $b \in R$ such that $rb = a$. It follows that R is a pure submodule of \widehat{R}/F . Since \widehat{R} is a pure-essential extension of R we deduce that $F = 0$. Let $x \in \widehat{R}$. We may assume that $x \notin R$. There exists $0 \neq r \in R$ such that $x \notin r\widehat{R}$. If $x \in R + r\widehat{R}$ then $x = a + ry$, with $a \in R$ and $y \in \widehat{R}$. We have $a \notin rR$ else $x \in r\widehat{R}$. So $r = pa$ for some $p \in P$. If $x \notin R + r\widehat{R}$ then, since R/Rr is self fp-injective, from Steps 1 and 2 we deduce that $x - a - paz \in r\widehat{R}$ for some $a \in R$, $p \in P$ and $z \in \widehat{R}$. It is obvious that $a \notin rR$. Now it is easy to conclude. \square

As in the domain case we have:

Proposition 1.2. *\widehat{R} is a faithfully flat R -module.*

Proof. Let $x \in \widehat{R}$ and $r \in R$ such that $rx = 0$. By Proposition 1.1 there exist $a \in R$, $p \in P$ and $y \in \widehat{R}$ such that $x = a + pay$. So $rpay \in R$. It follows that there exists $b \in R$ such that $ra(1+pb) = 0$. Hence $ra = 0$ and $r \otimes x = ra \otimes (1+py) = 0$. \square

2. PURE-INJECTIVE HULLS OF UNISERIAL MODULES

The following lemma and Proposition 2.2 will be useful to prove the pure-injectivity of some modules in the sequel.

Lemma 2.1. *Let U be a module and F a flat module. Then, for each $r, s \in R$, $F \otimes_R (sU :_U r) \cong (F \otimes_R sU :_{F \otimes_R U} r)$.*

Proof. We put $E = F \otimes_R U$. Let ϕ be the composition of the multiplication by r in U with the natural map $U \rightarrow U/sU$. Then $(sU :_U r) = \ker(\phi)$. It follows that $F \otimes_R (sU :_U r)$ is isomorphic to $\ker(\mathbf{1}_F \otimes \phi)$ since F is flat. We easily check that $\mathbf{1}_F \otimes \phi$ is the composition of the multiplication by r in E with the natural map $E \rightarrow E/sE$. It follows that $F \otimes_R (sU :_U r) \cong (sE :_E r)$. \square

Proposition 2.2. *Every pure-injective R -module F satisfies the following property: if $(x_i)_{i \in I}$ is a family of elements of F and $(A_i)_{i \in I}$ a family of ideals of R such that the family $\mathcal{F} = (x_i + A_i F)_{i \in I}$ has the finite intersection property, then \mathcal{F} has a non-empty intersection. The converse holds if F is flat.*

Proof. Let $i \in I$ such that A_i is not finitely generated. By [1, Lemma 29] either $A_i = Pr_i$ or $A_i = \bigcap_{c \in R \setminus A_i} cR$. If, $\forall i \in I$ such that A_i is not finitely generated, we replace $x_i + A_i F$ by $x_i + r_i F$ in the first case, and by the family $(x_i + cF)_{c \in R \setminus A_i}$ in the second case, we deduce from \mathcal{F} a family \mathcal{G} which has the finite intersection property. Since F is pure-injective, it follows that there exists $x \in F$ which belongs to each element of the family \mathcal{G} by [10, Theorem 4]. We may assume that the family $(A_i)_{i \in I}$ has no smallest element. So, if A_i is not finitely generated, there exists $j \in I$ such that $A_j \subset A_i$. Let $c \in A_i \setminus PA_j$ such that $x_j + cF \in \mathcal{G}$. Then $x - x_j \in cF \subseteq A_i F$ and $x_j - x_i \in A_j F$. Hence $x - x_i \in A_i F$ for each $i \in I$.

Conversely, if F is flat then by Lemma 2.1 we have $(sF :_F r) = (sR : r)F$ for each $s, r \in R$. We use [10, Theorem 4] to conclude. \square

Proposition 2.3. *Let U be a uniserial module and F a flat pure-injective module. Then $F \otimes_R U$ is pure-injective.*

Proof. Let $E = F \otimes_R U$. We use [10, Theorem 4] to prove that E is pure-injective. Let $(x_i)_{i \in I}$ be a family of elements of F such that the family $\mathcal{F} = (x_i + N_i)_{i \in I}$ has the finite intersection property, where $N_i = (s_i E :_E r_i)$ and $r_i, s_i \in R, \forall i \in I$.

First we assume that $U = R/A$ where A is a proper ideal of R . So $E \cong F/AF$. If $s_i \notin A$ then $N_i = (s_i F :_F r_i)/AF = (Rs_i : r_i)F/AF$. We set $A_i = (Rs_i : r_i)$ in this case. If $s_i \in A$ then $N_i = (AF :_F r_i)/AF = (A : r_i)F/AF$. We put $A_i = (A : r_i)$ in this case. For each $i \in I$, let $y_i \in F$ such that $x_i = y_i + AF$. It is obvious that the family $(y_i + A_i F)_{i \in I}$ has the finite intersection property. By Proposition 2.2 this family has a non-empty intersection. Then \mathcal{F} has a non-empty intersection too.

Now we assume that U is not finitely generated. It is obvious that \mathcal{F} has a non-empty intersection if $x_i + N_i = E, \forall i \in I$. Now assume there exists $i_0 \in I$ such that $x_{i_0} + N_{i_0} \neq E$. Let $I' = \{i \in I \mid N_i \subseteq N_{i_0}\}$ and $\mathcal{F}' = (x_i + N_i)_{i \in I'}$. Then \mathcal{F} and \mathcal{F}' have the same intersection. By Lemma 2.1 $N_{i_0} = F \otimes_R (s_{i_0} U :_U r_{i_0})$. It follows that $(s_{i_0} U :_U r_{i_0}) \subset U$ because $N_{i_0} \neq E$. Hence $\exists u \in U$ such that $x_{i_0} + N_{i_0} \subseteq F \otimes_R Ru$. Then, $\forall i \in I', x_i + N_i \subseteq F \otimes_R Ru$. We have $F \otimes_R Ru \cong F/(0 : u)F$. From the first part of the proof $F/(0 : u)F$ is pure-injective. So we may replace R with $R/(0 : u)$ and assume that $(0 : u) = 0$. Let $A_i = ((s_i U :_U r_i) : u), \forall i \in I'$. Thus $N_i = A_i F, \forall i \in I'$. By Proposition 2.2 \mathcal{F}' has a non-empty intersection. So \mathcal{F} has a non-empty intersection too. \square

Let U be an R -module. As in [5, p.338] we set

$$U_{\sharp} = \{s \in R \mid \exists u \in U, u \neq 0 \text{ and } su = 0\} \text{ and } U^{\sharp} = \{s \in R \mid sU \subset U\}.$$

Then U_{\sharp} and U^{\sharp} are prime ideals.

Now it is possible to determine the pure-injective hull of each uniserial module. We get a generalization of [5, Corollary XIII.5.5]

Theorem 2.4. *The following assertions hold:*

- (1) *Let U be a uniserial R -module and $J = U^{\sharp} \cup U_{\sharp}$. Then $\widehat{R}_J \otimes_R U$ is the pure-injective hull of U . Moreover \widehat{U} is an essential extension of U if $J = U_{\sharp}$.*
- (2) *For each proper ideal A of R , $\widehat{R}/A\widehat{R}$ is the pure-injective hull of R/A . Moreover $\widehat{R}/A\widehat{R} \cong E_{R/A}(R/A)$ if A is archimedean.*

Proof. (1) If $s \in R \setminus J$ then multiplication by s in U is bijective. So U is an R_J -module. After replacing R with R_J , we may assume that $J = P$. We put $\tilde{U} = \widehat{R}_J \otimes_R U$.

Suppose that $P = U^\sharp$. By [10, Proposition 6] $\tilde{U} = \widehat{U} \oplus V$ where V is a submodule of \tilde{U} . Let $v \in V$. Then $v = x \otimes u$ where $u \in U$ and $x \in \widehat{R}$. By Proposition 1.1 $x = a + pay$, where $a \in R$, $p \in P$ and $y \in \widehat{R}$. Since $pU \subset U$, $\exists u' \in U \setminus (Pu \cup pU)$. Then $u = cu'$ for some $c \in R$ and $v = cau' + pcay \otimes u'$. We have $y \otimes u' = z + w$ where $w \in V$ and $z \in \widehat{U}$. So $cau' + pcay \otimes u' = 0$. Since U is pure in \widehat{U} , there exists $z' \in U$ such that $cau' + pcay \otimes u' = 0$. If $v \neq 0$ the equality $v = (1 + py)(\otimes cau')$ implies $cau' \neq 0$. By [1, Lemma 5] we get that $u' \in pU$, whence a contradiction. Hence $V = 0$.

Now suppose that $P = U_\sharp$. If $0 \neq z \in \tilde{U}$ then $z = x \otimes u$ where $u \in U$ and $x \in \widehat{R}$. By Proposition 1.1 there exist $a \in R$, $p \in P$ and $y \in \widehat{R}$ such that $x = a + pay$. So $z = au + y \otimes pau$. Let $A = (0 : au)$. By [1, Lemma 26], $A^\sharp = P$. So $(0 : pau) = (A : p) \neq A$. Let $r \in (A : p) \setminus A$. Then $0 \neq rz \in U$.

(2) We apply the first assertion by taking $U = R/A$. In this case, $U^\sharp = P$. The pure-injective hull of R/A is the same over R and over R/A . Since R/A is self fp-injective when A is archimedean then we use [5, Lemma XIII.2.7] to prove the last assertion. \square

In the previous theorem, if U is not cyclic and if $U^\sharp \subseteq U_\sharp$ then \widehat{U} is not necessarily isomorphic to $E_{R/(0:U)}(U)$. For instance:

Example 2.5. Assume that $P = Z$ and P is faithful. We choose $U = P$. Then $U^\sharp = U_\sharp = P$, $\widehat{U} = P\widehat{R}$ and $E_R(U) = \widehat{R}$.

If U is a non-standard uniserial module over a valuation domain R then \widehat{U} is indecomposable by [3, Proposition 5.1] and there exists a standard uniserial module V such that $\widehat{U} \cong \widehat{V}$ by [5, Theorem XIII.5.9]. So, $\widehat{R} \otimes_R U \cong \widehat{R} \otimes_R V$ doesn't imply $U \cong V$. However, it is possible to get the following proposition:

Proposition 2.6. *Let U and V be uniserial modules and $J = U^\sharp \cup U_\sharp$. Assume that $\widehat{R} \otimes_R U \cong \widehat{R} \otimes_R V$. Then U and V are isomorphic if one of the following conditions is satisfied:*

- (1) $U^\sharp = J$ and $J \neq J^2$,
- (2) U is countably generated.

Proof. Let $\phi : \widehat{R} \otimes_R U \rightarrow \widehat{R} \otimes_R V$ be the isomorphism. Let $0 \neq u \in U$. Then $\phi(u) = x \otimes v$ for some $x \in \widehat{R}$ and $v \in V$. By proposition 1.1 we may assume that $x = 1 + py$ for some $p \in P$ and $y \in \widehat{R}$. First we shall prove that $(0 : u) = (0 : v)$. It is obvious that $(0 : v) \subseteq (0 : u)$. Let $r \in (0 : u)$. Then $x \otimes rv = 0$. From the flatness of \widehat{R} we deduce that there exist $s \in R$ and $z \in \widehat{R}$ such that $x = sz$ and $srv = 0$. If $s \in P$ then we get that $1 = qe$ for some $q \in P$ and $e \in \widehat{R}$. Since R is pure in \widehat{R} , it follows that $1 \in P$. This is absurd. Hence s is a unit and $r \in (0 : v)$.

Let v, v' be nonzero elements of V and $x, x' \in 1 + P\widehat{R}$ such that $x \otimes v = x' \otimes v'$. There exists $t \in R$ such that $v = tv'$. Now we shall prove that t is a unit of R . We get that $(x' - tx) \otimes v' = 0$. If $t \in P$, as above we deduce that $v' = 0$, whence a contradiction.

Let $u \in U$ and $v \in V$ as in the first part of the proof. By [1, Lemma 26] we have $U_\sharp = (0 : u)^\sharp = (0 : v)^\sharp = V_\sharp$. Let $p \in P$. We shall prove that $u \in pU$ if

and only if $v \in pV$. If $v = pw$ for some $w \in V$ then $\phi(u) = px \otimes w = p\phi(z)$ for some $z \in \widehat{R} \otimes_R U$. Since U is a pure submodule, then $u = pu'$ for some $u' \in U$. Conversely, if $u = pu'$ for some $u' \in U$ and $\phi(u') = x' \otimes v'$ where $v' \in V$ and $x' \in 1 + P\widehat{R}$, we get that $x' \otimes pv' = x \otimes v$. From above, we deduce that $v \in pV$. So, $U^\# = V^\#$.

Now we can prove that U and V are isomorphic when the first condition is satisfied. In this case U and V are modules over R_J . Since $J \neq J^2$, JR_J is a principal ideal of R_J . Since $JU \subset U$ and $JV \subset V$, U and V are cyclic over R_J . Let $u \in U$ and $v \in V$ as in the first part of the proof, and suppose that $U = R_J u$. If $v = rw$ for some $r \in R_J$ and $w \in V$ then we get, as above, that $u = ru'$ for some $u' \in U$. So r is a unit and U and V are isomorphic.

Let $\{u_i\}_{i \in I}$ be a spanning set of U . For each $i \in I$, let $v_i \in V$ and $x_i \in 1 + P\widehat{R}$ such that $\phi(u_i) = x_i \otimes v_i$. Suppose that $(0 : U) \subset (0 : u)$, $\forall u \in U$. From the first part of proof we deduce that $(0 : V) \subset (0 : v)$, $\forall v \in V$. We have $\bigcap_{i \in I} (0 : u_i) = (0 : U)$. Thus $\bigcap_{i \in I} (0 : v_i) = (0 : V)$. So, for each $v \in V$ there exists $i \in I$ such that $(0 : v_i) \subset (0 : v)$. Hence $v \in Rv_i$. Now, suppose $\exists u \in U$ such that $(0 : u) = (0 : U)$. By [5, Lemma X.1.4] $J = U^\#$. We may assume that $J = J^2$ and I is infinite. Then $JU = U$ and $JV = V$. Let $v \in V$. There exists $p \in J$ such that $v \in pV$. But there exists $i \in I$ such that $u_i \notin pU$. So, $v_i \notin pV$. Hence $v \in R_J v_i$. Now suppose that $I = \mathbb{N}$. Let $(a_n)_{n \in \mathbb{N}}$ be a sequence of elements of P such that $u_n = a_n u_{n+1}$, $\forall n \in \mathbb{N}$. We put $\varphi(u_0) = v_0$. Suppose that $\varphi(u_n) = s_n v_n$ where s_n is a unit. By the second part of the proof there exists a unit t_n such that $a_n v_{n+1} = t_n \varphi(u_n)$. Hence we set $\varphi(u_{n+1}) = t_n^{-1} v_{n+1}$. So, by induction on n , we get an isomorphism $\varphi : U \rightarrow V$. \square

Let $T = \text{End}_R(\widehat{R})$. Then T is a local ring by [3, Proposition 5.1] and [5, Theorem XIII.3.10]. For each R -module M , $\widehat{R} \otimes_R M$ is a left T -module. As in [4] we say that a left uniserial T -module F is **shrinkable** if there exists two T -submodules G and H of F such that $0 \subset H \subset G \subset F$ and $F \cong G/H$. Otherwise F is said to be **unshrinkable**.

Proposition 2.7. *Let U be a uniserial R -module. Then:*

- (1) $\widehat{R} \otimes_R U$ is a left unshrinkable uniserial T -module.
- (2) $\text{End}_T(\widehat{R} \otimes_R U)$ is a local ring.

Proof. (1) Let $x \in 1 + P\widehat{R}$. First we prove that Rx is a pure submodule of \widehat{R} . Let $a, b \in R$ and $y \in \widehat{R}$ such that $by = ax$. By Proposition 1.1 $y = c + pcz$ for some $c \in R$, $p \in P$ and $z \in \widehat{R}$. Suppose that $a \notin Rbc$. Then $bc = ra$ for some $r \in P$. If $x = 1 + qx'$ for some $q \in P$ and $x' \in \widehat{R}$, we get that $a(1 - r) = a(rpz - qx') = aty'$ for some $t \in P$ and $y' \in \widehat{R}$. Since R is a pure submodule of \widehat{R} there exists $s \in R$ such that $a(1 - r - ts) = 0$. We deduce that $a = 0$, whence a contradiction. So $a \in Rbc$. By using similar arguments we easily show that Rx is faithful.

Let $z, z' \in \widehat{R} \otimes_R U$. We have $z = x \otimes u$ and $z' = x' \otimes u'$ where $x, x' \in 1 + P\widehat{R}$ and $u, u' \in U$. Assume that $u' = ru$ for some $r \in R$. The homomorphism $\phi : Rx \rightarrow Rrx'$ such that $\phi(x) = rx'$ is well defined and can be extended to \widehat{R} . We get that $\phi z = z'$. Hence $\widehat{R} \otimes_R U$ is uniserial over T .

Suppose that $\widehat{R} \otimes_R U$ is shrinkable over T . By [4, Lemma 1.17] there exists $z \in \widehat{R} \otimes_R U$ such that Tz is shrinkable. We have $z = x \otimes u$ where $x \in 1 + P\widehat{R}$ and

$u \in U$. So $Tz = \widehat{R} \otimes_R Ru$. There exist $z' \in Tz$ and a non-injective T -epimorphism $\alpha : Tz' \rightarrow Tz$. Let $K = \text{Ker } \alpha$. We may assume that $\alpha(z') = z$. We have $z' = x' \otimes ru$ where $x' \in 1 + P\widehat{R}$ and $r \in R$. Let y be a nonzero element of K . Thus $y = tz' = ay' \otimes ru$ for some $t \in T$, $y' \in 1 + P\widehat{R}$ and $a \in R$. But there exist $s, s' \in T$ such that $x' = sy'$ and $y' = s'x'$. So $0 \neq ax' \otimes ru \in K$. Since $y \neq 0$ we have $aru \neq 0$. On the other hand $x \otimes aru = \alpha(ax' \otimes ru) = 0$. It follows that $aru = 0$ whence a contradiction. So $\widehat{R} \otimes_R U$ is unshrinkable.

(2) is an immediate of (1) and [4, Proposition 9.24]. \square

Proposition 2.8. *Let \mathfrak{c} be a cardinal. Consider a \mathfrak{c} -generated R -module M and U a pure uniserial R -submodule of M . Then U is \mathfrak{c} -generated.*

Proof. We easily check that $\widehat{R} \otimes_R U$ is a pure submodule of $\widehat{R} \otimes_R M$. By Proposition 2.3 $\widehat{R} \otimes_R U$ is pure-injective. Hence $\widehat{R} \otimes_R U$ is a summand of $\widehat{R} \otimes_R M$. On the other hand $\widehat{R} \otimes_R M$ is a \mathfrak{c} -generated T -module. Then $\widehat{R} \otimes_R U$ is also \mathfrak{c} -generated over T . We may assume that $\widehat{R} \otimes_R U$ is generated by $(1 \otimes u_i)_{i \in I}$, where I is a set whose cardinal is \mathfrak{c} and $u_i \in U$, $\forall i \in I$. Let V be the submodule of U generated by $(u_i)_{i \in I}$. Then the inclusion map $V \rightarrow U$ induces an isomorphism $\widehat{R} \otimes_R V \rightarrow \widehat{R} \otimes_R U$. Since \widehat{R} is faithfully flat it follows that $V = U$. \square

From Theorem 2.4 we deduce the following corollary on the structure of indecomposable injective modules.

Corollary 2.9. *Let E be an indecomposable injective module, $J = E_{\#}$ and $\mathcal{A}(E) = \{(0 :_{R_J} x) \mid 0 \neq x \in E\}$. Then:*

(1) $\forall A, B \in \mathcal{A}(E)$, $A \subseteq B$ there exists a monomorphism

$$\varphi_{A,B} : \widehat{R}_J / B\widehat{R}_J \rightarrow \widehat{R}_J / A\widehat{R}_J$$

such that $\varphi_{A,C} = \varphi_{A,B} \circ \varphi_{B,C}$, $\forall A, B, C \in \mathcal{A}(E)$, $A \subseteq B \subseteq C$.

(2) $E \cong \varinjlim \{(\widehat{R}_J / A\widehat{R}_J, \varphi_{A,B}) \mid A, B \in \mathcal{A}(E), A \subseteq B\}$.

(3) $E \cong \widehat{R}_J / (0 :_{R_J} e)\widehat{R}_J$ if $(0 :_{R_J} e) = (0 :_{R_J} E)$ for some $e \in E$.

(4) Suppose that E contains a uniserial R_J -module U such that $\mathcal{A}(E) = \mathcal{A}(U)^1$.

Then $E \cong \widehat{R}_J \otimes_R U$. Moreover, $\forall A, B \in \mathcal{A}(E)$, $A \subseteq B$, there exists $r \in R$ such that one can choose $\varphi_{A,B} = \mathbf{1}_{\widehat{R}_J} \otimes \bar{r}$ where $\bar{r} : R_J/B \rightarrow R_J/A$ is defined by $\bar{r}(a + B) = ar + A$, $\forall a \in R$.

Proof. (1) If $A \in \mathcal{A}(E)$ then $A^{\#} = J$ by [1, Lemma 26]. So A is an archimedean ideal of R_J . By Theorem 2.4 there exists an isomorphism

$$\phi_A : \widehat{R}_J / A\widehat{R}_J \rightarrow (0 :_E A).$$

Let $u_{A,B} : (0 :_E B) \rightarrow (0 :_E A)$ be the inclusion map, $\forall A, B \in \mathcal{A}(E)$, $A \subseteq B$. We set $\varphi_{A,B} = \phi_A^{-1} \circ u_{A,B} \circ \phi_B$. It is easy to check the first assertion.

(2) and (3) These assertions are now obvious.

(4) First we prove that U is fp-injective. Let $x \in E$ and $s \in R$ such that $0 \neq sx \in U$. We put $u = sx$. From $\mathcal{A}(E) = \mathcal{A}(U)$, it follows that $\exists v \in U$ such that $(0 :_{R_J} v) = (0 :_{R_J} x)$ and consequently $u = tv$ for some $t \in R$. We set $A = (0 :_{R_J} x)$. We get that $(0 :_{R_J} u) = (A :_{R_J} t) = (A :_{R_J} s)$. By [1, Lemma 26]

¹We know that this condition holds if R satisfies an additional hypothesis: see [1, Corollary 22], [8, Theorem 5.5] or Remark 3.6

$A^\sharp = E_\sharp = J$. It follows that $R_J s = R_J t$. So U is a pure submodule of E . We conclude by Theorem 2.4 and [5, Lemma XIII.2.7] that $E \cong \widehat{R}_J \otimes_R U$.

Let $u, v \in U$ such that $(0 :_{R_J} u) = A$ and $(0 :_{R_J} v) = B$. There exists $r \in R$ such that $v = ru$ and $B = (A : r)$ (if $A = B$ we take $v = u$ and $r = 1$). So \bar{r} is a monomorphism. \square

3. PURE-INJECTIVE HULLS OF POLYSERIAL MODULES

We say that a module M is **polyserial** if it has a pure-composition series

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M,$$

(i.e. M_k is a pure submodule of M , for each k , $0 \leq k \leq n$) where M_k/M_{k-1} is uniserial for each k , $1 \leq k \leq n$. By [5, Lemma I.7.8], if M is finitely generated, M has a pure-composition series, where $M_k/M_{k-1} \cong R/A_k$ and A_k is a proper ideal, for each k , $1 \leq k \leq n$. We denote by $\text{gen } M$ the minimal number of generators of M . By [5, Lemma V.5.3] $n = \text{gen } M$. The following sequence (A_1, \dots, A_n) is called the **annihilator sequence** of M and is uniquely determined by M , up to the order (see [5, Theorem V.5.5]).

Now we can extend the result obtained by Warfield[10] in the domain case for finitely generated modules.

Theorem 3.1. *Let M be a finitely generated R -module. Then $\widehat{R} \otimes_R M \cong \widehat{M}$. Moreover, $\widehat{M} \cong \widehat{R}/A_1 \widehat{R} \oplus \cdots \oplus \widehat{R}/A_n \widehat{R}$ where (A_1, \dots, A_n) is the annihilator sequence of M .*

Proof. It is easy to verify that M is a pure submodule of $\widehat{R} \otimes_R M$. We have that $\widehat{R} \otimes_R M_1$ is a pure submodule of $\widehat{R} \otimes_R M$ too. By Proposition 2.3 $\widehat{R} \otimes_R M_1$ is pure-injective. It follows that $\widehat{R} \otimes_R M \cong (\widehat{R} \otimes_R M_1) \oplus (\widehat{R} \otimes_R M/M_1)$. By induction on n we get that $\widehat{R} \otimes_R M \cong \widehat{R}/A_1 \widehat{R} \oplus \cdots \oplus \widehat{R}/A_n \widehat{R}$. So $\widehat{R} \otimes_R M$ is pure-injective. By [10, Proposition 6] \widehat{M} is a direct summand of $\widehat{R} \otimes_R M$. So $\widehat{R} \otimes_R M \cong \widehat{M} \oplus V$, where V is a submodule of $\widehat{R} \otimes_R M$. From Proposition 1.1 we deduce that, for each $x \in \widehat{R} \otimes_R M$, there exist $m \in M$, $p \in P$ and $y \in \widehat{R} \otimes_R M$ such that $x = m + py$. Assume that $x \in V$. There exists $z \in \widehat{M}$ and $v \in V$ such that $x = m + pz + pv$. It follows that $x = pv$, whence $V = PV$. On the other hand, $\widehat{R}/A \widehat{R}$ is indecomposable by [3, Proposition 5.1] and $\text{End}_R(\widehat{R}/A \widehat{R})$ is local by [11, Theorem 9] or [5, Theorem XIII.3.10], for every proper ideal A . By Krull-Schmidt Theorem $V \cong \widehat{R}/A_{k_1} \widehat{R} \oplus \cdots \oplus \widehat{R}/A_{k_p} \widehat{R}$ where $\{k_1, \dots, k_p\}$ is a subset of $\{1, \dots, n\}$. If $V \neq 0$, by Proposition 1.1 we get $V \neq PV$. This contradiction completes the proof. \square

The **Malcev rank** of a module N is defined as the cardinal number

$$\text{Mr } N = \sup\{\text{gen } M \mid M \subseteq N, \text{ gen } M < \infty\}.$$

The following proposition is identical to the first part of [5, Proposition XII.1.6]. Here we give a different proof.

Proposition 3.2. *The length of any pure-composition series of a polyserial module M equals $\text{Mr } M$.*

Proof. Let $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ be a pure-composition series of M with uniserial factors. As in [5, Corollary XII.1.5] we prove that $\text{Mr } M \leq n$. Equality holds for $n = 1$. From the pure-composition series of M , we deduce a pure-composition series of M/M_1 of length $n - 1$. By induction hypothesis M/M_1 contains a finitely generated submodule Y with $\text{gen } Y = n - 1$.

Assume that Y is generated by $\{y_2, \dots, y_n\}$. Let $x_2, \dots, x_n \in M$ such that $y_k = x_k + M_1$ and F be the submodule of M generated by x_2, \dots, x_n . If $F \cap M_1 = M_1$ then $M_1 \subseteq F$ and M_1 is a pure submodule of F . In this case M_1 is finitely generated by Proposition 2.8. It follows that the following sequence is exact:

$$0 \rightarrow \frac{M_1}{PM_1} \rightarrow \frac{F}{PF} \rightarrow \frac{Y}{PY} \rightarrow 0.$$

So we have $\text{gen } Y = \text{gen } F - \text{gen } M_1 \leq n - 2$. We get a contradiction since $\text{gen } Y = n - 1$. Hence $F \cap M_1 \neq M_1$. Let $x_1 \in M_1 \setminus F \cap M_1$. Let X be the submodule of M generated by x_1, \dots, x_n . Clearly $Rx_1 = M_1 \cap X$. We will show that $Px_1 = Rx_1 \cap PX$. Let $x \in Rx_1 \cap PX$. Then $x = p \sum_{k=1}^{k=n} a_k x_k = rx_1$ where $p \in P$ and r, a_1, \dots, a_n are elements of R . It follows that $p \sum_{k=2}^{k=n} a_k x_k = (r - pa_1)x_1$. So $(r - pa_1)x_1 \in M_1 \cap F \subset Rx_1$. We deduce that $r - pa_1 \in P$ whence $r \in P$. Hence $x \in Px_1$. Consequently the following sequence is exact:

$$0 \rightarrow \frac{Rx_1}{Px_1} \rightarrow \frac{X}{PX} \rightarrow \frac{Y}{PY} \rightarrow 0.$$

Then $\text{gen } X = n$. □

Now we study the pure-injective hulls of polyserial modules.

Theorem 3.3. *Let M be a polyserial module with the following pure-composition series:*

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

For each integer k , $1 \leq k \leq n$ we put $U_k = M_k/M_{k-1}$. Then:

- (1) There exists a subset I of $\{k \in \mathbb{N} \mid 1 \leq k \leq n\}$ such that $\widehat{M} \cong \bigoplus_{k \in I} \widehat{U}_k$.
- (2) $\widehat{R} \otimes_R M$ is pure-injective and isomorphic to $\bigoplus_{k=1}^{k=n} \widehat{R} \otimes_R U_k$.
- (3) The collection $(\widehat{R} \otimes_R U_k)_{1 \leq k \leq n}$ is uniquely determined by M .

Proof. (1) Let N be a pure submodule of M . The inclusion map $N \rightarrow \widehat{N}$ can be extended to $w : M \rightarrow \widehat{N}$. Let $f : M \rightarrow \widehat{N} \oplus \widehat{M/\overline{N}}$ defined by $f(x) = (w(x), x + N)$, for each $x \in M$. It is easy to verify that f is a pure monomorphism. It follows that \widehat{M} is a summand of $\widehat{N} \oplus \widehat{M/\overline{N}}$. So, by induction on n , we easily get that \widehat{M} is a summand of $\bigoplus_{k=1}^{k=n} \widehat{U}_k$. Since, $\forall k \in \mathbb{N}$, $1 \leq k \leq n$, \widehat{U}_k is indecomposable by [3, Proposition 5.1] and $\text{End}_R(\widehat{U}_k)$ is local by [11, Theorem 9] or [5, Theorem XIII.3.10], we apply Krull-Schmidt Theorem to conclude.

(2) We do as in the proof of Theorem 3.1.

(3) Since $\widehat{R} \otimes_R M$ and $\widehat{R} \otimes_R U_k$ are T -modules, we conclude by Proposition 2.7 and Krull-Schmidt theorem. □

Corollary 3.4. *If M is polyserial and countably generated then any two pure-composition series of M are isomorphic.*

Proof. By Theorem 3.3 the collection $(\widehat{R} \otimes_R U_k)_{1 \leq k \leq n}$ is uniquely determined by M . It remains to show that, if U and V are uniserial modules such that $\widehat{R} \otimes_R U \cong$

$\widehat{R} \otimes_R V$ then $U \cong V$. It is an immediate consequence of Proposition 2.8 and Proposition 2.6. \square

Recall that an R -module M is **finitely** (respectively **countably cogenerated**) if M is a submodule of a product of finitely (respectively countably) many injective hulls of simple modules.

The following proposition completes [1, Corollary 35].

Proposition 3.5. *The following conditions are equivalent:*

- (1) *Every finitely generated R -module is countably cogenerated and every ideal of R is countably generated.*
- (2) *For each prime ideal J which is the union of the set of primes properly contained in J there is a countable subset whose union is J , and for each prime ideal J which is the intersection of the set of primes containing properly J there is a countable subset whose intersection is J .*
- (3) *Each uniserial module is countably generated.*

(1) \Leftrightarrow (2) holds by [1, Corollary 35]

(3) \Rightarrow (2) Let J be a prime ideal. Then J and R_J are uniserial R -modules. So they are countably generated. If R_J is generated by $\{t_n^{-1} \mid n \in \mathbb{N}\}$, where $t_n \notin J \forall n \in \mathbb{N}$, then $J = \bigcap_{n \in \mathbb{N}} Rt_n$. Now it is easy to get the second condition.

(1) \Rightarrow (3) Let U be a uniserial module and $J = U^\# \cup U_\#$. Then U is an R_J -module. But R/J countably cogenerated is equivalent to R_J countably generated. Hence U is countably generated over R if and only if U is countably generated over R_J . So we may assume that $J = P$.

First assume that $U^\# = P$. If $PU \subset U$ then $U = Ru$ where $u \in U \setminus PU$. Now suppose that $PU = U$. Let $r, s \in P$ such that $rU \neq 0$. If $rU = rsU$ then by [1, Lemma 5] we have $U = sU$, hence a contradiction. Let $\{p_n \mid n \in \mathbb{N}\}$ be a spanning set of P such that $p_{n+1} \notin Rp_n$. Then $U = \bigcup_{n \in \mathbb{N}} p_n U$. We may assume that $p_n U \neq 0, \forall n \in \mathbb{N}$. So $p_n U \subset p_{n+1} U$ for each $n \in \mathbb{N}$. Let $u_n \in p_{n+1} U \setminus p_n U$ for each $n \in \mathbb{N}$. Then U is generated by $\{u_n \mid n \in \mathbb{N}\}$.

Now suppose that $U_\# = P$. Assume that $(0 : u) = (0 : U)$ for some $u \in U$. Let $v \in U$ such that $u = av$ for some $a \in R$. By [1, Lemma 2] $(0 : u) = ((0 : v) : a)$. We get that $(0 : v) = ((0 : v) : a) = (0 : U)$. Since $(0 : v)^\# = P$ by [1, Lemma 26] a is a unit, and consequently U is cyclic. Now we assume that $(0 : U) \subset (0 : u)$ for each $u \in U$. We have $(0 : U) = \bigcap_{u \in U} (0 : u)$. By [1, Lemma 30] there exists a countable family $(u_n)_{n \in \mathbb{N}}$ of elements of U such that $(0 : U) = \bigcap_{n \in \mathbb{N}} (0 : u_n)$ and $u_{n+1} \notin Ru_n, \forall n \in \mathbb{N}$. If $u \in U$, since $(0 : u) \neq (0 : U)$, then there exists $n \in \mathbb{N}$ such that $(0 : u_n) \subset (0 : u)$. Hence $u \in Ru_n$ and U is generated by $\{u_n \mid n \in \mathbb{N}\}$. \square

Remark 3.6. In the same way, one can prove that the two first conditions of [1, Proposition 32] (respectively [1, Corollary 34]) are equivalent to the following: each indecomposable injective module E such that $E_\# = P$ contains a uniserial pure submodule which is countably generated (respectively each indecomposable injective module contains a uniserial pure submodule which is countably generated).

Proposition 3.7. *Suppose that R satisfies the equivalent conditions of Proposition 3.5. Then any two pure-composition series of a polyserial R -module are isomorphic.*

Proof. It is an immediate consequence of Proposition 3.5 and Corollary 3.4. \square

4. TWO CRITERIA FOR MAXIMALITY OF R

By Theorem 3.1, if M is finitely generated, then \widehat{M} is a direct sum of $\text{gen } M$ indecomposable pure-injective modules and $\text{gen } M = \text{Mr } M$ by [5, Corollary XII.1.7]. But Theorem 4.5 proves that, if M is polyserial, then \widehat{M} is not necessarily a direct sum of $\text{Mr } M$ indecomposable pure-injective modules.

As in [7], if $x \in \widehat{R} \setminus R$, we say that $B(x) = \{r \in R \mid x \notin R + r\widehat{R}\}$ is the **breath ideal** of x . Then Proposition 4.2 is a generalization of [7, Proposition 1.4]. The following lemma is useful to prove this proposition.

Lemma 4.1. *Let J be a proper ideal such that $J = \bigcap_{c \notin J} cR$. Then $J\widehat{R} = \bigcap_{c \notin J} c\widehat{R}$.*

Proof. By Theorem 2.4 $\widehat{R}/J\widehat{R}$ is the pure-injective hull of R/J . In the proof of Step 3 of Proposition 1.1 it is already shown that $\bigcap_{a \neq 0} a\widehat{R} = 0$ if $\bigcap_{a \neq 0} aR = 0$. So we apply this result to R/J to get the lemma. \square

Recall that the **ideal topology** of R is the linear topology which has as a basis of neighborhoods of 0 the nonzero principal ideals.

Proposition 4.2. *Let A be a proper ideal. Then R/A is Hausdorff and non-complete in its ideal topology if and only if $A = B(x)$ for some x in $\widehat{R} \setminus R$.*

Proof. To show that $R/B(x)$ is Hausdorff, we do as in [7, Proposition 1.4], we prove that $a \notin B(x)$ implies that $pa \notin B(x)$ for some $p \in P$. We have $x = r + ay$ where $r \in R$ and $y \in \widehat{R}$. By Proposition 1.1, $\widehat{R} = R + P\widehat{R}$. So $y = s + pz$, for some $s \in R$, $p \in P$ and $z \in \widehat{R}$. Therefore we get $x = r + as + paz \in R + pa\widehat{R}$. For each $a \notin B(x)$, $x \in r_a + a\widehat{R}$ for some $r_a \in R$. If the family $(r_a + aR)_{a \notin B(x)}$ has a non-empty intersection then, by using Lemma 4.1, we get that $x \in R + B(x)\widehat{R}$, whence a contradiction. So $R/B(x)$ is non-complete.

Conversely, assume that R/A is Hausdorff and non-complete. Then there exists a family $(r_a + aR)_{a \notin A}$ which has the finite intersection and an empty total intersection. Since \widehat{R} is pure-injective, the total intersection of the family $(r_a + a\widehat{R})_{a \notin A}$ contains an element x which doesn't belong to R . Clearly $B(x) \subseteq A$. If $x = r + b\widehat{R}$ for some $r \in R$ and $b \in A$ then $r \in r_a + aR$ for each $a \notin A$, since R is a pure submodule of \widehat{R} . We get a contradiction. So $A = B(x)$. \square

The following lemma is a generalization of [7, Lemma 1.3]. It will be useful to prove Theorem 4.4.

Lemma 4.3. *Let $x \in \widehat{R}$ such that $x = r + ay$ for some $r, a \in R$ and $y \in \widehat{R}$. Then $B(y) = (B(x) : a)$.*

Proof. Let $t \notin B(y)$. Then $y = s + tz$ for some $s \in R$ and $z \in \widehat{R}$. It follows that $x = r + as + aty$. So $t \notin (B(x) : a)$.

Conversely, if $t \notin (B(x) : a)$ then we get the following equalities $x = r + ay = s + taz$ for some $s \in R$ and $z \in \widehat{R}$. Since R is a pure submodule of \widehat{R} it follows that $a(y - tz - b) = 0$ for some $b \in R$. From the flatness of \widehat{R} we deduce that $(y - tz - b) \in (0 : a)\widehat{R}$. But $ta \notin B(x)$ implies that $ta \neq 0$, whence $(0 : a) \subset Rt$. Hence $t \notin B(y)$. \square

Theorem 4.4. *Let N be the nilradical of R . Then R is maximal if and only if R/N and R_N are maximal.*

Proof. Suppose that R is maximal. It is obvious that R/N is maximal. By [6, Lemma 2] R_N is maximal too.

Conversely assume that R/N and R_N are maximal. Let K be the kernel of the natural map $R \rightarrow R_N$. Let $r \in K$. Thus there exists $s \in R \setminus N$ such that $sr = 0$. It follows that $K \subseteq N \subset (0 : r)$. Then $K^2 = 0$. So K is a uniserial R/K -module which is linearly compact if R/K is maximal. Consequently R is maximal if and only if R/K is maximal. In the sequel we may assume that $K = 0$. So $N = Z$ and it is an R_N -module. It is enough to show that N is a linearly compact module. Let $(A_i)_{i \in I}$ be a family of ideals contained in N and $(x_i)_{i \in I}$ a family of elements of N such that the family $\mathcal{F} = (x_i + A_i)_{i \in I}$ has the finite intersection property. We put $A = \bigcap_{i \in I} A_i$. We may assume that $A \subset A_i, \forall i \in I$.

First suppose that $N \subset A^\sharp$. Assume that the total intersection of \mathcal{F} is empty. Then R/A is non-complete in its ideal topology. By Proposition 4.2 there exists $x \in \widehat{R} \setminus R$ such that $B(x) = A$. Let $b \in A^\sharp \setminus N$. There exists $a \in (A : b) \setminus A$. Since $B(x) = A$ we have $x = r + ay$ for some $r \in R$ and $y \in \widehat{R}$. By Lemma 4.3 $B(y) = (A : a)$. Since $b \in B(y)$ we have $N \subset B(y)$. By Proposition 4.2 $R/B(y)$ is non-complete in its ideal topology. This contradicts that R/N is maximal. So the total intersection of \mathcal{F} is non-empty in this case.

Now we assume that $N = A^\sharp$. Then A is an ideal of R_N . By [1, Lemma 29] either $A = Na$ for some $a \in N$ or $A = \bigcap_{a \notin A} aR_N$.

First we assume that $A = Na$. We may suppose that $A_i \subseteq aR_N, \forall i \in I$. Since \mathcal{F} has the finite intersection property, $x_i + aR_N = x_j + aR_N, \forall i, j \in I$. Let $y \in x_i + aR_N$ for each $i \in I$. Then $(x_i - y + A_i)_{i \in I}$ is a family of cosets of aR_N which has the finite intersection property. But aR_N/aN is a uniserial module over R/N . Then aR_N/aN is linearly compact since R/N is maximal. Thus $\bigcap_{i \in I} (x_i - y + A_i) \neq \emptyset$. Hence the total intersection of \mathcal{F} is non-empty.

Now suppose that $A = \bigcap_{a \notin A} aR_N$. By Proposition 3.5 and [1, Lemma 30] there exists a countable family $(a_n)_{n \in \mathbb{N}}$ of elements of $N \setminus A$ such that $A = \bigcap_{n \in \mathbb{N}} a_n R_N$ and $a_n \notin a_{n+1} R_N, \forall n \in \mathbb{N}$. By induction on n we get a subfamily $(A_{i_n})_{n \in \mathbb{N}}$ of the family $(A_i)_{i \in I}$ such that $A_{i_n} \subset a_n R_N$ in the following way: we choose $i_0 \in I$ such that $A_{i_0} \subset a_0 R_N$ and, $\forall n \in \mathbb{N}$, we pick i_{n+1} such that $A_{i_{n+1}} \subset A_{i_n} \cap a_{n+1} R_N$. Then the family $(x_{i_n} + a_n R_N)_{n \in \mathbb{N}}$ has the finite intersection property. Since R_N is maximal there exists $x \in x_{i_n} + a_n R_N, \forall n \in \mathbb{N}$. But the equality $A = \bigcap_{a \notin A} aR_N$ implies that, $\forall n \in \mathbb{N}$, there exists an integer $m > n$ such that $a_m R_N \subseteq A_{i_n}$. Since $x - x_{i_m} \in a_m R_N$ and $x_{i_m} - x_{i_n} \in A_{i_n}$ we get that $x \in x_{i_n} + A_{i_n}, \forall n \in \mathbb{N}$. Hence \mathcal{F} has a non-empty total intersection. The proof is now complete. \square

Theorem 4.5. *Then R is maximal if and only if, for each polyserial R -module M , \widehat{M} is direct sum of Mr M indecomposable pure-injective modules.*

Proof. If R is maximal, then each polyserial module M is a direct sum of Mr M pure-injective uniserial modules by [5, Proposition XII.2.4] (even if R is not a domain, this proposition holds, with the same proof).

If R is not maximal then R/N or R_N is not maximal by Theorem 4.4.

Assume that $R' = R/N$ is not maximal. Then $E = \widehat{R'}/R'$ is a nonzero torsion-free R' -module. Let $x \in \widehat{R'} \setminus R'$, \bar{x} be its image in E and U the submodule of E such that $U/R'\bar{x}$ is the torsion submodule of $E/R'\bar{x}$. Then U is a pure submodule

of E , a rank one torsion-free module and a uniserial module. Let M be the inverse image of U by the natural map $\widehat{R}' \rightarrow E$. Then M is a pure submodule of \widehat{R}' and a non-uniserial polyserial module with the two following (standard) uniserial factors: R' and U . We have $\text{Mr } M = 2$. Let W be a submodule of \widehat{R}' such that $M \cap W = 0$ and $M \rightarrow \widehat{R}'/W$ is a pure monomorphism. Thus $R' \cap W = 0$ and $R' \rightarrow \widehat{R}'/W$ is a pure monomorphism too. Since R' is pure-essential in \widehat{R}' it follows that $W = 0$. We conclude that M is pure-essential in \widehat{R}' , so that $\widehat{M} = \widehat{R}' \subset \widehat{R}' \oplus \widehat{U}$. (Let us observe that M and U are not finitely generated by Theorem 3.1.)

Suppose that $R' = R_N$ is not maximal. After replacing R' with R'/rR' , where r is a non-unit of R' , we may assume that R' is coherent and self fp-injective by [1, Theorem 11]. Then $E = \widehat{R}'/R'$ is a nonzero fp-injective R' -module. By [2, Lemma 6] E contains a pure uniserial submodule U . We define M as above. Then $\text{Mr } M = 2$ and M is an essential submodule of \widehat{R}' . So $\widehat{M} = \widehat{R}'$. \square

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