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Gianmarco Capitanio

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# ON GEODESIC ENVELOPES AND CAUSTICS

Gianmarco Capitanio

## Abstract

We give a global description of the envelope of the geodesics tangent to a smooth curve in a Riemannian surface. We prove that the envelope is the union of the curve, its inflectional geodesics and its caustic.

KEYWORDS : Envelope theory, Geodesics, Caustics.

2000 MSC : 53C22, 58K30

## 1 Introduction

The *geodesic envelope* of a curve is the envelope of its tangent geodesics. In this note we describe, from a global viewpoint, the geodesic envelope of a curve in a Riemannian surface (which is not supposed to be convex). For this, we consider the *caustic* of a curve, formed by the conjugate points to those of the curve along its tangent geodesics. Our main result is the following.

**Theorem.** *The geodesic envelope of a curve in a complete Riemannian surface is the union of the curve, its inflectional geodesics and its caustic.*

This theorem generalizes to Riemannian manifolds some of Thom's results about envelopes of 1-parameter families of lines in the projective plane (see [6]).

The framework here is the same as several classical subjects in Geometry, Optics and Calculus of Variations, going back for instance to Archimedes, Huygens, Barrow and Jacobi, as the study of evolutes of curves and caustics of ray systems. The relation between singularities of ray systems, their caustics, wave fronts, Legendre transformations and reflection groups was discovered by Arnold in the 1972 paper [1], in the setting of Symplectic and Contact Geometry, and further developed by O.V. Lyashko, A.B. Givental, O.P. Shcherbak (see [2]).

## 2 Inflectional geodesics

In this section we show that the geodesic envelope of a curve contains its inflectional geodesics. For this, let us first recall some standard definitions in Envelope Theory.

Let  $(M, g)$  be a complete Riemannian surface and let  $\gamma : I \rightarrow (M, g)$  be a smooth parameterized curve, such that  $\gamma'$  never vanishes. Denote by  $\Gamma_\xi : \mathbb{R} \rightarrow M$  the geodesic, parameterized by arclength, oriented according to  $\gamma'$  and

tangent to  $\gamma$  at  $\gamma(\xi) = \Gamma_\xi(0)$ . We say that  $\gamma$  is convex at  $\gamma(\xi)$  if at this point the tangency order of  $\gamma$  with  $\Gamma_\xi$  is 1.

The *graph* of the family of geodesics  $\{\Gamma_\xi : \xi \in I\}$  is the surface

$$\Phi := \{(\xi, \Gamma_\xi(\mathbb{R})) : \xi \in I\} \subset I \times M ;$$

the *geodesic envelope* of  $\gamma$ ,  $\mathcal{E}(\gamma)$ , is the apparent contour of the graph in  $M$ , that is, the critical value set of the graph's projection  $(\xi, P) \mapsto P$ . Notice that  $\gamma \subset \mathcal{E}(\gamma)$ .

**Remark.** The graph  $\Phi$  is locally embedded, that is, for every  $(\xi, \Gamma_\xi(u)) \in \Phi$  and every small enough  $\varepsilon > 0$ , the surface

$$\{(\eta, \Gamma_\eta(v)) : |\xi - \eta| < \varepsilon, |u - v| < \varepsilon\}$$

is embedded in  $I \times M$ .

Consider a smooth branch  $\hat{\delta} \subset \Phi$  of the critical set of the graph's projection; denote by  $\delta \subset \mathcal{E}(\gamma)$  its projection in  $M$ . Let  $(\xi, P) \in \hat{\delta}$ . The branch  $\delta$  is said to be a *geometric envelope* at  $P$  if the curves  $(\xi, \Gamma_\xi)$  intersect transversally  $\hat{\delta}$  at  $(\xi, P)$ .

**Proposition 1.** *The geodesic envelope of a curve contains its inflectional geodesics (as non-geometric branches).*

In the general theory of tangential families (see [4]), the germs having non-geometric envelope branches form an infinite codimension submanifold in the manifold of tangential family germs.

*Proof.* Let  $P = \gamma(\xi_0)$  be a simple inflection of  $\gamma$ . The claim is obvious if  $P$  has no conjugate point along  $\Gamma_{\xi_0}$ . So, let  $Q = \Gamma_{\xi_0}(T)$  be the first conjugate point to  $P$  along  $\Gamma_{\xi_0}$ . Fix two geodesic balls  $B_P$  and  $B_Q$ , centered at  $P$  and  $Q$  respectively, of radius  $r > 0$  arbitrary small. For  $\xi \neq \xi_0$ ,  $\xi \rightarrow 0$ , the geodesics  $\Gamma_\xi$  meet  $\Gamma_{\xi_0}$  in  $B_P$  and  $\partial B_P$  at  $\Gamma_\xi(t_1(\xi))$ , where  $t_1(\xi) = r + o(\xi)$ ; all these intersections belong to the closure of the same connected component of  $\partial B_P \setminus \{\Gamma_{\xi_0}(\pm r)\}$ . Next, the geodesics  $\Gamma_\xi$  intersect  $\partial B_Q$  at  $\Gamma_\xi(t_2(\xi))$ , where  $t_2(\xi) = T - r + o(\xi)$ , and  $\Gamma_{\xi_0}$  in  $B_Q$ . Hence, the geodesic segments  $\Gamma_\xi([t_1(\xi), t_2(\xi)])$  do not intersect  $\Gamma_{\xi_0}([t, T - r])$ . Thus, their lifting in the graph  $\Phi$  form a smooth surface, whose apparent contour in  $M$  is  $\Gamma_{\xi_0}([r, T - r])$  for every  $r > 0$  arbitrary small. This yields  $\Gamma_{\xi_0}([0, T]) \subset \mathcal{E}(\gamma)$ . A similar argument shows that  $\mathcal{E}(\gamma)$  contains the  $\Gamma_{\xi_0}$  geodesic segment between any two consecutive conjugate points. Therefore,  $\Gamma_{\xi_0}(\mathbb{R}) \subset \mathcal{E}(\gamma)$ . Finally, if  $\gamma$  has non-simple inflections, we consider a deformation  $\gamma_\lambda$ , having only simple inflections for  $\lambda \neq 0$ ; by continuity we get the claim.  $\square$

Figure 1 shows the perestroika of the geodesic envelope, corresponding to a generic deformation of a curve with vanishing inflections. Notice that the multiplicity of an inflectional geodesic is equal to the order of the inflection.

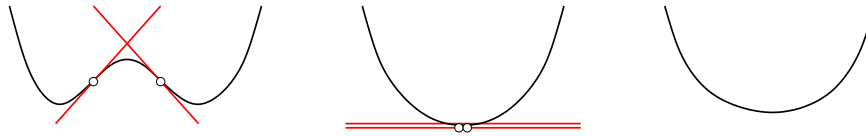


Figure 1: The geodesic envelope perestroika for vanishing inflections.

### 3 The caustic of a curve

In this section we introduce the caustic of a curve and we discuss its main properties.

Let us recall that the *caustic* of a point  $P \in M$ , denoted  $C(P)$ , is the locus of its conjugate points; the caustic of  $P$  splits into components, called  $n$ -caustics and denoted  $C_n(P)$ , formed by the  $n$ -th conjugate points to  $P$ .

Fix  $v \in T_P M$ ,  $\|v\| = 1$ . Consider “polar coordinates”  $\{u, \vartheta\} = \mathbb{R}^+ \times \mathbb{S}^1$  at  $P$ ;  $(u, \vartheta)$  represents the point  $c(u, \vartheta)$ , where  $c(\cdot, \vartheta)$  is the geodesic, parameterized by arclength, issuing from  $P$  and forming an angle  $\vartheta$  with the geodesic  $t \mapsto \exp_P(tv)$ . Actually,  $\{u, \vartheta\}$  are not a coordinate system, since in general  $\exp_P$  is not a global diffeomorphism. As shown in [5], the caustic of  $P$  has equation  $\lambda(u, \vartheta) = 0$ , where  $\lambda$  is the smooth function defined by the arclength element’s square  $dt^2 = du^2 + \lambda^2 d\vartheta^2$ .

For  $n \in \mathbb{N}$ , let us denote by  $\tau_{\pm n}(\xi)$  the arclength value for which  $\Gamma_\xi(\tau_{\pm n}(\xi))$  is the  $n$ -th conjugate points to  $\gamma(\xi)$  along  $\Gamma_\xi$  in the direction  $n\gamma'$ . We also set  $\tau_0(\xi) := \xi$  and  $\gamma(\xi)^{\pm n} := \Gamma_\xi(\tau_{\pm n}(\xi))$ .

**Definition.** The *caustic* of a curve  $\gamma$  is the curve

$$C(\gamma) := \{\gamma(\xi)^n : \xi \in I, n \in \mathbb{Z}\}$$

formed by the conjugates points to those of  $\gamma$  along its tangent geodesics.

Also the caustic of a curve splits into the  $n$ -caustics, formed by the  $n$ -th conjugate points (due to the positivity of the injectivity radius). Each  $n$ -caustic is a smooth parameterized curve, the parameter running in a subset of  $I$  (as follows from standard regularity results for solutions of parameter depending second order differential equations).

**Example.** If  $\gamma$  is a curve in a strictly convex compact surface, then  $C_n(\gamma)$  exists for every  $n \in \mathbb{N}$ ; moreover, if  $\gamma$  is closed, each  $n$ -caustic is closed. For example, if  $\gamma$  is a curve in the standard sphere  $\mathbb{S}^2$  and  $-\gamma$  is the antipodal curve, then  $C_n(\gamma) = (-1)^n \gamma$ .

Consider “fiber coordinates”  $\{u, \xi\} = \mathbb{R} \times I$  at  $\gamma$ :  $(u, \xi)$  represents the point  $\Gamma_\xi(u)$  (the “fiber coordinates” are not a coordinate system on  $M$ ; actually, they are a coordinate system on the graph of the geodesic family). Whenever we fix  $\xi \in I$ , the “polar coordinate” construction provides a smooth function  $\lambda_\xi(u, \vartheta)$ , taking  $\gamma'(\xi)/\|\gamma'(\xi)\|$  as fixed direction in  $T_{\gamma(\xi)}M$ . The equation of the  $\gamma$ -caustic in  $\mathbb{R} \times I$  is then  $\lambda_\xi(u, 0) = 0$ .

If  $\partial_u \lambda_\xi = 0$  at  $\xi = \xi_0$ ,  $u = \tau_n(\xi_0)$ ,  $\vartheta = 0$ , then  $\gamma(\xi_0)^n$  is a singular point of  $C_n(\gamma(\xi_0))$ ; the singularity is of type  $(p+1, p+2)$ ,  $p \in \mathbb{N}$ , thus generically a semicubic cusp. If  $\partial_\vartheta \lambda_\xi = 0$  at  $\xi = \xi_0$ ,  $u = \tau_n(\xi_0)$ ,  $\vartheta = 0$ , then  $C_n(\gamma(\xi_0))$  has an inflection at  $\gamma(\xi_0)^n$ . This case never arises if  $M$  is strictly convex. These facts are proven in [5].

**Definition.** A point  $\xi_0 \in I$  is said to be *n-special* if the product  $\partial_u \lambda_\xi \partial_\vartheta \lambda_\xi$  vanishes at  $\xi = \xi_0$ ,  $u = \tau_n(\xi_0)$  and  $\vartheta = 0$ .

Let us denote by  $X^n \subset I$  the union of the set of the *n-special* points of  $I$  with the set of the points  $\xi \in I$  such that  $\gamma$  has an inflection at  $\gamma(\xi)$ .

**Remark.**  $X^n$  is closed and generically discrete in  $I$ ; that is, if it is not, there exists a smooth perturbation  $\gamma_\lambda : I_\lambda \rightarrow (M_\lambda, g_\lambda)$  of  $\gamma : I \rightarrow (M, g)$ , such that the perturbed sets  $X_\lambda^n$  are discrete in  $I_\lambda$  for every  $\lambda \neq 0$ .

The caustic of a point can be viewed as the envelope of the geodesics issuing from it. We shall see now that a similar characterization holds for the caustic of a curve.

**Proposition 2.** *The geodesic envelope of a curve contains its caustics (as geometric branches).*

*Proof.* Suppose  $\xi_0 \in I \setminus X^n$ . For every  $\xi$  close enough to  $\xi_0$ , consider the map germ  $(\mathbb{R}, 0) \rightarrow (M, \gamma(\xi)^n)$  defined by  $\eta \mapsto \gamma(\xi + \eta)^n$ , whose image is the germ of an embedded curve (indeed,  $\xi \notin X^n$ ). Take  $\xi_1, \xi_2 \in I$ ,  $\xi_1 \neq \xi_2$ . Then the geodesic passing through  $\gamma(\xi_1)$  and  $\gamma(\xi_2)$  is tangent to  $C_n(\gamma(\xi_1))$  and  $C_n(\gamma(\xi_2))$ . Since  $\xi_0$  is not *n-special*, the caustic segment between  $\gamma(\xi_1)$  and  $\gamma(\xi_2)$  is an embedded curve, provided that  $\xi_1$  and  $\xi_2$  are small enough. This implies that the above  $\xi$ -pencil of germs has an embedded geometric envelope at  $\gamma(\xi_0)^n$ , tangent to  $C_n(\gamma(\xi_0))$  at this point. On the other hand,  $\Gamma_\xi$  is tangent to  $C_n(\gamma(\xi))$  at  $\gamma(\xi)^n$ , and then to  $C_n(\gamma)$ , provided that  $\xi$  is small enough. Thus,  $\gamma(\xi_0)^n \in \mathcal{E}(\gamma)$  and the envelope is geometric at this point.

If  $X^n$  is discrete in  $I$ , this yields  $C_n(\gamma) \subset \mathcal{E}(\gamma)$ , due to the closeness of the envelope. If  $X^n$  is not discrete in  $I$ , we consider a perturbation  $\gamma_\lambda : I_\lambda \rightarrow (M_\lambda, g_\lambda)$  of  $\gamma : I \rightarrow (M, g)$  such that  $C_n(\gamma_\lambda) \subset \mathcal{E}(\gamma_\lambda)$  for every  $\lambda \neq 0$ . Then, by continuity, we get the claim for  $\lambda = 0$ .  $\square$

**Remark.**  $C_n(\gamma)$  is locally embedded and convex at  $\gamma(\xi)^n$  for every  $\xi \in I \setminus X^n$ . One can also prove that if  $\gamma(\xi)$  is an inflection of  $\gamma$  and  $\xi$  is not *n-special*, then  $C_n(\gamma)$  has an inflection of the same order at  $\gamma(\xi)^n$ . Moreover, if  $C_n(\gamma)$  has a singularity at  $\gamma(\xi)$ , then  $C_n(\gamma(\xi))$  has a singularity at the same point.

## 4 Proof of the main theorem and further results

Here we prove our main theorem and discuss further properties of geodesic envelopes. We denote by  $\hat{\mathcal{E}}(\gamma)$  the geodesic envelope of  $\gamma$  in  $\mathbb{R} \times I$ , that is,  $\hat{\mathcal{E}}(\gamma)$  is the set of the pairs  $(u, \xi) \in \mathbb{R} \times I$  such that  $(\xi, \Gamma_\xi(u))$  is a critical point of

the graph's projection. Notice that the set of the points at which  $\hat{\mathcal{E}}(\gamma)$  is an embedded curve is generically dense in  $\hat{\mathcal{E}}(\gamma)$ . We complete now the proof of our main Theorem.

*Proof of the Theorem.* Let  $(u_0, \xi_0) \in \hat{\mathcal{E}}(\gamma)$ . Suppose first that  $\hat{\mathcal{E}}(\gamma)$  is embedded at this point. If the envelope is geometric at this point, then there exists an embedded curve  $\sigma$ , such that  $(u_0, \xi_0) \in \sigma \subset \hat{\mathcal{E}}(\gamma)$ , everywhere transversal to the fibers of the natural fibration  $\mathbb{R} \times I \rightarrow I$ . Hence, for every  $\xi$  close enough to  $\xi_0$ , the geodesics  $\Gamma_\xi$  and  $\Gamma_{\xi_0}$  intersect each other for an arclength value close to  $u_0$ ; i.e. there exists two smooth function germs  $t_1, t_1^0 : (I, \xi_0) \rightarrow (\mathbb{R}, u_0)$  such that  $\Gamma_{\xi_0} \circ t_1^0 \equiv \Gamma_\xi \circ t_1$ . Since  $\gamma$  is convex at  $\gamma(\xi_0)$ , these geodesics intersect each other also for an arclength value close to 0; that is, there exists two smooth function germs  $t_2, t_2^0 : (I, \xi_0) \rightarrow (\mathbb{R}, 0)$  such that  $\Gamma_{\xi_0} \circ t_2^0 \equiv \Gamma_\xi \circ t_2$ . Thus, we have that  $\Gamma_{\xi_0}(u_0)$  is conjugate to  $\gamma(\xi_0)$ . Therefore,  $\gamma(\xi_0)^n \in C(\gamma)$ . On the other hand, if the envelope is not geometric at  $(u_0, \xi_0)$ , then the argument used in the proof of Proposition 1 shows here that  $\hat{\mathcal{E}}(\gamma)$  contains the whole fiber  $(\mathbb{R}, \xi_0)$  and that  $\gamma$  can not be convex at  $\gamma(\xi_0)$ . Hence,  $\gamma(\xi_0)$  is an inflection of  $\gamma$ .

If  $\hat{\mathcal{E}}(\gamma)$  is almost everywhere embedded, then we are done due to the closeness of  $\hat{\mathcal{E}}(\gamma)$ . If it is not the case, we consider as before a perturbation and by continuity we get the claim.  $\square$

The following result is a consequence of the general classification of singularities of tangential families, carried out in [4].

**Proposition 3.** *The generic singularities of geodesic envelopes are semi-cubic cusps, transversal self-intersections and second order self-tangencies (depicted in figure 2); these singularities are stable.*

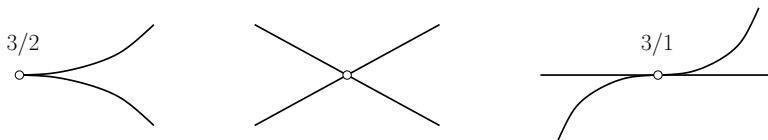


Figure 2: Generic (and stable) singularities of geodesic envelopes.

The stability is meant here with respect to small enough smooth deformations  $\gamma_\lambda : I_\lambda \rightarrow (M_\lambda, g_\lambda)$  of  $\gamma : I \rightarrow (M, g)$  among the curves and the Riemannian manifolds verifying the hypothesis listed in section 2. Notice that the semicubic cusps and the transversal self-intersections are the generic and stable singularities of envelopes of 1-parameter families of curves (see [6]). The second-order self-tangency is a new singularity, arising in the context of envelopes of tangential families. Remark that the generic (and stable) singularities of the caustic of a curve are only semicubic cusps and transversal self-intersections.

**Remark.** The evolutes of  $\gamma$  have singularities along  $\mathcal{E}(\gamma)$ . These singularities has been studied and classified near  $\gamma$  and its inflectional geodesics (see [2]). It would be interesting to study the evolutes' singularities along the caustic of  $\gamma$ .

Let us recall that the Last Geometrical Theorem of Jacobi (see e.g. [3]) states that the first caustic of a generic point in a generic strictly convex Riemannian surface has at least four cusps. We can translate this theorem in terms of geodesic envelopes as follows (see also [4]).

**Proposition 4.** *Let  $\gamma : \mathbb{S}^1 \rightarrow (M, g)$  be a smooth parameterized closed curve in a strictly convex Riemannian surface. Then the first caustics of  $\gamma$ ,  $C_{\pm 1}(\gamma)$ , have generically at least 4 cusps each one, provided that the length of  $\gamma$  is small enough.*

This statement is a consequence of the Theorem of Jacobi and of the stability of semicubic cusps as geodesic envelope singularities (Proposition 3).

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GIANMARCO CAPITANIO  
Université Paris VII – UFR de Mathématiques  
Equipe Géométrie et Dynamique  
Case 7012 – 2, place Jussieu  
75251 Paris Cedex 05 France  
e-mail : Gianmarco.Capitania@math.jussieu.fr  
[www.institut.math.jussieu.fr/~capitani](http://www.institut.math.jussieu.fr/~capitani)