PDE model with memory term applied to signal quantization. Mathematical and numerical analysis.

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Abstract

We presente a new method for signal restoration/quantization based on diffusion reaction model with memory term. We prove that the model is stable, with the existence and uniqueness results. We also propose a numerical approximation that we prove the convergence and present some experiments on noisy signals.

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1 Introduction

In recent years, partial differential equations have became a useful framework for signal restoration. Since the pioneerning works of Witkin [14] and Malik and Perona [11], signal restoration has profited of a great deal of ideas and advancements using partial differential equations tools.

The classical method to quantify a signal u_0 into a certain values $V_N = \{\lambda_1, \lambda_2, \ldots, \lambda_N\}$ is to considere a potential function $H(s)$ satisfying $H(+\infty) = H(-\infty) = +\infty$ having $\{\lambda_i\}$ as local minimas and to solve the minimum problem : "$\min W(u) = \int H(u)dx$", with $u = u_0$ as initial data. The minimum is interpreted as the stady state of the following ordinary differential equation :

$$\frac{\partial u}{\partial t} = -h(u) \qquad u(\cdot, 0) = u_0, \tag{1}$$

where $h = H'$. In the practice, the set of the values V_N depends on the initial data u_0 and the classical method to fixe V_N is given by Lloyd [10] which consists to estimate the quadratic error of u_0 with respect to V_N by minimizing the following energy :

$$E(V_N \cup W_N) = \sum_{i=1}^{N} \int_{\beta_i}^{\beta_{i+1}} (\lambda - \lambda_i)^2 df(\lambda),$$

where $f(\lambda) = P(u_0(s) < \lambda)$ is the the probability distribution assosiated to the signal u_0, and $W_N = \beta_0, \dots, \beta_{N+1}$ represents the set of separator such that $\beta_i < \lambda_i < \beta_{i+1}$. Alvarez and Esclarin [1] have improved the Lloyd's method by the introduction of two terms in the energy $E(V_N)$ in ordre to penalize quatization values for being too close. They propose the folowing :

$$E(V_N \cup W_N) = \sum_{i=1}^{N} \int_{\beta_i}^{\beta_{i+1}} (\lambda - \lambda_i)^2 df(\lambda) + C_1(\beta_{i+1} - \beta_i)^{-1} + C_2(\beta_i + \beta_{i+1} - 2\lambda_i)^2,$$

where C_1, C_2 are positif constants. The first added term mean that $|\beta_{i+1} - \beta_i|$ must be sufficiently large and the last term that β_i and β_{i+1} are symmetrical with respect to λ_i. This improvement removes some nonuniqueness problems that can appear when the Lloyd energy is used.

Since generaly signals are noisy, me must include a denoising process in the diffusion equation (1). This process must avoiding all introduction of blur effect or any other artifact in order to to preserve the local tendancy of the initial signal. In this sens one of the most popular model arising in signal and image restoration hase been proposed by Malik and Perona [11] by the following partial differential equation :

$$\frac{\partial u}{\partial t} = \operatorname{div}\left(g(|u'|^2)u'\right)' \qquad u(\cdot, 0) = u_0, \tag{2}$$

where g is a smooth non-increasing positive function with $g(0) = 1$ and $sg(s^2) \to 0$ at infinity. The idea of the equation (2) is that the restoration process obtained is conditional in the sens where : if $|u'(x)|$ is large then $g(|u'|^2) \simeq 0$ and the diffusion will be stoppped and if $|u'(x)|$ is small then $g(|u'|^2) \simeq 1$ and the diffusion will tend to smooth around x as the isotropic heat equation. This model (2) has been considered as an important improvement of the signal restoration and the edge detection theory [12].

Unfortunately, the Malik and Perona model is ill posed. Indeed, by writing the equation in the form :

$$\frac{\partial u}{\partial t} = (g(|u'|^2) + 2|u'|g'(|u'|^2))u'',$$

we observe that the diffusion is inverted in the regions where $|u'|$ is large and the process can be interpreted as a backward heat equation which is known to be ill posed. The ill posedness means that very close initial signals could produce divergent solutions [7].

and at understanding whether (2) can be given a "well-posedness" theory. We can refer to the papers of Kawohl and Kutev [8] and Kichenassamy [9] in which we find the confirmation that in general case, we have non existence of a weak solution and non uniqueness results.

The most interesting approach is to slightly modify the equation (2) by putting a regularized term in place of $|u'|^2$ in order to have a well-posed equation. There are essentially two propositions which we consider as a "direct derivation" from the Malik-Perona Model. The first, proposed by Catté, Lions, Morel and Coll, consists of special regularization

whereby $|u'|^2$ is replaced by $|\rho * u'|^2$, where ρ is a smooth kernel, for example $\rho = G_\sigma$ a Gaussian with variance σ. This term play an important role to reduce the noise by estimating of the variance σ (see [14]). In [5], the well posedness of the model is proved. The second proposition is time-delay regularization, where one replaces $|u'|^2$ by an average of its values from 0 to t. The idea of Nitzberg and Shiota [13] is to use an exponential kernel such that $|u'|^2$ is replaced by :

$$v(x,t) = e^{-t}v_0(x) + \int_0^t e^{(s-t)}|u'(x,s)|^2 \, ds,$$

where v_0 is an initial data (for example $v_0 = |u_0'|^2$). The avantage of this model is to make the inhibition term v "less sensible" with respect to the variation of the scale t, and then the diffusion process is more stable then (2). In [2] the author of the present paper have shown that this model, in any dimension, admits a unique classical solution (u, v) which can blow up in finite time, and with A. Chambolle [3], we study it from a numerical viewpoint (after a slight change). Unfortunately this model, as the Malik-Perona equation, is enable to reduce noise with large slop.

For this raisons our choice is to define v as a combination of the above two regularization types by the following :

$$v(x,t) = \ e^{-t}|\rho * u_0'|^2(x) + \int_0^t e^{(s-t)}|\rho * u'(x,s)|^2 \ ds. \tag{3}$$

that we associate to the denoising/quantization equation :

$$\frac{\partial u}{\partial t} = \big(g(v)u'\big)' - \theta(t)h(u)$$

where the term $(g(v)u')'$ is as Malik et Perona, v is given by (3), h is a quantization function and $\theta(t)$ is a function that performs a balance between denoising and quatization. The goal of the introduction of this parameter, $\theta(t)$, is to favorise the denoising process for small scales by the inhibition of the quantization function $h(u)$. The principal motivation is to avoid that the quatization effect enhance the noise before that the denoising process reduces it. For example by choosing $\theta(t) \simeq 0$ for small t and $\theta(t) \simeq 1$ if t is large. Therefore, the proposed diffusion-reaction process is described by the system :

$$\frac{\partial u}{\partial t} \ = \ \big(g(v)u'\big)' - \theta(t)h(u), \qquad u(\cdot,0) = u_0, \tag{4}$$

$$\frac{\partial v}{\partial t} \ = \ |\rho * u'|^2 - v, \qquad v(\cdot,0) = |\rho * u_0'|^2. \tag{5}$$

This paper is organized as follows : In section 2.1 we prove the consistency and the stability of the model. In section 2.2 we construct an iterative scheme that we considere as numerical approximation of (4)-(5) and we prove the convergence. Finally, in the section 3, we present a discretization of the model and in section 4 we show some experiments on noisy signals.

To close this section let us note that, in the image processing context, there exists some efficient methods [1, 6] using the diffusion reaction process that perform image quantization. The adaptation of this methods into signal processing (dimension one) is not "clear" since they use 2D-geometric arguments. Indeed, Cottet and Germain propose a diffusion tensor [6] using the projection of the gradient in an "appriori estimate" of its orthogonal direction, and Alvarez and Esclarin [1] use the mean curvature motion as denoising operator. As we know these operators make a sens only if the dimension is strictly higher then one.

2 Mathematical and numerical analysis

Let $T < \infty$ and $Q_T :=]0,1[\times]0,T[$. In this section we study and approximate numerically the system :

$$\frac{\partial u}{\partial t} \ = \ (g(v)u')' - \theta(t)h(u) \quad \text{in } Q_T, \ \ u(\cdot,0) = u_0 \ \text{ on }]0,1[, \quad u'(0) = u'(1) = 0, \tag{6}$$

$$\frac{\partial v}{\partial t} \ = \ |\rho' * u|^2 - v \quad \text{in } Q_T, \ \ v(\cdot,0) = v_0 \ \text{ on }]0,1[. \tag{7}$$

The function g is as the Malik and Perona Model and we choose : $g(s) = \big(1+s\big)^{-1}$. h is assumed to be continous and Lipschitz function satisfying $h(s) = s - M$ if $s > M$ and $h(s) = s - M$ if $s < m$, with $m = \lambda_1$ and $M = \lambda_N$ (see figure 1). $\theta(t)$ is a continous positif function such that $0 \le \theta(t) \le 1$. ρ is a positif relguar kernel, for instance a gaussian with variance $\sigma > 0$. In the term $|\rho' * u|$, $u(\cdot,t)$ is assumed be extended linearly and continously in all $\mathbb{R}$.

2.1 Consistency and stability of the model

Theorem 1. *Let $T > 0$ and $(u_0, v_0) \in \big(H^1(0,1)\big)^2$.*

(i) There exists a unique weak solution $(u,v) \in \big(H^1(Q_T) \cap L^\infty(Q_T)\big)^2$ of system (6)-(7).

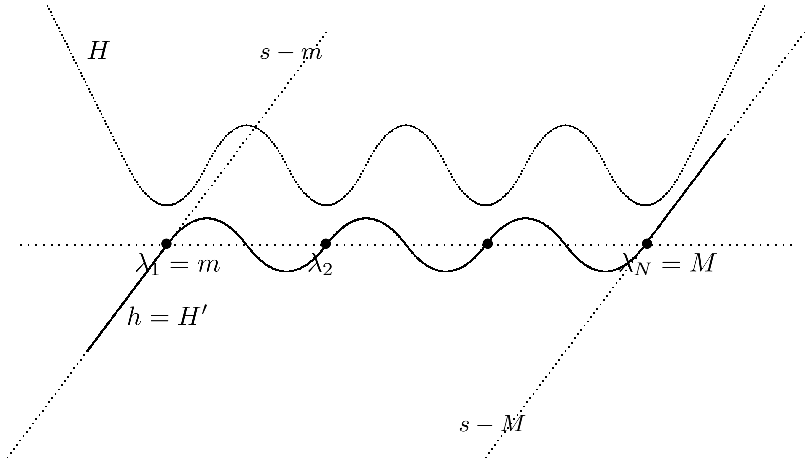

Figure 1: The potential function H and its derivative.

(ii) u satisfys the maximum principle :

$$\min\{m, \min u_0\} \leq u(x,t) \leq \max\{M, \max u_0\} \quad \textit{a.e. in} \quad Q_T. \tag{8}$$

(iii) Let $(\overline{u}, \overline{v})$ a solution of system (6)-(7) with an other initial that $(\overline{u}_0, \overline{v}_0) \in \big(H^1(0,1)\big)^2$. Then there exists a constant C which depends only on $\|v_0\|_\infty$, $\|u_0\|_\infty$, $\|\overline{u}_0\|_\infty$, ρ, g, h and T such that

$$\|u(\cdot,t) - \overline{u}(\cdot,t)\|^2_{L^2(0,1)} \leq C\big(\|u_0 - \overline{u}_0\|^2_{L^2(0,1)} + \|v_0 - \overline{v}_0\|^2_{L^\infty(0,1)}\big).$$

Proof : The existence part is given in the section (2.2) and the uniqueness is a consequence of (iii).

First we prove the maximum principle (8) by using the truncation Stampacchia method [4]. Let $\psi \in C^1(\mathbb{R})$ a bounded function such that $\psi' > 0$ on $]0,+\infty[$, $\psi \equiv 0$ on $]-\infty, 0[$, and Ψ the primitive of ψ such that $\Psi(0) = 0$. Define $K = \max\{\max u_0, M\}$ and $\phi(t) = \int_0^1 \Psi(u(x,t) - K)\, dx$. We easily prove that ϕ is positive function, belonging in $C([0,+\infty[;\mathbb{R}^+) \cap C^1((0,+\infty[;\mathbb{R}^+)$, $\phi(0) = 0$ and

$$\begin{aligned}
\phi'(t) &= \int_0^1 \psi(u(x,t) - K)\frac{\partial u}{\partial t}\, dx \\
&= -\int_0^1 \psi'(u(x,t) - K) g(v) |u'(x,t)|^2\, dx - \theta(t)\int_0^1 \psi(u(x,t) - K) h(u(x,t))\, dx \\
&\leq \theta(t)\int_0^1 \psi(u(x,t) - K)(u(x,t) - K - h(u(x,t)))\, dx - \theta(t)\int_0^1 \psi(u(x,t) - K)(u(x,t) - K)\, dx.
\end{aligned}$$

Since we have $s - K - h(s) \leq s - M - h(s) \leq 0$ and $s\psi(s) \geq 0$ we deduce that $\phi' \leq 0$ in $]0,1[$, consequently $\phi \equiv 0$. Thus for all $t > 0$ we have $u(\cdot,t) \leq K$ a.e. in $]0,1[$.

As consequence of (8), v is bounded in $L^\infty(Q_T)$ and we have $\|v\|_{L^\infty(Q_T)} \leq \max(\|v_0\|_{L^\infty(0,1)}, C\|u\|^2_{L^\infty(Q_T)})$. (To simplify the notations, we denote by C all constant depending only on $\|v_0\|_\infty$, $\|u_0\|_\infty$, $\|\overline{u}_0\|_\infty$, ρ, g, h and T.) We also deduce that u is bounded in $L^2(0,T;H^1(0,1))$. Indeed, multiplying (6) by u and integrating on $]0,1[$, we get

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2(0,1)} = -\int_0^1 g(v)|u'|^2\, dx - \theta(t)\int_0^1 h(u)u\, dx.$$

Then by integration on $[0,T]$ and using the fact that $0 < g(\|v\|_{L^\infty(Q_T)}) \leq g(v)$ and $\theta(t)|h(u)u| \leq C\|u\|_{L^\infty(Q_T)}$, we obtain

$$\|u'\|^2_{L^2(Q_T)} \leq C(\|u_0\|^2_{L^2(0,1)} - \|u(\cdot,T)\|^2_{L^2(0,1)} + C\|u\|_{L^\infty(Q_T)}) \leq C\|u\|_{L^\infty(Q_T)}.$$

Now we prove the stability claim (iii). Denote $E = u - \overline{u}$ and $W = v - \overline{v}$. Then we easily write

$$\frac{\partial E}{\partial t} - \big(g(v)E'\big)' = \big((g(\overline{v}) - g(v))u'\big)' - \theta(t)\big(h(u) - h(\overline{u})\big). \tag{9}$$

Multiplying (9) by E and integrating on $]0,1[$,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|E\|^2_{L^2(0,1)} + C\|E'\|^2_{L^2(0,1)} &\leq \|W\|_{L^\infty(0,1)}\|u'\|_{L^2(0,1)}\|E'\|_{L^2(0,1)} - \theta(t)\int_0^1 (h(u) - h(\overline{u}))E\,dx, \\
&\leq \frac{1}{2}\big(\frac{1}{C}\|W\|^2_{L^\infty(0,1)}\|u'\|^2_{L^2(0,1)} + C\|E'\|^2_{L^2(0,1)}\big) + C\|E\|^2_{L^2(0,1)}.
\end{aligned}$$

Then we obtain the inequality :

$$\frac{d}{dt}\|E\|^2_{L^2(0,1)} \leq \frac{1}{C}\|W\|^2_{L^\infty(0,1)}\|u'\|^2_{L^2(0,1)} + C\|E\|^2_{L^2(0,1)}. \tag{10}$$

In other hand we have

$$\begin{aligned}
|W|(\cdot,t) &\leq \|W_0\|_{L^\infty(0,1)} + \Big|\int_0^t e^{t-s}\big(\rho' * u(\cdot,s) + \rho' * \overline{u}(\cdot,s)\big)\big(\rho' * u(\cdot,s) - \rho' * \overline{u}(\cdot,s)\big)\,ds\Big| \\
&\leq \|W_0\|_{L^\infty(0,1)} + C\|u + \overline{u}\|_{L^\infty(0,T;L^2(0,1))}\int_0^t \|E(\cdot,s)\|_{L^2(0,1)}\,ds.
\end{aligned}$$

Using the last inequality and the fact that $u, \overline{u}, u'$ are in $L^2(Q_T)$ and after integrating (10) over $[0,1]$, we can write it in the form :

$$\|E(\cdot,t)\|^2_{L^2(0,1)} \leq \|E_0\|^2_{L^2(0,1)} + C\Big(\|W_0\|_{L^\infty(0,1)} + C\int_0^t \|E(\cdot,s)\|_{L^2(0,1)}\,ds\Big)^2 + C\int_0^t \|E(\cdot,s)\|^2_{L^2(0,1)}\,ds.$$

Then by the inequalities $(x+y)^2 \leq 3/2(x^2+y^2)$ and $(\int |f|dx)^2 \leq C\int |f|^2dx$ we obtain :

$$\|E(\cdot,t)\|^2_{L^2(0,1)} \leq \|E_0\|^2_{L^2(0,1)} + C\|W_0\|^2_{L^\infty(0,1)} + C\int_0^t \|E(\cdot,s)\|^2_{L^2(0,1)}\,ds.$$

This implies, using gronwall's Lemma, that

$$\|E(\cdot,t)\|^2_{L^2(0,1)} \leq C\big(\|E_0\|^2_{L^2(0,1)} + \|W_0\|^2_{L^\infty(0,1)}\big)$$

which is the desired estimate.□

2.2 Numerical approximation

For all fixed $\delta > 0$, we define the sequence $(u_{(\delta,n)}, v_{(\delta,n)})_n$ by the following interative scheme :

$$(u_{(\delta,0)}, v_{(\delta,0)}) = (u_0, v_0) \in \big(H^1(0,1)\big)^2, \quad v_0 \geq 0, \tag{11}$$

$$\frac{u_{(\delta,n+1)} - u_{(\delta,n)}}{\delta} = (g(v_{(\delta,n)})u'_{(\delta,n+1)})' - \theta(n\delta)h(u_{(\delta,n)}), \qquad u'_{(\delta,n+1)}(0) = u'_{(\delta,n+1)}(1) = 0, \tag{12}$$

$$\frac{v_{(\delta,n+1)} - v_{(\delta,n)}}{\delta} = |\rho' * u_{(\delta,n+1)}|^2 - v_{(\delta,n+1)}. \tag{13}$$

First remark that if we have $0 \leq v_{(\delta,n)} \in L^\infty(0,1)$ then the equation (12) is strictly elliptic and we know that there exists a unique solution $u_{(\delta,n+1)}$ in $H^1(0,1)$ given by the following minimum problem :

$$u_{(\delta,n+1)} := \text{Argmin}_{w\in H^1(0,1)}\left\{\int_0^1 g(v_{(\delta,n)})|w'|^2\,dx + \frac{1}{2\delta}\int_0^1 |w - u_{(\delta,n)}|^2\,dx + \theta(n\delta)\int_0^1 H(w)\,dx\right\}. \tag{14}$$

Therefore $u_{(\delta,n+1)}$ satisfys the maximum principle given by the following lemma :

Lemma 1. *Assume that $0 \leq v_{(\delta,n)} \in L^\infty(0,1)$ and $\delta \sup |h'| < 1$, then we have :*

$$\min\{m, \min u_{(\delta,n)}\} \leq u_{(\delta,n+1)} \leq \max\{M, \max u_{(\delta,n)}\}. \tag{15}$$

Remarks 1. *Since $0 \leq v_0 \in L^\infty(0,1)$ and using Lemma 1, its clear that $u_1 \in L^\infty(0,1)$. In other hand equation (13) allows to write v_1 explicitly as linear combinaison of $|\rho' * u_1|^2$ and v_0, then we easily obtain that $0 \leq v_1 \leq \max(C_\rho \|u_1\|^2_{L^\infty(0,1)}, \|v_0\|_{L^\infty(0,1)})$. Therefore, by induction we deduce that $0 \leq v_{(\delta,n)} \in L^\infty(0,1)$ for all n, and consequently Lemma 1 holds for all n. It follows that :*

$$\min\{m, \min u_0\} \leq u_{(\delta,n)} \leq \max\{M, \max u_0\}. \tag{16}$$

Proof of lemma 1. Let ψ as in the proof of Theorem 1 and define $\beta_n = \max\{\sup u_{(\delta,n)}, M\}$. Multiplying by $\psi(u_{(\delta,n+1)} - \beta_n)$ the equation (12) and integrating on $]0,1[$ we get

$$\begin{aligned}\frac{1}{\delta}\int_0^1 (u_{(\delta,n+1)} - u_{(\delta,n)})\psi(u_{(\delta,n+1)} - \beta_n)\,dx &= -\int_0^1 g(v_{(\delta,n)})|u'_{(\delta,n+1)}|^2\psi'(u_{(\delta,n+1)} - \beta_n)\,dx \\ &- \theta(n\delta)\int_0^1 h(u_{(\delta,n)})\psi(u_{(\delta,n+1)} - \beta_n)\,dx.\end{aligned}$$

After simple calculus we obtain :

$$\begin{aligned}\int_0^1 (u_{(\delta,n+1)} - \beta_n)\psi(u_{(\delta,n+1)} - \beta_n)\,dx &= -\delta\int_0^1 g(v_{(\delta,n)})|u'_{(\delta,n+1)}|^2\psi'(u_{(\delta,n+1)} - \beta_n)\,dx \\ &+ \int_0^1 (u_{(\delta,n)} - \delta\theta(n\delta)h(u_{(\delta,n)}) - \beta_n)\psi(u_{(\delta,n+1)} - \beta_n)\,dx\end{aligned}$$

Remark that the inequality $\delta \sup |h'| < 1$ implies that $s \to s - \delta\theta(n\delta)h(s)$ is a non decreasing function, and since we have $h(s) > 0$ if $s > M$, we obtain $u_{(\delta,n)} - \delta\theta(n\delta)h(u_{(\delta,n)}) \leq \beta_n - \delta\theta(n\delta)h(\beta_n) \leq \beta_n$. Thus

$$\int_0^1 (u_{(\delta,n+1)} - \beta_n)\psi(u_{(\delta,n+1)} - \beta_n)\,dx \leq 0.$$

Using the fact that $s\psi(s)$ is a positive function we obtain that $(u_{(\delta,n+1)} - \beta_n)\psi(u_{(\delta,n+1)} - \beta_n) = 0$ a.e. in $]0,1[$. Then $u_{(\delta,n+1)} \leq \beta_n$ a.e. in $]0,1[$.

With the same arguments, we prove the "inf" part by using $\psi(\alpha_n - u_{(\delta,n+1)})$ where $\alpha_n = \min\{\inf u_{(\delta,n)}, m\}$ and since $u_{(\delta,n+1)}$ is continuous in $[0,1]$ ($H^1(0,1) \subset C([0,1])$), we can replace "sup" by "max" and "inf" by "min". $\square$

We define the piecewise constant (in time) interpolations ($[\cdot]$ denotes the integer part) $u_\delta(x,t) = u_{(\delta,[t/\delta]+1)}(x)$, $v_\delta(x,t) = v_{(\delta,[t/\delta]+1)}(x)$ and $\theta_\delta(t) = \theta(([t/\delta]+1)\delta)$ and the piecewise affine interpolations $\widehat{u}_\delta(x,t) = (1-\eta)u_{(\delta,[t/\delta])}(x) + \eta u_{(\delta,[t/\delta]+1)}(x)$ and $\widehat{v}_\delta(x,t) = (1-\eta)v_{(\delta,[t/\delta])}(x) + \eta v_{(\delta,[t/\delta]+1)}(x)$, where ($\eta = t/\delta - [t/\delta] \in [0,1)$). Then the discrete system (12)-(13) can be written in the form ($\tau^{-\delta}$ is defined by $\tau^{-\delta}f(\cdot,t) = f(\cdot, t-\delta)$.)

$$\frac{\partial \widehat{u}_\delta}{\partial t} = (g(\tau^{-\delta}v_\delta)u'_\delta)' - (\tau^{-\delta}\theta_\delta(t))h(\tau^{-\delta}u_\delta), \qquad u'_\delta(0,\cdot) = u'_\delta(1,\cdot) = 0, \tag{17}$$

$$\frac{\partial \widehat{v}_\delta}{\partial t} = |\rho' * u_\delta|^2 - v_\delta. \tag{18}$$

The main result of this section is the following theorem:

Theorem 2. *Let $T > 0$. There exists (u,v) a weak solution of the system (6)-(7) in $\big(H^1(Q_T) \cap L^\infty(Q_T)\big)^2$ such that, we have the convergences, as $\delta \to 0$:*

$$\widehat{u}_\delta, u_\delta \longrightarrow u \qquad \textit{strongly in } L^2(0,T;H^1(0,1)), \tag{19}$$

$$\widehat{v}_\delta, v_\delta \longrightarrow u \qquad \textit{strongly in } L^2(0,T;H^1(0,1)), \tag{20}$$

$$(\widehat{u}_\delta, \widehat{v}_\delta) \rightharpoonup (u,v) \qquad \textit{weakly in } H^1(Q_T). \tag{21}$$

The proof of this theorem will be given in section 2.3.

2.3 A priori estimates and convergence

Lemma 2. *The sequence (u'_δ) is bounded in $L^2(Q_T)$ and we have :*

$$\int_{Q_T} |u'_\delta|^2\,dx\,dt \le \big(1+\|v_0\|_{L^\infty(0,1)}\big)\big(\|u_0\|^2_{L^2(0,1)} + 2T\|h\|_{L^\infty(I)}\|u_\delta\|_{L^\infty(Q_T)}\big).$$

where $I=[\min\{m,\min u_0\},\max\{M,\max u_0\}]$.

Proof : Multiplying equation (12) by $u_{(\delta,n+1)}$ and integrating on $]0,1[$, we get

$$\begin{aligned}\int_0^1 |u_{(\delta,n+1)}|^2 &= -\delta\int_0^1 g(v_{(\delta,n)})|u'_{(\delta,n+1)}|^2dx + \int_0^1 (u_{(\delta,n)} - \delta\theta(n\delta)h(u_{(\delta,n)}))u_{(\delta,n+1)}dx, \\ &\le \int_0^1 (u_{(\delta,n)} - \delta\theta(n\delta)h(u_{(\delta,n)}))u_{(\delta,n+1)}dx.\end{aligned} \tag{22}$$

Then using Hölder inequality its clear that we obtain

$$\|u_{(\delta,n+1)}\|_{L^2} \le \|u_{(\delta,n)} - \delta\theta(n\delta)h(u_{(\delta,n)})\|_{L^2}, \tag{23}$$

and

$$\begin{aligned}\int_0^1 u_{(\delta,n)}u_{(\delta,n+1)}dx &\le \|u_{(\delta,n)}\|_{L^2}\|u_{(\delta,n)} - \delta\theta(n\delta)h(u_{(\delta,n)})\|_{L^2} \\ &\le \|u_{(\delta,n)}\|^2_{L^2} + \delta\|u_{(\delta,n)}\|_{L^2}\|h(u_{(\delta,n)})\|_{L^2}.\end{aligned}$$

Therefore we can write, using (22)

$$\delta\int_0^1 g(v_{(\delta,n)})|u'_{(\delta,n+1)}|^2dx \le \|u_{(\delta,n)}\|^2_{L^2} - \|u_{(\delta,n+1)}\|^2_{L^2} + \delta\|h(u_{(\delta,n)})\|_{L^2}\big(\|u_{(\delta,n)}\|_{L^2} + \|u_{(\delta,n+1)}\|_{L^2}\big),$$

and we obtain for all $t\in(0,T)$

$$\begin{aligned}\int_0^t\!\!\int_0^1 |u'_\delta|^2\,dx\,ds &= \int_0^{\delta[t/\delta]}\int_0^1 |u'_\delta|^2\,dx\,ds + \int_{\delta[t/\delta]}^t\int_0^1 |u'_\delta|^2\,dx\,ds \\ &\le (1+\|v_\delta\|_{L^\infty})\Big(\int_0^{\delta[t/\delta]}\int_0^1 \frac{1}{1+v_\delta}|u'_\delta|^2\,dx\,ds + \int_{\delta[t/\delta]}^t\int_0^1 \frac{1}{1+v_\delta}|u'_\delta|^2\,dx\,ds\Big) \\ &\le (1+\|v_\delta\|_{L^\infty})\Big((\|u_0\|^2_{L^2} - \|u_{(\delta,[t/\delta])}\|^2_{L^2}) + \frac{t-\delta[t/\delta]}{\delta}(\|u_{(\delta,[t/\delta])}\|^2_{L^2} - \|u_{(\delta,[t/\delta]+1)}\|_{L^2} \\ &\quad + \int_0^t \|h(\tau^{-\delta}u_\delta)\|_{L^2}\big(\|\tau^{-\delta}u_\delta\|_{L^2} + \|u_\delta\|_{L^2}\big)ds\Big).\end{aligned}$$

Finally, since $\delta[t/\delta]\le t\le\delta([t/\delta]+1)$, we obtain for all $t\in(0,T)$

$$\int_0^t\!\!\int_0^1 |u'_\delta|^2\,dx\,dt \le (1+\|v_\delta\|_{L^\infty})\Big(\|u_0\|^2_{L^2(0,1)} + \int_0^t \|h(\tau^{-\delta}u_\delta)\|_{L^2}\big(\|\tau^{-\delta}u_\delta\|_{L^2} + \|u_\delta\|_{L^2}\big)ds\Big).$$

This prove the Lemma.□

For simplicity, we introduce the notations :

$$A_{(k,l)} := \int_0^1 \frac{1}{1+v_{(\delta,k)}}|u'_{(\delta,l)}|^2\,dx,\quad B_{(k,l)} := \int_0^1 |u_{(\delta,k)} - u_{(\delta,l)}|^2\,dx \quad\text{and}\quad C_n = \theta(n\delta)\int_0^1 H(u_{(\delta,n)})\,dx, \tag{24}$$

and remark that the minimum problem (14) allows to obtain :

$$\frac{1}{2\delta}B_{(n,n+1)} \le (A_{(n,n)} - A_{(n,n+1)}) + (C_n - C_{n+1}). \tag{25}$$

Lemma 3. *The following assertions holds :*

(i) $\big(\partial\widehat{v}_\delta/\partial t\big)$ *is uniformly bounded in* $L^\infty(Q_T)$*, and we have :*

$$\Big\|\frac{\partial\widehat{v}_\delta}{\partial t}\Big\|_{L^\infty(Q_T)} \le \max(C_\rho\|u_\delta\|^2_{L^\infty(Q_T)}, \|v_\delta\|_{L^\infty(Q_T)}). \tag{26}$$

(ii) $(\widehat{u}_\delta)$ *is uniformly bounded in* $H^1\big(Q_T\big)$ *and satisfies the inequality*

$$\Big\|\frac{\partial\widehat{u}_\delta}{\partial t}\Big\|^2_{L^2(Q_T)} \le 2\Big(\|u_0'\|^2_{L^2(0,1)} + \Big\|\frac{\partial\widehat{v}_\delta}{\partial t}\Big\|_{L^\infty(Q_T)}\|u_\delta'\|^2_{L^2(Q_T)} + \theta(0)\int_0^1 H(u_0)\,dx\Big). \tag{27}$$

(iii) (v_δ) *is uniformly bounded in* $L^\infty(0,T;H^1(0,1))$ *and we have*

$$\|v_\delta'\|_{L^\infty((0,T);L^2(0,1))} \le CT\|u_\delta\|_{L^\infty(Q_T)}\|u_\delta\|_{L^\infty((0,T);L^2(0,1))} + \|v_0'\|_{L^2(0,1)}, \tag{28}$$

with $C = \|\rho'\|_{L^\infty(0,1)}\|\rho''\|_{L^\infty(0,1)}$.

Proof : Since $v_\delta \ge 0$ its clear that $-v_\delta \le \partial\widehat{v}_\delta/\partial t \le |\rho' * u_\delta|^2$ in Q_T from which we easily deduce (26). To prove (ii) we use the notations (24) and inequality (25), then we can write (we denote $\delta_T = T/\delta - [T/\delta]$)

$$\begin{aligned}
\iint_{Q_T}\Big(\frac{\partial\widehat{u}_\delta}{\partial t}\Big)^2 dt\,dx &= \int_0^{\delta[T/\delta]}\int_0^1\Big(\frac{\partial\widehat{u}_\delta}{\partial t}\Big)^2 dt\,dx + \int_{\delta[T/\delta]}^T\int_0^1\Big(\frac{\partial\widehat{u}_\delta}{\partial t}\Big)^2 dt\,dx \\
&= \sum_{n=0}^{[T/\delta]-1}\frac{1}{\delta}B_{(n,n+1)} + \delta_T\ \frac{1}{\delta}B_{([T/\delta],[T/\delta]+1)} \\
&\le 2\Big(\sum_{n=0}^{[T/\delta]-1}(A_{(n,n)} - A_{(n,n+1)}) + \delta_T\big(A_{([T/\delta],[T/\delta])} - A_{([T/\delta],[T/\delta]+1)}\big)\Big) \qquad (29) \\
&\qquad + \ 2\Big(\sum_{n=0}^{[T/\delta]-1}(C_n - C_{n+1}) + \delta_T\big(C_{[T/\delta]} - C_{[T/\delta]+1}\big)\Big). \qquad (30)
\end{aligned}$$

To estimate (29), we use the fact that $A_{(\cdot,\cdot)} \ge 0$ and $0 \le \delta_T \le 1$, and we write

$$\begin{aligned}
&\sum_{n=0}^{[T/\delta]-1}(A_{(n,n)} - A_{(n,n+1)}) + \delta_T\big((A_{([T/\delta],[T/\delta])} - A_{([T/\delta],[T/\delta]+1)})\big) \\
= \ & A_{(0,0)} + \sum_{n=1}^{[T/\delta]-1}(A_{(n,n)} - A_{(n-1,n)}) - A_{([T/\delta]-1,[T/\delta])} + \delta_T\big(A_{([T/\delta],[T/\delta])} - A_{([T/\delta],[T/\delta]+1)}\big) \\
\le \ & A_{(0,0)} + \sum_{n=1}^{[T/\delta]-1}(A_{(n,n)} - A_{(n-1,n)}) + (A_{([T/\delta],[T/\delta])} - A_{([T/\delta]-1,[T/\delta])}) \\
\le \ & \|u_0'\|^2_{L^2(0,1)} + \Big\|\frac{\partial\widehat{v}_\delta}{\partial t}\Big\|_{L^\infty(Q_T)}\|u_\delta'\|^2_{L^2(Q_T)}
\end{aligned}$$

and to estimate (30), we use the fact that $C_n \ge 0$ and we write

$$\begin{aligned}
\sum_{n=0}^{[T/\delta]-1}(C_n - C_{n+1}) + \big(T/\delta - [T/\delta]\big)\big(C_{[T/\delta]} - C_{[T/\delta]+1}\big) &= C_0 - C_{[T/\delta]} + \delta_T\big(C_{[T/\delta]} - C_{[T/\delta]+1}\big) \\
&\le C_0.
\end{aligned}$$

Then we obtain (27) by replacing the two last inequalities in (29) and (30).

To prove (iii), we derive equation (13) and obtain

$$v'_{(\delta,n+1)} = \frac{\delta}{1+\delta} 2(\rho' * u_{(\delta,n+1)})(\rho" * u_{(\delta,n+1)}) + \frac{1}{1+\delta} v'_{(\delta,n)}.$$

Then by using the L^2 norm, we get

$$\|v'_{(\delta,n+1)}\|_{L^2(0,1)} \leq \frac{\delta}{1+\delta} C\|u_\delta\|_{L^\infty(Q_T)} \|u_{(\delta,n+1)}\|_{L^2(0,1)} + \frac{1}{1+\delta} \|v'_{(\delta,n)}\|_{L^2(0,1)}.$$

with $C = \|\rho'\|_{L^\infty(0,1)} \|\rho"\|_{L^\infty(0,1)}$.Then by induction we get for all n

$$\begin{aligned}
\|v'_{(\delta,n)}\|_{L^2(0,1)} &\leq \frac{C\|u_\delta\|_{L^\infty(Q_T)}}{1+\delta} \sum_{j=1}^{n} \left(\frac{1}{1+\delta}\right)^{n-j} \delta \|u_{\delta,j}\|_{L^2(0,1)} + \left(\frac{1}{1+\delta}\right)^{n} \|v'_0\|_{L^2(0,1)} \\
&\leq C\|u_\delta\|_{L^\infty(Q_T)} \sum_{j=1}^{n} \delta \|u_{(\delta,j)}\|_{L^2(0,1)} + \|v'_0\|_{L^2(0,1)}
\end{aligned}$$

from which we easily deduce (28).□

Proof of theorem 2. According to the lemma 1, 2 and 3 we know that there exists two subsequences still denoted by (u_δ), $(\widehat{u}_\delta)$ (resp. (v_δ), $(\widehat{v}_\delta)$), and a function u (resp. v) $\in H^1(Q_T) \cap L^\infty(Q_T)$ such that $\widehat{u}_\delta, u_\delta \to u$ (resp. $\widehat{v}_\delta, v_\delta \to v$) strongly in $L^2(Q_T)$, weakly in $L^2(0,T;H^1(0,1))$ and $\widehat{u}_\delta \to u$ (resp. $\widehat{v}_\delta \to v$) weakly in $H^1(Q_T)$. We can also assume that up a subsequence we have that (u_δ) and $(\tau^{-\delta} u_\delta) \to u$ a.e. in Q_T (resp. (v_δ) and $(\tau^{-\delta} v_\delta) \to v$ a.e. in Q_T).

Now Multipying the equation (17) by $\phi \in C^\infty(\overline{Q_T})$ and integrating the result in Q_T, we get

$$\int_0^T\int_0^1 \frac{\partial \widehat{u}_\delta}{\partial t} \phi \, dx\, dt = -\int_0^T\int_0^1 \frac{u'_\delta \phi'}{1+\tau^{-\delta} v_\delta} dx\, dt + \int_0^T\int_0^1 (\tau^{-\delta}\theta_\delta) h(\tau^{-\delta} u_\delta) \phi \, dx\, dt.$$

Using the weak convergence $\partial \widehat{u}_\delta/\partial t \rightharpoonup \partial u/\partial t$ in $L^2(Q_T)$, we obtain

$$\int_0^T\int_0^1 \frac{\partial \widehat{u}_\delta}{\partial t} \phi \, dx\, dt \quad \to \quad \int_0^T\int_0^1 \frac{\partial u}{\partial t} \phi \, dx\, dt \quad \text{as} \quad \delta \to 0.$$

Combining the weak convergence $u'_\delta \to u'$ in $L^2(Q_T)$ with the strong convergence $(1 + \tau^{-h} v_\delta)^{-1} \phi' \to (1+v)^{-1}\phi'$ in $L^2(Q_T)$ (this convergence is obtained by using the Lebesgue Theorem), we deduce that

$$\int_0^T\int_0^1 \frac{u'_\delta \phi'}{1+\tau^{-\delta} v_\delta} dx\, dt \quad \to \quad \int_0^T\int_0^1 \frac{u'\phi'}{1+v} dx\, dt \quad \text{as} \quad \delta \to 0.$$

Since we have $(\tau^{-\delta}\theta_\delta(t)) h(\tau^{-\delta} u) \to \theta(t) h(u)$ a.e. in Q_T and $|(\tau^{-\delta}\theta_\delta(t)) h(\tau^{-\delta} u)\phi| \leq \|h\|_\infty |\phi| \in L^2(Q_T)$, its also clear that

$$\int_0^T\int_0^1 (\tau^{-\delta}\theta_\delta(t)) h(\tau^{-\delta} u) \phi \, dt dx \quad \to \quad \int_0^T\int_0^1 \theta(t) h(u) \phi \, dt dx \quad \text{as} \quad \delta \to 0.$$

Then we obtain

$$\int_0^T\int_0^1 \frac{\partial u}{\partial t} \phi \, dx\, dt = -\int_0^T\int_0^1 \frac{u'\phi'}{1+v} dx\, dt + \int_0^T\int_0^1 \theta(t) h(u) \phi \, dt dx. \tag{31}$$

which mains that u is a weak solution of (6) and since we know that $\rho' * u_\delta$ uniformly converges to $\rho' * u$ in Q_T, we also deduce that v is a solution of (7).

Finally, by uniqueness of the solution (u, v) of the system (6)-(7), we know that the whole sequence (u_δ, v_δ) weakly converges in $L^2(0,T;H^1(0,1))^2$ to (u, v). □

3 Discretization

First we describe how we discretize the coupled system (12)-(13). We denote by u_i^n (resp. v_i^n) the approximation of u (resp. v) at point (ih) $(0 \leq i \leq N)$ and time $t = n\,\delta$, where the size of the initial signal u_0 is aqual to N and $h = 1/N$. Using the classical finite-differences, we write the approximation of $(g(v)u')'$ at point ih and at scale $t = (n+1)\,\delta$ by:

$$\frac{1}{h^2}\Big(\big(g(v_i^n)(u_{i+1}^{n+1} - u_i^{n+1})\big) - \big(g(v_{i-1}^n)(u_i^{n+1} - u_{i-1}^{n+1})\big)\Big).$$

Then the equation (12) becomes:

$$\frac{u_i^{n+1} - u_i^n}{\delta} = \frac{1}{h^2}\Big(\big(g(v_i^n)(u_{i+1}^{n+1} - u_i^{n+1})\big) - \big(g(v_{i-1}^n)(u_i^{n+1} - u_{i-1}^{n+1})\big)\Big) - \theta(n\delta)\,h(u_i^n), \tag{32}$$

with the Neumann boundary condition $u_0^{n+1} = u_1^{n+1}$ and $u_{N-1}^{n+1} = u_N^{n+1}$. Rearranging the right hand side of (32), we get

$$h^2\Big(\frac{u^{n+1} - u^n}{\delta} + \theta(n\delta)\,h(u_i^n)\Big) + A(v^n)u^{n+1} = 0, \tag{33}$$

where the matrix $A(v^n)$ is tridiagonal and positive defined. By classical arguments we know that $[I + \delta h^{-2}A(v^n)]$ is invertible. The discretization of (13) is written by :

$$v_i^{n+1} = \frac{1}{1+\delta}\big(\delta\,h^{-2}|(\rho' * u^n)_i|^2 + v_i^n\big).$$

4 Experiments

Figure 2 shows a noisy signal and its quantized version by different models. The original signal 2-(a) is piecewise constant taking the values $m = \lambda_1 = 50, \lambda_2 = 100, \lambda_3 = 150, M = \lambda_4 = 200$ and the signal 2-(b) is obtained by adding a gaussian noise with variance 20% to the signal 2-(a). We then construct the function h by using the values $\{\lambda_i\}$ as indicated in the begining of the section 2 (see also figure 1), by :

$$h(x) = \begin{cases} x - \lambda_1 \text{ if } x \leq \lambda_1, \\ \alpha\Big((x-\lambda_i)(x-\lambda_{i+1})^2 + (x-\lambda_i)^2(x-\lambda_{i+1})\Big) \text{ if } x \in [\lambda_i, \lambda_{i+1}], \\ x - \lambda_4 \text{ if } x \geq \lambda_4, \end{cases}$$

where $\alpha > 0$ is choosen such that $\delta \sup |h'| < 1$, here we use $\delta = 0.1$. The signal 2-(c) is obtained without denoising process by using only the evolution equation $du/dt = -h(u)$. As we remark, in this case, the quantization enhance the noise since the attraction of the value of the signal at a point x depends only on $\min_i |\lambda_i - u_0(x)|$ and not on the local analysis of the signal. The signal 2-(d) is obtained with Laplacian as denoising process $du/dt = \Delta u - h(u)$. In this case we observe that the regularization effect of the heat equation combined with the quantization function creat a sort of staircase in the slopes going from λ_i into λ_{i+k} with $|k| \geq 2$, for example around $x = 100$. In the signal (e), we clearly remark that our model (used with $\theta(t) = 1$ for all t) performs the required quantization. All the experiments have been done at scale $\sigma = 100$ which correponds to the evolution time $t = 5000$ ($t = 0.5\sigma^2$).

In figure 3 we show the evolution steps with our model of the piecewise signal 2-(a) where 50% of gaussian noise is added. In this experiments we have used $\theta(t) = (t/T)^2$. This choice clearly favorise the denoising process for small scales as we remark in the figures 3-(b) at $t = 10$, 3-(c) at $t = 25$ and 3-(d) at $t = 50$, and yields a quantized signal 3-(f) at $t = T$.

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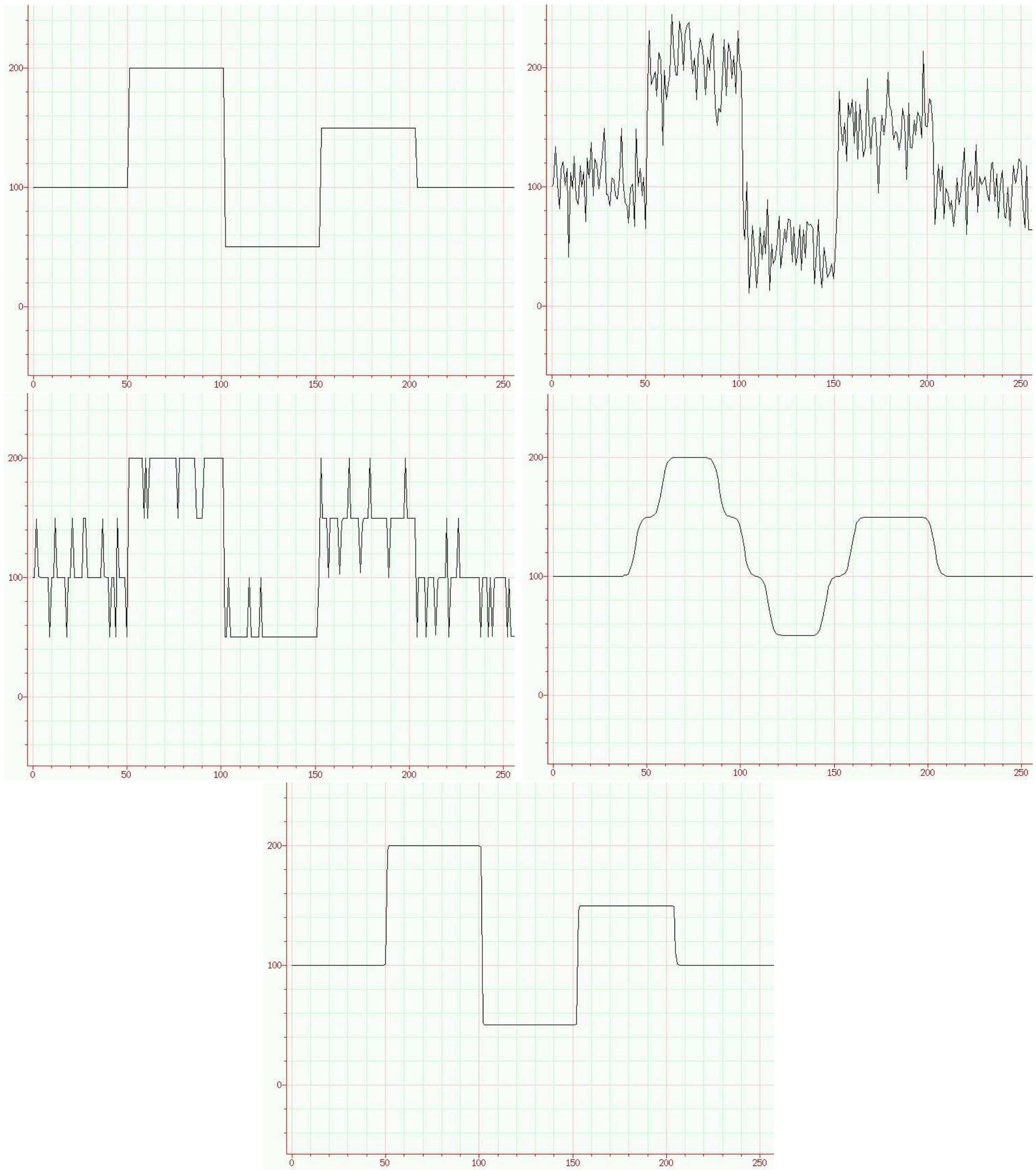

Figure 2: Top Left: (a) piecewise constant signal. Top Right: (b) Signal (a) with 20% of gaussian noise. Middle Left: (c) The quantized version of (b) without denoising operator. Middle Right: (d) Signal (b) quantized by equation $du/dt = \Delta u - h(u)$. Bottom : (e) Signal (b) restored and quantized by our model.

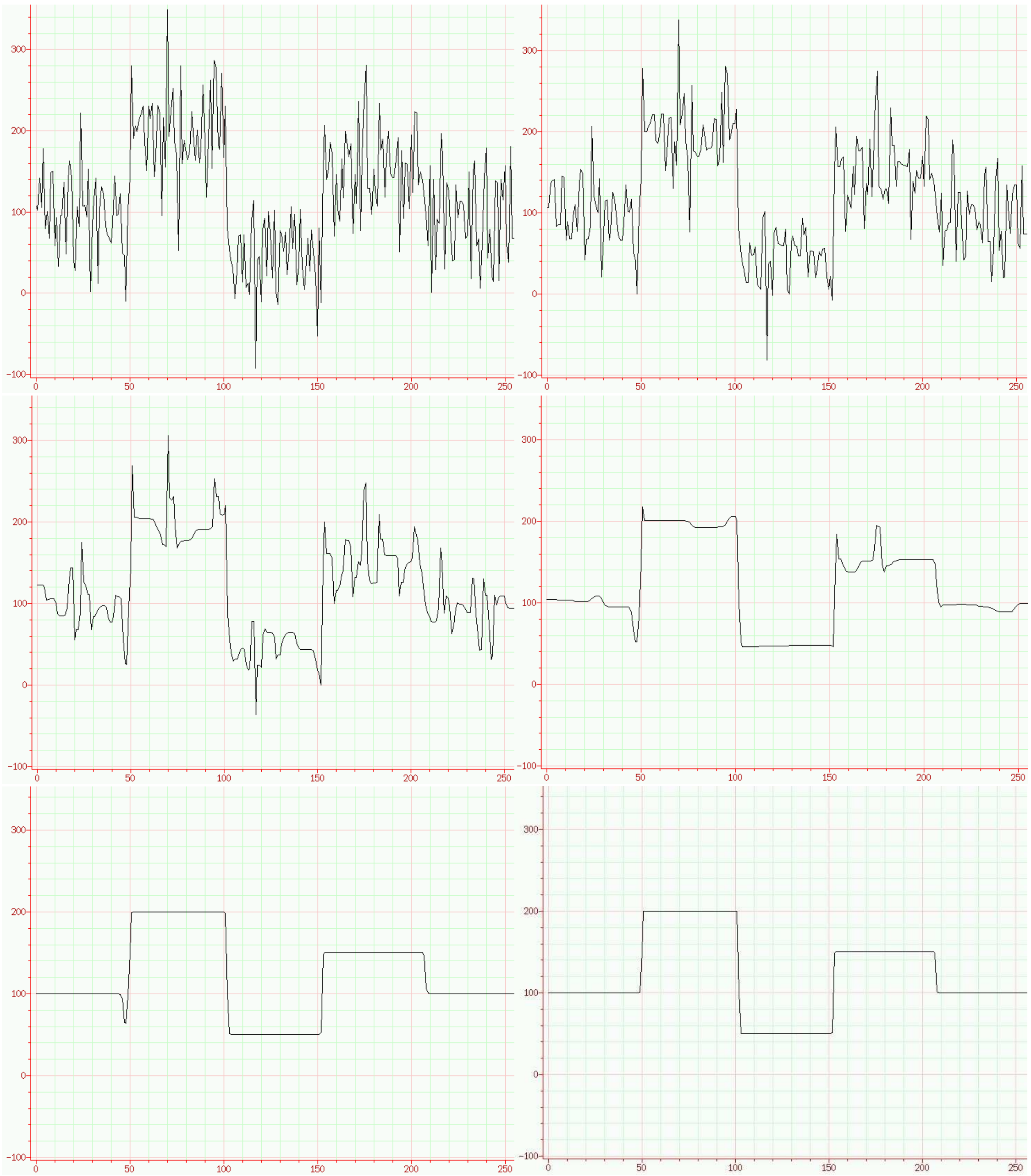

Figure 3: Top Left: (a) Signal (Figure 2-a) with 50% of gaussian noise. Top Right: (b) $t = 10$. Middle Left: (c) $t = 25$. Middle Right: (d) $t = 50$. Bottom Left: (e) $t = 75$. Bottom Left: (f) $t = 100$.

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