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On bialgebras associated with paths and essential paths on ADE graphs

R. Coquereaux[1] and A. O. García[2*]

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[1]Centre de Physique Théorique, Case 907, Luminy, 13009, Marseille, France.

[2]Max-Planck-Institut für Physik, Föhringer Ring 6, 80805 München, Germany, and Forschungszentrum Karlsruhe, Postfach 3640, 76021 Karlsruhe, Germany**.

Abstract

We define a graded multiplication on the vector space of essential paths on a graph G (a tree) and show that it is associative. In most interesting applications, this tree is an ADE Dynkin diagram. The vector space of length preserving endomorphisms of essential paths has a grading obtained from the length of paths and possesses several interesting bialgebra structures. One of these, the Double Triangle Algebra (DTA) of A. Ocneanu, is a particular kind of quantum groupoid (a weak Hopf algebra) and was studied elsewhere; its coproduct gives a filtrated convolution product on the dual vector space. Another bialgebra structure is obtained by replacing this filtered convolution product by a graded associative product. It can be obtained from the former by projection on a subspace of maximal grade, but it is interesting to define it directly, without using the DTA. What is obtained is a weak bialgebra, not a weak Hopf algebra.

Introduction

Paths of a given length n between two vertices a, b of a Dynkin diagram (extended or not) can be interpreted in terms of classical or quantum $SU(2)$ intertwiners between representations $a \otimes \tau^n$ and b, where τ is the fundamental (spin 1/2). In a similar way, essential paths are associated with (classical or quantum) morphisms between $a \otimes \tau_n$ and b, where τ_n denotes an irreducible representation.

We consider the graded vector space of essential paths (defined by A. Ocneanu for quite general graphs) and its algebra of grade-preserving endomorphisms. The corresponding associative product called *composition product* is denoted $\circ$.

* Email: coque@cpt.univ-mrs.fr, garcia@iwr.fzk.de
** Present address.

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We first show that the space of essential paths carries an associative algebra structure (denoted $\bullet$) compatible with its natural grading. Its definition involves the usual concatenation product of paths, but the situation is not trivial since the concatenation product of two essential paths is usually not essential. This is actually our main result, and it seems to be new (the existing literature is more concerned with the algebra structures that can be defined at the level of the *graded* tensor square of this vector space).

Using this "improved" concatenation product between essential paths, one can then define –besides the composition product– two other interesting algebra structures on the algebra of grade-preserving endomorphisms. One of these algebra structures (denoted $\star$) is associated with a filtrated convolution product and gives rise to a weak Hopf algebra structure: this is the Double Triangle Algebra (DTA) introduced by A. Ocneanu in [9]. It was studied elsewhere (see [11], [12], [6]). Another algebra structure, that we call the graded convolution product or simply[1] *graded product*, and again denote by the symbol $\bullet$, can be obtained from the former product by projection on its component of highest degree. However, it is possible and useful to study it directly without making any reference to its filtered relative. This is what we do.

Both products $\bullet$ and $\star$ are compatible with the composition of endomorphisms $\circ$. Compatibility here means that the associated coproducts are algebra homomorphisms with respect to the composition product. The use of a particular scalar product allows one to study these three product structures on the same underlying vector space (the diagonal graded tensor square of the space of essential paths). The bialgebra associated with the pair $(\circ, \star)$ is known to be a particular kind of quantum groupoid. However, in this paper we are interested in the bialgebra associated with the pair $(\circ, \bullet)$, and we show that it has a weaker structure: it is a weak bialgebra but not a weak Hopf (bi)-algebra.

The whole theory should apply when the diagrams that we consider (usually ADE Dynkin diagrams) are replaced by members of higher Coxeter-Dynkin systems [7, 10]: the vector space spanned by the vertices of the chosen diagram is, in particular, a module over the graph algebra associated with a Weyl alcove of $SU(N)$ at some level —such generalised $\mathcal{A}$ diagrams are indeed obtained by truncation of the Weyl chamber of $SU(N)$. These systems admit also orbifolds —$\mathcal{D}$ diagrams— and exceptionnals. In the higher cases, the grading does not refer to the positive integer that measures the length of a path, but to a particular Young diagram. Therefore the grading is defined with respect to a more general monoid (actually an integral positive cone), and the adjective "filtrated" should be understood accordingly.

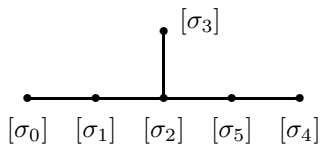
Our paper is organized as follows. In the first section we consider the vector space of all paths on a graph, and show that it is a non-unital bialgebra. In section 2 we restrict our attention to the subspace of essential paths and show that we need to introduce a new associative multiplication, $\bullet$, involving an appropriate projection operator, in order to insure stability. This vector space of essential paths is an algebra, but not a bialgebra. In the third section we show that the graded algebra of endomorphisms of essential paths can be endowed with a new product compatible with the grading, for which this space

[1] Both $\bullet$ and $\star$ can be understood as convolution products. Given two elements of grades p and q, the composition product is trivial unless $p = q$, the graded product gives an element of grade $p + q$ whereas the "filtered product" can be decomposed along vector subspaces of all grades $p + q, p + q - 2, p + q - 4, \ldots$

is a weak bialgebra. The non trivial condition insuring compatibility of the coproduct with the multiplication of endomorphisms is exemplified at the end of section 3, in the case of the graph E_6. The equation expressing this general condition is obtained in appendix A, and the general proof showing that such a compatibility condition always holds in our situation is given in section 4. In the fifth section we illustrate, in the case of the graph A_2, the fact that the two bialgebra structures respectively associated with the products $(\circ, \star)$ and $(\circ, \bullet)$ differ. Appendix A is quite general: we consider an arbitrary algebra A endowed with a scalar product and we show that although its endomorphism algebra can be given a coalgebra structure, some non trivial relation has to be satisfied in order for this space to be a bialgebra —the coproduct on $End(A)$ should be an homomorphism. We also study what happens when the algebra A is graded and when we replace $End(A)$ by a graded diagonal sum of endomorphisms.

1 The space of paths on a graph

Take a connected and simply connected graph G. For the time being, we do not assume any other extra requirements. At a later stage we will take G to be a tree, and, even more precisely, a Dynkin diagram of type ADE. For instance, a possible graph could be $G = E_6$.



Consider the set of elementary paths on G. These are just ordered lists of *neighboring* points a_i (or edges ξ_k joining two neighboring points) of the graph,

$$[a_0, a_1, a_2, \cdots, a_{L-1}, a_L] \qquad a_i \in G$$

This is clearly a path of length L, starting at a_0 and ending at a_L. Build a vector space, called $Paths$, by simply considering formal linear combinations over $\mathbb{C}$ of elementary paths. Now define the product of elementary paths by concatenation, ie, by joining the matching endpoints of the two paths (say, of lengths L and K) one after the other,

$$[a_0, a_1, \cdots, a_L]\,[b_0, b_1, \cdots, b_K] \quad = \quad \begin{cases} [a_0, a_1, \cdots, a_L, b_1, \cdots, b_K] & \text{if } a_L = b_0 \\ 0 & \text{otherwise} \end{cases}$$

Such an operation creates another elementary path of length $L + K$. This product extends by linearity to the whole vector space, and is associative (this is trivial to see). Moreover, the resulting algebra is *graded* by the length of the paths.

Consider additionally the zero-length paths $[a_0]$, there will be one such for each point a_0 of the graph. If the graph is finite, the sum over all points of the graph of the corresponding zero-length paths will be a (left and right) unit for this algebra,

$$\mathbf{1} = \sum_{a_0 \in G} [a_0]$$

Therefore $Paths$ is a graded associative algebra with unit.

We could also define a coalgebra structure on this space, introducing a coproduct that would be group-like for all elementary paths p:

$$\Delta p = p \otimes p$$

and extending it by linearity. It is straightforward to see that it is coassociative and that it is an algebra homomorphism, $\Delta(pp') = \Delta p \, \Delta p'$. Additionally, the (linear) operation

$$\epsilon(p) = 1 \qquad \text{for all elementary } p$$

is a counit for Δ.

The above defined unit is not compatible with the coproduct ($\Delta \mathbf{1} \neq \mathbf{1} \otimes \mathbf{1}$). $Paths$ is therefore a (non-unital) bialgebra. It is infinite dimensional even if the graph G is finite, as paths can be made arbitrarily long by backtracking on G.

However, as we shall see in the next section, the space $\mathcal{E}$ of essential paths that we consider in this paper is only a vector subspace but not a subalgebra of $Paths$. For this reason, a different approach will be required.

2 The algebra $\mathcal{E}$ of essential paths

2.1 Essential paths on a graph

We will now briefly introduce essential paths on the given graph G. Consider first the adjacency matrix of the graph, and call β its maximal eigenvalue. Also call $\vec{\mu} = (\mu_0 = 1, \mu_1, \cdots, \mu_N)$ the corresponding eigenvector, normalized such that the entry μ_0 associated to a distinguished point 0 of G is equal to 1 (this component is minimal). $\vec{\mu}$ is called the Perron-Frobenius eigenvector, and all its components are strictly positive. The next step is to introduce the linear operators

$$C_k : Paths \longmapsto Paths \qquad\qquad k = 1, 2, 3, \cdots$$

which act on elementary paths as follows: on a path of length $L \leq k$, C_k gives zero, otherwise ($L > k$) its action is given by,

$$\begin{aligned} & C_k\left([a_0, a_1, \cdots, a_{k-1}, a_k, a_{k+1}, a_{k+2}, \cdots, a_L]\right) \\ & \qquad = \delta_{a_{k-1}, a_{k+1}} \sqrt{\frac{\mu_{a_k}}{\mu_{a_{k-1}}}} \quad [a_0, a_1, \cdots, a_{k-1}, a_{k+2}, \cdots, a_L] \end{aligned}$$

These operators obviously preserve the end-points of the paths they act upon, and shorten their length by 2 units —removing a backtrack in the path at position k, if any, and giving 0 otherwise.

The *essential paths* are defined as those elements of $Paths$ annihilated by all the C_k's. They constitute, of course, a vector subspace[2] $\mathcal{E} \subset Paths$:

$$\mathcal{E} = \{p \in Paths \,/\, C_k p = 0 \; \forall k\}$$

We will use $\mathcal{E}_l$ to denote the subspace of essential paths of length l, and $\mathcal{E}(a \xrightarrow{l} b)$ if we want to further restrict the set to those paths with definite starting point a and ending point b.

[2] If the graph is a Dynkin diagram of type ADE then $\beta < 2$ and $\mathcal{E}$ is finite dimensional, as there are essential paths up to a certain length only, namely from 0 to $\kappa - 1$, where κ is the Coxeter number of the diagram defined by $\beta = 2\, cos(\pi/\kappa)$.

On the whole $Paths$ there is a natural scalar product, defined on elementary paths p, p' by

$$\langle p, p' \rangle = \delta_{p,p'} \qquad\qquad (p, p' \text{ elementary})$$

and consequently also an orthogonal projector

$$P : Paths \longmapsto \mathcal{E}$$

As paths with different lengths or end-points are orthogonal, P can be decomposed as a sum of projectors on each subspace,

$$\begin{aligned} P &= \sum_{\substack{a,b \in G \\ l \in \mathbb{N}}} P^l_{ab} \\ P^l_{ab} &: \quad Paths(a \xrightarrow{l} b) \longmapsto \mathcal{E}(a \xrightarrow{l} b) \end{aligned}$$

We had on $Paths$ an algebra structure, but actually $\mathcal{E}$ is only a vector subspace and not a subalgebra of $Paths$. Therefore, a new product has to be found on $\mathcal{E}$ if we want to endow it with an algebra structure. The simplest one (it must also be somehow related to the one on $Paths$!), is:

$$e \bullet e' \equiv P(ee') \tag{1}$$

where e, e' are essential paths, P is the above orthogonal projector and the product ee' is the concatenation product in $Paths$. We shall prove below the associativity property and find a unit element for this product.

2.1.1 The grading of $\mathcal{E}$

As we did with $Paths$, the space of essential paths can be graded by the length of the paths,

$$\mathcal{E} = \bigoplus_{l \in \mathbb{N}} \mathcal{E}_l$$

The product $\bullet$ is clearly compatible with this grading because $\langle\ ,\ \rangle$ is null for paths with different lengths, hence the projector P also preserves the length. For this reason, we shall call it the <u>graded</u> product on $\mathcal{E}$. As stated in the Introduction, it is possible to define also a <u>filtered</u> product on the same space (which is called $\times$ in [4]), such that $p \times p'$ can be decomposed on paths of lengths smaller or equal to $length(p) + length(p')$. Moreover, the graded product $\bullet$ could be obtained from the filtered one by restriction to the component of highest length, although this approach will not be followed here.

2.1.2 Example of essential paths on E_6

The space $\mathcal{E}(E_6)$ can be constructed using the above definitions, and is of dimension 156. More precisely, the dimensions of the graded components are $(6, 10, 14, 18, 20, 20, 20, 18, 14, 10, 6)$. For instance, the subspace $\mathcal{E}_2$ of paths of length 2 has dimension 14. It is composed of a subspace corresponding to paths with different endpoints plus a 4-dimensional subspace of paths with coinciding

ends. Inside the latter there is a 2-dimensional subspace of paths which start and end at the point 2, which is generated by:

$$\begin{aligned}
\mathcal{E}(2 \xrightarrow{2} 2) &= \left\{ \frac{1}{N_1}\left([2,3,2] - \sqrt{\frac{\mu_3}{\mu_1}}\,[2,1,2] \right), \right. \\
&\quad \left. \frac{1}{N_2}\left([2,5,2] - \frac{\sqrt{\mu_3\mu_5}}{\mu_1+\mu_3}\,[2,3,2] - \frac{\sqrt{\mu_1\mu_5}}{\mu_1+\mu_3}\,[2,1,2] \right) \right\} \\
&= \left\{ \frac{1}{N_1}\left([2,3,2] - \sqrt{-1+\sqrt{3}}\,[2,1,2] \right) \right. \\
&\quad \left. \frac{1}{N_2}\left([2,5,2] - \frac{1}{\sqrt{3}}\sqrt{-1+\sqrt{3}}\,[2,3,2] - \frac{1}{\sqrt{3}}\,[2,1,2] \right) \right\}
\end{aligned}$$

These paths are orthogonal, and can be normalized with an appropriate choice of the coefficients N_i.

2.2 Associativity

The product $\bullet$ in $\mathcal{E}$ is associative. In fact, we will prove a stronger condition for the operator P, which implies associativity of $\bullet$:

$$P(P(p_1)P(p_2)) = P(p_1p_2) \qquad \text{for any} \quad p_i \in Paths \tag{2}$$

To see this take $e, e', e'' \in \mathcal{E}$ then

$$\begin{aligned}
(e \bullet e') \bullet e'' &= P(P(ee')\,e'') = P(P(ee')\,P(e'')) = P((ee')\,e'') \\
&= P(e\,(e'e'')) = P(P(e)\,P(e'e'')) = P(e\,P(e'e'')) \\
&= e \bullet (e' \bullet e'')
\end{aligned}$$

The condition (2) may also be rewritten in the completely equivalent way

$$\begin{aligned}
P(P(p_1)P(p_2)) = P(p_1p_2) \quad &\Longleftrightarrow \quad P(P(p_1)P(p_2) - p_1p_2) = 0 \\
&\Longleftrightarrow \quad I \equiv \langle e, p_1p_2 - P(p_1)P(p_2)\rangle = 0 \qquad \text{for all} \quad e \in \mathcal{E}
\end{aligned}$$

Now we have to show that $I = 0$ for any $p_i \in Paths$:

- If $p_1, p_2 \in \mathcal{E} \subset Paths$ then $P(p_i) = p_i \implies p_1p_2 - P(p_1)P(p_2) = 0 \implies I = 0$.
- If $p_1 \equiv e_1 \in \mathcal{E}$ but $p_2 \in Paths$ then

  $$I = \langle e, e_1(p_2 - P(p_2))\rangle = \langle e, e_1 n\rangle$$

  Here $n \equiv p_2 - P(p_2) \in \mathcal{E}^{\perp}$ is orthogonal to $\mathcal{E}$.

  Without loss of generality, we may assume that the paths involved in I have well defined end-points and length (it is enough to show associativity for such paths, then associativity for linear combinations of those follows immediately):

  $$\begin{aligned}
  p_1 = e_1 = e_1(a \xrightarrow{l_1} b) \\
  p_2 = p_2(b' \xrightarrow{l_2} c) \quad \Rightarrow \quad n = n(b' \xrightarrow{l_2} c)
  \end{aligned}$$

To get a non-trivial scalar product in I we must also take $b' = b$ and

$$e = e(a \xrightarrow{l_1+l_2} c)$$

As it will be proven in subsection 4.1, such an essential path e can always be decomposed as:

$$e = \sum_{\substack{v \in G \\ i_v}} e'_{i_v}(a \xrightarrow{l_1} v)\, e''_{i_v}(v \xrightarrow{l_2} c)$$

where the sum runs over all intermediate points v appearing in e after l_1 steps, and possibly several e'_{i_v}, e''_{i_v} for each v. Essentiality of e and linear independence of paths of different end-points imply that all the e'_{i_v} and e''_{i_v} are also essential. But now it is easy to see that

$$\begin{aligned} I = \langle e, e_1 n\rangle &= \left\langle \sum_{v,i_v} e'_{i_v}(a \xrightarrow{l_1} v)\, e''_{i_v}(v \xrightarrow{l_2} c)\ ,\ e_1(a \xrightarrow{l_1} b)\, n(b \xrightarrow{l_2} c) \right\rangle \\ &= \sum_{i_b} \left\langle e'_{i_b}(a \xrightarrow{l_1} b)\, e''_{i_b}(b \xrightarrow{l_2} c)\ ,\ e_1 n \right\rangle \\ &= \sum_{i_b} \left\langle e'_{i_b}\ ,\ e_1 \right\rangle \left\langle e''_{i_b}\ ,\ n \right\rangle \end{aligned}$$

Therefore we get $I = 0$ because $n \perp \mathcal{E}$, so $\left\langle e''_{i_b}\ ,\ n \right\rangle = 0$.

- If both $p_1, p_2 \in Paths$ then $p_i = e_i + n_i$ with $P(p_i) = e_i$

  Therefore

$$\begin{aligned} I &= \langle e, (e_1 + n_1)(e_2 + n_2) - e_1 e_2\rangle = \langle e, e_1 n_2 + n_1 e_2 + n_1 n_2\rangle \\ &= 0 \end{aligned}$$

  due to the previous case.

2.3 Unit element

The algebra $\mathcal{E}$ is unital, and the unit element is clearly the same as the one in $Paths$, explicitly given by

$$\mathbf{1}_{\mathcal{E}} = \sum_{v \in G} e(v \xrightarrow{0} v) \tag{3}$$

where the sum extends over all the points of the graph, and the essential paths $e(v \xrightarrow{0} v)$ are obviously nothing more than the trivial paths $e(v \xrightarrow{0} v) \equiv [v]$.

Concluding this section, we emphasize that $\mathcal{E}$ is not only a vector space but also an associative algebra. Moreover, it is endowed with a (canonical) scalar product obtained by restriction from the one on $Paths$. It has therefore also a coalgebra structure[3], which is not a priori very interesting since the comultiplication will not be an algebra homomorphism in general. The coproduct that

[3]Identify elements with their duals, and map the product to the dual coproduct.

we had defined for *Paths* does not work either (the compatibility property with the product does not hold) since the product itself was modified. Therefore, contrary to *Paths*, the vector space $\mathcal{E}$ endowed with the graded multiplication $\bullet$ does not have a bialgebra structure.

3 The weak-$*$-bialgebra $End_{\#}(\mathcal{E})$

We have already shown in section 2 that the space $\mathcal{E}$ of essential paths constitutes a graded unital associative algebra. Applying the general construction of Appendix A (see in particular Eq. (28)) to the particular case of the graded algebra $A = \mathcal{E}$, we show now that a corresponding weak bialgebra structure on the space of its graded endomorphisms does exist. Moreover, we shall see that it has a compatible star operation.

We remind again the reader that the product $\bullet$ that we consider now on $End_{\#}(\mathcal{E})$ is graded but that it is possible to construct another product (called $\star$) on the same vector space, which is filtered rather than graded. Moreover, the structure corresponding to the pair $(\circ, \star)$ is a weak Hopf algebra. This other construction is not studied in the present paper. What we obtain here instead, is a weak bialgebra structure for the pair $(\circ, \bullet)$.

3.1 Product and coproduct

$\mathcal{E}$ being a graded algebra, its endomorphisms can also be graded. We therefore consider the space $\mathcal{B}$ of length preserving endomorphisms on $\mathcal{E}$, namely

$$\begin{aligned} \mathcal{B} \equiv End_{\#}(\mathcal{E}) &= \bigoplus_n End(\mathcal{E}_n) \\ &\overset{iso}{\simeq} \bigoplus_n \mathcal{E}_n \otimes \mathcal{E}_n^* \end{aligned}$$

As discussed in section A.3, we now consider the convolution product $\bullet$ on the space of these endomorphisms. Recalling (18) we see that it is determined by the product on the algebra $\mathcal{E}$, which we had also denoted by $\bullet$, meaning concatenation of paths plus re-projection on the essential subspace. Explicitly, on monomials we have

$$(e_i \otimes e^j) \bullet (e_k \otimes e^l) = e_i \bullet e_k \otimes e^j \bullet e^l$$

We also take the coproduct (26), which reads

$$\Delta\left(e_i \otimes e^j\right) = \sum_I \left(e_i \otimes e^{(n)I}\right) \otimes \left(e_I^{(n)} \otimes e^j\right) \qquad \text{whenever} \quad e_i \in \mathcal{E}_n,\ e^j \in \mathcal{E}_n^*$$

but remark that the compatibility condition (28) still remains to be verified. This will be done for a general graph later (see section 4), but for any given graph it is interesting to explicitly check equation (28); we illustrate this below in the case of the graph E_6.

3.1.1 Case E_6

As an example, we look at the highly non-trivial case of the graph E_6. We shall consider normalized essential paths of length 4 on E_6 and show how they appear in the $\bullet$ products of essential paths of length 2 (this is just one possibility among others, of course). We have a natural coproduct on the dual but also, using the chosen scalar product, a coproduct on the same space of essential paths. Hence, we may use the previous calculation to find the expression of the coproduct D of a particular essential path of length 4 —at least, that part which decomposes on the tensor products of essential paths of length 2. Finally, we check that the compatibility condition described by Eq. (28) is satisfied, so that we can be sure, in advance, that the corresponding graded endomorphism algebra is indeed a week bialgebra.

The subspace $\mathcal{E}(2 \xrightarrow{4} 2)$ of essential paths of length 4 is 3-dimensional and generated by the orthonormalized essential paths $e_1(2 \xrightarrow{4} 2)$, $e_2(2 \xrightarrow{4} 2)$ and $e_3(2 \xrightarrow{4} 2)$. With our convention for choosing the basis the first two read explicitly, up to a normalization factor,

$$\begin{aligned}
e_1(2 \xrightarrow{4} 2) \quad &\propto \quad \frac{1}{\sqrt{2}}\sqrt{1+\sqrt{3}}\,([2,3,2,1,2]-[2,3,2,5,2]) \\
&\qquad -([2,5,2,1,2]-[2,5,2,5,2])-\sqrt{1+\sqrt{3}}\,[2,5,4,5,2] \\
e_2(2 \xrightarrow{4} 2) \quad &\propto \quad \sqrt{1+\sqrt{3}}\,[2,1,0,1,2]-([2,1,2,1,2]-[2,1,2,5,2]) \\
&\qquad +\frac{\sqrt{3}}{2}\sqrt{-1+\sqrt{3}}\,([2,3,2,1,2]-[2,3,2,5,2]) \\
&\qquad +\frac{1}{2}\left(-1+\sqrt{3}\right)([2,5,2,1,2]-[2,5,2,5,2]) \\
&\qquad +\frac{1}{\sqrt{2}}\sqrt{-1+\sqrt{3}}\,[2,5,4,5,2]
\end{aligned}$$

The generator $e_2(2 \xrightarrow{4} 2)$ appears as a component in some products of essential paths of length 2, namely in those products involving paths which have the point 2 as one of the endpoints. These are:

$$\begin{aligned}
e(0 \xrightarrow{2} 2) &= [0,1,2] \\
e(2 \xrightarrow{2} 0) &= [2,1,0] \\
\\
e(2 \xrightarrow{2} 4) &= [2,5,4] \\
e(4 \xrightarrow{2} 2) &= [4,5,2] \\
\\
e_1(2 \xrightarrow{2} 2) &\propto -\sqrt{-1+\sqrt{3}}\,[2,1,2]+[2,3,2] \\
e_2(2 \xrightarrow{2} 2) &\propto -[2,1,2]-\sqrt{-1+\sqrt{3}}\,[2,3,2]+\sqrt{3}\,[2,5,2]
\end{aligned}$$

The non-trivial products having a contribution in the direction $e_2(2 \xrightarrow{4} 2)$ are

$$
\begin{aligned}
e(2 \xrightarrow{2} 0) \bullet e(0 \xrightarrow{2} 2) &= \sqrt{1-\frac{1}{\sqrt{3}}}\, e_2(2 \xrightarrow{4} 2) + \cdots \\
e_1(2 \xrightarrow{2} 2) \bullet e_1(2 \xrightarrow{2} 2) &= -\frac{1}{\sqrt{6\sqrt{3}}}\, e_2(2 \xrightarrow{4} 2) + \cdots \\
e_1(2 \xrightarrow{2} 2) \bullet e_2(2 \xrightarrow{2} 2) &= -\frac{1}{3}\sqrt{\frac{3}{2}+\sqrt{3}}\, e_2(2 \xrightarrow{4} 2) + \cdots \\
e_2(2 \xrightarrow{2} 2) \bullet e_1(2 \xrightarrow{2} 2) &= -\sqrt{-\frac{4}{3}+\frac{7}{3\sqrt{3}}}\, e_2(2 \xrightarrow{4} 2) + \cdots \\
e_2(2 \xrightarrow{2} 2) \bullet e_2(2 \xrightarrow{2} 2) &= -\frac{1}{3}\sqrt{-3+2\sqrt{3}}\, e_2(2 \xrightarrow{4} 2) + \cdots \\
e(2 \xrightarrow{2} 4) \bullet e(4 \xrightarrow{2} 2) &= \sqrt{\frac{3}{2}-\frac{5}{2\sqrt{3}}}\, e_2(2 \xrightarrow{4} 2) + \cdots
\end{aligned}
$$

The factors preceding $e_2(2 \xrightarrow{4} 2)$ in the above formulas are the coefficients m_{ij}^k which enter (13) and (27). The sum of the squares of the above six coefficients equals 1, and this shows, in a particular example, how condition Eq. (28) can be checked (remember that it should be satisfied for each definite grading of the coproducts of all elements).

Using (16) we may also write $De_2(2 \xrightarrow{4} 2)$ as

$$
\begin{aligned}
&\sqrt{1-\frac{1}{\sqrt{3}}}\, e(2 \xrightarrow{2} 0) \otimes e(0 \xrightarrow{2} 2) - \frac{1}{\sqrt{6\sqrt{3}}}\, e_1(2 \xrightarrow{2} 2) \otimes e_1(2 \xrightarrow{2} 2) \\
&-\frac{1}{3}\sqrt{\frac{3}{2}+\sqrt{3}}\, e_1(2 \xrightarrow{2} 2) \otimes e_2(2 \xrightarrow{2} 2) - \sqrt{-\frac{4}{3}+\frac{7}{3\sqrt{3}}}\, e_2(2 \xrightarrow{2} 2) \otimes e_1(2 \xrightarrow{2} 2) \\
&-\frac{1}{3}\sqrt{-3+2\sqrt{3}}\, e_2(2 \xrightarrow{2} 2) \otimes e_2(2 \xrightarrow{2} 2) + \sqrt{\frac{3}{2}-\frac{5}{2\sqrt{3}}}\, e(2 \xrightarrow{2} 4) \otimes e(4 \xrightarrow{2} 2) \\
&+\cdots
\end{aligned}
$$

where the missing terms include tensor products of paths of lengths $(3,1)$, $(1,3)$, $(0,4)$, and $(4,0)$. The last two are clearly $[2] \otimes e_2(2 \xrightarrow{4} 2) + e_2(2 \xrightarrow{4} 2) \otimes [2]$.

As we will show explicitly[4] in section 4, this also means that the path

[4] This also follows immediately from (28) once this requirement is checked

$e_2(2 \xrightarrow{4} 2)$ itself can be decomposed as

$$
\begin{aligned}
e_2(2 \xrightarrow{4} 2) \quad = \quad & \sqrt{1-\frac{1}{\sqrt{3}}}\, e(2 \xrightarrow{2} 0) \bullet e(0 \xrightarrow{2} 2) \\
& -\frac{1}{\sqrt{6\sqrt{3}}}\, e_1(2 \xrightarrow{2} 2) \bullet e_1(2 \xrightarrow{2} 2) \\
& -\frac{1}{3}\sqrt{\frac{3}{2}+\sqrt{3}}\, e_1(2 \xrightarrow{2} 2) \bullet e_2(2 \xrightarrow{2} 2) \\
& -\sqrt{-\frac{4}{3}+\frac{7}{3\sqrt{3}}}\, e_2(2 \xrightarrow{2} 2) \bullet e_1(2 \xrightarrow{2} 2) \\
& -\frac{1}{3}\sqrt{-3+2\sqrt{3}}\, e_2(2 \xrightarrow{2} 2) \bullet e_2(2 \xrightarrow{2} 2) \\
& +\sqrt{\frac{3}{2}-\frac{5}{2\sqrt{3}}}\, e(2 \xrightarrow{2} 4) \bullet e(4 \xrightarrow{2} 2)
\end{aligned}
$$

We could write a similar decomposition using instead products of paths of lengths 1 and 3, or 3 and 1, or even the trivial ones 0 and 4, or 4 and 0.

3.2 Unit and counit

There is an obvious unit for the product $\bullet$, which works in both the graded and non-graded versions of the endomorphisms of $\mathcal{E}$. Using (3), and the dualization map associated with the scalar product (see (10)), it can be written as

$$\mathbf{1}_{\mathcal{B}} \equiv \mathbf{1}_{\mathcal{E}} \otimes \sharp\,(\mathbf{1}_{\mathcal{E}}) \tag{4}$$

As we already have a coproduct, we can find the counit using the axioms it satisfies. In particular

$$(id \otimes \epsilon)\,\Delta(a \otimes u) = a \otimes u$$

requires

$$\epsilon(a \otimes u) \equiv u(a) \tag{5}$$

or, equivalently, $\epsilon(\rho) = \mathrm{Tr}(\rho)$.

3.3 Comonoidality

The algebra $\mathcal{B} \equiv End_{\#}(\mathcal{E})$ we have defined is not a bialgebra in the usual sense, since

$$\Delta\mathbf{1}_{\mathcal{B}} \quad \neq \quad \mathbf{1}_{\mathcal{B}} \otimes \mathbf{1}_{\mathcal{B}} \tag{6}$$

therefore $\mathcal{B}$ is a *weak bialgebra*. It is, however, comonoidal, which means that it satisfies both the left and right comultiplicativity conditions of the unit [8, 1],

$$
\begin{aligned}
\Delta^2\mathbf{1}_{\mathcal{B}} &= (\Delta\mathbf{1}_{\mathcal{B}} \otimes \mathbf{1}_{\mathcal{B}}) \bullet (\mathbf{1}_{\mathcal{B}} \otimes \Delta\mathbf{1}_{\mathcal{B}}) \\
\Delta^2\mathbf{1}_{\mathcal{B}} &= (\mathbf{1}_{\mathcal{B}} \otimes \Delta\mathbf{1}_{\mathcal{B}}) \bullet (\Delta\mathbf{1}_{\mathcal{B}} \otimes \mathbf{1}_{\mathcal{B}})
\end{aligned}
$$

The important consequence of this property is that the category of $End_{\#}(\mathcal{E})$-comodules is a monoidal category.

We will check explicitly the first property. Using (3) and (4) with $e_v^{(0)} \equiv [v]$ and $e^{(0)v}$ its dual, the LHS becomes

$$\begin{aligned}\Delta^2 \mathbf{1}_{\mathcal{B}} &= (\Delta \otimes id)\, \Delta \mathbf{1}_{\mathcal{B}} \\ &= \sum_{v,w,x,y \in G} \left(e_v^{(0)} \otimes e^{(0)x}\right) \otimes \left(e_x^{(0)} \otimes e^{(0)y}\right) \otimes \left(e_y^{(0)} \otimes e^{(0)w}\right)\end{aligned}$$

This has to be compared with the RHS

$$\begin{aligned}(\Delta \mathbf{1}_{\mathcal{B}} \otimes \mathbf{1}_{\mathcal{B}}) \bullet (\mathbf{1}_{\mathcal{B}} \otimes \Delta \mathbf{1}_{\mathcal{B}}) &= \mathbf{1}_{(1)} \otimes \left(\mathbf{1}_{(2)} \bullet \mathbf{1}_{(1)'}\right) \otimes \mathbf{1}_{(2)'} \\ = \sum_{\substack{v,w,x \\ v',w',x'}} & \left(e_v^{(0)} \otimes e^{(0)x}\right) \otimes \left[\left(e_x^{(0)} \otimes e^{(0)w}\right) \bullet \left(e_{v'}^{(0)} \otimes e^{(0)x'}\right)\right] \otimes \left(e_{x'}^{(0)} \otimes e^{(0)w'}\right)\end{aligned}$$

Considering that the product in square brackets above is

$$\begin{aligned}\left(e_x^{(0)} \otimes e^{(0)w}\right) \bullet \left(e_{v'}^{(0)} \otimes e^{(0)x'}\right) &= e_x^{(0)} e_{v'}^{(0)} \otimes e^{(0)w} e^{(0)x'} \\ &= \delta_{x,v'}\, \delta_{w,x'}\, e_x^{(0)} \otimes e^{(0)w}\end{aligned}$$

we conclude that

$$(\Delta \mathbf{1}_{\mathcal{B}} \otimes \mathbf{1}_{\mathcal{B}}) \bullet (\mathbf{1}_{\mathcal{B}} \otimes \Delta \mathbf{1}_{\mathcal{B}}) = \sum_{\substack{v,w,x \\ w'}} \left(e_v^{(0)} \otimes e^{(0)x}\right) \otimes \left(e_x^{(0)} \otimes e^{(0)w}\right) \otimes \left(e_w^{(0)} \otimes e^{(0)w'}\right)$$

which obviously coincides with the expression we got above for $\Delta^2 \mathbf{1}_{\mathcal{B}}$ after an index relabeling.

The check of the right comonoidality property is just a trivial variation of the above. Weak multiplicativity of the counit (the "dual" property) does not hold in general.

3.4 Non-existence of an antipode

Given an algebra or coalgebra, the unit and counit must be unique if they exist at all, and this is so for the weak bialgebra $(\mathcal{B} = End_{\#}(\mathcal{E}), \bullet)$. One could hope to find a corresponding antipode to turn this bialgebra into a weak Hopf algebra but this is not possible, as we will show now.

We refer the reader to [8, 1, 2] for axioms concerning the antipode in weak Hopf algebras. There are slight variations among these references, for instance [8] defines first left and right pre-antipodes, as an intermediate step to have an antipode. This is not relevant here, as the axioms for an antipode in any of [8, 1, 2] necessarily imply that S must be such that

$$S(x_{(1)})\, x_{(2)} = \mathbf{1}_{(1)}\, \epsilon\left(x\, \mathbf{1}_{(2)}\right) \tag{7}$$

for any element x of the Hopf algebra. Therefore, we can assume that this holds for an element $\rho \in End(\mathcal{E}_n)$ of the form

$$\rho = a \otimes u \qquad \text{with} \quad a = a^{(n)} \in \mathcal{E}_n\ , \quad u = u^{(n)} \in \mathcal{E}_n^*\ , \quad n \geq 1$$

Using $\Delta\rho = \sum_I \left(a \otimes e^{(n)I}\right) \otimes \left(e_I^{(n)} \otimes u\right)$ and replacing it in (7) we get

$$\sum_I S\left(a \otimes e^{(n)I}\right) \bullet \left(e_I^{(n)} \otimes u\right)$$

on the LHS and

$$\sum_{v,w,x\in G} \left(e_v^{(0)} \otimes e^{(0)x}\right) \epsilon \left[\left(a \bullet e_x^{(0)}\right) \otimes \left(u \bullet e^{(0)w}\right)\right]$$

on the RHS. In this last term the sum over points x, w of the graph contributes only when x is the ending point a_f of the path a, and w is the ending point of (the dual of) u. Therefore, we must have

$$\sum_I S\left(a \otimes e^{(n)I}\right) \bullet \left(e_I^{(n)} \otimes u\right) \quad = \quad u(a) \left(\sum_v e_v^{(0)}\right) \otimes e^{(0)a_f}$$

We see now that this is not possible, as the LHS gives tensor product factors of grading $\geq n$ —the product of whatever comes out of the antipode times $e_I^{(n)}$ will always be a path of length at least n, or the null element— whereas the RHS involves paths of length zero and is non-null in the general case. Hence, it is not possible to find an operator S which could satisfy the axiom (7).

3.5 The star operation

We can define a star operation $\star$ on $Paths$ and $\mathcal{E}$ just by reversing the orientation of the paths:

$$p^\star \quad = \quad [a_L, a_{L-1}, \cdots, a_1, a_0] \equiv \tilde{p} \qquad \text{if} \qquad p = [a_0, a_1, \cdots, a_L]$$

and extending it by anti-linearity. Of course, if e is essential then $e^\star$ will also be essential, and a basis of $\mathcal{E}$ can always be chosen so as to have both a vector e_i and its conjugate in the basis, thus $e_i^\star \equiv e_j$ for some j.

The antilinear mapping $\star$ turns $(\mathcal{E}, \bullet)$ into a $\star$-algebra, because $P\star = \star P$; therefore

$$(a \bullet b)^\star = b^\star \bullet a^\star$$

and

$$(\mathbf{1}_\mathcal{E})^\star = \mathbf{1}_\mathcal{E}$$

We can also introduce a conjugation on the algebra $\mathcal{B} = End_\#(\mathcal{E})$ by making use of the above one, defining

$$\star : End_\#(\mathcal{E}) \quad \longmapsto \quad End_\#(\mathcal{E})$$

on monomials by

$$(a \otimes u)^\star \equiv a^\star \otimes u^\star$$

This operation trivially verifies

$$\begin{aligned} (\mathbf{1}_\mathcal{B})^\star &= \mathbf{1}_\mathcal{B} \\ \epsilon\left(\rho^\star\right) &= \overline{\epsilon\left(\rho\right)} \end{aligned}$$

and

$$(\rho \bullet \rho')^\star = \rho'^\star \bullet \rho^\star$$

To prove that

$$\Delta\left(\rho^\star\right) = \left(\Delta\rho\right)^{\star\otimes\star}$$

one should only note that

$$\sum_J \left(e^J\right)^\star \otimes e_J^\star = \sum_J e^J \otimes e_J$$

for each orthonormal sub-basis $\{e_J\} = \left\{e_J^{(n)}\right\}$ of definite grading n, which holds because we can always choose the e_J such that $e_J^\star = e_I$ for some I. This star operation is a normal (non-twisted) one, however it would also be possible to introduce a twisted [5] version.

4 Proof of the weak bialgebra compatibility condition

We prove in this section that, in the case of the algebra of graded endomorphisms of essential paths $\mathcal{B} = End_\#(\mathcal{E})$ the condition (28) holds. This condition, as we have seen, insures the homomorphism property of the coproduct. Some auxiliary but relevant results are obtained first.

4.1 Decomposition of essential paths

An essential path of well defined endpoints a, b and length L,

$$e = e(a \xrightarrow{L} b)$$

is necessarily a linear combination

$$\sum_p \alpha_p \, p(a \xrightarrow{L} b)$$

where all the p are elementary paths from a to b. Of course we can now take $0 \leq l \leq L$ and rewrite each p using subpaths of lengths l, $L - l$, namely $p(a \xrightarrow{L} b) = p'(a \xrightarrow{l} v)\, p''(v \xrightarrow{L-l} b)$ for some $v \in G$, and p', p'' elementary too. Therefore,

$$e = \sum_{v\in G} \sum_{p',p''} \alpha_{vp'p''} \, p'(a \xrightarrow{l} v)\, p''(v \xrightarrow{L-l} b)$$

As e is essential, in particular it must happen that $C_k e = 0$ for $k = 1, 2, \cdots, l-1$. But for these values of k

$$0 = C_k e = \sum_{v\in G} \sum_{p''} C_k \left(\sum_{p'} \alpha_{vp'p''} \, p'(a \xrightarrow{l} v) \right) p''(v \xrightarrow{L-l} b)$$

and using the linear independence of the elementary paths p'' we see that for each of the possible p'' the linear combination in parentheses must be essential:

$$\sum_{p'} \alpha_{vp'p''} \, p'(a \xrightarrow{l} v) \equiv \sum_i \beta_{vip''} \, e_i'(a \xrightarrow{l} v)$$

Here the index i runs over a basis of essential paths of definite endpoints a, v and length l. Getting this back into e, we get

$$e = \sum_{v \in G} \sum_{i,p''} \beta_{vip''}\, e'_i(a \xrightarrow{l} v)\, p''(v \xrightarrow{L-l} b)$$

We now use that $C_k e = 0$ for $k = l+1, \cdots, L-1$, so

$$0 = C_k e = \sum_{v,i} e'_i(a \xrightarrow{l} v)\, C_{k-l} \left(\sum_{p''} \beta_{vip''}\, p''(v \xrightarrow{L-l} b) \right)$$

and due to the linear independence of the basis $\{e'_i\}$ of essential paths we conclude again that for any value of i and v the term in parentheses must be essential:

$$\sum_{p''} \beta_{vip''}\, p''(v \xrightarrow{L-l} b) \equiv \sum_j \gamma_{vij}\, e''_j(v \xrightarrow{L-l} b)$$

Putting this back into e, and using $P(e) = e$, we obtain the desired factorization:

$$\begin{aligned} e &= \sum_{v,i,j} \gamma_{vij}\, P\left(e'_i(a \xrightarrow{l} v)\, e''_j(v \xrightarrow{L-l} b) \right) \\ &= \sum_{v,i,j} \gamma_{vij}\, e'_i(a \xrightarrow{l} v) \bullet e''_j(v \xrightarrow{L-l} b) \end{aligned}$$

The cases $l = 0, L$ are completely trivial. We can formulate this intermediate result as a lemma.

Lemma Any essential path $e(a \xrightarrow{L} b)$ of well defined endpoints a, b and length L can be decomposed, for any fixed given positive value $l < L$, as a linear combination of products of shorter essential paths

$$e(a \xrightarrow{L} b) = \sum_{v,i,j} \gamma_{vij}\, e'_i(a \xrightarrow{l} v) \bullet e''_j(v \xrightarrow{L-l} b) \tag{8}$$

where the sum extends over all possible points v of the graph which can be reached from a and b with essential paths of length l and $L-l$, respectively. If we assume that both (sub)basis $\{e'_i\}$ and $\{e''_j\}$ are orthonormal then also

$$\sum_{v,i,j} |\gamma_{vij}|^2 = \|e\|^2 \tag{9}$$

■

Note that the decomposition (8) can be used to build the essential paths recursively. With regard to the dimensionality of this space, remark that when G is a Dynkin diagram of type ADE, the following result (that we do not prove here) is known: The vector space spanned by the vertices $a, b, \cdots$ of G is a module over the graph algebra of A_n, where $n+1$ is the Coxeter number of G and A_n is the commutative algebra with generators $N_0, N_1, \cdots, N_{n-1}$ obeying the following relations: N_0 is the unit, N_1 is the (algebraic) generator with $N_1 N_p = N_{p-1} + N_{p+1}$, if $p < n-1$, and $N_1 N_{n-1} = N_{n-2}$. If s denotes the

number of vertices of G, this module action is encoded by n matrices F_p of size $s \times s$. They are related to the previous generators by $N_p a = \sum_b (F_p)_{ab} b$. The number of essential paths of length p on the graph G is equal to the sum of the matrix elements of F_p.

4.2 The weak bialgebra condition

The coefficients $m^{(n+m)K}_{nI,mJ}$ that enter the weak bialgebra condition (27) are just the components of products $e^{(n)}_I \bullet e^{(m)}_J$ of essential paths of lengths n, m respectively, along the directions $e^{(n+m)}_K$. Using a more explicit notation than above, the non-trivial contributions are

$$m^{e_k(a \xrightarrow{L} b)}_{e_n(a \xrightarrow{l} c)\,,\, e_r(c \xrightarrow{L-l} b)} \quad \equiv \quad \langle e_k \,,\, e_n \bullet e_r \rangle \;=\; \langle e_k \,,\, e_n\, e_r \rangle$$

where we have used the definition (1) for the product, self-adjointness of the operator P, and the fact that e_k is essential so $P(e_k) = e_k$. Taking $e = e_k$ in the decomposition (8) we can now write

$$\begin{aligned} m^{e_k}_{e_n\,,\,e_r} &= \sum_{v,i,j} \gamma^{(k)}_{vij} \left\langle e'_i(a \xrightarrow{l} v)\,,\, e_n(a \xrightarrow{l} c) \right\rangle \left\langle e''_j(v \xrightarrow{L-l} b)\,,\, e_r(c \xrightarrow{L-l} b) \right\rangle \\ &= \sum_{v,i,j} \gamma^{(k)}_{vij}\, \delta_{vc}\, \delta_{in}\, \delta_{jr} \;=\; \gamma^{(k)}_{cnr} \end{aligned}$$

Therefore the coefficients $\gamma^{(k)}_{cnr}$ that enter the decomposition of e_k are the same that those involved in the product. The weak bialgebra condition (27) reduces now to the orthonormality condition (9) of the e_k , that is

$$\begin{aligned} \sum_{nr} \overline{m^{e_k(a \xrightarrow{L} b)}_{e_n(a \xrightarrow{l} c)\,,\, e_r(c \xrightarrow{L-l} b)}}\; m^{e_{k'}(a \xrightarrow{L} b)}_{e_n(a \xrightarrow{l} c)\,,\, e_r(c \xrightarrow{L-l} b)} &= \sum_{c,n,r} \overline{\gamma^{(k)}_{cnr}}\, \gamma^{(k')}_{cnr} \\ &= \langle e_k \,,\, e_{k'} \rangle \;=\; \delta_{kk'} \end{aligned}$$

5 Comparison of the two bialgebra structures for the A_2 diagram

The graph A_2 gives rise to the simplest non-trivial example, an 8-dimensional algebra (whereas A_3 already produces a 34-dimensional one). It consists of two points and one (bi-oriented) edge. The only essential paths are: $a_1 \equiv [1]$, $a_2 \equiv [2]$, and the right and left oriented paths $r \equiv [1,2]$ and $l \equiv [2,1]$ respectively.

We shall compare, for this example, the two bialgebra structures mentioned in the text. The first, the graded one, is a weak bialgebra, semi-simple but not co-semi-simple. The second, the filtrated one, is a weak Hopf algebra; it is both simple and co-semi-simple.

5.1 The graded bialgebra structure

The products in $\mathcal{E}(A_2)$ (corresponding to (13)) are:

$$\begin{aligned} a_i \bullet a_j &= \delta_{ij} a_i & r^2 = l^2 = r \bullet l = l \bullet r &= 0 \\ a_1 \bullet r = r \bullet a_2 &= r & a_2 \bullet r = r \bullet a_1 &= 0 \\ a_2 \bullet l = l \bullet a_1 &= l & a_1 \bullet l = l \bullet a_2 &= 0 \end{aligned}$$

The dual operation in $\mathcal{E}(A_2)$, the coproduct corresponding to (16)), is

$$\begin{aligned} Da_1 &= a_1 \otimes a_1 & Da_2 &= a_2 \otimes a_2 \\ Dr &= a_1 \otimes r + r \otimes a_2 & Dl &= a_2 \otimes l + l \otimes a_1 \end{aligned}$$

Now we consider $E \equiv End_{\#}(\mathcal{E}(A_2))$: we call ρ_{ij} the endomorphism of paths of length zero taking a_j into a_i, which we also identify using the map $\sharp$ as $\rho_{ij} = a_i \otimes a_j$. We also have the $\rho_{rr}, \rho_{rl}, \rho_{lr}, \rho_{ll}$ acting on the space of paths of length 1. Thus E has dimension 8 as a vector space.

The product in E is the usual composition product, so

$$\begin{aligned} \rho_{ij} \circ \rho_{kl} &= \delta_{jk} \rho_{il} & i, j = 1, 2 \\ \rho_{ij} \circ \rho_{**} = \rho_{**} \circ \rho_{ij} &= 0 & * = r, l \\ \rho_{d_1 d_2} \circ \rho_{d_3 d_4} &= \delta_{d_2 d_3} \rho_{d_1 d_4} & d_i = r, l \end{aligned}$$

Obviously, $(E, \circ)$ is the direct sum of two subalgebras, namely $End(\mathcal{E}_{0,1}(A_2))$, the endomorphisms of paths of length i, both isomorphic to $M_{2x2}(\mathbb{C})$.

Regarding the coproduct on E, remember that for the graded case we defined

$$\Delta\rho = (P \otimes P)(1 \otimes \tau \otimes 1)(D_A \otimes D_{A^*})\rho$$

In our present example this implies

$$\begin{aligned} \Delta\rho_{ij} &= \rho_{ij} \otimes \rho_{ij} \\ \\ \Delta\rho_{rr} &= \rho_{11} \otimes \rho_{rr} + \rho_{rr} \otimes \rho_{22} \\ \Delta\rho_{ll} &= \rho_{22} \otimes \rho_{ll} + \rho_{ll} \otimes \rho_{11} \\ \Delta\rho_{rl} &= \rho_{12} \otimes \rho_{rl} + \rho_{rl} \otimes \rho_{21} \\ \Delta\rho_{lr} &= \rho_{21} \otimes \rho_{lr} + \rho_{lr} \otimes \rho_{12} \end{aligned}$$

Indeed, in the first case, for example, the calculation reads

$$\begin{aligned} \Delta\rho_{rr} &= \Delta(r \otimes r) = (P \otimes P)(1 \otimes \tau \otimes 1)\left((a_1 \otimes r + r \otimes a_2) \otimes (a_1 \otimes r + r \otimes a_2)\right) \\ &= (P \otimes P)\left(a_1 \otimes a_1 \otimes r \otimes r + r \otimes r \otimes a_2 \otimes a_2 + a_1 \otimes r \otimes r \otimes a_2 + r \otimes a_1 \otimes a_2 \otimes r\right) \\ &= a_1 \otimes a_1 \otimes r \otimes r + r \otimes r \otimes a_2 \otimes a_2 = \rho_{11} \otimes \rho_{rr} + \rho_{rr} \otimes \rho_{22} \end{aligned}$$

because the terms $a_1 \otimes r \otimes r \otimes a_2 + r \otimes a_1 \otimes a_2 \otimes r$ do not belong to $E \otimes E$ and get projected out by the operator $P \otimes P$. It is easy to check that Δ is both coassociative and an algebra homomorphism for the product $\circ$. Therefore, E is a bialgebra. The element $\mathbb{1} = \rho_{11} + \rho_{22} + \rho_{rr} + \rho_{ll}$ is a unit for $\circ$ but its coproduct is not $\mathbb{1} \otimes \mathbb{1}$.

If we declare the elementary paths a_1, a_2, r, l orthonormal, we obtain an induced scalar product on the space of endomorphisms. We can use it to map the above coproduct to a product that we call $\bullet$.

The first algebra (product $\circ$) is isomorphic, by construction, with the semi-simple algebra $M_2(\mathbb{C}) \oplus M_2(\mathbb{C})$. The matrix units, or "elementary matrices", are realized as follows. Each entry denotes a single matrix unit (replace the chosen generator by 1 and set the others entries to zero): $\begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} \oplus \begin{pmatrix} \rho_{rr} & \rho_{rl} \\ \rho_{lr} & \rho_{ll} \end{pmatrix}$

The graded algebra (product $\bullet$) is not semi-simple. It can be realized[5] as a direct sum of two algebras of matrices 2×2 with entries in the ring of Grassman numbers with generators $\{1, \theta\}$, $\theta^2 = 0$. Indeed, the basis vectors $\{\rho_{11}, \rho_{rr}, \rho_{ll}, \rho_{22}\}$ generate an algebra isomorphic with $\begin{pmatrix} a & b\,\theta \\ c\,\theta & d \end{pmatrix}$, where a, b, c, d are complex numbers. Vectors $\{\rho_{12}, \rho_{rl}, \rho_{lr}, \rho_{21}\}$ generate another copy of the same four-dimensional algebra. The eight generators can be realized as (dots stand for the number 0):

$$\rho_{11} = \begin{pmatrix} 1 & \cdot \\ \cdot & \cdot \end{pmatrix} \oplus \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix} \qquad \rho_{rr} = \begin{pmatrix} \cdot & \theta \\ \cdot & \cdot \end{pmatrix} \oplus \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix}$$

$$\rho_{ll} = \begin{pmatrix} \cdot & \cdot \\ \theta & \cdot \end{pmatrix} \oplus \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix} \qquad \rho_{22} = \begin{pmatrix} \cdot & \cdot \\ \cdot & 1 \end{pmatrix} \oplus \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix}$$

$$\rho_{12} = \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix} \oplus \begin{pmatrix} \cdot & -\theta \\ \theta & 1 \end{pmatrix} \qquad \rho_{rl} = \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix} \oplus \begin{pmatrix} \cdot & \cdot \\ \theta & \cdot \end{pmatrix}$$

$$\rho_{lr} = \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix} \oplus \begin{pmatrix} \cdot & \theta \\ \cdot & \cdot \end{pmatrix} \qquad \rho_{21} = \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix} \oplus \begin{pmatrix} 1 & \theta \\ -\theta & \cdot \end{pmatrix}$$

5.2 The filtrated bialgebra structure

The filtrated bialgebra structure associated with A_2 (see also [4]) uses the same composition product $\circ$ but the second product $\star$ is different from $\bullet$. Actually, the case A_2 is rather special, in the following sense: there exists an associative structure (call it also $\star$) on the space of essential paths $\mathcal{E}(A_2)$ such that the filtrated algebra structure that we consider on the eight dimensional space E coincides with the tensor square of the later. This is (unfortunately) not so for other ADE diagrams, not even for the A_N when $N > 2$. The product $\star$ on $\mathcal{E}(A_2)$ is :

$$\begin{aligned} a_i \star a_j &= \delta_{ij} a_i & r^2 &= l^2 = 0 \\ & & r \star l &= a_1\,, \quad l \star r = a_2 \\ a_1 \star r &= r \star a_2 = r & a_2 \star r &= r \star a_1 = 0 \\ a_2 \star l &= l \star a_1 = l & a_1 \star l &= l \star a_2 = 0 \end{aligned}$$

Comparing with the multiplication $\bullet$ of the previous section, we see that the difference lies in the values of $r \star l$ and $l \star r$ that, here, do not vanish. The product $\star$ in E is:

$$(u \otimes v) \star (u' \otimes v') \doteq (u \star u') \otimes (v \star v')$$

It is easy to write the multiplication table and to see that this algebra is semi-simple and isomorphic, like $(E, \circ)$, with the direct sum of two full matrix

[5] So, there are four projective irreducible modules and the radical is $\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$

algebras 2×2 over the complex numbers. However, the eight generators are represented in a very different way. With the same reading convention as before, the matrix units are given by: $\begin{pmatrix} \rho_{11} & \rho_{rr} \\ \rho_{ll} & \rho_{22} \end{pmatrix} \oplus \begin{pmatrix} \rho_{12} & \rho_{rl} \\ \rho_{lr} & \rho_{21} \end{pmatrix}$. The corresponding coproducts (compare with the previous section) read as follow: $\Delta\rho_{u,v}$, when $u, v = r, l$ are as before, but the $\Delta\rho_{i,j}$, $i, j = 1, 2$ are different[6]:

$$\begin{aligned}
\Delta\rho_{11} &= \rho_{11}\otimes\rho_{11} + rr\otimes ll \\
\Delta\rho_{12} &= \rho_{12}\otimes\rho_{12} + rl\otimes lr \\
\Delta\rho_{21} &= \rho_{21}\otimes\rho_{21} + lr\otimes rl \\
\Delta\rho_{22} &= \rho_{22}\otimes\rho_{22} + ll\otimes rr
\end{aligned}$$

A The bialgebra of endomorphisms of an algebra

We describe the (weak) bialgebra structure of the space of endomorphisms of the algebra $\mathcal{E}$. Actually, only the fact that $\mathcal{E}$ possesses an algebra structure is needed here, so we may start from an arbitrary algebra that we call A.

A.1 A and A^*

Take A an associative algebra, with or without unit, and finite dimensional. Call $m_A : A\otimes A \to A$ its product. Now introduce the linear dual vector space $A^* = \{A \xrightarrow{lin} \mathbb{C}\}$, which can be automatically endowed with a (coassociative) coalgebra structure in the standard way. We call $D_{A^*} : A^* \to A^*\otimes A^*$ its coproduct (of course, $D_{A^*}(u)(a\otimes b) = u(ab)$). Now *choose* a scalar product $\langle\ ,\ \rangle$ on A: this defines an antilinear isomorphism[7] $\sharp$ between A and A^*, given by

$$\begin{aligned}
\sharp : A &\longrightarrow A^* \\
a &\longrightarrow \sharp(a) = \langle a,\ \rangle
\end{aligned} \tag{10}$$

Its inverse is usually called $\flat \equiv \sharp^{-1}$. Use this identification $\sharp$ to define a comultiplication on A

$$\begin{aligned}
D_A &: A \longmapsto A\otimes A \\
D_A(a) &\equiv (\flat\otimes\flat)\, D_{A^*}(\sharp(a))
\end{aligned} \tag{11}$$

and its dual, a product m_{A^*} on A^*:

$$\begin{aligned}
m_{A^*} &: A^*\otimes A^* \longmapsto A^* \\
m_{A^*}(u\otimes v) &\equiv \sharp\circ m_A\circ(\flat\otimes\flat)(u\otimes v) = \sharp\,(\flat(u)\flat(v))
\end{aligned} \tag{12}$$

That is, we choose m_{A^*} in such a way that $\sharp$ becomes an algebra homomorphism. Note that if $\{e_i\}$ is an orthonormal basis of A and

$$e_i e_j = \sum_k m_{ij}^k\, e_k \tag{13}$$

[6] Notice that $\Delta\mathbf{1} = (11+ll)\otimes(11+rr) + (rr+22)\otimes(ll+22)$

[7] $\sharp$ goes from A to A^*, but in the same way one can define $\sharp^*$ from A^* to A^{**}. As both A and A^{**} can be identified in the finite dimensional case, $\sharp$ is invertible.

then using $e^i \equiv \sharp(e_i)$ we have

$$D_{A^*}(e^k) = \sum_{ij} m^k_{ij}\, e^i \otimes e^j \tag{14}$$

$$e^i e^j \equiv m_{A^*}(e^i \otimes e^j) = \sum_k \overline{m^k_{ij}}\, e^k \tag{15}$$

$$D_A(e_k) = \sum_{ij} \overline{m^k_{ij}}\, e_i \otimes e_j \tag{16}$$

Having the same coefficients m^k_{ij} as m_A, the operations D_A and m_{A^*} are automatically (co)associative. Neither A nor A^* are a priori bialgebras, so there is no reason for D_A or D_{A^*} to be algebra homomorphisms with respect to m_A or m_{A^*}.

A.2 $End(A)$ - the non-graded case

Let us now see what we can do on $End(A)$. Notice that we are not considering any graduation on A (in this subsection, elements of $End(A)$ are general endomorphisms). We know that

$$End(A) \simeq A \otimes A^*$$

; the algebra structure of $End(A)$ is given by the associative composition product

$$\rho \circ \rho' = (a \otimes u) \circ (a' \otimes u') \equiv a \otimes u(a')\, u' \qquad \text{whenever } \rho = a \otimes u\,,\ \rho' = a' \otimes u' \tag{17}$$

and uses nothing more than the vector space structure of A.

If Φ, Φ' are vectorial homomorphisms from a coalgebra A to an algebra B, one can define a convolution product; whereas if Φ is a vectorial homomorphism from an algebra A to a coalgebra B, a convolution coproduct can be given on Φ. This comes from the fact that $Hom(A, B) \simeq B \otimes A^*$. Coming back to our case, taking $B = A$ equipped with both a product m_A and a coproduct[8] D_A (ergo also their dual operations D_{A^*}, m_{A^*} on A^*), the convolution product[9] is

$$(\rho \bullet \rho')(a) \equiv \rho(a_1).\rho'(a_2) \qquad \text{where } D_A(a) = a_1 \otimes a_2 \tag{18}$$

This is just the natural multiplication in the tensor product of algebras of A and A^*:

$$\begin{aligned} (a \otimes u) \bullet (a' \otimes u') &= aa' \otimes uu' \\ \rho \bullet \rho' &= (m_A \otimes m_{A^*})\,(1 \otimes \tau \otimes 1)(\rho \otimes \rho') \end{aligned} \tag{19}$$

where $a, a' \in A$, $u, u' \in A^*$ and τ is the twist permuting two factors of a tensor product. In particular if we start from a vector space A endowed with both an algebra and a coalgebra structure (it may be, or not, a bialgebra), the above

[8] either because we started from an algebra A, selected a scalar product, and applied the procedure of the previous subsection, or because we started from a genuine bialgebra A.

[9] Warning: calling $\bullet$ a convolution product may be misleading since the already mentioned filtrated multiplication is also of the same type. The reader will certainly understand which is which from the context.

construction gives two distinct multiplicative structures to the space $End(A)$ —the composition product $\circ$ and the product $\bullet$. Equivalently, it gives two distinct comultiplicative structures to the space $End(A^*)$. Now, if we want to consider $End(A)$ both as an algebra and a coalgebra, one has somehow to identify $End(A^*)$ with $End(A)$, and, for this reason, we need to choose some scalar product.

Let us therefore consider an algebra A endowed with some given scalar product, and "dualize" one of the two products on $End(A)$ —either $\circ$ or $\bullet$— to get a coproduct on $End(A)^*$. Finally, we map the latter to a comultiplication on $End(A)$ simply by using the isomorphism $\flat \otimes \sharp$. For later convenience we choose to dualize the composition product $\circ$. In this way we obtain the "composition" coproduct Δ on $End(A)$,

$$\Delta \quad : \quad End(A) \longmapsto End(A) \otimes End(A) \tag{20}$$

$$\Delta(a \otimes u) \quad = \quad \sum_i \left(a \otimes e^i\right) \otimes (e_i \otimes u) \quad \in (A \otimes A^*) \otimes (A \otimes A^*)$$

where the sum runs over an orthonormal basis of A (for the chosen scalar product) as in the previous subsection. This coproduct Δ is trivially coassociative, and the product $\bullet$ is associative due to the corresponding properties of m_A and m_{A^*}.

Now we want $End(A)$ to be a bialgebra but the comultiplication Δ that we just considered has a priori no reason to be an algebra homomorphism[10] for $\bullet$, ie, in general $\Delta(\rho \bullet \rho') \neq \Delta\rho\,(\bullet \otimes \bullet)\,\Delta\rho'$. Let us, however, analyze the terms separately:

$$\begin{aligned} \Delta(\rho \bullet \rho') \quad &= \quad \Delta((a \otimes u) \bullet (a' \otimes u')) = \Delta(aa' \otimes uu') \\ &= \quad \sum_i \left(aa' \otimes e^i\right) \otimes (e_i \otimes uu') \end{aligned} \tag{21}$$

On the other hand,

$$\begin{aligned} \Delta\rho\,(\bullet \otimes \bullet)\,\Delta\rho' \quad &= \quad \sum_{ij} \left[\left(a \otimes e^i\right) \otimes (e_i \otimes u)\right]\ (\bullet \otimes \bullet)\ \left[\left(a' \otimes e^j\right) \otimes (e_j \otimes u')\right] \\ &= \quad \sum_{ij} \left(aa' \otimes e^i e^j\right) \otimes (e_i e_j \otimes uu') \end{aligned}$$

Using the explicit expressions (13),(15) this becomes

$$\Delta\rho\,(\bullet \otimes \bullet)\,\Delta\rho' \quad = \quad \sum_{kl} \left(\sum_{ij} \overline{m^k_{ij}}\, m^l_{ij} \right) \left(aa' \otimes e^k\right) \otimes (e_l \otimes uu') \tag{22}$$

$(End(A), \bullet, \Delta)$ would be a bialgebra only if (21) and (22) coincide. As the elements ρ and ρ' (in fact a, a', u, u') can be chosen arbitrarily, this requires

$$\sum_{kl} \left(\sum_{ij} \overline{m^k_{ij}}\, m^l_{ij} \right) e^k \otimes e_l = \sum_k e^k \otimes e_k$$

[10] This would still be the case even if A were a true bialgebra. However, we are not making this hypothesis here.

namely

$$\sum_{ij} \overline{m^k_{ij}}\, m^l_{ij} = \delta^{kl} \qquad\qquad \forall k,l \tag{23}$$

This requirement can be rewritten also as

$$m_A\left(D_A(a)\right) = a \qquad\qquad \forall a \in A \tag{24}$$

It is a necessary condition for $(End(A), \bullet, \Delta)$ to be a bialgebra. One may be surprised to see that this condition does not seem to involve the chosen scalar product on A... but it does, since D_A itself involves it (and if there would be no chosen scalar product, the composition coproduct Δ would only be defined on the dual of $End(A)$, so that one could not even ask for this compatibility requirement). Remark: we used here the composition coproduct Δ and the convolution product $\bullet$, but exactly the same can be done in the dual picture, ie, taking the convolution coproduct $\Delta_\bullet$ and the composition product $\circ$. The resulting condition is exactly the same.

A.3 $End_\#(A)$ - the graded case

In this subsection we particularize the above discussion to the case of a *graded algebra* A, where $A = \bigoplus_n A_n$ for the underlying vector space, and $m_A : A_n \otimes A_m \to A_{n+m}$. Now we restrict the endomorphisms to be grade-preserving:

$$\begin{aligned} End_\#(A) &= \bigoplus_n End(A_n) \\ &\overset{iso}{\simeq} \bigoplus_n A_n \otimes (A_n)^* \end{aligned}$$

Hence $End_\#(A)$ is a graded space, and the composition product preserves this grading,

$$\circ \,:\, End(A_n) \otimes End(A_k) \longmapsto \delta_{nk}\, End(A_n)$$

As A is graded, its dual A^* can be decomposed as $A^* = \bigoplus_n (A_n)^*$. We can also write, for instance,

$$D_A \,:\, A_n \longmapsto \bigoplus_{k=0,\cdots,n} A_{n-k} \otimes A_k$$

The convolution product (18) turns $End_\#(A)$ into a graded algebra,

$$\bullet \,:\, End(A_n) \otimes End(A_k) \longmapsto End(A_{n+k})$$

as it is easy to see from the explicit expression (19): take a, u of grading n, and a', u' of grading k, thus both aa' and uu' have grading $n+k$.

The composition coproduct defined by (20) has to be restricted with a projector

$$\begin{aligned} P_\# \quad &: \quad End(A) \longmapsto End_\#(A) \\ P_\#\left(a^{(n)} \otimes u^{(k)}\right) &= \delta_{nk}\, a^{(n)} \otimes u^{(k)} \qquad\qquad \forall n,k,\ a^{(n)} \in A_n\,,\ u^{(k)} \in A^*_k \end{aligned}$$

if we want its image to be inside $End_{\#}(A) \otimes End_{\#}(A)$. This comes from the fact that the dual product may be defined just on elements of $\bigoplus_n A_n^* \otimes A_n$ or extended to the whole $A^* \otimes A$. Therefore

$$\begin{aligned} \Delta \quad &: \quad End_{\#}(A) \longmapsto End_{\#}(A) \otimes End_{\#}(A) \\ \Delta(a \otimes u) \quad &= \quad \sum_i P_{\#}\left(a \otimes e^i\right) \otimes P_{\#}\left(e_i \otimes u\right) \end{aligned} \tag{25}$$

where we assumed the basis elements e_i to have definite grade. Writing explicitly the grading of each e_i as $e_I^{(n)} \in A_n$ (now I runs over a basis of A_n) and using the projectors $P_{\#}$ we get

$$\Delta\left(a^{(n)} \otimes u^{(n)}\right) = \sum_I \left(a^{(n)} \otimes e^{(n)I}\right) \otimes \left(e_I^{(n)} \otimes u^{(n)}\right) \qquad a^{(n)} \in A_n\ , \quad u^{(n)} \in A_n^* \tag{26}$$

As before, Δ and $\bullet$ are co/associative, but not necessarily compatible (we want Δ to be an algebra homomorphism for $\bullet$). The necessary condition is a slight modification of the one presented for the non-graded case in the previous section. Take two endomorphisms of definite grade, $\rho_n = a^{(n)} \otimes u^{(n)} \in A_n \otimes A_n^*$ and $\rho'_k = a'^{(k)} \otimes u'^{(k)} \in A_k \otimes A_k^*$, and redo (21) and (22) explicitly incorporating the grading in the notation. Then

$$\begin{aligned} \Delta(\rho \bullet \rho') \quad &= \quad \Delta\left(\left(a^{(n)} \otimes u^{(n)}\right) \bullet \left(a'^{(k)} \otimes u'^{(k)}\right)\right) = \Delta\left(a^{(n)} a'^{(k)} \otimes u^{(n)} u'^{(k)}\right) \\ &= \quad \sum_I \left(a^{(n)} a'^{(k)} \otimes e^{(n+k)I}\right) \otimes \left(e_I^{(n+k)} \otimes u^{(n)} u'^{(k)}\right) \\ \Delta\rho\,(\bullet \otimes \bullet)\,\Delta\rho' \quad &= \quad \sum_{IJ} \left[\left(a^{(n)} \otimes e^{(n)I}\right) \otimes \left(e_I^{(n)} \otimes u^{(n)}\right)\right] (\bullet \otimes \bullet) \\ &\qquad\qquad \left[\left(a'^{(k)} \otimes e^{(k)J}\right) \otimes \left(e_J^{(k)} \otimes u'^{(k)}\right)\right] \\ &= \quad \sum_{IJ} \left(a^{(n)} a'^{(k)} \otimes e^{(n)I} e^{(k)J}\right) \otimes \left(e_I^{(n)} e_J^{(k)} \otimes u^{(n)} u'^{(k)}\right) \end{aligned}$$

Expanding $e_I^{(n)} e_J^{(k)} = \sum_L m_{nI,kJ}^{(n+k)L}\, e_L^{(n+k)}$ (see (13), (15)) and equating both parts we obtain

$$\sum_{KL} \left(\sum_{IJ} \overline{m_{nI,kJ}^{(n+k)K}}\, m_{nI,kJ}^{(n+k)L}\right) e^{(n+k)K} \otimes e_L^{(n+k)} = \sum_K e^{(n+k)K} \otimes e_K^{(n+k)} \qquad \forall n, k$$

This necessary condition for $(End_{\#}(A), \bullet, \Delta)$ to be a bialgebra translates into

$$\sum_{IJ} \overline{m_{nI,kJ}^{(n+k)K}}\, m_{nI,kJ}^{(n+k)L} = \delta^{KL} \qquad \forall n, k \tag{27}$$

It can be written as

$$m_A\left((D_A a)^{(n-k,k)}\right) = a \qquad \forall a = a^{(n)} \in A_n, \quad 0 \le k \le n \tag{28}$$

where $(D_A a)^{(n-k,k)} \in A_{n-k} \otimes A_k$ is the term of the coproduct having first (resp. second) factor of grading $n-k$ (resp. k). The condition (24) should therefore be satisfied *for each term of definite grading* of the coproduct of a.

A.3.1 The case of *Paths*

The above equation (28) is verified in the case of the algebra *Paths*. This is easy to see[11] as $D_A a$ —being dual to m_A— gives all the possible "cuts" of a. Taking a to be an elementary path of length (grading) n, $(D_A a)^{(n-k,k)}$ is simply the cut where the second factor has length k. Concatenating back both factors we re-obtain a again. Thus we have a (graded) bialgebra structure on $End_{\#}(Paths)$. What is less evident is that we have also the same property for $End_{\#}(\mathcal{E})$, when it is endowed with the *graded* multiplication $\bullet$, as proven in this paper (sec. 4).

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