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# On a multiple harmonic power series.

Michel Émery \*

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## Abstract

If  $\text{Li}_s$  denotes the polylogarithm of order  $s$ , where  $s$  is a natural number, and if  $z$  belongs to the unit disk,

$$\text{Li}_s \left( \frac{-z}{1-z} \right) = - \sum_{1 \leq i_1 \leq \dots \leq i_s} \frac{z^{i_s}}{i_1 i_2 \dots i_s}.$$

In particular,

$$\sum_{n \geq 1} \frac{(-1)^{n+1}}{n^s} = \sum_{1 \leq i_1 \leq \dots \leq i_s} \frac{1}{i_1 \dots i_s 2^{i_s}}.$$

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**Keywords:** Multiple harmonic series, Lerch function, Polylogarithm.

## Introduction

Equalities and identities between multiple harmonic series and polylogarithms have been investigated by many authors; see for instance [1] and the references therein. These series usually involve summations over all  $s$ -tuples  $(i_1, \dots, i_s)$  of natural numbers such that  $i_1 < \dots < i_s$ , where  $s$  is fixed. We shall be concerned with an instance where the summation indices may be equal to each other, that is, a sum over all integers  $i_1, \dots, i_s$  verifying  $1 \leq i_1 \leq \dots \leq i_s$ .

**Definition.** For  $\alpha \in \mathbb{C} \setminus \{-1, -2, \dots\}$  and  $s \in \{1, 2, \dots\}$ , the Lerch function of order  $s$  with shift  $\alpha$  is defined by

$$\text{Li}_s^\alpha(w) = \sum_{n \geq 1} \frac{w^n}{(\alpha + n)^s}.$$

The power series converges in the unit disk only, but the analytic function  $\text{Li}_s^\alpha$  extends to the whole complex plane minus a cut along for instance the

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half-line  $[1, \infty)$ ; see § 1.11 of [2]. When  $\alpha = 0$ ,  $\text{Li}_s^\alpha$  is just  $\text{Li}_s$ , the usual polylogarithm of order  $s$ . We shall be interested in the values of  $\text{Li}_s^\alpha(w)$  in the half-plane  $\text{Re}(w) < \frac{1}{2}$  only; they are given by the next proposition (which establishes anew the existence of the analytic extension of  $\text{Li}_s^\alpha$  to that half-plane).

**Proposition.** *For  $\alpha \in \mathbb{C} \setminus \{-1, -2, \dots\}$  and  $s \in \{1, 2, \dots\}$ , one has the following power series expansion, which converges for  $|z| < 1$ :*

$$\begin{aligned} \text{Li}_s^\alpha \left( \frac{-z}{1-z} \right) &= - \sum_{1 \leq i_1 \leq \dots \leq i_s} \frac{(i_s - 1)!}{(\alpha+1)(\alpha+2) \dots (\alpha+i_s)} \frac{z^{i_s}}{(\alpha+i_1)(\alpha+i_2) \dots (\alpha+i_{s-1})}. \end{aligned}$$

When  $\alpha = 0$ , the right-hand side becomes much simpler:

**Corollary.** *Fix  $s \in \{1, 2, \dots\}$ . For  $z$  in the unit disk,*

$$\text{Li}_s \left( \frac{-z}{1-z} \right) = - \sum_{1 \leq i_1 \leq \dots \leq i_s} \frac{z^{i_s}}{i_1 i_2 \dots i_s}.$$

Equivalently, for  $\text{Re } w < \frac{1}{2}$ ,

$$\text{Li}_s(w) = - \sum_{1 \leq i_1 \leq \dots \leq i_s} \frac{1}{i_1 i_2 \dots i_s} \left( \frac{-w}{1-w} \right)^{i_s}.$$

For instance, choosing  $z = \frac{1}{2}$  in the corollary gives

$$\sum_{n \geq 1} \frac{(-1)^{n+1}}{n^s} = \sum_{1 \leq i_1 \leq \dots \leq i_s} \frac{1}{i_1 \dots i_s 2^{i_s}};$$

as the left-hand side is  $(1 - 2^{1-s})\zeta(s)$ , and the right-hand one is  $\sum a_p / (p2^p)$  with  $a_p \leq (1 + \log p)^{s-1}$ , this is a reasonably fast series expansion of  $\zeta(s)$ . More generally, taking  $z = \frac{1}{2}$  in the proposition gives a more rapidly convergent series for  $\sum_n (-1)^n (\alpha + n)^{-s}$ .

### Proof of the proposition

Observe first that the right-hand side in the statement of the proposition is a power series  $\sum c_p z^p$ , with

$$\begin{aligned} |c_p| &\leq \frac{(p-1)!}{|\alpha+1| \dots |\alpha+p|} \sum_{1 \leq i_1 \leq \dots \leq i_{s-1} \leq p} \frac{1}{|\alpha+i_1| \dots |\alpha+i_{s-1}|} \\ &\leq \frac{(p-1)!}{|\alpha+1| \dots |\alpha+p|} \frac{p^{s-1}}{C(\alpha)^{s-1}}, \end{aligned}$$

where  $C(\alpha) = \inf \{|\alpha+1|, |\alpha+2|, \dots\} > 0$ . By d'Alembert's test, the series  $\sum |c_p z^p|$  converges in the unit disk; so does also  $\sum c_p z^p$ . Consequently, to prove the proposition, it suffices to show that both sides of the claimed identity are equal for  $|z|$  small enough. We shall take  $|z| < \frac{1}{2}$ ; for such a  $z$ , one has  $|-z/(1-z)| < 1$ , whence

$$\text{Li}_s^\alpha \left( \frac{-z}{1-z} \right) = \sum_{n \geq 1} \frac{(-1)^n}{(\alpha+n)^s} \left( \frac{z}{1-z} \right)^n = \sum_{n \geq 1} \frac{(-1)^n}{(\alpha+n)^s} \sum_{p \geq n} \binom{p-1}{n-1} z^p.$$

Exchanging the summations gives

$$(*) \quad \text{Li}_s^\alpha \left( \frac{-z}{1-z} \right) = \sum_{p \geq 1} z^p \sum_{n=1}^p \binom{p-1}{n-1} \frac{(-1)^n}{(\alpha+n)^s}.$$

This exchange is licit because, setting  $K(\alpha) = 1/\inf_{n \geq 1} |1+\alpha/n| < \infty$ , one has the estimate

$$\sum_{p \geq n \geq 1} \left| \binom{p-1}{n-1} \frac{(-1)^n z^p}{(\alpha+n)^s} \right| \leq K(\alpha)^s \sum_{p \geq n \geq 1} \binom{p-1}{n-1} \frac{|z|^p}{n^s} = K(\alpha)^s \sum_{n \geq 1} \frac{1}{n^s} \left( \frac{|z|}{1-|z|} \right)^n,$$

which is finite because  $0 \leq |z|/(1-|z|) < 1$ .

To establish the proposition, it remains to compute the coefficient of  $z^p$  in (\*), and more precisely to show that, for all  $\alpha \notin \{-1, -2, \dots\}$ ,  $s \in \{1, 2, \dots\}$  and  $p \geq 1$ ,

$$\sum_{n=1}^p \binom{p-1}{n-1} \frac{(-1)^n}{(\alpha+n)^s} = \begin{cases} -\frac{(p-1)!}{(\alpha+1)_p} \sum_{1 \leq i_1 \leq \dots \leq i_{s-1} \leq p} \prod_{r=1}^{s-1} \frac{1}{\alpha+i_r} & \text{if } s \geq 2, \\ -\frac{(p-1)!}{(\alpha+1)_p} & \text{if } s = 1, \end{cases}$$

where  $(x)_p$  is the Pochhammer symbol standing for  $\prod_{j=0}^{p-1} (x+j)$ .

Putting  $q = p-1$ ,  $m = n-1$  and  $\beta = \alpha+1$ , it suffices to prove the following lemma:

**Lemma.** For all  $\beta \in \mathbb{C} \setminus \{0, -1, \dots\}$  and all integers  $q \geq 0$  and  $s \geq 1$ , one has

$$\sum_{m=0}^q \binom{q}{m} \frac{(-1)^m}{(\beta+m)^s} = \begin{cases} \frac{q!}{(\beta)_{q+1}} \sum_{0 \leq i_1 \leq \dots \leq i_{s-1} \leq q} \prod_{r=1}^{s-1} \frac{1}{\beta+i_r} & \text{if } s \geq 2; \\ \frac{q!}{(\beta)_{q+1}} & \text{if } s = 1. \end{cases}$$

**Proof of the lemma**

Call  $L(q, \beta)$  and  $R(q, \beta)$  the left- and right-hand sides in this statement. First, for  $q = 0$  and  $q = 1$ , the lemma is easily verified:

$$L(0, \beta) = \frac{1}{\beta^s} = R(0, \beta);$$

$$L(1, \beta) = \frac{1}{\beta^s} - \frac{1}{(\beta+1)^s} = \frac{1}{\beta(\beta+1)} \sum_{u+v=s-1} \frac{1}{\beta^u} \frac{1}{(\beta+1)^v} = R(1, \beta).$$

Next, observe that

$$\begin{aligned} L(q+1, \beta) &= \sum_m \binom{q+1}{m} \frac{(-1)^m}{(\beta+m)^s} = \sum_m \left[ \binom{q}{m} + \binom{q}{m-1} \right] \frac{(-1)^m}{(\beta+m)^s} \\ &= \sum_m \binom{q}{m} \left[ \frac{(-1)^m}{(\beta+m)^s} + \frac{(-1)^{m+1}}{(\beta+m+1)^s} \right] \\ &= L(q, \beta) - L(q, \beta+1). \end{aligned}$$

The rest of the proof will consist in verifying that, for  $q \geq 1$ , the right-hand side satisfies the same relation:

$$(**) \quad R(q+1, \beta) = R(q, \beta) - R(q, \beta+1).$$

When this is done, (\*\*) immediately implies that the property

$$\forall \beta \notin \{0, -1, \dots\} \quad L(q, \beta) = R(q, \beta)$$

extends by induction from  $q = 1$  to all  $q \geq 2$ , thus proving the lemma.

The following notation will simplify the proof of (\*\*): for all integers  $n \geq 0$ ,  $t \geq 1$ ,  $a \geq 0$  and  $b \geq a$ , set

$$f_n = \frac{1}{\beta+n}; \quad S_a^b(t) = \sum_{a \leq i_1 \leq \dots \leq i_t \leq b} \prod_{r=1}^t f_{i_r}; \quad S_a^b(0) = 1;$$

and remark that, for  $t \geq 0$  and  $0 \leq a \leq b < c$ ,

$$S_a^c(t) = \sum_{u+v=t} S_a^b(u) S_{b+1}^c(v);$$

similarly, for  $1 \leq a \leq b$ ,

$$S_{a-1}^{b+1}(t) = \sum_{u+v+w=t} S_{a-1}^{a-1}(u) S_a^b(v) S_{b+1}^{b+1}(w) = \sum_{u+v+w=t} f_{a-1}^u S_a^b(v) f_{b+1}^w.$$

Keeping these remarks in mind, the proof of (\*\*) goes as follows. Put  $t = s - 1$  and write

$$\begin{aligned} R(q, \beta) - R(q, \beta+1) &= \frac{q!}{(\beta)_{q+1}} S_0^q(t) - \frac{q!}{(\beta+1)_{q+1}} S_1^{q+1}(t) \\ &= \frac{q!}{(\beta+1)_q} \left[ \frac{1}{\beta} S_0^q(t) - \frac{1}{\beta+q+1} S_1^{q+1}(t) \right]. \end{aligned}$$

The quantity in square brackets can be rewritten as

$$\begin{aligned}
& f_0 \sum_{v=0}^t S_0^0(t-v) S_1^q(v) - f_{q+1} \sum_{v=0}^t S_1^q(v) S_{q+1}^{q+1}(t-v) \\
&= \sum_{v=0}^t S_1^q(v) (f_0^{t-v+1} - f_{q+1}^{t-v+1}) = \sum_{v=0}^t S_1^q(v) (f_0 - f_{q+1}) \sum_{u+w=t-v} f_0^u f_{q+1}^w \\
&= (f_0 - f_{q+1}) \sum_{u+v+w=t} f_0^u S_1^q(v) f_{q+1}^w = (f_0 - f_{q+1}) S_0^{q+1}(t) \\
&= \frac{q+1}{\beta(\beta+q+1)} S_0^{q+1}(t).
\end{aligned}$$

Finally, the difference  $R(q, \beta) - R(q, \beta+1)$  amounts to

$$\frac{q!}{(\beta+1)_q} \frac{q+1}{\beta(\beta+q+1)} S_0^{q+1}(t) = \frac{(q+1)!}{(\beta)_{q+2}} S_0^{q+1}(t) = R(q+1, \beta).$$

This proves (\*\*), and at the same time the lemma and the proposition.

**Remark.** In the particular case when  $\alpha = 0$  and  $z = \frac{1}{2}$ , (\*) is exactly formula (4) of J. Sondow [3], obtained by accelerating convergence of the alternating zeta series. What the lemma does is computing the numerator in that formula. To that end, (\*\*) is needed for  $\beta \in \{1, 2, \dots\}$  only; but the general proof is just as easy.

## References

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