



HAL
open science

Verma modules and preprojective algebras

Christof Geiss, Bernard Leclerc, Jan Schröer

► **To cite this version:**

Christof Geiss, Bernard Leclerc, Jan Schröer. Verma modules and preprojective algebras. Nagoya Mathematical Journal, 2006, 182, pp.241-258. hal-00003229v2

HAL Id: hal-00003229

<https://hal.science/hal-00003229v2>

Submitted on 29 Jul 2005

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Verma modules and preprojective algebras

Christof GEISS ^{*}, Bernard LECLERC [†] and Jan SCHRÖER [‡]

Abstract

We give a geometric construction of the Verma modules of a symmetric Kac-Moody Lie algebra \mathfrak{g} in terms of constructible functions on the varieties of nilpotent finite-dimensional modules of the corresponding preprojective algebra Λ .

1 Introduction

Let \mathfrak{g} be the symmetric Kac-Moody Lie algebra associated to a finite unoriented graph Γ without loop. Let \mathfrak{n}_- denote a maximal nilpotent subalgebra of \mathfrak{g} . In [Lu1, §12], Lusztig has given a geometric construction of $U(\mathfrak{n}_-)$ in terms of certain Lagrangian varieties. These varieties can be interpreted as module varieties for the preprojective algebra Λ attached to the graph Γ by Gelfand and Ponomarev [GP]. In Lusztig's construction, $U(\mathfrak{n}_-)$ gets identified with an algebra $(\mathcal{M}, *)$ of constructible functions on these varieties, where $*$ is a convolution product inspired by Ringel's multiplication for Hall algebras.

Later, Nakajima gave a similar construction of the highest weight irreducible integrable \mathfrak{g} -modules $L(\lambda)$ in terms of some new Lagrangian varieties which differ from Lusztig's ones by the introduction of some extra vector spaces W_k for each vertex k of Γ , and by considering only stable points instead of the whole variety [Na, §10].

The aim of this paper is to extend Lusztig's original construction and to endow \mathcal{M} with the structure of a Verma module $M(\lambda)$.

To do this we first give a variant of the geometrical construction of the integrable \mathfrak{g} -modules $L(\lambda)$, using functions on some natural open subvarieties of Lusztig's varieties instead of functions on Nakajima's varieties (Theorem 1). These varieties have a simple description in terms of the preprojective algebra Λ and of certain injective Λ -modules q_λ .

Having realized the integrable modules $L(\lambda)$ as quotients of \mathcal{M} , it is possible, using the co-multiplication of $U(\mathfrak{n}_-)$, to construct geometrically the raising operators $E_i^\lambda \in \text{End}(\mathcal{M})$ which make \mathcal{M} into the Verma module $M(\lambda)$ (Theorem 2). Note that we manage in this way to realize Verma modules with arbitrary highest weight (not necessarily dominant).

Finally, we dualize this setting and give a geometric construction of the dual Verma module $M(\lambda)^*$ in terms of the delta functions $\delta_x \in \mathcal{M}^*$ attached to the finite-dimensional nilpotent Λ -modules x (Theorem 3).

^{*}C. Geiss acknowledges support from DGAPA grant IN101402-3.

[†]B. Leclerc is grateful to the GDR 2432 and the GDR 2249 for their support.

[‡]J. Schröer was supported by a research fellowship from the DFG (Deutsche Forschungsgemeinschaft).

2 Verma modules

2.1 Let \mathfrak{g} be the symmetric Kac-Moody Lie algebra associated with a finite unoriented graph Γ without loop. The set of vertices of the graph is denoted by I . The (generalized) Cartan matrix of \mathfrak{g} is $A = (a_{ij})_{i,j \in I}$, where $a_{ii} = 2$ and, for $i \neq j$, $-a_{ij}$ is the number of edges between i and j .

2.2 Let $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ be a Cartan decomposition of \mathfrak{g} , where \mathfrak{h} is a Cartan subalgebra and $(\mathfrak{n}, \mathfrak{n}_-)$ a pair of opposite maximal nilpotent subalgebras. Let $\mathfrak{b} = \mathfrak{n} \oplus \mathfrak{h}$. The Chevalley generators of \mathfrak{n} (*resp.* \mathfrak{n}_-) are denoted by e_i ($i \in I$) (*resp.* f_i) and we set $h_i = [e_i, f_i]$.

2.3 Let α_i denote the simple root of \mathfrak{g} associated with $i \in I$. Let $(-; -)$ be a symmetric bilinear form on \mathfrak{h}^* such that $(\alpha_i; \alpha_j) = a_{ij}$. The lattice of integral weights in \mathfrak{h}^* is denoted by P , and the sublattice spanned by the simple roots is denoted by Q . We put

$$P_+ = \{\lambda \in P \mid (\lambda; \alpha_i) \geq 0, (i \in I)\}, \quad Q_+ = Q \cap P_+.$$

2.4 Let $\lambda \in P$ and let $M(\lambda)$ be the Verma module with highest weight λ . This is the induced \mathfrak{g} -module defined by $M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C} u_\lambda$, where u_λ is a basis of the one-dimensional representation of \mathfrak{b} given by

$$h u_\lambda = \lambda(h) u_\lambda, \quad n u_\lambda = 0, \quad (h \in \mathfrak{h}, n \in \mathfrak{n}).$$

As a P -graded vector space $M(\lambda) \cong U(\mathfrak{n}_-)$ (up to a degree shift by λ). $M(\lambda)$ has a unique simple quotient denoted by $L(\lambda)$, which is integrable if and only if $\lambda \in P_+$. In this case, the kernel of the \mathfrak{g} -homomorphism $M(\lambda) \rightarrow L(\lambda)$ is the \mathfrak{g} -module $I(\lambda)$ generated by the vectors

$$f_i^{(\lambda; \alpha_i)+1} \otimes u_\lambda, \quad (i \in I).$$

3 Constructible functions

3.1 Let X be an algebraic variety over \mathbb{C} endowed with its Zariski topology. A map f from X to a vector space V is said to be constructible if its image $f(X)$ is finite, and for each $v \in f(X)$ the preimage $f^{-1}(v)$ is a constructible subset of X .

3.2 By $\chi(A)$ we denote the Euler characteristic of a constructible subset A of X . For a constructible map $f : X \rightarrow V$ one defines

$$\int_{x \in X} f(x) = \sum_{v \in V} \chi(f^{-1}(v)) v \in V.$$

More generally, for a constructible subset A of X we write

$$\int_{x \in A} f(x) = \sum_{v \in V} \chi(f^{-1}(v) \cap A) v.$$

4 Preprojective algebras

4.1 Let Λ be the preprojective algebra associated to the graph Γ (see for example [Ri, GLS]). This is an associative \mathbb{C} -algebra, which is finite-dimensional if and only if Γ is a graph of type A, D, E . Let s_i denote the simple one-dimensional Λ -module associated with $i \in I$, and let p_i be its projective cover and q_i its injective hull. Again, p_i and q_i are finite-dimensional if and only if Γ is a graph of type A, D, E .

4.2 A finite-dimensional Λ -module x is nilpotent if and only if it has a composition series with all factors of the form s_i ($i \in I$). We will identify the dimension vector of x with an element $\beta \in Q_+$ by setting $\mathbf{dim}(s_i) = \alpha_i$.

4.3 Let q be an injective Λ -module of the form

$$q = \bigoplus_{i \in I} q_i^{\oplus a_i}$$

for some nonnegative integers a_i ($i \in I$).

Lemma 1 *Let x be a finite-dimensional Λ -module isomorphic to a submodule of q . If $f_1 : x \rightarrow q$ and $f_2 : x \rightarrow q$ are two monomorphisms, then there exists an automorphism $g : q \rightarrow q$ such that $f_2 = gf_1$.*

Proof — Indeed, q is the injective hull of its socle $b = \bigoplus_{i \in I} s_i^{\oplus a_i}$. Let c_j ($j = 1, 2$) be a complement of $f_j(\text{socle}(x))$ in b . Then $c_1 \cong c_2$ and the maps

$$h_j := f_j \oplus \text{id} : x \oplus c_j \rightarrow q, \quad (j = 1, 2)$$

are injective hulls. The result then follows from the unicity of the injective hull. \square

Hence, up to isomorphism, there is a unique way to embed x into q .

4.4 Let \mathcal{M} be the algebra of constructible functions on the varieties of finite-dimensional nilpotent Λ -modules defined by Lusztig [Lu2] to give a geometric realization of $U(\mathfrak{n}_-)$. We recall its definition.

For $\beta = \sum_{i \in I} b_i \alpha_i \in Q_+$, let Λ_β denote the variety of nilpotent Λ -modules with dimension vector β . Recall that Λ_β is endowed with an action of the algebraic group $G_\beta = \prod_{i \in I} GL_{b_i}(\mathbb{C})$, so that two points of Λ_β are isomorphic as Λ -modules if and only if they belong to the same G_β -orbit. Let $\widetilde{\mathcal{M}}_\beta$ denote the vector space of constructible functions from Λ_β to \mathbb{C} which are constant on G_β -orbits. Let

$$\widetilde{\mathcal{M}} = \bigoplus_{\beta \in Q_+} \widetilde{\mathcal{M}}_\beta.$$

One defines a multiplication $*$ on $\widetilde{\mathcal{M}}$ as follows. For $f \in \widetilde{\mathcal{M}}_\beta$, $g \in \widetilde{\mathcal{M}}_\gamma$ and $x \in \Lambda_{\beta+\gamma}$, we have

$$(f * g)(x) = \int_U f(x')g(x''), \quad (1)$$

where the integral is over the variety of x -stable subspaces U of x of dimension γ , x'' is the Λ -submodule of x obtained by restriction to U and $x' = x/x''$. In the sequel in order to simplify

notation, we will not distinguish between the subspace U and the submodule x'' of x carried by U . Thus we shall rather write

$$(f * g)(x) = \int_{x''} f(x/x'')g(x''), \quad (2)$$

where the integral is over the variety of submodules x'' of x of dimension γ .

For $i \in I$, the variety Λ_{α_i} is reduced to a single point : the simple module s_i . Denote by $\mathbf{1}_i$ the function mapping this point to 1. Let $\mathcal{G}(i, x)$ denote the variety of all submodules y of x such that $x/y \cong s_i$. Then by (2) we have

$$(\mathbf{1}_i * g)(x) = \int_{y \in \mathcal{G}(i, x)} g(y). \quad (3)$$

Let \mathcal{M} denote the subalgebra of $\widetilde{\mathcal{M}}$ generated by the functions $\mathbf{1}_i$ ($i \in I$). By Lusztig [Lu2], $(\mathcal{M}, *)$ is isomorphic to $U(\mathfrak{n}_-)$ by mapping $\mathbf{1}_i$ to the Chevalley generator f_i .

4.5 In the identification of $U(\mathfrak{n}_-)$ with \mathcal{M} , formula (3) represents the left multiplication by f_i . In order to endow \mathcal{M} with the structure of a Verma module we need to introduce the following important definition. For $\nu \in P_+$, let

$$q_\nu = \bigoplus_{i \in I} q_i^{\oplus(\nu; \alpha_i)}.$$

Lusztig has shown [Lu3, §2.1] that Nakajima's Lagrangian varieties for the geometric realization of $L(\nu)$ are isomorphic to the Grassmann varieties of Λ -submodules of q_ν with a given dimension vector.

Let x be a finite-dimensional nilpotent Λ -module isomorphic to a submodule of the injective module q_ν . Let us fix an embedding $F : x \rightarrow q_\nu$ and identify x with a submodule of q_ν via F .

Definition 1 For $i \in I$ let $\mathcal{G}(x, \nu, i)$ be the variety of submodules y of q_ν containing x and such that y/x is isomorphic to s_i .

This is a projective variety which, by 4.3, depends only (up to isomorphism) on i , ν and the isoclass of x .

5 Geometric realization of integrable irreducible \mathfrak{g} -modules

5.1 For $\lambda \in P_+$ and $\beta \in Q_+$, let Λ_β^λ denote the variety of nilpotent Λ -modules of dimension vector β which are isomorphic to a submodule of q_λ . Equivalently Λ_β^λ consists of the nilpotent modules of dimension vector β whose socle contains s_i with multiplicity at most $(\lambda; \alpha_i)$ ($i \in I$). This variety has been considered by Lusztig [Lu4, §1.5]. In particular it is known that Λ_β^λ is an open subset of Λ_β , and that the number of its irreducible components is equal to the dimension of the $(\lambda - \beta)$ -weight space of $L(\lambda)$.

5.2 Define $\widetilde{\mathcal{M}}_\beta^\lambda$ to be the vector space of constructible functions on Λ_β^λ which are constant on G_β -orbits. Let $\mathcal{M}_\beta^\lambda$ denote the subspace of $\widetilde{\mathcal{M}}_\beta^\lambda$ obtained by restricting elements of \mathcal{M}_β to Λ_β^λ .

Put $\widetilde{\mathcal{M}}^\lambda = \bigoplus_\beta \widetilde{\mathcal{M}}_\beta^\lambda$ and $\mathcal{M}^\lambda = \bigoplus_\beta \mathcal{M}_\beta^\lambda$. For $i \in I$ define endomorphisms E_i, F_i, H_i of $\widetilde{\mathcal{M}}^\lambda$ as follows:

$$(E_i f)(x) = \int_{y \in \mathcal{G}(x, \lambda, i)} f(y), \quad (f \in \widetilde{\mathcal{M}}_\beta^\lambda, x \in \Lambda_{\beta - \alpha_i}^\lambda), \quad (4)$$

$$(F_i f)(x) = \int_{y \in \mathcal{G}(i, x)} f(y), \quad (f \in \widetilde{\mathcal{M}}_\beta^\lambda, x \in \Lambda_{\beta + \alpha_i}^\lambda), \quad (5)$$

$$(H_i f)(x) = (\lambda - \beta; \alpha_i) f(x), \quad (f \in \widetilde{\mathcal{M}}_\beta^\lambda, x \in \Lambda_\beta^\lambda). \quad (6)$$

Theorem 1 *The endomorphisms E_i, F_i, H_i of $\widetilde{\mathcal{M}}^\lambda$ leave stable the subspace \mathcal{M}^λ . Denote again by E_i, F_i, H_i the induced endomorphisms of \mathcal{M}^λ . Then the assignments $e_i \mapsto E_i, f_i \mapsto F_i, h_i \mapsto H_i$, give a representation of \mathfrak{g} on \mathcal{M}^λ isomorphic to the irreducible representation $L(\lambda)$.*

5.3 The proof of Theorem 1 will involve a series of lemmas.

5.3.1 For $\mathbf{i} = (i_1, \dots, i_r) \in I^r$ and $\mathbf{a} = (a_1, \dots, a_r) \in \mathbb{N}^r$, define the variety $\mathcal{G}(x, \lambda, (\mathbf{i}, \mathbf{a}))$ of flags of Λ -modules

$$\mathfrak{f} = (x = y_0 \subset y_1 \subset \dots \subset y_r \subset q_\lambda)$$

with $y_k/y_{k-1} \cong s_{i_k}^{\oplus a_k}$ ($1 \leq k \leq r$). As in Definition 1, this is a projective variety depending (up to isomorphism) only on (\mathbf{i}, \mathbf{a}) , λ and the isoclass of x and not on the choice of a specific embedding of x into q_λ .

Lemma 2 *Let $f \in \widetilde{\mathcal{M}}_\beta^\lambda$ and $x \in \Lambda_{\beta - a_1 \alpha_{i_1} - \dots - a_r \alpha_{i_r}}^\lambda$. Put $E_i^{(a)} = (1/a!) E_i^a$. We have*

$$(E_{i_r}^{(a_r)} \dots E_{i_1}^{(a_1)} f)(x) = \int_{\mathfrak{f} \in \mathcal{G}(x, \lambda, (\mathbf{i}, \mathbf{a}))} f(y_r).$$

The proof is standard and will be omitted.

5.3.2 By [Lu1, 12.11] the endomorphisms F_i satisfy the Serre relations

$$\sum_{p=0}^{1-a_{ij}} (-1)^p F_j^{(p)} F_i F_j^{(1-a_{ij}-p)} = 0$$

for every $i \neq j$. A similar argument shows that

Lemma 3 *The endomorphisms E_i satisfy the Serre relations*

$$\sum_{p=0}^{1-a_{ij}} (-1)^p E_j^{(p)} E_i E_j^{(1-a_{ij}-p)} = 0$$

for every $i \neq j$.

Proof— Let $f \in \widetilde{\mathcal{M}}_\beta^\lambda$ and $x \in \Lambda_{\beta - \alpha_i - (1-a_{ij})\alpha_j}^\lambda$. By Lemma 2,

$$(E_j^{(p)} E_i E_j^{(1-a_{ij}-p)} f)(x) = \int_{\mathfrak{f}} f(y_3)$$

the integral being taken on the variety of flags

$$\mathfrak{f} = (x \subset y_1 \subset y_2 \subset y_3 \subset q_\lambda)$$

with $y_1/x \cong s_j^{\oplus 1-a_{ij}-p}$, $y_2/y_1 \cong s_i$ and $y_3/y_2 \cong s_j^{\oplus p}$. This integral can be rewritten as

$$\int_{y_3} f(y_3) \chi(\mathcal{F}[y_3; p])$$

where the integral is now over all submodules y_3 of q_λ of dimension β containing x and $\mathcal{F}[y_3; p]$ is the variety of flags \mathfrak{f} as above with fixed last step y_3 . Now, by moding out the submodule x at each step of the flag, we are reduced to the same situation as in [Lu1, 12.11], and the same argument allows to show that

$$\sum_{p=0}^{1-a_{ij}} \chi(\mathcal{F}[y_3; p]) = 0,$$

which proves the Lemma. \square

5.3.3 Let $x \in \Lambda_\beta^\lambda$. Let $\varepsilon_i(x)$ denote the multiplicity of s_i in the head of x . Let $\varphi_i(x)$ denote the multiplicity of s_i in the socle of q_λ/x .

Lemma 4 Let $i, j \in I$ (not necessarily distinct). Let y be a submodule of q_λ containing x and such that $y/x \cong s_j$. Then

$$\varphi_i(y) - \varepsilon_i(y) = \varphi_i(x) - \varepsilon_i(x) - a_{ij}.$$

Proof — We have short exact sequences

$$0 \rightarrow x \rightarrow q_\lambda \rightarrow q_\lambda/x \rightarrow 0, \quad (7)$$

$$0 \rightarrow y \rightarrow q_\lambda \rightarrow q_\lambda/y \rightarrow 0, \quad (8)$$

$$0 \rightarrow x \rightarrow y \rightarrow s_j \rightarrow 0, \quad (9)$$

$$0 \rightarrow s_j \rightarrow q_\lambda/x \rightarrow q_\lambda/y \rightarrow 0. \quad (10)$$

Clearly, $\varepsilon_i(x) = |\mathrm{Hom}_\Lambda(x, s_i)|$, the dimension of $\mathrm{Hom}_\Lambda(x, s_i)$. Similarly $\varepsilon_i(y) = |\mathrm{Hom}_\Lambda(y, s_i)|$, $\varphi_i(x) = |\mathrm{Hom}_\Lambda(s_i, q_\lambda/x)|$, $\varphi_i(y) = |\mathrm{Hom}_\Lambda(s_i, q_\lambda/y)|$. Hence we have to show that

$$|\mathrm{Hom}_\Lambda(x, s_i)| - |\mathrm{Hom}_\Lambda(y, s_i)| = |\mathrm{Hom}_\Lambda(s_i, q_\lambda/x)| - |\mathrm{Hom}_\Lambda(s_i, q_\lambda/y)| - a_{ij}. \quad (11)$$

In our proof, we will use a property of preprojective algebras proved in [CB, §1], namely, for any finite-dimensional Λ -modules m and n there holds

$$|\mathrm{Ext}_\Lambda^1(m, n)| = |\mathrm{Ext}_\Lambda^1(n, m)|. \quad (12)$$

(a) If $i = j$ then $a_{ij} = 2$, $|\mathrm{Hom}_\Lambda(s_j, s_i)| = 1$ and $|\mathrm{Ext}_\Lambda^1(s_j, s_i)| = 0$ since Γ has no loops. Applying $\mathrm{Hom}_\Lambda(-, s_i)$ to (9) we get the exact sequence

$$0 \rightarrow \mathrm{Hom}_\Lambda(s_j, s_i) \rightarrow \mathrm{Hom}_\Lambda(y, s_i) \rightarrow \mathrm{Hom}_\Lambda(x, s_i) \rightarrow 0,$$

hence

$$|\mathrm{Hom}_\Lambda(x, s_i)| - |\mathrm{Hom}_\Lambda(y, s_i)| = -1.$$

Similarly applying $\text{Hom}_\Lambda(s_i, -)$ to (10) we get an exact sequence

$$0 \rightarrow \text{Hom}_\Lambda(s_i, s_j) \rightarrow \text{Hom}_\Lambda(s_i, q_\lambda/x) \rightarrow \text{Hom}_\Lambda(s_i, q_\lambda/y) \rightarrow 0,$$

hence

$$|\text{Hom}_\Lambda(s_i, q_\lambda/x)| - |\text{Hom}_\Lambda(s_i, q_\lambda/y)| = 1,$$

and (11) follows.

(b) If $i \neq j$, we have $|\text{Hom}_\Lambda(s_i, s_j)| = 0$ and $|\text{Ext}_\Lambda^1(s_i, s_j)| = |\text{Ext}_\Lambda^1(s_j, s_i)| = -a_{ij}$. Applying $\text{Hom}_\Lambda(s_i, -)$ to (9) we get an exact sequence

$$0 \rightarrow \text{Hom}_\Lambda(s_i, x) \rightarrow \text{Hom}_\Lambda(s_i, y) \rightarrow 0,$$

hence

$$|\text{Hom}_\Lambda(s_i, x)| - |\text{Hom}_\Lambda(s_i, y)| = 0. \quad (13)$$

Moreover, by [Bo, §1.1], $|\text{Ext}_\Lambda^2(s_i, s_j)| = 0$ because there are no relations from i to j in the defining relations of Λ . (Note that the proof of this result in [Bo] only requires that $I \subseteq J^2$ (here we use the notation of [Bo]). One does not need the additional assumption $J^n \subseteq I$ for some n . Compare also the discussion in [BK].)

Since q_λ is injective $|\text{Ext}_\Lambda^1(s_i, q_\lambda)| = 0$, thus applying $\text{Hom}_\Lambda(s_i, -)$ to (7) we get an exact sequence

$$0 \rightarrow \text{Hom}_\Lambda(s_i, x) \rightarrow \text{Hom}_\Lambda(s_i, q_\lambda) \rightarrow \text{Hom}_\Lambda(s_i, q_\lambda/x) \rightarrow \text{Ext}_\Lambda^1(s_i, x) \rightarrow 0,$$

hence

$$|\text{Hom}_\Lambda(s_i, x)| - |\text{Hom}_\Lambda(s_i, q_\lambda)| + |\text{Hom}_\Lambda(s_i, q_\lambda/x)| - |\text{Ext}_\Lambda^1(s_i, x)| = 0. \quad (14)$$

Similarly, applying $\text{Hom}_\Lambda(s_i, -)$ to (8) we get

$$|\text{Hom}_\Lambda(s_i, y)| - |\text{Hom}_\Lambda(s_i, q_\lambda)| + |\text{Hom}_\Lambda(s_i, q_\lambda/y)| - |\text{Ext}_\Lambda^1(s_i, y)| = 0. \quad (15)$$

Subtracting (14) from (15) and taking into account (12) and (13) we obtain

$$|\text{Ext}_\Lambda^1(x, s_i)| - |\text{Ext}_\Lambda^1(y, s_i)| = |\text{Hom}_\Lambda(s_i, q_\lambda/x)| - |\text{Hom}_\Lambda(s_i, q_\lambda/y)|. \quad (16)$$

Now applying $\text{Hom}_\Lambda(-, s_i)$ to (9) we get the long exact sequence

$$0 \rightarrow \text{Hom}_\Lambda(y, s_i) \rightarrow \text{Hom}_\Lambda(x, s_i) \rightarrow \text{Ext}_\Lambda^1(s_j, s_i) \rightarrow \text{Ext}_\Lambda^1(y, s_i) \rightarrow \text{Ext}_\Lambda^1(x, s_i) \rightarrow 0,$$

hence

$$|\text{Hom}_\Lambda(y, s_i)| - |\text{Hom}_\Lambda(x, s_i)| - a_{ij} - |\text{Ext}_\Lambda^1(y, s_i)| + |\text{Ext}_\Lambda^1(x, s_i)| = 0,$$

thus, taking into account (16), we have proved (11). \square

Lemma 5 *With the same notation we have*

$$\varphi_i(x) - \varepsilon_i(x) = (\lambda - \beta; \alpha_i).$$

Proof— We use an induction on the height of β . If $\beta = 0$ then x is the zero module and $\varepsilon_i(x) = 0$. On the other hand $q_\lambda/x = q_\lambda$ and $\varphi_i(x) = (\lambda; \alpha_i)$ by definition of q_λ . Now assume that the lemma holds for $x \in \Lambda_\beta^\lambda$ and let $y \in \Lambda_{\beta+\alpha_j}^\lambda$ be a submodule of q_λ containing x . Using Lemma 4 we get that

$$\varphi_i(y) - \varepsilon_i(y) = (\lambda - \beta; \alpha_i) - a_{ij} = (\lambda - \beta - \alpha_j; \alpha_i),$$

as required, and the lemma follows. \square

Lemma 6 *Let $f \in \widetilde{\mathcal{M}}_\beta^\lambda$. We have*

$$(E_i F_j - F_j E_i)(f) = \delta_{ij}(\lambda - \beta; \alpha_i) f.$$

Proof— Let $x \in \Lambda_{\beta-\alpha_i+\alpha_j}^\lambda$. By definition of E_i and F_j we have

$$(E_i F_j f)(x) = \int_{\mathfrak{p} \in \mathfrak{P}} f(y)$$

where \mathfrak{P} denotes the variety of pairs $\mathfrak{p} = (u, y)$ of submodules of q_λ with $x \subset u, y \subset u, u/x \cong s_i$ and $u/y \cong s_j$. Similarly,

$$(F_j E_i f)(x) = \int_{\mathfrak{q} \in \mathfrak{Q}} f(y)$$

where \mathfrak{Q} denotes the variety of pairs $\mathfrak{q} = (v, y)$ of submodules of q_λ with $v \subset x, v \subset y, x/v \cong s_j$ and $y/v \cong s_i$.

Consider a submodule y such that there exists in \mathfrak{P} (resp. in \mathfrak{Q}) at least one pair of the form (u, y) (resp. (v, y)). Clearly, the subspaces carrying the submodules x and y have the same dimension d and their intersection has dimension at least $d - 1$. If this intersection has dimension exactly $d - 1$ then there is a unique pair (u, y) (resp. (v, y)), namely $(x + y, y)$ (resp. $(x \cap y, y)$). This means that

$$\int_{\mathfrak{p} \in \mathfrak{P}; y \neq x} f(y) = \int_{\mathfrak{q} \in \mathfrak{Q}; y \neq x} f(y).$$

In particular, since when $i \neq j$ we cannot have $y = x$, it follows that

$$(E_i F_j - F_j E_i)(f) = 0, \quad (i \neq j).$$

On the other hand if $i = j$ we have

$$((E_i F_i - F_i E_i)(f))(x) = f(x)(\chi(\mathfrak{P}') - \chi(\mathfrak{Q}'))$$

where \mathfrak{P}' is the variety of submodules u of q_λ containing x such that $u/x \cong s_i$, and \mathfrak{Q}' is the variety of submodules v of x such that $x/v \cong s_i$. Clearly we have $\chi(\mathfrak{Q}') = \varepsilon_i(x)$ and $\chi(\mathfrak{P}') = \varphi_i(x)$. The result then follows from Lemma 5. \square

5.3.4 The following relations for the endomorphisms E_i, F_i, H_i of $\widetilde{\mathcal{M}}^\lambda$ are easily checked

$$[H_i, H_j] = 0, \quad [H_i, E_j] = a_{ij} E_j, \quad [H_i, F_j] = -a_{ij} F_j.$$

The verification is left to the reader. Hence, using Lemmas 3 and 6, we have proved that the assignments $e_i \mapsto E_i, f_i \mapsto F_i, h_i \mapsto H_i$, give a representation of \mathfrak{g} on $\widetilde{\mathcal{M}}^\lambda$.

Lemma 7 *The endomorphisms E_i, F_i, H_i leave stable the subspace \mathcal{M}^λ .*

Proof — It is obvious for H_i , and it follows from the definition of \mathcal{M}^λ for F_i . It remains to prove that if $f \in \mathcal{M}_\beta^\lambda$ then $E_i f \in \mathcal{M}_{\beta-\alpha_i}^\lambda$. We shall use induction on the height of β . We can assume that f is of the form $F_j g$ for some $g \in \mathcal{M}_{\beta-\alpha_j}^\lambda$. By induction we can also assume that $E_i g \in \mathcal{M}_{\beta-\alpha_i-\alpha_j}^\lambda$. We have

$$E_i f = E_i F_j g = F_j E_i g + \delta_{ij}(\lambda - \beta + \alpha_j; \alpha_i)g,$$

and the right-hand side clearly belongs to $\mathcal{M}_{\beta-\alpha_i}^\lambda$. \square

Lemma 8 *The representation of \mathfrak{g} carried by \mathcal{M}^λ is isomorphic to $L(\lambda)$.*

Proof — For all $f \in \mathcal{M}_\beta$ and all $x \in \Lambda_{\beta+(a_i+1)\alpha_i}^\lambda$ we have $f * \mathbf{1}_i^{*(a_i+1)}(x) = 0$. Indeed, by definition of Λ^λ the socle of x contains s_i with multiplicity at most a_i . Therefore the left ideal of \mathcal{M} generated by the functions $\mathbf{1}_i^{*(a_i+1)}$ is mapped to zero by the linear map $\mathcal{M} \rightarrow \mathcal{M}^\lambda$ sending a function f on Λ_β to its restriction to Λ_β^λ . It follows that for all β the dimension of $\mathcal{M}_\beta^\lambda$ is at most the dimension of the $(\lambda - \beta)$ -weight space of $L(\lambda)$.

On the other hand, the function $\mathbf{1}_0$ mapping the zero Λ -module to 1 is a highest weight vector of \mathcal{M}^λ of weight λ . Hence $\mathbf{1}_0 \in \mathcal{M}^\lambda$ generates a quotient of the Verma module $M(\lambda)$, and since $L(\lambda)$ is the smallest quotient of $M(\lambda)$ we must have $\mathcal{M}^\lambda = L(\lambda)$. \square

This finishes the proof of Theorem 1.

6 Geometric realization of Verma modules

6.1 Let $\beta \in Q_+$ and $x \in \Lambda_{\beta-\alpha_i}$. Let $q = \bigoplus_{i \in I} q_i^{\oplus a_i}$ be the injective hull of x . For every $\nu \in P_+$ such that $(\nu; \alpha_i) \geq a_i$ the injective module q_ν contains a submodule isomorphic to x . Hence, for such a weight ν and for any $f \in \mathcal{M}_\beta$, the integral

$$\int_{y \in \mathcal{G}(x, \nu, i)} f(y)$$

is well-defined.

Proposition 1 *Let $\lambda \in P$ and choose $\nu \in P_+$ such that $(\nu; \alpha_i) \geq a_i$ for all $i \in I$. The number*

$$\int_{y \in \mathcal{G}(x, \nu, i)} f(y) - (\nu - \lambda; \alpha_i) f(x \oplus s_i) \tag{17}$$

does not depend on the choice of ν . Denote this number by $(E_i^\lambda f)(x)$. Then, the function

$$E_i^\lambda f : x \mapsto (E_i^\lambda f)(x)$$

belongs to $\mathcal{M}_{\beta-\alpha_i}$.

Denote by E_i^λ the endomorphism of \mathcal{M} mapping $f \in \mathcal{M}_\beta$ to $E_i^\lambda f$. Notice that Formula (5), which is nothing but (3), also defines an endomorphism of \mathcal{M} independent of λ which we again denote by F_i . Finally Formula (6) makes sense for any λ , not necessarily dominant, and any $f \in \mathcal{M}_\beta$. This gives an endomorphism of \mathcal{M} that we shall denote by H_i^λ .

Theorem 2 *The assignments $e_i \mapsto E_i^\lambda$, $f_i \mapsto F_i$, $h_i \mapsto H_i^\lambda$, give a representation of \mathfrak{g} on \mathcal{M} isomorphic to the Verma module $M(\lambda)$.*

The rest of this section is devoted to the proofs of Proposition 1 and Theorem 2.

6.2 Denote by e_i^λ the endomorphism of the Verma module $M(\lambda)$ implementing the action of the Chevalley generator e_i . Let \mathcal{E}_i^λ denote the endomorphism of $U(\mathfrak{n}_-)$ obtained by transporting e_i^λ via the natural identification $M(\lambda) \cong U(\mathfrak{n}_-)$. Let Δ be the comultiplication of $U(\mathfrak{n}_-)$.

Lemma 9 For $\lambda, \mu \in P$ and $u \in U(\mathfrak{n}_-)$ we have

$$\Delta(\mathcal{E}_i^{\lambda+\mu}u) = (\mathcal{E}_i^\lambda \otimes 1 + 1 \otimes \mathcal{E}_i^\mu)\Delta u.$$

Proof — By linearity it is enough to prove this for u of the form $u = f_{i_1} \cdots f_{i_r}$. A simple calculation in $U(\mathfrak{g})$ shows that

$$\begin{aligned} e_i f_{i_1} \cdots f_{i_r} &= f_{i_1} \cdots f_{i_r} e_i + \sum_{k=1}^r \delta_{ii_k} f_{i_1} \cdots f_{i_{k-1}} h_i f_{i_{k+1}} \cdots f_{i_r} \\ &= f_{i_1} \cdots f_{i_r} e_i + \sum_{k=1}^r \delta_{ii_k} \left(f_{i_1} \cdots f_{i_{k-1}} f_{i_{k+1}} \cdots f_{i_r} h_i - \left(\sum_{s=k+1}^r a_{ii_s} \right) f_{i_1} \cdots f_{i_{k-1}} f_{i_{k+1}} \cdots f_{i_r} \right). \end{aligned}$$

It follows that, for $\nu \in P$,

$$\mathcal{E}_i^\nu(f_{i_1} \cdots f_{i_r}) = \sum_{k=1}^r \delta_{ii_k} \left((\nu; \alpha_i) - \sum_{s=k+1}^r a_{ii_s} \right) f_{i_1} \cdots f_{i_{k-1}} f_{i_{k+1}} \cdots f_{i_r}.$$

Now, using that Δ is the algebra homomorphism defined by $\Delta(f_i) = f_i \otimes 1 + 1 \otimes f_i$, one can finish the proof of the lemma. Details are omitted. \square

6.3 We endow $U(\mathfrak{n}_-)$ with the Q_+ -grading given by $\deg(f_i) = \alpha_i$. Let u be a homogeneous element of $U(\mathfrak{n}_-)$. Write $\Delta u = u \otimes 1 + u^{(i)} \otimes f_i + A$, where A is a sum of homogeneous terms of the form $u' \otimes u''$ with $\deg(u'') \neq \alpha_i$. This defines $u^{(i)}$ unambiguously.

Lemma 10 For $\lambda, \mu \in P$ we have

$$\mathcal{E}_i^{\lambda+\mu}u = \mathcal{E}_i^\lambda u + (\mu; \alpha_i) u^{(i)}.$$

Proof — We calculate in two ways the unique term of the form $E \otimes 1$ in $\Delta(\mathcal{E}_i^{\lambda+\mu}u)$. On the one hand, we have obviously $E \otimes 1 = \mathcal{E}_i^{\lambda+\mu}u \otimes 1$. On the other hand, using Lemma 9, we have

$$E \otimes 1 = \mathcal{E}_i^\lambda u \otimes 1 + (1 \otimes \mathcal{E}_i^\mu)(u^{(i)} \otimes f_i) = \mathcal{E}_i^\lambda u \otimes 1 + (\mu; \alpha_i) u^{(i)} \otimes 1.$$

Therefore,

$$E = \mathcal{E}_i^{\lambda+\mu}u = \mathcal{E}_i^\lambda u + (\mu; \alpha_i) u^{(i)}.$$

\square

6.4 Now let us return to the geometric realization \mathcal{M} of $U(\mathfrak{n}_-)$. Let E_i^λ denote the endomorphism of \mathcal{M} obtained by transporting e_i^λ via the identification $M(\lambda) \cong \mathcal{M}$.

Lemma 11 *Let $\lambda \in P_+$, $f \in \mathcal{M}_\beta$ and $x \in \Lambda_{\beta-\alpha_i}^\lambda$. Then*

$$(E_i^\lambda f)(x) = \int_{y \in \mathcal{G}(x, \lambda, i)} f(y).$$

Proof — Let $r_\lambda : \mathcal{M} \rightarrow \mathcal{M}^\lambda$ be the linear map sending $f \in \mathcal{M}_\beta$ to its restriction to Λ_β^λ . By Theorem 1, this is a homomorphism of $U(\mathfrak{n}_-)$ -modules mapping the highest weight vector of $\mathcal{M} \cong M(\lambda)$ to the highest weight vector of $\mathcal{M}^\lambda \cong L(\lambda)$. It follows that r_λ is in fact a homomorphism of $U(\mathfrak{g})$ -modules, hence the restriction of $E_i^\lambda f$ to $\Lambda_{\beta-\alpha_i}^\lambda$ is given by Formula (4) of Section 5. \square

Let again $\lambda \in P$ be arbitrary, and pick $f \in \mathcal{M}_\beta$. It follows from Lemma 10 that for any $\mu \in P$

$$E_i^{\lambda+\mu} f - (\mu; \alpha_i) f^{(i)} = E_i^\lambda f.$$

Let $x \in \Lambda_{\beta-\alpha_i}$. Choose $\nu = \lambda + \mu$ sufficiently dominant so that x is isomorphic to a submodule of q_ν . Then by Lemma 11, we have

$$(E_i^\nu f)(x) = \int_{y \in \mathcal{G}(x, \nu, i)} f(y).$$

On the other hand, by the geometric description of Δ given in [GLS, §6.1], if we write

$$\Delta f = f \otimes 1 + f^{(i)} \otimes \mathbf{1}_i + A$$

where A is a sum of homogeneous terms of the form $f' \otimes f''$ with $\deg(f'') \neq \alpha_i$, we have that $f^{(i)}$ is the function on $\Lambda_{\beta-\alpha_i}$ given by $f^{(i)}(x) = f(x \oplus s_i)$. Hence we obtain that for $x \in \Lambda_{\beta-\alpha_i}$

$$(E_i^\lambda f)(x) = \int_{y \in \mathcal{G}(x, \nu, i)} f(y) - (\nu - \lambda; \alpha_i) f(x \oplus s_i).$$

This proves both Proposition 1 and Theorem 2. \square

6.5 Let $\lambda \in P_+$. We note the following consequence of Lemma 11.

Proposition 2 *Let $\lambda \in P_+$. The linear map $r_\lambda : \mathcal{M} \rightarrow \mathcal{M}^\lambda$ sending $f \in \mathcal{M}_\beta$ to its restriction to Λ_β^λ is the geometric realization of the homomorphism of \mathfrak{g} -modules $M(\lambda) \rightarrow L(\lambda)$. \square*

7 Dual Verma modules

7.1 Let S be the anti-automorphism of $U(\mathfrak{g})$ defined by

$$S(e_i) = f_i, \quad S(f_i) = e_i, \quad S(h_i) = h_i, \quad (i \in I).$$

Recall that, given a left $U(\mathfrak{g})$ -module M , the dual module M^* is defined by

$$(u\varphi)(m) = \varphi(S(u)m), \quad (u \in U(\mathfrak{g}), m \in M, \varphi \in M^*).$$

This is also a left module. If M is an infinite-dimensional module with finite-dimensional weight spaces M_ν , we take for M^* the graded dual $M^* = \bigoplus_{\nu \in P} M_\nu^*$.

For $\lambda \in P$ we have $L(\lambda)^* \cong L(\lambda)$, hence the quotient map $M(\lambda) \rightarrow L(\lambda)$ gives by duality an embedding $L(\lambda) \rightarrow M(\lambda)^*$ of $U(\mathfrak{g})$ -modules.

7.2 Let $\mathcal{M}^* = \bigoplus_{\beta \in Q_+} \mathcal{M}_\beta^*$ denote the vector space graded dual of \mathcal{M} . For $x \in \Lambda_\beta$, we denote by δ_x the delta function given by

$$\delta_x(f) = f(x), \quad (f \in \mathcal{M}_\beta).$$

Note that the map $\delta : x \mapsto \delta_x$ is a constructible map from Λ_β to \mathcal{M}_β^* . Indeed the preimage of δ_x is the intersection of the constructible subsets

$$\mathcal{M}_{(i_1, \dots, i_r)} = \{y \in \Lambda_\beta \mid (\mathbf{1}_{i_1} * \dots * \mathbf{1}_{i_r})(y) = (\mathbf{1}_{i_1} * \dots * \mathbf{1}_{i_r})(x)\}, \quad (\alpha_{i_1} + \dots + \alpha_{i_r} = \beta).$$

7.3 We can now dualize the results of Sections 5 and 6 as follows. For $\lambda \in P$ and $x \in \Lambda_\beta$ put

$$(E_i^*)(\delta_x) = \int_{y \in \mathcal{G}(i, x)} \delta_y, \quad (18)$$

$$(F_i^{\lambda*})(\delta_x) = \int_{y \in \mathcal{G}(x, \nu, i)} \delta_y - (\nu - \lambda; \alpha_i) \delta_{x \oplus s_i}, \quad (19)$$

$$(H_i^{\lambda*})(\delta_x) = (\lambda - \beta; \alpha_i) \delta_x, \quad (20)$$

where in (19) the weight $\nu \in P_+$ is such that x is isomorphic to a submodule of q_ν . The following theorem then follows immediately from Theorems 1 and 2.

Theorem 3 (i) *The formulas above define endomorphisms $E_i^*, F_i^{\lambda*}, H_i^{\lambda*}$ of \mathcal{M}^* , and the assignments $e_i \mapsto E_i^*, f_i \mapsto F_i^{\lambda*}, h_i \mapsto H_i^{\lambda*}$, give a representation of \mathfrak{g} on \mathcal{M}^* isomorphic to the dual Verma module $M(\lambda)^*$.*

(ii) *If $\lambda \in P_+$, the subspace $\mathcal{M}^{\lambda*}$ of \mathcal{M}^* spanned by the delta functions δ_x of the finite-dimensional nilpotent submodules x of q_λ carries the irreducible submodule $L(\lambda)$. For such a module x , Formula (19) simplifies as follows*

$$(F_i^{\lambda*})(\delta_x) = \int_{y \in \mathcal{G}(x, \lambda, i)} \delta_y.$$

□

Example 1 Let \mathfrak{g} be of type A_2 . Take $\lambda = \varpi_1 + \varpi_2$, where ϖ_i is the fundamental weight corresponding to $i \in I$. Thus $L(\lambda)$ is isomorphic to the 8-dimensional adjoint representation of $\mathfrak{g} = \mathfrak{sl}_3$.

A Λ -module x consists of a pair of linear maps $x_{21} : V_1 \rightarrow V_2$ and $x_{12} : V_2 \rightarrow V_1$ such that $x_{12}x_{21} = x_{21}x_{12} = 0$. The injective Λ -module $q = q_\lambda$ has the following form :

$$q = \begin{pmatrix} u_1 & \longrightarrow & u_2 \\ v_1 & \longleftarrow & v_2 \end{pmatrix}$$

This diagram means that (u_1, v_1) is a basis of V_1 , that (u_2, v_2) is a basis of V_2 , and that

$$q_{21}(u_1) = u_2, \quad q_{21}(v_1) = 0, \quad q_{12}(v_2) = v_1, \quad q_{12}(u_2) = 0.$$

Using the same type of notation, we can exhibit the following submodules of q :

$$x_1 = (v_1), \quad x_2 = (u_2), \quad x_3 = (v_1 \quad u_2), \quad x_4 = (u_1 \longrightarrow u_2), \quad x_5 = (v_1 \longleftarrow v_2),$$

$$x_6 = \begin{pmatrix} u_1 & \longrightarrow & u_2 \\ v_1 & & \end{pmatrix}, \quad x_7 = \begin{pmatrix} & & u_2 \\ v_1 & \longleftarrow & v_2 \end{pmatrix}.$$

This is not an exhaustive list. For example, $x'_4 = ((u_1 + v_1) \longrightarrow u_2)$ is another submodule, isomorphic to x_4 . Denoting by $\mathbf{0}$ the zero submodule, we see that δ_0 is the highest weight vector of $L(\lambda) \subset M(\lambda)^*$. Next, writing for simplicity δ_i instead of δ_{x_i} and F_i instead of F_i^λ , Theorem 3 (ii) gives the following formulas for the action of the F_i 's on $L(\lambda)$.

$$F_1\delta_0 = \delta_1, \quad F_2\delta_0 = \delta_2, \quad F_1\delta_2 = \delta_3 + \delta_4, \quad F_2\delta_1 = \delta_3 + \delta_5,$$

$$F_1\delta_3 = F_1\delta_4 = \delta_6, \quad F_2\delta_3 = F_2\delta_5 = \delta_7, \quad F_2\delta_3 = F_1\delta_6 = \delta_q, \quad F_1\delta_q = F_2\delta_q = 0.$$

Now consider the Λ -module $x = s_1 \oplus s_1$. Since x is not isomorphic to a submodule of q_λ , the vector δ_x does not belong to $L(\lambda)$. Let us calculate $F_i\delta_x$ ($i = 1, 2$) by means of Formula (19). We can take $\nu = 2\varpi_1$. The injective Λ -module q_ν has the following form :

$$q_\nu = \begin{pmatrix} w_1 & \longleftarrow & w_2 \\ v_1 & \longleftarrow & v_2 \end{pmatrix}$$

It is easy to see that the variety $\mathcal{G}(x, \nu, 2)$ is isomorphic to a projective line \mathbb{P}_1 , and that all points on this line are isomorphic to

$$y = \begin{pmatrix} w_1 & \\ v_1 & \longleftarrow & v_2 \end{pmatrix}$$

as Λ -modules. Hence,

$$F_2\delta_x = \chi(\mathbb{P}_1) \delta_y - (\nu - \lambda; \alpha_2) \delta_{x \oplus s_2} = 2\delta_y + \delta_{s_1 \oplus s_1 \oplus s_2}.$$

On the other hand, $\mathcal{G}(x, \nu, 1) = \emptyset$, so that

$$F_1\delta_x = -(\nu - \lambda; \alpha_1) \delta_{x \oplus s_1} = -\delta_{s_1 \oplus s_1 \oplus s_1}.$$

References

- [Bo] K. BONGARTZ, *Algebras and quadratic forms*, J. London Math. Soc. **28** (1983), 461–469.
- [BK] M. C. R. BUTLER, A. D. KING, *Minimal resolutions of algebras*, J. Algebra **212** (1999), 323–362.
- [CB] W. CRAWLEY-BOEVEY, *On the exceptional fibres of Kleinian singularities*, Amer. J. Math. **122** (2000), 1027–1037.
- [GLS] C. GEISS, B. LECLERC, J. SCHRÖER, *Semicanonical bases and preprojective algebras*, Ann. Scient. Éc. Norm. Sup. **38** (2005), 193–253.
- [GP] I. M. GELFAND, V. A. PONOMAREV, *Model algebras and representations of graphs*, Funct. Anal. Appl. **13** (1980), 157–166.
- [Lu1] G. LUSZTIG, *Quivers, perverse sheaves, and quantized enveloping algebras*, J. Amer. Math. Soc. **4** (1991), 365–421.
- [Lu2] G. LUSZTIG, *Semicanonical bases arising from enveloping algebras*, Adv. Math. **151** (2000), 129–139.
- [Lu3] G. LUSZTIG, *Remarks on quiver varieties*, Duke Math. J. **105** (2000), 239–265.
- [Lu4] G. LUSZTIG, *Constructible functions on varieties attached to quivers*, in Studies in memory of Issai Schur 177–223, Progress in Mathematics **210**, Birkhäuser 2003.
- [Na] H. NAKAJIMA, *Instantons on ALE spaces, quiver varieties, and Kac-Moody algebras*, Duke Math. J. **76** (1994), 365–416.
- [Ri] C. M. RINGEL, *The preprojective algebra of a quiver*, in Algebras and modules II (Geiranger, 1966), 467–480, CMS Conf. Proc. **24**, AMS 1998.

Christof GEISS : Instituto de Matemáticas, UNAM
Ciudad Universitaria, 04510 Mexico D.F., Mexico
email : christof@math.unam.mx

Bernard LECLERC : LMNO, Université de Caen,
14032 Caen cedex, France
email : leclerc@math.unicaen.fr

Jan SCHRÖER : Department of Pure Mathematics, University of Leeds,
Leeds LS2 9JT, England
email : jschroer@maths.leeds.ac.uk