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Clifton-Pohl torus and geodesic completeness by a 'complex' point of view ^{*}

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Abstract

We show that a natural complexification and a mild generalization of the idea of completeness guarantee geodesic completeness of Clifton-Pohl torus; we explicitly compute all of its geodesics.

Keywords: Clifton-Pohl torus, geodesic completeness, holomorphic metric, analytical continuation, elliptic functions.

1 Foreword

The geodesic equations of the so-called 'Clifton-Pohl torus' $\mathbf{T}$ (a compact, geodesically incomplete, Lorentz manifold, see [ONE], 7.16), can be naturally thought of as a system of ordinary differential equations in the complex domain. Thus they yield in fact holomorphic germs of solutions: it is therefore a natural idea to 'complexify' the environment to make all this sound.

Therefore, we propose a natural, and mild, generalization of the idea of geodesic completeness, amounting to conjugating analytical continuation with complexification of the environment. We shall show that all geodesics of $\mathbf{T}$ are 'complete' by this point of view: more precisely, their germs of solution will be shown to admit 'endless' analytical continuability on $\mathbb{C}$.

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We shall use the concept of a *holomorphic metric* on a complex manifold $\mathbf{M}$ (see e.g. [LEB]): it amounts to a nondegenerating symmetric section of the twice covariant holomorphic tensor bundle $\mathbf{T}_0^2\mathbf{M}$. Of course, it carries no 'signature'; however, by symmetry, it induces a canonical Levi-Civita's connexion on $\mathbf{M}$, allowing geodesics to be defined as auto-parallel paths; in this paper we shall not fathom these issues beyond: the reader is referred also to [MAN]. Finally, if $\mathbf{M}$ arises as a 'complexification' of a semi-Riemannian manifold $\mathbf{N}$, it is easily seen that the real geodesics of $\mathbf{N}$ are restrictions to the real axis of the complex ones of $\mathbf{M}$ and vice versa (see [LEB]). This fact sometimes allows us to 'flank' isolated singularities on the real line by running along complex trips, i.e. to 'connect' geodesics which, in the usual sense, are completely unrelated. In other cases, geodesics which end at finite time by the usual point of view, could be continued only by admitting 'complexified' values.

We suggest an idea of our notion of *completeness* (see also definition 2): a complex curve γ from a Riemann surface (S, π) over $\mathbb{C}$ into a complex manifold $\mathbf{M}$ will be told to be **complex-complete** if $\mathbb{C} \setminus \pi(S)$ is a discrete set. In other words, up to changing branch, γ is analytically continuable everywhere, except at most at a discrete set in the complex plane.

En passant, our procedure will yield explicit solutions for all geodesics of $\mathbf{T}$, mainly by means of elliptic functions.

2 Basic definitions and lemmata

In the following, $\mathcal{U}$ will be a region in the complex plane and $\mathbf{M}$ a complex manifold.

Definition 1 *A holomorphic metric on $\mathbf{M}$ is an everywhere maximum-rank symmetric section of the twice covariant holomorphic tensor bundle $\mathbf{T}_0^2\mathbf{M}$. A holomorphic Riemannian manifold is a complex manifold endowed with a holomorphic metric.*

The idea of the analytical continuation of a holomorphic mapping element (or of a germ) $f: \mathcal{U} \rightarrow \mathbf{M}$ is well known and amounts to a quadruple $Q_{\mathbf{M}} = (S, \pi, j, F)$, where S is a connected Riemann surface over a region of $\mathbb{C}$, $\pi: S \rightarrow \mathbb{C}$ is a nonconstant holomorphic mapping such that $U \subset \pi(S)$, $j: U \rightarrow S$ is a holomorphic immersion such that $\pi \circ j = id|_U$ and $F: S \rightarrow \mathbf{M}$ is a holomorphic mapping such that $F \circ j = f$. We do not consider branch

points here; it is a well known result that there exists a unique maximal analytical continuation, called the (regular) *Riemann surface*, of $(\mathcal{U}, f)$, which is made up by all elements which are analytical continuations of f . As general references, see e.g. [AHL], [CAS],[MAL], [NAR], [PAL].

Definition 2 *A complex curve $F : S \rightarrow \mathbf{M}$ defined on a Riemann surface (S, π, j, F) over $\mathbb{C}$ is **complex-complete** provided that $\mathbb{C} \setminus \pi(S)$ is a discrete set in the complex plane; a real-analytic Lorentz manifold $\mathbf{N}$ is (weakly) complex-complete provided that, for every geodesic γ in $\mathbf{N}$, there exists a complexification $\mathbf{M}_\gamma$ of $\mathbf{N}$, with embedding mapping $\zeta_\gamma : \mathbf{N} \rightarrow \mathbf{M}_\gamma$, such that the Riemann surface of $\zeta \circ \gamma$ is complex-complete.*

We conclude this section by remarking that, as a consequence of the existence-and-uniqueness theorem of o.d.e.'s theory in the complex domain (see e.g. [HIL], th 2.2.2, [INC] p.281-284), for each point p in a holomorphic Riemannian manifold and each holomorphic tangent vector X at p , there exists a unique holomorphic geodesic element starting at p with velocity X .

3 Clifton-Pohl torus

Let now $\mathbf{N} := \mathbb{R}^2 \setminus \{0\}$ be endowed with the Lorentz metric $du \odot dv / (u^2 + v^2)$; the group D generated by scalar multiplication by 2 is a group of isometries of $\mathbf{N}$; its action is properly discontinuous, hence $\mathbf{T} = \mathbf{N}/D$ is a Lorentz surface. Now, $\mathbf{T}$ is topologically equivalent to the closed annulus $1 \leq \varrho \leq 2$, with boundaries identified by the action of D , i.e. a torus; notwithstanding, $\mathbf{T}$ is geodesically incomplete, since $t \mapsto (1/(1-t), 0)$ is a geodesic of $\mathbf{N}$ (see [ONE]). This example suggest us (in fact rather weakly) to extend the domain of the definition of the natural parameter t to the complex plane: of course, this in turn requires a complexification of $\mathbf{T}$ or, equivalently, $\mathbf{N}$. Doing this would render the above curve complex-complete, according to definition 2. In the following, we shall indeed study the holomorphic Riemannian manifold $\mathbf{M} = [\mathbb{C}^2 \setminus ((1, i)\mathbb{C} \cup (1, -i)\mathbb{C}), du \odot dv / (u^2 + v^2)]$, which is a natural complexification of $\mathbf{N}$: by methonymy, we shall use the name 'Clifton-Pohl torus' for $\mathbf{N}$ and $\mathbf{M}$ too, since our completeness theorems can be easily be 'restricted' to the real slice $\mathbf{N}$ and then pushed down from $\mathbf{N}$ to $\mathbf{T}$ with respect to the action of D . Here is our main result.

Theorem 3 *Each geodesic γ of $\mathbf{M}$ is complex-complete.*

Proof. The geodesic equations of $\mathbf{M}$ are:

$$\ddot{u} = 2u/(u^2 + v^2) \dot{u}^2, \quad \ddot{v} = 2v/(u^2 + v^2) \dot{v}^2. \quad (1)$$

Let us first study null geodesics of $\mathbf{M}$: without loss of generality, it is enough to deal with the case $v \equiv A$. Equations (1) imply $\ddot{u} = 2u/(u^2 + A^2) \dot{u}^2$, which is solved by $t \mapsto (C - Bt)^{-1}$ if $A = 0$ and by $t \mapsto \tan(At + B)$ if $A \neq 0$, for suitable complex constants B and C . The above functions are meromorphic, hence complex-complete by definition 2. We turn to nonnull geodesics: let γ start at (α, β) , with velocity (x, y) ; we may suppose, without loss of generality, $\alpha \neq 0$ and $\beta \neq 0$. Moreover, we have $x \neq 0$ and $y \neq 0$ otherwise γ would be null. The equations (1) can be integrated once to yield:

$$\dot{u} \dot{v} = A(u^2 + v^2), \quad u/\dot{u} + v/\dot{v} = B, \quad (2)$$

where $A = xy/(\alpha^2 + \beta^2)$ and $B = \alpha/x + \beta/y$.

Now we single out all geodesics such that $u\dot{v} \equiv v\dot{u}$: by also keeping into account the equations (2), they are easily seen to be of the form $t \mapsto (a_1 \exp(bt), a_2 \exp(bt))$ for suitable complex constants a_1, a_2 and b : these curves are clearly complex-complete.

Therefore, from now on, we may suppose $\alpha y \neq \beta x$, without loss of generality: this implies, by easy calculations,

$$AB^2 \operatorname{Ch} \log(\alpha/\beta) \neq 2, \quad (3)$$

for any branch of the logarithm.

Now let us choose a branch $\log$ of the logarithm, defined on a connected neighbourhood of both α and β and perform the local change of coordinates $\omega = \log u$, $\eta = \log v$: the equations (2) are turned into

$$\dot{\omega} \dot{\eta} = 2A \operatorname{Ch}(\omega - \eta), \quad 1/\dot{\omega} + 1/\dot{\eta} = B. \quad (4)$$

We can solve (4) with respect to $\dot{\omega}$ and $\dot{\eta}$, getting

$$\begin{cases} \dot{\omega} = 2 \left(B - \sqrt{B^2 - 2/[A \operatorname{Ch}(\omega - \eta)]} \right)^{-1} \\ \dot{\eta} = 2 \left(B + \sqrt{B^2 - 2/[A \operatorname{Ch}(\omega - \eta)]} \right)^{-1} \end{cases}. \quad (5)$$

Subtract and set $\varphi := \omega - \eta$; this yields

$$\dot{\varphi} = 2\sqrt{(A\text{Ch}\varphi)^2 - 2A\text{Ch}\varphi} \quad (6)$$

with the initial value $\varphi(0) = \log(\alpha/\beta)$: by (3), the differential equation (6) is regular at $\varphi(0)$, and we can choose a branch of SettTh at ϕ_0 and make the further change of variable $\varphi = 2\text{SettTh}\psi$, getting, by easy calculations,

$$\dot{\psi} = \sqrt{A^2B^2 - 2A}\sqrt{(1 + \psi^2)\left(1 - \frac{A^2B^2 + 2A}{2A - A^2B^2}\psi^2\right)} \quad (7)$$

with the (nonsingular) initial condition $\psi(0) = \psi_0 := 2\text{SettTh}\varphi(0)$.

By separation of variables we can bring back this equation to the calculation of an elliptic integral, depending on the parameters A and B : as it is shown e.g. in [AHL] p.240, this implies that ψ is an elliptic function $\mathbf{e}_{A,B}$, so that φ is analytically continuable everywhere, except at most at the discrete set $\mathbf{e}_{A,B}^{-1}(\{\pm 1\})$. Therefore, φ is complex-complete.

From (5), we get now that both $\dot{\eta}$ and $\dot{\omega}$ is complex-complete; since integration clearly preserves completeness, also η and ω are complex-complete: but $u = \exp(\omega)$ and $v = \exp(\eta)$: this eventually implies, by analytical continuation, that γ is complex-complete. ■

Final Remarks: the concern naturally arises whether there exists a minimal complexification of $\mathbf{T}$, or rather of $\mathbf{N}$, which guarantees geodesic completeness: this seems to be a harder question.

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