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Daniel Arnaudon, Nicolas Crampé, Anastasia Doikou, Luc Frappat, Eric Ragoucy

Analytical Bethe Ansatz for closed and open $gl(\mathcal{N})$-spin chains in any representation

D. Arnaudon[a], N. Crampé[ab], A. Doikou[a], L. Frappat[ac] and É. Ragoucy[a] [1]

[a] *Laboratoire d'Annecy-le-Vieux de Physique Théorique*
LAPTH, CNRS, UMR 5108, Université de Savoie
B.P. 110, F-74941 Annecy-le-Vieux Cedex, France

[b] *University of York, Department of mathematics*
Heslington, York YO10 5DD, United Kingdom

[c] *Member of Institut Universitaire de France*

Abstract

We present an "algebraic treatment" of the analytical Bethe Ansatz. For this purpose, we introduce abstract monodromy and transfer matrices which provide an algebraic framework for the analytical Bethe Ansatz. It allows us to deal with a generic $gl(\mathcal{N})$-spin chain possessing on each site an arbitrary $gl(\mathcal{N})$-representation. For open spin chains, we use the classification of the reflection matrices to treat all the diagonal boundary cases.
As a result, we obtain the Bethe equations in their full generality for closed and open spin chains. The classifications of finite dimensional irreducible representations for the Yangian (closed spin chains) and for the reflection algebras (open spin chains) are directly linked to the calculation of the transfer matrix eigenvalues.
As examples, we recover the usual closed and open spin chains, we treat the alternating spin chains and the closed spin chain with impurity.

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[1] daniel.arnaudon@lapp.in2p3.fr, nc501@york.ac.uk, anastasia.doikou@lapp.in2p3.fr, luc.frappat@lapp.in2p3.fr, eric.ragoucy@lapp.in2p3.fr

1 Introduction

The investigation of the integrable quantum spin chains was initiated by H. Bethe in 1931 [1] where he studied the closed spin 1/2 Heisenberg chain [2]. Since then, numerous generalisations of this spin chain have been introduced: anisotropic XXZ spin chain [3–5]; spin 1 chains [6, 7]; alternating spin chains [8, 9]; spin chains with higher spins [5, 10–15]; spin 1/2 chains with spin 1 impurities [16–18]. Correlation functions of this type of spin chains have been computed in e.g. [19–25]. The framework of integrable open spin chains has been developed in [26–29].

There exist different motivations to study generalisations of the integrable spin chains. First, they describe dynamics which can be computed exactly of quantum mechanical models. Indeed, new models have been investigated to describe theoretically crystalline material in order to compare with the experimental data (e.g. for the crystals $MnCu(S_2C_2O_2)_2(H_2O)_3$ see [30], $(VO)_2P_2O_7$ [31–35] or $Cu_2(OH)_2CO_3$ [36]). Spin chains allows one to treat some limits of other models as the Hubbard model [37], the quantum chromodynamics theory [38] or integrable relativistic quantum field theories [39–42]. Finally, recent developments in AdS/CFT correspondence have also put spin chain models in foreground of string theory [43–45].

The increasing number of applications urge us to seek for a complete treatment of (closed and open) spin chain models. Different schemes exist for dealing with these problems, most of them relying on the Bethe Ansatz (coordinate, algebraic, analytical or thermodynamical for the main ones). We present here a formulation of the analytical Bethe Ansatz for $gl(\mathcal{N})$ (closed and open) spin chains whatever the representation at each site the quantum spins belong to. In particular, we unify by this way all the generalisations of the XXX model.

The main results of this paper are the following:

- The determination of the Bethe equations for a closed spin chain model where each quantum spin is represented in an arbitrary representation of $gl(\mathcal{N})$ (called generic closed spin chain).

- The computation of Bethe equations for any open spin chain constructed from an arbitrary diagonal reflection matrix and a generic closed spin chain.

- For each of the above mentioned models, the calculation of the underlying symmetry and the integrability of the models.

This paper consists of two main sections. The first one is devoted to the study of closed spin chain, while the second one deals with the case of boundaries. The structure of these two parts is similar. We recall first the algebraic settings (Yangians or boundary algebras) for the monodromy matrix. Then, we use the classification of the representations of these algebras to compute a represented transfer

matrix. Next, we use generalisation of analytical Bethe Ansatz to obtain the Bethe equations. Finally, we work out some examples.

2 Closed spin chain

2.1 The R matrix

We will consider the $gl(\mathcal{N})$ invariant R matrices [46, 47]

$$R_{ab}(\lambda) = \mathbb{I}_{\mathcal{N}} \otimes \mathbb{I}_{\mathcal{N}} - \frac{\hbar\, \mathcal{P}_{ab}}{\lambda} \,, \tag{2.1}$$

where $\mathcal{P}_{ab}$ is the permutation operator

$$\mathcal{P}_{ab} = \sum_{i,j=1}^{\mathcal{N}} E_{ij} \otimes E_{ji} \tag{2.2}$$

and $\hbar$ is the deformation parameter. It is usually set to 1 in the context of quantum groups (Yangians), and to $-i$ when dealing with spin chain models: here, we leave it free. E_{ij} are the elementary matrices with 1 in position (i, j) and 0 elsewhere. From the algebraic point of view, the value of $\hbar$ is irrelevant (provided non-vanishing). It is in general set to 1 when studying Yangians, while it is set to $-i$ in the spin chains context. Here, we leave it free.

This R matrix satisfies the following properties

(i) Yang–Baxter equation [4, 5, 46–48]

$$R_{ab}(\lambda_a - \lambda_b)\; R_{ac}(\lambda_a)\; R_{bc}(\lambda_b) = R_{bc}(\lambda_b)\; R_{ac}(\lambda_a)\; R_{ab}(\lambda_a - \lambda_b) \tag{2.3}$$

(ii) Unitarity

$$R_{ab}(\lambda)\; R_{ba}(-\lambda) = \zeta(\lambda)\, \mathbb{I}_{\mathcal{N}} \otimes \mathbb{I}_{\mathcal{N}} \,, \tag{2.4}$$

where $R_{ba}(\lambda) = \mathcal{P}_{ab} R_{ab}(\lambda) \mathcal{P}_{ab} = R_{ab}^{t_a t_b}(\lambda) = R_{ab}(\lambda)$ and

$$\zeta(\lambda) = \left(1 - \frac{\hbar}{\lambda}\right)\left(1 + \frac{\hbar}{\lambda}\right) \,. \tag{2.5}$$

It obeys $[A_a A_b,\; R_{ab}(\lambda)] = 0$ for $A \in End(\mathbb{C}^{\mathcal{N}})$.

The R matrix can be interpreted physically as a scattering matrix [3, 4, 49] describing the interaction between two solitons (viewed in this framework as low level excited states in a thermodynamical limit of a spin chain) that carry the fundamental representation of $gl(\mathcal{N})$.

2.2 Yangian $\mathcal{Y}(gl(\mathcal{N}))$

We present in this section some definitions and properties of the Yangian [50] associated to the Lie algebra $gl(\mathcal{N})$ that will be used in the following.

The Yangian $\mathcal{Y}(gl(\mathcal{N}))$ is the complex associative unital algebra with the generators $\{T_{ij}^{(n)} | 1 \leq i,j \leq \mathcal{N}, n \in \mathbb{Z}_{\geq 0}\}$ subject to the defining relations

$$[T_{ij}^{(r+1)}\,,\,T_{kl}^{(s)}] - [T_{ij}^{(r)}\,,\,T_{kl}^{(s+1)}] = T_{kj}^{(r)}\,T_{il}^{(s)} - T_{kj}^{(s)}\,T_{il}^{(r)}\;, \tag{2.6}$$

where $r,s \in \mathbb{Z}_{\geq 0}$ and $T_{ij}^{(0)} = \delta_{ij}$.
The R matrix previously introduced allows us to encode the Yangian defining relations in a simple equation, called FRT exchange relation [51]

$$R_{ab}(\lambda_a - \lambda_b)\;\mathcal{T}_a(\lambda_a)\;\mathcal{T}_b(\lambda_b) = \mathcal{T}_b(\lambda_b)\;\mathcal{T}_a(\lambda_a)\;R_{ab}(\lambda_a - \lambda_b)\;, \tag{2.7}$$

where the generators are gathered in the following matrix (belonging to $End(\mathbb{C}^{\mathcal{N}}) \otimes \mathcal{Y}(gl(\mathcal{N}))[[\lambda^{-1}]]$)

$$\mathcal{T}(\lambda) = \sum_{i,j=1}^{\mathcal{N}} E_{ij} \otimes T_{ij}(\lambda) = \sum_{i,j=1}^{\mathcal{N}} E_{ij} \otimes \sum_{r\geq 0} \frac{\hbar^r}{\lambda^r}\;T_{ij}^{(r)} = \sum_{r\geq 0} \frac{\hbar^r}{\lambda^r}\;\mathcal{T}^{(r)}\;. \tag{2.8}$$

Using the commutation relations (2.7), it is easy to show that $\mathcal{T}^{(1)}$ generates a $gl(\mathcal{N})$ algebra.
In order to construct representations of $\mathcal{Y}(gl(\mathcal{N}))$, the following algebra homomorphism from $\mathcal{Y}(gl(\mathcal{N}))$ to $\mathcal{U}(gl(\mathcal{N}))$ (universal enveloping algebra of $gl(\mathcal{N})$) will be used[2]

$$T_{ij}(\lambda) \longmapsto \delta_{ij} - \frac{\hbar\; e_{ji}}{\lambda}\;, \tag{2.9}$$

where $\{e_{ij}\}$ is a basis of the Lie algebra $gl(\mathcal{N})$. The Yangian of $gl(\mathcal{N})$ is a Hopf algebra with the coproduct given by

$$\begin{aligned} \Delta : \mathcal{Y}(gl(\mathcal{N})) &\longrightarrow \mathcal{Y}(gl(\mathcal{N})) \otimes \mathcal{Y}(gl(\mathcal{N})) \\ T_{ij}(\lambda) &\longmapsto \sum_{k=1}^{\mathcal{N}} T_{ik}(\lambda) \otimes T_{kj}(\lambda)\;. \end{aligned} \tag{2.10}$$

The coproduct is the cornerstone to deal with the tensor product of representations. We define also by recursion $\Delta^{(n)} = (\Delta \otimes id^{\,\otimes\, n-2})\Delta^{(n-1)}$ for $n > 2$ and $\Delta^{(2)} = \Delta$.

The quantum determinant $qdet\,\mathcal{T}(\lambda)$ is a formal series in λ^{-1} with coefficients in $\mathcal{Y}(gl(\mathcal{N}))$ defined as follows

$$qdet\,\mathcal{T}(\lambda) = \sum_{\sigma\in\mathfrak{S}_{\mathcal{N}}} sgn(\sigma)\;T_{1,\sigma(1)}(\lambda - \hbar\mathcal{N} + \hbar)\cdots T_{\mathcal{N},\sigma(\mathcal{N})}(\lambda)\;, \tag{2.11}$$

[2]To be compatible with the pseudo-vacuum as usually defined in the study of spin chain models, the convention used here for the homomorphism differs from the one introduced in [52]. The link between the two conventions is provided by the Yangian automorphism $T(\lambda) \longmapsto T^t(-\lambda)$, where ^{t} is the usual transposition.

where $\mathfrak{S}_{\mathcal{N}}$ is the permutation group of $\mathcal{N}$ indices. A well-known result (see e.g. [53]) establishes that the coefficients of $qdet\ \mathcal{T}(\lambda)$ are algebraically independent and generate the centre of $\mathcal{Y}(gl(\mathcal{N}))$. It is important for the following to realise that the quantum determinant represented in any finite-dimensional irreducible representation will be proportional to the identity matrix.

There exists an equivalent definition of the quantum determinant which will be used in the following as well. Let A_m be the antisymmetriser operator in $(\mathbb{C}^{\mathcal{N}})^{\otimes m}$, i.e.

$$A_m(e_{i_1}\otimes\cdots\otimes e_{i_m}) = \frac{1}{m!}\sum_{\sigma\in\mathfrak{S}_m} sgn(\sigma)\ e_{i_{\sigma(1)}}\otimes\cdots\otimes e_{i_{\sigma(m)}}\ , \tag{2.12}$$

where $\{e_i|1\leq i\leq\mathcal{N}\}$ is the canonical basis of $\mathbb{C}^{\mathcal{N}}$ and $1\leq i_1,\ldots,i_m\leq\mathcal{N}$. The antisymmetriser is a projector in $(\mathbb{C}^{\mathcal{N}})^{\otimes m}$. It has the remarkable property:

Proposition 2.1 ***[54]*** *The following identities hold*

$$\begin{aligned} A_m\ \mathcal{T}_1(\lambda)\cdots\mathcal{T}_m(\lambda-m\hbar+\hbar)A_m &= \mathcal{T}_m(\lambda-m\hbar+\hbar)\cdots\mathcal{T}_1(\lambda)\ A_m & (2.13)\\ &= A_m\ \mathcal{T}_1(\lambda)\cdots\mathcal{T}_m(\lambda-m\hbar+\hbar)\ . & (2.14)\end{aligned}$$

When $m=\mathcal{N}$, the antisymmetriser becomes a one-dimensional projector and one can show [54]

$$qdet\ \mathcal{T}(\lambda)\ A_{\mathcal{N}} = \mathcal{T}_{\mathcal{N}}(\lambda-\hbar\mathcal{N}+\hbar)\cdots\mathcal{T}_1(\lambda)\ A_{\mathcal{N}}\ . \tag{2.15}$$

The relation (2.15) can be used as an equivalent definition of the quantum determinant.
To study spin chains, we will use the following automorphisms of $\mathcal{Y}(gl(\mathcal{N}))$
(i) Inversion

$$inv:\ \mathcal{T}(\lambda)\longmapsto\mathcal{T}^{-1}(-\lambda) \tag{2.16}$$

(ii) Shift

$$s_a:\ \mathcal{T}(\lambda)\longmapsto\mathcal{T}(\lambda+a)\,,\ a\in\mathbb{C}\ . \tag{2.17}$$

One can compute the elements of $\mathcal{T}^{-1}(\lambda)$ in terms of $\mathcal{T}(\lambda)$ using the following formula

$$\mathcal{T}^{-1}(\lambda-\hbar\mathcal{N}+\hbar) = \big(qdet\ \mathcal{T}(\lambda)\big)^{-1}\ \mathcal{T}^*(\lambda)\ , \tag{2.18}$$

where $\mathcal{T}^*(\lambda)$ is the quantum comatrix, i.e. its entries $T^*_{ij}(\lambda)$ are $(-1)^{i+j}$ times the quantum determinants of the submatrices of $\mathcal{T}(\lambda)$ obtained by removing the i^{th} column and j^{th} row.

2.3 Algebraic transfer matrix

In the following, in order to construct spin chains, it will be necessary to deal with the tensor product of ℓ copies of the Yangian. For $1\leq i\leq\ell$, we denote by $\mathcal{L}_{ai}(\lambda)\in End(\mathbb{C}^{\mathcal{N}})\otimes\mathcal{Y}(gl(\mathcal{N}))$ one copy of the Yangian which acts non trivially on the i^{th} space only. The space a, always isomorphic to

$End(\mathbb{C}^{\mathcal{N}})$ in the present paper, is called auxiliary space whereas the space i is called quantum space. Obviously, $\mathcal{L}_{ai}(\lambda)$ satisfies the defining relations of the Yangian

$$R_{ab}(\lambda_a - \lambda_b)\ \mathcal{L}_{ai}(\lambda_a)\ \mathcal{L}_{bi}(\lambda_b) = \mathcal{L}_{bi}(\lambda_b)\ \mathcal{L}_{ai}(\lambda_a)\ R_{ab}(\lambda_a - \lambda_b)\ . \tag{2.19}$$

Let us stress that the matrix $\mathcal{L}_{ai}(\lambda)$ is local, i.e. it contains only the i^{th} copy of the Yangian. On the contrary, thanks to the coproduct, one constructs a non-local algebraic object, the monodromy matrix

$$\mathcal{T}_a(\lambda) = \Delta^{(\ell)}(\mathcal{L}(\lambda)) = \mathcal{L}_{a1}(\lambda)\ \mathcal{L}_{a2}(\lambda)\ \ldots\ \mathcal{L}_{a\ell}(\lambda) \in End(\mathbb{C}^{\mathcal{N}}) \otimes \left(\mathcal{Y}(gl(\mathcal{N}))\right)^{\otimes \ell}\ . \tag{2.20}$$

Let us remark that the quantum spaces are omitted in the LHS of (2.20), as usual in the notation of the monodromy matrix. The entries of the monodromy matrix $\mathcal{T}_a(\lambda)$ are given by

$$T_{ij}(\lambda) = \sum_{k_1,\ldots,k_{\ell-1}=1}^{\mathcal{N}} L_{ik_1}(\lambda)\ \otimes\ L_{k_1k_2}(\lambda)\ \otimes\ \ldots\ \otimes\ L_{k_{\ell-1}j}(\lambda)\ . \tag{2.21}$$

Since the coproduct is a morphism, $\mathcal{T}_a(\lambda)$ also satisfies the defining relations of the Yangian

$$R_{ab}(\lambda_a - \lambda_b)\ \mathcal{T}_a(\lambda_a)\ \mathcal{T}_b(\lambda_b) = \mathcal{T}_b(\lambda_b)\ \mathcal{T}_a(\lambda_a)\ R_{ab}(\lambda_a - \lambda_b)\ . \tag{2.22}$$

Now, we can introduce the main object for the study of spin chains, i.e. the transfer matrix

$$t(\lambda) = tr_a\left(\mathcal{T}_a(\lambda)\right) = \sum_{i=1}^{\mathcal{N}} T_{ii}(\lambda)\ . \tag{2.23}$$

Equation (2.22) immediately implies

$$[\ t(\lambda)\ ,\ t(\mu)\ ] = 0 \tag{2.24}$$

which will guarantee the integrability of the models (see section 2.6.1).

Let us remark that, at that point, the monodromy and transfer matrices are algebraic objects (in $\left(\mathcal{Y}(gl(\mathcal{N}))\right)^{\otimes \ell}$), and, as such, play the rôle of generating functions for the construction of monodromy and transfer matrices as they are usually introduced in spin chain models. The latter will be constructed from the former using representations of the Yangian, as it will be done below.

2.4 Symmetry

The algebraic structure defined above is sufficient to determine the symmetry of the transfer matrix. Indeed, we have:

Proposition 2.2 *The $gl(\mathcal{N})$ algebra is a symmetry of $t(\lambda)$. Its generators are expressed in terms of the local $gl(\mathcal{N})$ generators as*

$$T_{ij}^{(1)} = L_{ij}^{(1)}\ \otimes\ 1^{\otimes \ell-1} + 1\ \otimes\ L_{ij}^{(1)}\ \otimes\ 1^{\otimes \ell-2} + \cdots + 1^{\otimes \ell-1} \otimes\ L_{ij}^{(1)}\ \ . \tag{2.25}$$

Proof: Taking the trace in space a of the exchange relations (2.22), we obtain

$$(\lambda_a - \lambda_b)\,[\,t(\lambda_a)\,,\,\mathcal{T}(\lambda_b)\,] = \hbar\,[\,\mathcal{T}(\lambda_a)\,,\,\mathcal{T}(\lambda_b)\,]\,. \tag{2.26}$$

Then, the λ_b free term reads

$$\big[\,t(\lambda_a)\,,\,\mathcal{T}^{(1)}\,\big] = 0\,, \tag{2.27}$$

which proves that $\mathcal{T}^{(1)} = \sum_{i,j=1}^{\mathcal{N}} E_{ij} \otimes T_{ij}^{(1)}$ is a symmetry of the transfer matrix. These generators generates the $gl(\mathcal{N})$ Lie algebra. ∎

Thus, anticipating the spin chain interpretation, we can deduce that <u>all</u> the integrable models constructed in the usual way from $t(\lambda)$ (such as the ones presented in section 2.7) possess a $gl(\mathcal{N})$ symmetry. In other words, the $gl(\mathcal{N})$ symmetry is valid whatever the Yangian representations are. Depending on the model considered (i.e. the choice of representations), we will get the expression of the symmetry generators by evaluating the relation (2.25) in the representations under consideration.

2.5 Representations

As already mentioned, spin chain models will be obtained through the evaluation of the algebraic monodromy and transfer matrices in Yangian representations. We thus present here some basic results on the classification of finite-dimensional irreducible representations of $\mathcal{Y}(gl(\mathcal{N}))$.

2.5.1 Evaluation representations

Keeping in mind the forthcoming spin chains interpretation, we choose for each local $\mathcal{Y}(gl(\mathcal{N}))$ algebra an irreducible finite-dimensional evaluation representation.

We start with a finite-dimensional irreducible representation of $gl(\mathcal{N})$, $M(\boldsymbol{\alpha})$, with highest weight $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_{\mathcal{N}})$ and associated to the highest weight vector v. This highest weight vector obeys

$$e_{kj}\; v = 0 \qquad ,\quad 1 \le k < j \le \mathcal{N} \tag{2.28}$$

$$e_{kk}\; v = \alpha_k\; v \qquad ,\quad 1 \le k \le \mathcal{N}\,, \tag{2.29}$$

where $\alpha_1, \ldots, \alpha_{\mathcal{N}}$ are integers with $\alpha_{k+1} \le \alpha_k$. Indeed the constraints on the parameters α_k are criteria so that the representation be finite-dimensional and irreducible. Similar criteria will be given in Theorem 2.4 for the Yangian.

The evaluation representation $M_\lambda(\boldsymbol{\alpha})$ of $\mathcal{Y}(gl(\mathcal{N}))$ is built from $M(\boldsymbol{\alpha})$ and follows from the homomorphism (2.9), according to

$$L_{jk}(\lambda)\; v = 0 \qquad ,\quad 1 \le k < j \le \mathcal{N} \tag{2.30}$$

$$L_{kk}(\lambda)\; v = \left(1 - \frac{\hbar\,\alpha_k}{\lambda}\right)\; v \qquad ,\quad 1 \le k \le \mathcal{N}\,. \tag{2.31}$$

It is important for the following to remark that the previous relations imply that the entries of the matrix $\lambda\mathcal{L}(\lambda)$ are analytical.

The representation $M_\lambda((1,0,\ldots,0))$, associated to the $gl(\mathcal{N})$ fundamental representation, of $\mathcal{L}(\lambda)$ provides the R matrix (2.1).
Let us remark that $M_{\lambda+a}(\boldsymbol{\alpha})$ $(a \in \mathbb{C})$ defines also a representation of the Yangian, which is isomorphic to $M_\lambda(\boldsymbol{\alpha})$, according to the shift automorphism (2.17).

2.5.2 Representations of the monodromy matrix

The evaluation representations of $\mathcal{L}(\lambda)$ allow us to build a representation of the monodromy matrix. Indeed, evaluating each of the local $\mathcal{L}_{a,n}(\lambda)$ in a representation $M_{\lambda+a_n}(\boldsymbol{\alpha^n})$ for $1 \leq n \leq \ell$, the tensor product built on

$$M_{\lambda+a_1}(\boldsymbol{\alpha^1}) \otimes \cdots \otimes M_{\lambda+a_\ell}(\boldsymbol{\alpha^\ell}) \tag{2.32}$$

provides, via (2.21), a finite-dimensional representation for $\mathcal{T}(\lambda)$.
Denoting by v^n the highest weight vector associated to $\boldsymbol{\alpha^n} = (\alpha_1^n, \ldots, \alpha_{\mathcal{N}}^n)$, the vector

$$v^+ = v^1 \otimes \cdots \otimes v^\ell \tag{2.33}$$

is the highest weight vector of the representation (2.32) i.e.

$$T_{jk}(\lambda)\ v^+ = 0 \qquad , \quad 1 \leq k < j \leq \mathcal{N} \tag{2.34}$$

$$T_{kk}(\lambda)\ v^+ = \prod_{n=1}^{\ell} \left(1 - \frac{\hbar\ \alpha_k^n}{\lambda + a_n}\right)\ v^+ \qquad , \quad 1 \leq k \leq \mathcal{N}\ . \tag{2.35}$$

We will be interested only in the irreducible finite-dimensional representations of the monodromy matrix. When the representation is reducible, the Bethe Ansatz does not give all the eigenvalues of the transfer matrix.

There exists a necessary and sufficient criteria for a tensor product of Yangian representations to be irreducible. It uses the following definition

Definition 2.3 *Let X and Y two disjoint finite subsets of $\mathbb{Z}$. X and Y are crossing if there exists $x_1, x_2 \in X$ and $y_1, y_2 \in Y$ such that*

$$x_1 < y_1 < x_2 < y_2 \qquad or \quad y_1 < x_1 < y_2 < x_2\ . \tag{2.36}$$

Otherwise X and Y are non-crossing.

We associate to each highest weight $\boldsymbol{\alpha}$ the following subset of $\mathbb{Z}$

$$X_{\boldsymbol{\alpha}} = \{\alpha_1, \alpha_2 - 1, \ldots, \alpha_{\mathcal{N}} - \mathcal{N} + 1\}\ . \tag{2.37}$$

The theorem giving the criteria to obtain irreducible representations states:

Theorem 2.4 *[55]*

1. *The tensor product* $M_{\lambda+a_1}(\boldsymbol{\alpha^1})\otimes\cdots\otimes M_{\lambda+a_\ell}(\boldsymbol{\alpha^\ell})$ *is irreducible if and only if the tensor product*
$$M_\lambda(\boldsymbol{\alpha^1}-\frac{a_1}{\hbar}\mathbf{1})\otimes\cdots\otimes M_\lambda(\boldsymbol{\alpha^\ell}-\frac{a_\ell}{\hbar}\mathbf{1})$$
is irreducible, with $\mathbf{1}=(1,\ldots,1)$.

2. *The tensor product* $M_\lambda(\boldsymbol{\alpha^1})\otimes\cdots\otimes M_\lambda(\boldsymbol{\alpha^\ell})$ *is irreducible if and only if all the tensor products* $M_\lambda(\boldsymbol{\alpha^p})\otimes M_\lambda(\boldsymbol{\alpha^q})$ *with* $p<q$ *are irreducible.*

3. *The tensor product* $M_\lambda(\boldsymbol{\alpha})\otimes M_\lambda(\boldsymbol{\beta})$ *is irreducible if and only if the sets* $X_{\boldsymbol{\alpha}}\backslash X_{\boldsymbol{\beta}}$ *and* $X_{\boldsymbol{\beta}}\backslash X_{\boldsymbol{\alpha}}$ *are non-crossing.*

Note that if $\boldsymbol{\alpha^1}=\boldsymbol{\alpha^2}=\ldots=\boldsymbol{\alpha^\ell}\equiv\boldsymbol{\alpha}$ the tensor product $M_\lambda(\boldsymbol{\alpha})^{\otimes\ell}$ is irreducible. This special case is the one generally used for spin chains models (see examples below).

2.6 Analytical Bethe Ansatz

We now use the above mathematical framework to study general closed spin chains. We will be able to construct and study a spin chain with arbitrary (and not necessarily identical) representations of $gl(\mathcal{N})$ on each site of the chains. Put in other words, the algebraic set-up given above allows us to treat simultaneously all the possible spin chain models built in this way. In particular, we will obtain the Bethe equations for all these models.

2.6.1 Hamiltonian of the model

From now on, we use as local and monodromy matrices the following elements:

$$\widehat{\mathcal{L}}_{a,n}(\lambda)=(\lambda+a_n)\mathcal{L}_{a,n}(\lambda)\qquad\text{and}\quad\widehat{\mathcal{T}}(\lambda)=\prod_{n=1}^{\ell}(\lambda+a_n)\,\mathcal{T}(\lambda)\,. \tag{2.38}$$

The normalisation of the monodromy matrix (2.38) ensures its analyticity. Such a condition is crucial for the analytical Bethe Ansatz method. The transfer matrix will be accordingly normalised: $\widehat{t}(\lambda)=tr_a\widehat{\mathcal{T}}_a(\lambda)$.

The properly normalised transfer matrix is a monic polynomial in λ of degree ℓ: $\widehat{t}(\lambda)=\lambda^\ell+\sum_{n=0}^{\ell-1}H_n\lambda^n$. The ℓ generalised Hamiltonians H_n are in involution (see equation (2.24)) and algebraically independent (proved by looking at the number of involved sites in each H_n). The Hamiltonian of the spin chain model under consideration will be constructed as a polynomial in the generalised Hamiltonians H_n and will be then integrable.

Usually, in the spin chain context, we deal with Hamiltonian describing a local interaction, i.e. an interaction between nearest neighbour. Unfortunately, at this stage, there is no explicit formula to compute this type of Hamiltonian from the transfer matrix. Note however that when all the quantum spaces correspond to the same representation, an approach using the fusion of auxiliary spaces can be applied [10]. Explicit forms of Hamiltonians will be also given for various models in section 2.7.

2.6.2 Highest weight vector / Pseudo-vacuum

We now compute the eigenvalues of the transfer matrix $\widehat{t}(\lambda)$. As a by-product, they will provide the Hamiltonian eigenvalues. The procedure consists in three steps.

The first step consists in finding a particular eigenvector (so-called pseudo-vacuum) of the transfer matrix and in computing the corresponding eigenvalue. We get

$$\widehat{T}_{jk}(\lambda)\ v^+ = 0 \qquad , \quad 1 \leq k < j \leq \mathcal{N} \tag{2.39}$$

$$\widehat{T}_{kk}(\lambda)\ v^+ = \prod_{n=1}^{\ell} (\lambda + a_n - \hbar\ \alpha_k^n)\ v^+ \qquad , \quad 1 \leq k \leq \mathcal{N}\ , \tag{2.40}$$

where v_+ is given in (2.33).
In the following, we use the following notation, for $1 \leq k \leq \mathcal{N}$

$$P_k(\lambda) = \prod_{n=1}^{\ell} (\lambda + a_n - \hbar\ \alpha_k^n)\ . \tag{2.41}$$

These polynomials, called Drinfel'd polynomials, are usually introduced to classify the representations of Yangians.

The highest weight vector (2.33) is obviously an eigenvector of the transfer matrix. Indeed, one gets

$$\widehat{t}(\lambda)\ v^+ = \sum_{k=1}^{\mathcal{N}}\ \widehat{T}_{kk}(\lambda)\ v^+ = \Lambda^0(\lambda)\ v^+ \tag{2.42}$$

with

$$\Lambda^0(\lambda) = \sum_{k=1}^{\mathcal{N}}\ P_k(\lambda)\ . \tag{2.43}$$

Note that $\Lambda^0(\lambda)$ is analytical. In the context of the spin chains, the highest weight vector v^+ is called the pseudo-vacuum. The second step consists in the Ansatz itself which provides all the eigenvalues of $\widehat{t}(u)$ from $\Lambda^0(\lambda)$.

2.6.3 Dressing functions

We make the following assumption for the structure of all the eigenvalues of $\widehat{t}(u)$

$$\Lambda(\lambda) = \sum_{k=1}^{\mathcal{N}}\ P_k(\lambda)\ D_k(\lambda)\ , \tag{2.44}$$

where $D_k(\lambda)$, the so-called dressing functions, have to be determined. At that point, the irreducibility of the representation is a necessary criteria for the completeness of the spectrum obtained by dressing. From the asymptotic behaviour ($\lambda \to +\infty$) of $\widehat{t}(\lambda)$, we deduce that, for $1 \leq k \leq \mathcal{N}$

$$D_k(\lambda) \xrightarrow[\lambda \to +\infty]{} 1\ . \tag{2.45}$$

We suppose that the dressing functions are rational functions of the form

$$D_k(\lambda) = \prod_{n=1}^{M^{(k-1)}} \frac{\lambda + u_n^{(k-1)}}{\lambda - \lambda_n^{(k-1)} - \frac{\hbar\,(k-1)}{2}} \prod_{n=1}^{M^{(k)}} \frac{\lambda + v_n^{(k)}}{\lambda - \lambda_n^{(k)} - \frac{\hbar\,k}{2}}\ , \tag{2.46}$$

where $M^{(0)} = M^{(\mathcal{N})} = 0$.

Remarks

1. The relation between $D_k(\lambda)$ and $D_{k+1}(\lambda)$ poles is the basic ingredient of the analytical Bethe Ansatz. This pole structure is the simplest one which ensures the analyticity of the eigenvalues.

2. We introduced shifts in the denominators for later convenience.

3. The Lie algebra $gl(\mathcal{N})$ being an invariance of the transfer matrix (see proposition 2.2), the transfer matrix eigenvectors are indeed eigenvectors of the $gl(\mathcal{N})$ Cartan generators. The numbers $M^{(k)}$ $(1 \leq k \leq \mathcal{N}-1)$ are deduced from the action of these Cartan generators on the eigenvector of eigenvalue $\Lambda(\lambda)$.

We now tackle the third step, which consists in finding constraints to determine $u_n^{(k)}$ and $v_n^{(k)}$ in terms of $\lambda_n^{(n)}$.

2.6.4 Fusion procedure

We shall use the fusion introduced previously in [56,57] to obtain constraints on the dressing functions.

Let $A_{\mathcal{N}}$ be the antisymmetriser defined by the relation (2.12) which acts on auxiliary spaces $a_1, \ldots, a_{\mathcal{N}}$. Then, from the following relation

$$\widehat{\mathcal{T}}_{a_{\mathcal{N}}}(\lambda - \hbar\mathcal{N} + \hbar) \cdots \widehat{\mathcal{T}}_{a_1}(\lambda) \;=\; qdet\ \widehat{\mathcal{T}}(\lambda)\ A_{\mathcal{N}} + \widehat{\mathcal{T}}_{a_{\mathcal{N}}}(\lambda - \hbar\mathcal{N} + \hbar) \cdots \widehat{\mathcal{T}}_{a_1}(\lambda)\ (1 - A_{\mathcal{N}})\ , \tag{2.47}$$

we deduce, by taking the trace in the spaces $a_1, \ldots, a_{\mathcal{N}}$, that

$$\widehat{t}(\lambda - \hbar\mathcal{N} + \hbar)\ \widehat{t}(\lambda - \hbar\mathcal{N} + 2\hbar)\ \ldots\ \widehat{t}(\lambda) = qdet\ \widehat{\mathcal{T}}(\lambda) + \widehat{t}_{\mathfrak{f}}(\lambda)\ , \tag{2.48}$$

where $\widehat{t}_{\mathfrak{f}}(\lambda) = tr_{a_1 \ldots a_{\mathcal{N}}} \widehat{\mathcal{T}}_{a\mathcal{N}}(\lambda - \hbar\mathcal{N} + \hbar) \cdots \widehat{\mathcal{T}}_{a_1}(\lambda)(1 - A_{\mathcal{N}})$ is the so-called fused transfer matrix.
We can compute the value of the quantum determinant using (2.11) and the properties of the highest weight. Indeed,

$$qdet\ \widehat{\mathcal{T}}(\lambda)\ v^+ \;=\; \sum_{\sigma \in \mathfrak{S}_{\mathcal{N}}} sgn(\sigma)\ \widehat{T}_{1,\sigma(1)}(\lambda - \hbar\mathcal{N} + \hbar) \cdots \widehat{T}_{\mathcal{N},\sigma(\mathcal{N})}(\lambda)\ v^+ \tag{2.49}$$

$$\;=\; \prod_{k=1}^{\mathcal{N}} P_k(\lambda - \hbar\mathcal{N} + \hbar k)\ v^+\ . \tag{2.50}$$

The quantum determinant being central, the above relation implies that

$$qdet\ \widehat{\mathcal{T}}(\lambda) = \prod_{k=1}^{\mathcal{N}} P_k(\lambda - \hbar\mathcal{N} + \hbar k)\,. \tag{2.51}$$

Then, acting with any eigenvector v with eigenvalue $\Lambda(\lambda)$ on relation (2.48), one obtains

$$\Lambda(\lambda - \hbar\mathcal{N} + \hbar) \dots \Lambda(\lambda) = \prod_{k=1}^{\mathcal{N}} P_k(\lambda - \hbar\mathcal{N} + \hbar k) + \Lambda_{\mathfrak{f}}(\lambda) \ , \tag{2.52}$$

where $\Lambda_{\mathfrak{f}}(\lambda)\ v = \widehat{t}_{\mathfrak{f}}(\lambda)\ v$. Let us remark that this relation shows that v is also an eigenvector of $\widehat{t}_{\mathfrak{f}}(\lambda)$, in accordance with the commutator

$$[\ \widehat{t}_{\mathfrak{f}}(\lambda)\ ,\ \widehat{t}(\mu)\ ] = 0 \ . \tag{2.53}$$

Finally, picking the term proportional to $\prod_{k=1}^{\mathcal{N}} P_k(\lambda - \hbar\mathcal{N} + \hbar k)$ in the relation (2.52), we deduce a constraint between the dressing functions, namely

$$D_1(\lambda - \hbar\mathcal{N} + \hbar) \dots D_{\mathcal{N}}(\lambda) = 1 \ . \tag{2.54}$$

This constraint allows us to express the parameters $u_n^{(k)}$ and $v_n^{(k)}$ in terms of $\lambda_n^{(k)}$. We conclude that the dressing functions take the following form

$$D_k(\lambda) = \prod_{n=1}^{M^{(k-1)}} \frac{\lambda - \lambda_n^{(k-1)} - \frac{\hbar\,(k+1)}{2}}{\lambda - \lambda_n^{(k-1)} - \frac{\hbar\,(k-1)}{2}} \prod_{n=1}^{M^{(k)}} \frac{\lambda - \lambda_n^{(k)} - \frac{\hbar\,(k-2)}{2}}{\lambda - \lambda_n^{(k)} - \frac{\hbar\,k}{2}} \ . \tag{2.55}$$

2.6.5 Universal Bethe equations

We have chosen the normalisation of the matrix $\widehat{\mathcal{T}}(\lambda)$ in such a way that its entries are analytical. Then, the eigenvalues of $\widehat{t}(\lambda)$ are also analytical, since $\widehat{t}(\lambda)$ can be diagonalised by a constant matrix (see equation (2.24)).

Theorem 2.5 *The Bethe equations read, for* $1 \leq k \leq \mathcal{N} - 1$ *and* $1 \leq n \leq M^{(k)}$

$$\prod_{m=1}^{M^{(k-1)}} e_{-1}\left(\lambda_n^{(k)} - \lambda_m^{(k-1)}\right) \prod_{\substack{m=1 \\ m\neq n}}^{M^{(k)}} e_2\left(\lambda_n^{(k)} - \lambda_m^{(k)}\right) \prod_{m=1}^{M^{(k+1)}} e_{-1}\left(\lambda_n^{(k)} - \lambda_m^{(k+1)}\right) = \frac{P_k\left(\lambda_n^{(k)} + \frac{\hbar\,k}{2}\right)}{P_{k+1}\left(\lambda_n^{(k)} + \frac{\hbar\,k}{2}\right)} \tag{2.56}$$

where

$$e_x(\lambda) \ = \ \frac{\lambda - \frac{\hbar\,x}{2}}{\lambda + \frac{\hbar\,x}{2}} \ . \tag{2.57}$$

The left hand side of (2.56) depends only on the choice of the algebra (the indices of the function $e_x(\lambda)$ *describe the entries of the Cartan matrix of* $gl(\mathcal{N})$*), while the right hand side depends on the choice of the representation.*

Proof: By imposing that the $\Lambda(\lambda)$ residue vanishes at $\lambda = \lambda_n^{(k)} + \frac{\hbar\,k}{2}$, we find (2.56). ∎

The RHS of (2.56) can be written in terms of the functions $e_x(\lambda)$, using the expression of the highest weights. These Bethe equations have been computed in [12], however the method and the starting hypotheses are different. The identity between the results appears as a ground for this

Ansatz.
It should be clear that the Bethe equations (2.56), and the dressing of the eigenvalues, (2.44) and (2.55), are valid whatever the expression of the Drinfel'd polynomial is, and as such are universal. The dressing functions (and thus the expression of the eigenvalues) appear formally independent from the choice of the representations. However, the Bethe equations depending on the representations, their resolution will lead to different eigenvalues.
The choice of a closed spin chain model amounts to the choice of the $gl(\mathcal{N})$ representation $M(\boldsymbol{\alpha^k})$ for spins at sites k, $1 \leq k \leq \ell$. This will fix the evaluation representations $M_\lambda(\boldsymbol{\alpha^k})$, hence the polynomials $P_k(\lambda)$. Then, the eigenvalues and the Bethe equations follow. We now illustrate this procedure by employing spin chain models.

Remark: reducible representations
When the representation is reducible, the above calculations are still valid, but they do not lead to a complete set of eigenvalues for the transfer matrix. In fact, one gets in that case all the eigenvalues associated to the irreducible subrepresentation with highest weight v^+. A simple indication for that is the eq. (2.50) which now implies (2.51) only on this irreducible subrepresentation.

2.7 Examples

Choosing appropriate representations, we shall recover known results associated with the fundamental representation, generalise the relations about the alternating spin chains and provide new integrable models (such as general impurity spin chains). For simplicity, we will most of the time set the inhomogeneous parameters a_n to zero. However, our formalism easily deals with these inhomogeneous parameters, as we shall see in the next example.

2.7.1 Closed spin chain in the fundamental representation

The usual closed spin chain corresponds to spins in the fundamental representation. The Hamiltonian is given by the well-known formula

$$H = \frac{d}{d\lambda} \left(ln\ \widehat{t}(\lambda) \right) \Big|_{\lambda=0} . \tag{2.58}$$

In this case, we have $\boldsymbol{\alpha}^n = (1, 0, \ldots, 0)$, for $1 \leq n \leq \ell$. Then, the Drinfel'd polynomials read

$$P_k(\lambda) = \begin{cases} \prod_{j=1}^{\ell} (\lambda + a_j - \hbar) & , \quad k = 1 \\ \prod_{j=1}^{\ell} (\lambda + a_j) & , \quad k \neq 1 \end{cases} \tag{2.59}$$

so that

$$\frac{P_k\left(\lambda_n^{(k)} + \frac{\hbar\, k}{2}\right)}{P_{k+1}\left(\lambda_n^{(k)} + \frac{\hbar\, k}{2}\right)} = \begin{cases} \prod_{j=1}^{\ell} e_1\left(\lambda_n^{(k)} + a_j\right) & , \quad k = 1 \\ 1 & , \quad k \neq 1 \end{cases} . \tag{2.60}$$

Plugging these expressions in the Bethe equations (2.56), we recover the usual Bethe equations for closed spin chains[3]

$$\prod_{m=1}^{M^{(k-1)}} e_{-1}\left(\lambda_n^{(k)}-\lambda_m^{(k-1)}\right)\prod_{\substack{m=1\\m\neq n}}^{M^{(k)}} e_2\left(\lambda_n^{(k)}-\lambda_m^{(k)}\right)\prod_{m=1}^{M^{(k+1)}} e_{-1}\left(\lambda_n^{(k)}-\lambda_m^{(k+1)}\right)=\begin{cases}\displaystyle\prod_{j=1}^{\ell} e_1\left(\lambda_n^{(k)}+a_j\right) & , \quad k=1\\ 1 & , \quad k\neq 1\end{cases}$$

Since the value of the local operator $\mathcal{L}_{ij}(\lambda)$ at $\lambda=0$ is the permutation operator $\mathcal{P}_{ij}$ between spaces i and j (see eq. (2.2)), we can construct a local Hamiltonian by the relation (2.58) (when $a_n=0$)

$$H\propto\sum_{n=1}^{\ell}\mathcal{P}_{n-1,n}\qquad\text{with}\quad\mathcal{P}_{01}=\mathcal{P}_{\ell 1}\,. \tag{2.61}$$

In the case of $gl(2)$, one recovers the celebrated XXX Hamiltonian

$$H=\frac{1}{2}\sum_{n=1}^{\ell}\ \left(\ \sigma_{n-1}^x\sigma_n^x+\sigma_{n-1}^y\sigma_n^y+\sigma_{n-1}^z\sigma_n^z+1\ \right)\ , \tag{2.62}$$

where σ_n^t, $t=x,y,z$ are the Pauli matrices at site n, and $\sigma_0^t=\sigma_\ell^t$.

If we take $a_p\neq 0$ for a particular site p (and $a_n=0$ for $n\neq p$), we obtain a Hamiltonian with one impurity

$$H\propto\sum_{\substack{n=1\\n\neq p,p+1}}^{\ell}\mathcal{P}_{n-1,n}-\frac{\hbar}{a_p-\hbar}+\frac{1}{a_p^2-\hbar^2}\left(\ a_p^2\,\mathcal{P}_{p-1,p+1}-\hbar^2\,\mathcal{P}_{p+1,p}\ \right)+\frac{\hbar\,a_p}{a_p^2-\hbar^2}\mathcal{P}_{p-1,p+1}(\mathcal{P}_{p-1,p}-\mathcal{P}_{p+1,p})\,.$$

2.7.2 Closed spin chain for non-fundamental representations

One can generalise the above example to the case where all the spins belong to the same (not necessarily fundamental) representation, given by

$$\boldsymbol{\alpha^1}=\boldsymbol{\alpha^2}=\ldots=\boldsymbol{\alpha^\ell}=(\alpha_1,\alpha_2,\ldots,\alpha_{\mathcal{N}})\,. \tag{2.63}$$

In particular, we recover the result given in [10, 11, 23, 58, 59] about the XXX higher spin chains.

We will use the variables

$$\alpha_k\pm\alpha_{k+1}=\beta_k^{\pm}$$

which are integers, since we consider $gl(\mathcal{N})$ irreducible finite-dimensional representations. This leads to the following Drinfel'd polynomials

$$P_k(\lambda)=(\lambda-\hbar\alpha_k)^\ell\qquad\text{so that}\quad\frac{P_k\left(\lambda_n^{(k)}+\frac{\hbar\,k}{2}\right)}{P_{k+1}\left(\lambda_n^{(k)}+\frac{\hbar\,k}{2}\right)}=\left[e_{\beta_k^-}\left(\lambda_n^{(k)}+\hbar\frac{k-\beta_k^+}{2}\right)\right]^\ell\,. \tag{2.64}$$

[3]We remind that $\hbar=-i$ when dealing with spin chain models.

For instance, if we particularise to the $gl(2)$ spin chain in the spin s representation, we get as Bethe equations

$$\prod_{\substack{m=1\\ m\neq n}}^{M} e_2\left(\lambda_n-\lambda_m\right)=\left[e_{2s}\left(\lambda_n+\hbar\frac{1-2s}{2}\right)\right]^{\ell}\ . \tag{2.65}$$

The construction of a local Hamiltonian cannot be repeated from section 2.7.1 because there is no particular parameter where the local operator $\mathcal{L}(\lambda)$ is the permutation. However, a local Hamiltonian can be constructed by using the fusion method introduced in [10, 11, 23, 58] or by evaluating the universal R-matrix, see e.g. [59]. It takes the form

$$H\propto\sum_{n=1}^{\ell}Q_{2s}\left(\ s^x_{n-1}s^x_n+s^y_{n-1}s^y_n+s^z_{n-1}s^z_n\ \right)\ ,\qquad\text{where}\quad Q_{2s}(x)=\sum_{j=1}^{2s}\left(\sum_{k=1}^{j}\frac{1}{k}\right)\prod_{\substack{l=0\\ l\neq j}}^{2s}\frac{x-x_l}{x_j-x_l}\,. \tag{2.66}$$

In the above formula, $x_l=\frac{1}{2}(l(l+1)-2s(s+1))$, s^t_n, $t=x,y,z$ are the $gl(2)$ generators in the spin s representation acting in the quantum space n, and satisfying $s^t_0=s^t_\ell$. The energy spectrum is then given by

$$E=-\sum_{j=1}^{\ell}\frac{s}{\lambda_j+s^2}\ , \tag{2.67}$$

where λ_j are solutions of the Bethe equations (2.65).

2.7.3 Alternating spin chains

In alternating spin chains, the spins along the chain belong alternatively to two different given representations. We take the particular example of the alternating spin chain with the number of sites $\ell=2\widetilde{\ell}$ even. The spins of even sites are represented in the fundamental representation whereas the spins of the odd sites are in another representation. We take the following particular example where the highest weights are given by

$$\boldsymbol{\alpha}^n=\begin{cases}(1,0,\ldots,0), & 1\leq n\leq\ell \quad \text{and} \quad n \text{ even}\\ (2,0,\ldots,0), & 1\leq n\leq\ell \quad \text{and} \quad n \text{ odd}\end{cases}\ . \tag{2.68}$$

Then, the left hand side of the Bethe equations read, for $1\leq n\leq M^{(k)}$, $1\leq k\leq\mathcal{N}-1$ and $\mathcal{N}>2$

$$\frac{P_k\left(\lambda^{(k)}_n+\frac{\hbar\,k}{2}\right)}{P_{k+1}\left(\lambda^{(k)}_n+\frac{\hbar\,k}{2}\right)}=\begin{cases}\left(e_1\left(\lambda^{(1)}_n\right)e_2\left(\lambda^{(1)}_n-\frac{\hbar}{2}\right)\right)^{\widetilde{\ell}} & ,\quad k=1\\ 1 & ,\quad 1<k<\mathcal{N}\end{cases}\ . \tag{2.69}$$

We recover the Bethe equation given in [8] (see also [60]) for $gl(2)$. In the case $gl(2)$, we can compute a Hamiltonian by the usual formula (2.58) which contains both nearest and next-to-nearest neighbour

interactions with periodic boundary conditions. The explicit form of this Hamiltonian is given by [8]

$$H \;\propto\; \sum_{j=1}^{\widetilde{\ell}} \{\; (2\boldsymbol{\sigma}_{2j}\cdot\boldsymbol{s}_{2j+1}+1)(2\boldsymbol{\sigma}_{2j+2}\cdot\boldsymbol{s}_{2j+1}+3) \\ +(2\boldsymbol{\sigma}_{2j}\cdot\boldsymbol{s}_{2j-1}+1)\left[(\boldsymbol{s}_{2j-1}\cdot\boldsymbol{s}_{2j+1}+1)(2\boldsymbol{\sigma}_{2j}\cdot\boldsymbol{s}_{2j+1}+1)+2\right]\;\} \tag{2.70}$$

where $\boldsymbol{\sigma}=(\sigma^x,\sigma^y,\sigma^z)$ are the Pauli matrices and $\boldsymbol{s}=(s^x,s^y,s^z)$ are the generators of $sl(2)$ in the spin 1 representation.

We can also recover the results of [9] where another type of alternating spin chains has been studied for $su(3)$.

2.7.4 Impurity

We consider now a spin chain with one site (the impurity) in a representation different from the others. Let us take as example a spin chain where all sites are represented in the fundamental representation except for the p^{th} which is associated to the representation of highest weight $\boldsymbol{\alpha}^p$. In this case, the left hand side of (2.56) becomes

$$\frac{P_k\left(\lambda_n^{(k)}+\frac{\hbar\,k}{2}\right)}{P_{k+1}\left(\lambda_n^{(k)}+\frac{\hbar\,k}{2}\right)}=\begin{cases} e_1\left(\lambda_n^{(k)}\right)^{\ell-1}\dfrac{\lambda_n^{(k)}+\frac{\hbar}{2}-\hbar\,\alpha_1^p}{\lambda_n^{(k)}+\frac{\hbar}{2}-\hbar\,\alpha_2^p} & ,\quad k=1 \\ \dfrac{\lambda_n^{(k)}+\frac{\hbar\,k}{2}-\hbar\,\alpha_k^p}{\lambda_n^{(k)}+\frac{\hbar\,k}{2}-\hbar\,\alpha_{k+1}^p} & ,\quad k\neq 1 \end{cases} \tag{2.71}$$

The Hamiltonian can be written as

$$H\propto\sum_{\substack{n=1\\ n\neq p,p+1}}^{\ell}\mathcal{P}_{n-1,n}+\;\left(\;\mathcal{P}_{p-1,p+1}\;\widehat{\mathcal{L}}_{p+1,p}(0)-\hbar\;\right)\;\widehat{\mathcal{L}}^{-1}_{p-1,p}(0) \tag{2.72}$$

where $\mathcal{P}_{01}=\mathcal{P}_{\ell 1}$, $\widehat{\mathcal{L}}_{n,p}(\lambda)=\sum_{i,j}E_{ij}\otimes(\lambda-\hbar\mathcal{E}_{ji})$. In the last formula, E_{ij} belongs to the space n (fundamental representation), while $(\lambda-\hbar\mathcal{E}_{ji})$ is in the particular space p where the generators of $gl(\mathcal{N})$, $\mathcal{E}_{ij}$, are in the representation with the highest weight $\boldsymbol{\alpha}^p$.

2.7.5 Generalisation to tensor products of representation on each site

Up to now, we have assumed that on each site of the spin chain, only one evaluation representation occurs. This assumption is natural from the spin chain point of view, since one can interpret the underlying $gl(\mathcal{N})$ representation as carrying the spin. However, the algebraic framework we have presented can deal with more general representations of the Yangians, provided they are irreducible and finite-dimensional. Using the theorem 2.4, the irreducible representations will be constructed from tensor products of evaluation representations. Let us stress that, generically, this tensor product of evaluation representations is irreducible, although for the transfer matrix symmetry algebra $gl(\mathcal{N})$ these representations are fully reducible.

From the physical point of view, the model will describe a spin chain possessing on each site a quantum space which is a tensor product of evaluation representations of the Yangian. However, this model (in particular the transfer matrix) can be reinterpreted as a usual spin chain model but with a higher number of sites, each of them associated to only one evaluation representation.

Finally, let us remark that this construction is in essence opposite to the fusion procedure. Indeed, for the fusion, one takes particular points (described in theorem 2.4) where the tensor product of evaluation representations is reducible.

3 Open spin chains with preserving boundary conditions

In this section, we compute, along the lines described in the previous section, the Bethe equations for the open spin chains with soliton preserving boundary conditions [61–64]. For such a purpose, we first need to introduce some new algebraic objects such as the reflection algebra or the K matrix.

3.1 Reflection K matrix

In the case of soliton preserving boundary conditions, we need to introduce numerical matrices, called K matrices, which are solutions of the reflection (boundary Yang–Baxter) equations [27]:

$$R_{ab}(\lambda_a - \lambda_b)\; K_a(\lambda_a)\; R_{ba}(\lambda_a + \lambda_b)\; K_b(\lambda_b) = K_b(\lambda_b)\; R_{ab}(\lambda_a + \lambda_b)\; K_a(\lambda_a)\; R_{ba}(\lambda_a - \lambda_b)\;. \tag{3.1}$$

The K matrix is interpreted as the reflection of a soliton on the boundary, coming back as a soliton. The solutions of the equation (3.1) have been classified in [65]:

Proposition 3.1 *Any invertible solution of the soliton preserving reflection equation (3.1) takes the form* $K(\lambda) = U\left(\mathbb{E} + \dfrac{\xi}{\lambda}\,\mathbb{I}_{\mathcal{N}}\right)U^{-1}$ *where either*

(i) $\mathbb{E}$ *is diagonal and* $\mathbb{E}^2 = \mathbb{I}_{\mathcal{N}}$ *(diagonalisable solutions)*

(ii) $\mathbb{E}$ *is strictly triangular and* $\mathbb{E}^2 = 0$ *(non-diagonalisable solutions)*

The matrix U *is an element of the group* $GL(\mathcal{N})$ *and* ξ *a free parameter. The classification is done up to multiplication by a function of the spectral parameter.*
Note that all the K *matrices (but zero) obey a relation* $K(\lambda)K(-\lambda) = f(\lambda)\,\mathbb{I}_{\mathcal{N}}$ *for some non-zero even function* f*.*

A suitable relabelling of the indices allows us to choose the matrix $\mathbb{E}$ in (i) of proposition 3.1 as

$$\mathbb{E} = diag(\underbrace{1,\dots,1}_{\mathcal{M}},\underbrace{-1,\dots,-1}_{\mathcal{N}-\mathcal{M}})\;, \tag{3.2}$$

with $0 \leq \mathcal{M} \leq \mathcal{N}$. In the following we will only deal with diagonal solutions of the form

$$\widehat{K}(\lambda) = diag(\underbrace{\lambda+\xi,\dots,\lambda+\xi}_{\mathcal{M}}\;,\;\underbrace{-\lambda+\xi,\dots,-\lambda+\xi}_{\mathcal{N}-\mathcal{M}})\;. \tag{3.3}$$

We normalise the K matrix so that its entries be analytical.

Case of non-diagonal reflection matrices: The general treatment of non-diagonal reflection matrices is yet an open problem. In the case where each spin is represented in the fundamental representation, the problem has been solved in [56, 57] for $K^+ = 1$ and in [66] for simultaneously diagonalisable reflection matrices K^+ and K^-. In the case of the XXZ model, the procedure given in [56, 57] to treat the non-diagonal reflection matrices does not work. However, interesting developments have been done in [67,68] to attempt a general treatment of non-diagonal reflection matrices.

3.2 Reflection algebra

The reflection algebras are constructed as subalgebras of a Yangian, which is here $\mathcal{Y}(gl(\mathcal{N}))$. Starting from the generators $\mathcal{T}(\lambda)$ of $\mathcal{Y}(gl(\mathcal{N}))$ introduced in (2.8), we define

$$\mathcal{B}(\lambda) = \mathcal{T}(\lambda)\, K(\lambda)\, \mathcal{T}(-\lambda)^{-1} \,. \tag{3.4}$$

$\mathcal{B}(\lambda)$ generates an algebra, denoted $\mathfrak{B}(\mathcal{N}, \mathcal{M})$, whose exchange relations are given by

$$R_{ab}(\lambda_a - \lambda_b)\ \mathcal{B}_a(\lambda_a)\ R_{ba}(\lambda_a + \lambda_b)\ \mathcal{B}_b(\lambda_b) = \mathcal{B}_b(\lambda_b)\ R_{ab}(\lambda_a + \lambda_b)\ \mathcal{B}_a(\lambda_a)\ R_{ba}(\lambda_a - \lambda_b) \,. \tag{3.5}$$

Writing $\mathcal{B}(\lambda)$ as

$$\mathcal{B}(\lambda) = \sum_{i,j=1}^{\mathcal{N}} E_{ij} \otimes B_{ij}(\lambda) = \sum_{n=0}^{+\infty} \frac{\mathcal{B}^{(n)}}{\lambda^n}$$

one can show that $\mathcal{B}^{(1)}$ generates a $gl(\mathcal{M}) \oplus gl(\mathcal{N} - \mathcal{M})$ subalgebra in $\mathfrak{B}(\mathcal{N}, \mathcal{M})$.
Another reflection matrix $K^+(\lambda)$, solution of an equation dual to (3.1), is usually introduced to study open spin chains [28]. For simplicity, we will take here $K^+(\lambda) = \mathbb{I}_{\mathcal{N}}$.

The (algebraic) monodromy matrix used to construct open spin chain is obtained from the local operators $\mathcal{L}_{a\,j}(\lambda)$ of the Yangian (2.20). It takes the following form

$$\mathcal{B}_a(\lambda) = \mathcal{L}_{a1}(\lambda)\ \ldots\ \mathcal{L}_{a\ell}(\lambda)\ K_a(\lambda)\ \mathcal{L}^{-1}_{a\ell}(-\lambda)\ \ldots\ \mathcal{L}^{-1}_{a1}(-\lambda) \,. \tag{3.6}$$

The transfer matrix becomes

$$b(\lambda) = tr_a\left(\mathcal{B}_a(\lambda)\right) = \sum_{i=1}^{\mathcal{N}} B_{ii}(\lambda) \tag{3.7}$$

and, as in the Yangian case, the commutation relations defining the algebra allow us to show

$$[b(\lambda), b(\mu)] = 0 \,. \tag{3.8}$$

This relation (3.8) guarantees the integrability of the model, usually described by the following Hamiltonian

$$H = -\frac{1}{2}\frac{d}{d\lambda} b(\lambda)\Big|_{\lambda=0} \,. \tag{3.9}$$

Anticipating again the physical spin chain interpretation, one can compute the symmetry of these models

Proposition 3.2 *The transfer matrix $b(\lambda)$ describing open spin chain models admits an $gl(\mathcal{M}) \oplus gl(\mathcal{N}-\mathcal{M})$ symmetry.*

Proof: Following the steps given for the closed spin chains (see proof of proposition 2.2), one shows that $[\mathcal{B}^{(1)}\,,\, b(\lambda)] = 0$. Since $\mathcal{B}^{(1)}$ generates a $gl(\mathcal{M}) \oplus gl(\mathcal{N}-\mathcal{M})$ algebra, this ends the proof. ∎

3.3 Representations of $\mathfrak{B}(\mathcal{N},\mathcal{M})$

3.3.1 Representation of $\mathcal{T}^{-1}(\lambda)$

In order to study the representations of $\mathfrak{B}(\mathcal{N},\mathcal{M})$, we start from the representations of the Yangian introduced in the section 2.5. Let $M_\lambda(\boldsymbol{\alpha})$ be an evaluation representation of $\mathcal{L}(\lambda)$ with the highest weight vector v. We can show that v is also a highest vector of $\mathcal{L}^{-1}(\lambda)$ with

$$L'_{jk}(\lambda)\; v = 0 \qquad , \quad 1 \leq k < j \leq \mathcal{N} \tag{3.10}$$

$$L'_{kk}(\lambda)\; v = \lambda\, \frac{(\lambda+\hbar-\hbar\,\alpha_1)\cdots(\lambda+k\hbar-\hbar-\hbar\,\alpha_{k-1})}{(\lambda-\hbar\,\alpha_1)\cdots(\lambda+k\hbar-\hbar-\hbar\,\alpha_k)}\; v \qquad , \quad 1 \leq k \leq \mathcal{N}\,, \tag{3.11}$$

where $L'_{jk}(\lambda)$ are the matrix elements of $\mathcal{L}^{-1}(\lambda)$. The values appearing in (3.11) are computed from the relation (2.18).
The relations (3.10) and (3.11) imply that v^+ as given in (2.33) is the highest weight vector of $\mathcal{T}^{-1}(\lambda)$ with

$$T'_{jk}(\lambda)\; v^+ = 0 \qquad , \quad 1 \leq k < j \leq \mathcal{N} \tag{3.12}$$

$$T'_{kk}(\lambda)\; v^+ = \lambda^\ell \frac{P_1(\lambda+\hbar)\cdots P_{k-1}(\lambda+k\hbar-\hbar)}{P_1(\lambda)\cdots P_k(\lambda+k\hbar-\hbar)}\; v^+ \qquad , \quad 1 \leq k \leq \mathcal{N}\,, \tag{3.13}$$

where $T'_{jk}(\lambda)$ are the matrix elements of the matrix $\mathcal{T}^{-1}(\lambda)$ and $P_k(\lambda)$ are defined in (2.41).
Let us remark that

$$\widetilde{\mathcal{T}}(\lambda) = \frac{P_1(-\lambda)\,P_2(-\lambda+\hbar)\cdots P_{\mathcal{N}}(-\lambda+\hbar\mathcal{N}-\hbar)}{(-\lambda)^{\mathcal{N}\ell}}\,\mathcal{T}^{-1}(-\lambda) \tag{3.14}$$

can be understood as the Yangian generators represented in the following tensor product of evaluation representations (as defined in section 2.5.2)

$$\bigotimes_{n=1}^{\ell}\Big(M_\lambda(\boldsymbol{\beta^{1,n}})\otimes\cdots\otimes M_\lambda(\boldsymbol{\beta^{\mathcal{N},n}})\Big) \tag{3.15}$$

where $\boldsymbol{\beta^{k,n}} = (1-\alpha_1^n,\ldots,k-1-\alpha_{k-1}^n,0,k-\alpha_{k+1}^n,\ldots,\mathcal{N}-1-\alpha_{\mathcal{N}}^n)$. This shows that the matrix $(-\lambda)^{(\mathcal{N}-1)\ell}\,\widetilde{\mathcal{T}}(\lambda)$ is analytical.

3.3.2 Representation of the monodromy matrix $\mathcal{B}(\lambda)$

We can now describe the representations of the monodromy matrix $\mathcal{B}(\lambda)$ defined by (3.6). It is known [69] that any finite-dimensional representation of $\mathfrak{B}(\mathcal{N},\mathcal{M})$ is a highest weight representation. They can be constructed in the following way

Theorem 3.3 *Let us consider the Yangian highest weight representation $M_\lambda(\boldsymbol{\alpha^1})\otimes\ldots\otimes M_\lambda(\boldsymbol{\alpha^\ell})$ with highest weight vector $v^+=v^1\otimes\ldots\otimes v^\ell$. Then the realisation (3.6) generates a $\mathfrak{B}(\mathcal{N},\mathcal{M})$ highest weight representation, whose highest weight vector is also v^+ with*

$$B_{jk}(\lambda)\ v^+=0\,,\qquad 1\le k<j\le\mathcal{N} \tag{3.16}$$

$$B_{kk}(\lambda)\ v^+=\left(\sum_{j=1}^{k-1}a_j(\lambda)\ \mu_j(\lambda)+\frac{2\lambda}{2\lambda-k\hbar+\hbar}\,\mu_k(\lambda)\right)\ v^+\,,\qquad 1\le k\le\mathcal{N}\,, \tag{3.17}$$

where, for $1\le j\le\mathcal{N}$,

$$a_j(\lambda)=\frac{-2\lambda}{(2\lambda-j\hbar+\hbar)(2\lambda-j\hbar)} \tag{3.18}$$

and

$$\mu_k(\lambda)=\begin{cases}(-1)^\ell(\lambda+\xi)P_k(\lambda)\dfrac{P_1(-\lambda+\hbar)\cdots P_{k-1}(-\lambda+k\hbar-\hbar)}{P_1(-\lambda)\cdots P_k(-\lambda+k\hbar-\hbar)} & \text{for}\quad 1\le k\le\mathcal{M}\\[2ex] (-1)^\ell(-\lambda+\xi+\mathcal{M}\hbar)P_k(\lambda)\dfrac{P_1(-\lambda+\hbar)\cdots P_{k-1}(-\lambda+k\hbar-\hbar)}{P_1(-\lambda)\cdots P_k(-\lambda+k\hbar-\hbar)} & \mathcal{M}+1\le k\le\mathcal{N}\end{cases} \tag{3.19}$$

Proof: A direct calculation (similar to the one done in [69]) leads to, for $1\le k\le\mathcal{M}$,

$$\frac{2\lambda-k\hbar+\hbar}{2\lambda}B_{kk}(\lambda)v^+ + \frac{\hbar}{2\lambda}\sum_{j=1}^{k-1}B_{jj}(\lambda)v^+=(\lambda+\xi)\ T_{kk}(\lambda)T'_{kk}(-\lambda)v^+ \tag{3.20}$$

and for $\mathcal{M}+1\le k\le\mathcal{N}$

$$\frac{2\lambda-k\hbar+\hbar}{2\lambda}B_{kk}(\lambda)v^+ + \frac{\hbar}{2\lambda}\sum_{j=1}^{k-1}B_{jj}(\lambda)v^+=(-\lambda+\xi+\mathcal{M}\hbar)\ T_{kk}(\lambda)T'_{kk}(-\lambda)v^+\ . \tag{3.21}$$

Then, inverting these formulae and using the expressions (2.35) and (3.13), one gets the expression (3.17). ∎

3.4 Analytical Bethe Ansatz

The analytical Bethe Ansatz method is based upon the analyticity of the represented generators of the algebra. It is ensured by a suitable normalisation given in the following proposition

Proposition 3.4 *Let*

$$\widehat{\mathcal{B}}(\lambda) = (-1)^{\ell}\ P_1(-\lambda)\cdots P_{\mathcal{N}}(-\lambda+\mathcal{N}\hbar-\hbar)\ \mathcal{B}(\lambda)\,. \tag{3.22}$$

Then, $\widehat{\mathcal{B}}(\lambda)$ is analytical (in λ).

Proof: $\widehat{\mathcal{B}}(\lambda)$ can be rewritten as

$$\widehat{\mathcal{B}}(\lambda) = \widehat{\mathcal{T}}(\lambda)\ \times\ \widehat{K}(\lambda)\ \times\ \Big((-\lambda)^{(\mathcal{N}-1)\ell}\,\widetilde{\mathcal{T}}(\lambda)\Big)\ .$$

The three terms of this product are analytical. ∎
From now on, we will use $\widehat{\mathcal{B}}(\lambda)$ instead of $\mathcal{B}(\lambda)$ to ensure, as in the closed spin chain case, the analyticity of the eigenvalues of the transfer matrix

$$\widehat{b}(\lambda) = tr_a\,\widehat{\mathcal{B}}_a(\lambda)\,. \tag{3.23}$$

3.4.1 Pseudo-vacuum

As in the case of the closed spin chain, the first step of the analytical Bethe Ansatz consists in finding a particular eigenvalue of the transfer matrix. This eigenvalue is computed thanks to the highest weight vector v^+. Indeed, one gets

$$\widehat{b}(\lambda)\ v^+ = \sum_{k=1}^{\mathcal{N}} \widehat{B}_{kk}(\lambda)v^+ = \Lambda^0(\lambda)\ v^+ \tag{3.24}$$

where

$$\Lambda^0(\lambda) = \sum_{k=1}^{\mathcal{N}} g_k(\lambda)\ \beta_k(\lambda)\ . \tag{3.25}$$

The functions $g_k(\lambda)$ depends only on the boundary matrix. They are given by

$$g_k(\lambda) = \frac{2\lambda(2\lambda-\mathcal{N}\hbar)}{(2\lambda-k\hbar+\hbar)(2\lambda-k\hbar)} \times \begin{cases} \lambda+\xi & \text{for} \quad 1\le k\le \mathcal{M} \\ -\lambda+\xi+\mathcal{M}\hbar & \text{for} \quad \mathcal{M}+1\le k\le \mathcal{N}\,. \end{cases} \tag{3.26}$$

The functions $\beta_k(\lambda)$ depend on the choice of the representation:

$$\beta_k(\lambda) = P_1(-\lambda+\hbar)\cdots P_{k-1}(-\lambda+k\hbar-\hbar)P_k(\lambda)P_{k+1}(-\lambda+k\hbar)\cdots P_{\mathcal{N}}(-\lambda+\mathcal{N}\hbar-\hbar)\,. \tag{3.27}$$

Let us remark that $\Lambda^0(\lambda)$ is analytical and in particular its residue for $\lambda = k\hbar/2$ vanishes ($0\le k\le\mathcal{N}$).

3.4.2 Dressing functions

The central hypothesis of the analytical Bethe Ansatz is that all the eigenvalues of $\widehat{b}(\lambda)$ can be written

$$\Lambda(\lambda) = \sum_{k=1}^{\mathcal{N}} g_k(\lambda)\ \beta_k(\lambda)\ D_k(\lambda)\ , \tag{3.28}$$

where the dressing functions $D_k(\lambda)$ are rational functions and need to be determined while $g_k(\lambda)$ and $\beta_k(\lambda)$ are given by (3.26) and (3.27), respectively. The vanishing of the residues of $\Lambda(\lambda)$ at $\lambda = k\hbar/2$ implies that

$$D_k(k\hbar/2) = D_{k+1}(k\hbar/2) \qquad \text{for} \quad 1 \leq k \leq \mathcal{N}-1\ . \tag{3.29}$$

Starting from the expression (2.46) for the dressing functions, one can show that the $M^{(k)}$'s are even, and that (up to a rescaling $M^{(k)} \to M^{(k)}/2$) the dressing functions read

$$\begin{aligned} D_k(\lambda) \ = \ & \prod_{n=1}^{M^{(k-1)}} \frac{\lambda+\lambda_n^{(k-1)}-\frac{\hbar(k+1)}{2}}{\lambda+\lambda_n^{(k-1)}-\frac{\hbar\,(k-1)}{2}} \quad \frac{\lambda-\lambda_n^{(k-1)}-\frac{\hbar(k+1)}{2}}{\lambda-\lambda_n^{(k-1)}-\frac{\hbar\,(k-1)}{2}} \\ & \times \prod_{n=1}^{M^{(k)}} \frac{\lambda+\lambda_n^{(k)}-\frac{\hbar k}{2}+\hbar}{\lambda+\lambda_n^{(k)}-\frac{\hbar\,k}{2}} \quad \frac{\lambda-\lambda_n^{(k)}-\frac{\hbar k}{2}+\hbar}{\lambda-\lambda_n^{(k)}-\frac{\hbar\,k}{2}}\ , \end{aligned} \tag{3.30}$$

where $M^{(0)} = M^{(\mathcal{N})} = 0$.

3.4.3 Bethe equations

The normalisation of the matrix $\widehat{\mathcal{B}}(\lambda)$ has been chosen in such a way that its entries are analytical. Then, the eigenvalues of the transfer matrix $b(\lambda)$ are also analytical (since the diagonalisation matrix does not depend on λ).

Theorem 3.5 *The Bethe equations read, for* $1 \leq k \leq \mathcal{N}-1$ *and* $1 \leq n \leq M^{(k)}$

$$\begin{aligned} & \prod_{m=1}^{M^{(k-1)}} \widetilde{e}_{-1}\left(\lambda_n^{(k)}, \lambda_m^{(k-1)}\right) \prod_{\substack{m=1 \\ m\neq n}}^{M^{(k)}} \widetilde{e}_{2}\left(\lambda_n^{(k)}, \lambda_m^{(k)}\right) \prod_{m=1}^{M^{(k+1)}} \widetilde{e}_{-1}\left(\lambda_n^{(k)}, \lambda_m^{(k+1)}\right) \\ & \qquad = \ \frac{\beta_k\left(\lambda_n^{(k)}+\frac{\hbar\,k}{2}\right)}{\beta_{k+1}\left(\lambda_n^{(k)}+\frac{\hbar\,k}{2}\right)} \times \begin{cases} -e_{-\mathcal{M}-2\xi/\hbar}\left(\lambda_m^{(\mathcal{M})}\right) & \text{if} \quad k=\mathcal{M} \\ 1 \qquad \text{otherwise} \end{cases} \end{aligned} \tag{3.31}$$

where

$$\widetilde{e}_x(\lambda,\mu) = e_x\left(\lambda-\mu\right) e_x\left(\lambda+\mu\right)\ , \tag{3.32}$$

the functions $e_x(\lambda)$ *are defined by (2.57) and* $M^{(0)} = M^{(\mathcal{N})} = 0$.

Proof: By imposing the vanishing of the $\Lambda(\lambda)$ residue at $\lambda = \lambda_n^{(k)} + \frac{\hbar\,k}{2}$, we obtain (3.31). ∎

As in the case of the Yangian, the left hand side of (3.31) depends only on the choice of the algebra whereas the right hand side depends on the choice of the representation and the K matrix.

3.5 Examples

3.5.1 Generalities

All the cases presented in section 2.7 can be treated in a similar way for the open spin chain, using the usual formula given in [28] for the Hamiltonian.

As a basic example, one can easily check that the present approach reproduces correctly the results obtained for the open $gl(\mathcal{N})$-spin chain with generic boundary [57].

As more involved examples, we can generalise directly the spin s chain and the alternating spin chain (see sections 2.7.2 and 2.7.3) by adding a boundary with the procedure given above, extending the results obtained in [70].

3.5.2 Boundaries with operators

One may wonder whether the boundary matrices $K_\pm$ can be promoted to operators. Indeed this amounts to "fuse" the boundary to the last site to get a dynamical boundary. This was considered for instance in [28, 71–74].

We treat here an example suggested by K. Zarembo. We study the $gl(2)$ spin chain with $\ell-2$ spins 1 in the bulk and for two spins 1/2 the boundaries. For this spin chain, we represent the monodromy matrix where the highest weights are given by

$$\boldsymbol{\alpha^n} = \begin{cases} (1,0) & , \quad n=1,\ell \\ (2,0) & , \quad 1<n<\ell \end{cases} \tag{3.33}$$

The corresponding integrable Hamiltonian is given by (up to an irrelevant overall normalisation)

$$H = 2\boldsymbol{\sigma}_1\cdot\boldsymbol{S}_2 + 2\boldsymbol{S}_{L-1}\cdot\boldsymbol{\sigma}_L + \sum_{i=2}^{\ell-2}\left(\boldsymbol{S}_i\cdot\boldsymbol{S}_{i+1} - \frac{1}{4}\left(\boldsymbol{S}_i\cdot\boldsymbol{S}_{i+1}\right)^2\right) \tag{3.34}$$

with the following conventions:

$$\sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{3.35}$$

$$S^+ = \sqrt{2}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad S^- = \sqrt{2}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad S^z = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \tag{3.36}$$

and

$$\boldsymbol{A}\cdot\boldsymbol{B} = 2A^+B^- + 2A^-B^+ + A^zB^z\ . \tag{3.37}$$

The $\boldsymbol{S}$ used here is actually equal to $2\boldsymbol{s}$ of section 2.7 so that $\boldsymbol{\sigma}$ and $\boldsymbol{S}$ have the same commutation relations.

This Hamiltonian comes from the monodromy matrix (3.4) with the following prescriptions: the auxiliary space is three-dimensional (spin 1 representation of $gl(2)$); the reflection matrices $K^\pm$ are

taken to be identity matrices.
Hence $\tilde{b}(\lambda) = \mathrm{Tr}_a\, \mathcal{T}_a(\lambda)\mathcal{T}_a(-\lambda)^{-1}$ with

$$\mathcal{T}(\lambda) = R_{a\ell}^{(1,\frac{1}{2})}(\lambda) R_{a,\ell-1}^{(1,1)}(\lambda) \cdots R_{a2}^{(1,1)}(\lambda) R_{a1}^{(1,\frac{1}{2})}(\lambda) \tag{3.38}$$

and

$$R_{aj}^{(1,\frac{1}{2})}(\lambda) = \lambda \mathbb{I}_3 \otimes \mathbb{I}_2 - \frac{\hbar}{2} \boldsymbol{S}_a \cdot \boldsymbol{\sigma}_j \qquad j = 1, \ell\ , \tag{3.39}$$

$$R_{aj}^{(1,1)}(\lambda) = \frac{(\lambda+\hbar)(\lambda-2\hbar)}{2\hbar^2} \mathbb{I}_3 \otimes \mathbb{I}_3 - \frac{\lambda-\hbar}{4\hbar} \boldsymbol{S}_a \cdot \boldsymbol{S}_j + \frac{1}{16} \left(\boldsymbol{S}_a \cdot \boldsymbol{S}_j\right)^2\ , \quad j = 2, \ldots, \ell-1\ . \tag{3.40}$$

These R-matrices can be derived from $R_{ij}^{(\frac{1}{2},\frac{1}{2})}$ given by (2.1) using the usual fusion procedure [75]. As a consequence, the transfer matrix $\tilde{b}(\lambda)$ (and the Hamiltonian) commutes with the transfer matrix $b(\lambda)$ built with the same quantum spaces and spin 1/2 auxiliary space.

Then, using (3.31), the Bethe equations are, for $1 \leq n \leq M$

$$\prod_{\substack{m=1 \\ m \neq n}}^{M} e_2\left(\lambda_n - \lambda_m\right) e_2\left(\lambda_n + \lambda_m\right) = e_1\left(\lambda_n\right)^{\ell+2} e_3\left(\lambda_n\right)^{\ell-2} \tag{3.41}$$

The bulk part of (3.34) is the mixing matrix for some sort of gluon operators in large-N QCD, see [45]. The spin chain boundary term in (3.34) corresponds to the quark-gluon operators.

Perspectives

A natural development of this work is the generalisation to the trigonometric case, which will be presented in a further publication. Soliton non-preserving boundary conditions will also be studied in this framework.

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