

Holomorphic Cliffordian Functions

by

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Abstract.- The aim of this paper is to put the foundations of a new theory of functions, called holomorphic Cliffordian, which should play an essential role in the generalization of holomorphic functions to higher dimensions.

Let $\mathbb{R}_{0,2m+1}$ be the Clifford algebra of \mathbb{R}^{2m+1} with a quadratic form of negative signature, $D = \sum_{j=0}^{2m+1} e_j \frac{\partial}{\partial x_j}$ be the usual operator for monogenic functions and Δ the ordinary Laplacian.

The holomorphic Cliffordian functions are functions $f : \mathbb{R}^{2m+2} \rightarrow \mathbb{R}_{0,2m+1}$, which are solutions of $D\Delta^m f = 0$.

Here, we will study polynomial and singular solutions of this equation, we will obtain integral representation formulas and deduce the analogous of the Taylor and Laurent expansions for holomorphic Cliffordian functions.

In a following paper, we will put the foundations of the Cliffordian elliptic function theory.

0. Introduction

The classical theory of holomorphic functions of one complex variable has been generalized in two directions. The first is the theory of holomorphic functions of several complex variables : in this case we keep the field \mathbb{C} and take the system of partial differential operators $\partial/\partial\bar{z}_i$, $i = 1, \dots, n$. The second direction is the theory of monogenic functions : in this case we take the Clifford algebra and take the operator $D = \sum_{j=0}^m e_j \partial/\partial x_j$ ($\{e_i\}$ orthogonal basis).

Here we follow a different path : we think that the most important thing in the theory of one complex variable is the fact that the identity (i.e. z) and its powers (i.e. z^n) are holomorphic.

1. Notations

Let $\mathbb{R}_{0,2m+1}$ be the Clifford algebra of the real vector space V of dimension $2m + 1$, provided with a quadratic form of negative signature,

$m \in \mathbb{N}$. Denote by S the set of the scalars in $\mathbb{R}_{0,2m+1}$, which can be identified to \mathbb{R} . Let $\{e_i\}, i = 1, 2, \dots, 2m+1$ be an orthonormal basis of V and let $e_0 = 1$.

A point $x = (x_0, x_1, \dots, x_{2m+1})$ of \mathbb{R}^{2m+2} could be also considered as an element of $S \oplus V$, namely $x = \sum_{i=0}^{2m+1} e_i x_i$. So, x , being in $S \oplus V$, is in the Clifford algebra $\mathbb{R}_{0,2m+1}$ and we can act on him by the principal involution in $\mathbb{R}_{0,2m+1}$, which will coincide with a kind of ‘‘conjugation’’ :

$$x^* = x_0 - \sum_{i=1}^{2m+1} e_i x_i.$$

It is remarkable that

$$xx^* = x^*x = |x|^2,$$

where $|x|$ denotes the usual euclidean norm of x in \mathbb{R}^{2m+2} .

Sometime, if necessary, we will resort to the notation $x = x_0 + \vec{x}$, where \vec{x} is the vector part of x , namely $\vec{x} = \sum_{i=1}^{2m+1} e_i x_i$.

2. General definitions

Let Ω be an open set of $S \oplus V$. We will be interested in functions $f : \Omega \rightarrow \mathbb{R}_{0,2m+1}$. It should be noted that one might consider only functions $f : \Omega \rightarrow S \oplus V$. The last ones generate the previous by means of (right) linear combinations.

It is well known that the following operator, named Cauchy (or Fueter, or Dirac) operator ([1], [2], [3], [4]) lies on the basis of the theory of (left) monogenic functions :

$$(1) \quad D = \sum_{i=0}^{2m+1} e_i \frac{\partial}{\partial x_i}.$$

A function $f : \Omega \rightarrow \mathbb{R}_{0,2m+1}$ is said to be (left) monogenic in Ω if and only if :

$$Df(x) = 0$$

for each x on Ω .

It is important to note that the operator D possesses a conjugate operator D^* :

$$(2) \quad D^* = \frac{\partial}{\partial x_0} - \sum_{i=1}^{2m+1} e_i \frac{\partial}{\partial x_i},$$

and that $DD^* = D^*D = \Delta$, where Δ is the ordinary Laplacian.

Now let us state the following :

Définition.- A function $f : \Omega \rightarrow \mathbb{R}_{0,2m+1}$ is said to be (left) holomorphic Cliffordian in Ω if and only if :

$$D\Delta^m f(x) = 0$$

for each x of Ω . Here Δ^m means the m times iterated Laplacian Δ .

Remark.- The set of holomorphic Cliffordian functions is wider than the set of monogenic functions in the sense that every monogenic function is also a holomorphic Cliffordian, but the reciproque is false. Indeed, if $Df = 0$, then $D\Delta^m f = \Delta^m Df = 0$ because the operator Δ^m is a scalar operator.

The simplest example of a function which is holomorphic Cliffordian, but not monogenic is the identity, $\text{id} : x \mapsto x$, for which $Dx = -2m \neq 0$ and clearly $D\Delta^m x = 0$.

Later, we will be able to prove that all entire powers of x are holomorphic Cliffordians, while they are not monogenics.

Remark.- f is (left) holomorphic Cliffordian if and only if $\Delta^m f$ is (left) monogenic.

3. Some properties of the holomorphic Cliffordian functions

(i) All the components of the so called scalar, vector, bivector, \dots , up to the pseudo-scalar parts of a holomorphic Cliffordian function f are polyharmonics of order $m + 1$. This is obvious taking into account that, if $D\Delta^m f = 0$, then applying D^* , one get $\Delta^{m+1} f = 0$ and the result follows because Δ^{m+1} is a scalar operator.

(ii) If f is a polyharmonic function of order $m + 1$, i.e. $\Delta^{m+1} f = 0$, then the function D^*f is holomorphic Cliffordian. Indeed, $\Delta^{m+1} f = DD^* \Delta^m f = D\Delta^m(D^*f) = 0$.

This property will play an important role in the next part of this paper because it is a good machinery for generating holomorphic Cliffordian functions.

(iii) Let us compute $\Delta(xg)$, where $g : S \oplus V \rightarrow \mathbb{R}_{0,2m+1}$ is sufficiently smooth. One has :

$$\begin{aligned}\Delta(xg) &= \sum_{i=0}^{2m+1} \frac{\partial^2}{\partial x_i^2}(xg) = \sum_{i=0}^{2m+1} \frac{\partial}{\partial x_i} \left[\left(\frac{\partial}{\partial x_i}(x_0 + \bar{x}) \right) g + x \frac{\partial g}{\partial x_i} \right] \\ &= \sum_{i=0}^{2m+1} \frac{\partial}{\partial x_i} \left(e_i g + x \frac{\partial g}{\partial x_i} \right) = \sum_{i=0}^{2m+1} \left(e_i \frac{\partial g}{\partial x_i} + \frac{\partial x}{\partial x_i} \frac{\partial g}{\partial x_i} + x \frac{\partial^2 g}{\partial x_i^2} \right) = 2Dg + x\Delta g.\end{aligned}$$

Thus, we have :

$$\begin{aligned}2Dg &= \Delta(xg) - x\Delta g \\ x\Delta g &= \Delta(xg) - 2Dg\end{aligned}$$

Now, if we compute : $2D\Delta g = \Delta(x\Delta g) - x\Delta^2 g = \Delta(\Delta(xg) - 2Dg) - x\Delta^2 g$
 $= \Delta^2(xg) - 2D\Delta g - x\Delta^2 g$. In this way, we get

$$4D\Delta g = \Delta^2(xg) - x\Delta^2 g.$$

Using a recurrence process, we obtain :

$$2(p+1)D\Delta^p g = \Delta^{p+1}(xg) - x\Delta^{p+1}g,$$

for every $p \in \mathbb{N}$. Putting in the last formula, $p = m$, one deduces :

$$(3) \quad 2(m+1)D\Delta^m g = \Delta^{m+1}(xg) - x\Delta^{m+1}g.$$

which gives a sufficient condition for g to be holomorphic Cliffordian, namely g and xg have to be polyharmonics of order $(m+1)$.

But this condition is also necessary. If g is holomorphic Cliffordian, $D\Delta^m g = 0$ and, using (3) one has :

$$(4) \quad \Delta^{m+1}(xg) = x\Delta^{m+1}g.$$

Now, compute the right hand side : $x\Delta^{m+1}g = xD^*(D\Delta^m g) = 0$. So, xg is polyharmonic. From (4), again, it follows that g is also polyharmonic.

(iv) The equation $D\Delta^m f = 0$ is equivalent to the system

$$\begin{cases} Df_{(2p+1)} = f_{(2p+2)} \\ D^* f_{(2p+2)} = f_{(2p+3)} \\ Df_{(2m+1)} = 0. \end{cases}$$

with $p = 0, 1, \dots, m-1$ and $f_{(1)} = f$.

4. Some examples of holomorphic Cliffordian functions

Let us start with the following lemma :

Lemma.- If $f : \mathbb{R}^2 \rightarrow \mathbb{R}$, is harmonic, then $f(x_0, |\vec{x}|)$, where $x = x_0 + \vec{x}$ and $|\vec{x}|^2 = \sum_{i=1}^{2m+1} x_i^2$, is $(m+1)$ -harmonic, that is :

$$\Delta^{m+1} f(x_0, |\vec{x}|) = 0.$$

Proof – Set $r = |\vec{x}|$. Thus, the Laplacian could be written as :

$$\Delta = \frac{\partial^2}{\partial x_0^2} + \frac{\partial^2}{\partial r^2} + \frac{2m}{r} \frac{\partial}{\partial r}.$$

But $f(x_0, r)$ is harmonic, so $\frac{\partial^2 f}{\partial x_0^2} + \frac{\partial^2 f}{\partial r^2} = 0$ and hence :

$$\Delta f(x_0, |\vec{x}|) = \frac{2m}{r} \frac{\partial f}{\partial r}.$$

Now, compute the first iteration :

$$\begin{aligned} \frac{1}{2m} \Delta^2 f(x_0, |\vec{x}|) &= \frac{1}{r} \frac{\partial^3 f}{\partial x_0^2 \partial r} + \frac{2}{r^3} \frac{\partial f}{\partial r} - \frac{2}{r^2} \frac{\partial^2 f}{\partial r^2} \\ &+ \frac{1}{r} \frac{\partial^3 f}{\partial r^3} + \frac{2m}{r} \left(-\frac{1}{r^2} \frac{\partial f}{\partial r} + \frac{1}{r} \frac{\partial^2 f}{\partial r^2} \right) = \\ &= \frac{2m-2}{r^2} \frac{\partial^2 f}{\partial r^2} - \frac{2m-2}{r^3} \frac{\partial f}{\partial r}. \end{aligned}$$

Here, we have take into account that $\frac{1}{r} \frac{\partial}{\partial r} \left(\frac{\partial^2 f}{\partial x_0^2} + \frac{\partial^2 f}{\partial r^2} \right) = 0$.

Thus, we get :

$$\frac{1}{2m} \cdot \frac{1}{2m-2} \Delta^2 f(x_0, |\vec{x}|) = \frac{1}{r^2} \frac{\partial^2 f}{\partial r^2} - \frac{1}{r^3} \frac{\partial f}{\partial r}$$

It is easy to show that :

$$\frac{1}{2m} \cdot \frac{1}{2m-2} \Delta^2 f(x_0, |\vec{x}|) = \left(\frac{1}{r} \frac{\partial}{\partial r} \right)^2 f.$$

Using a recurrence process, it is possible to prove that

$$(5) \quad \Delta^k f(x_0, |\vec{x}|) = 2m(2m-2) \cdots (2m-2k+2) \left(\frac{1}{r} \frac{\partial}{\partial r} \right)^k f,$$

for $k \in \mathbb{N}$. In fact, one needs also a preliminary formula :

$$\frac{\partial^2}{\partial r^2} \left(\frac{1}{r} \frac{\partial}{\partial r} \right)^k = -2k \left(\frac{1}{r} \frac{\partial}{\partial r} \right)^{k+1} + \left(\frac{1}{r} \frac{\partial}{\partial r} \right)^k \frac{\partial^2}{\partial r^2}$$

the proof of which is also achieved by a recurrence argument. The end of the proof of the lemma would be performed setting in (5), $k = m + 1$. ■

Now, combining this lemma with the property (ii), we get a nice process for generating holomorphic Cliffordian functions. Let us illustrate this by the following :

Proposition.- Let $x = x_0 + \vec{x} = x_0 + \sum_{i=1}^{2m+1} e_i x_i$, $\lambda \in \mathbb{R}$ and $n \in \mathbb{N}$. Then, the functions $x \mapsto e^{\lambda x}$ and $x \mapsto x^n$ are holomorphic Cliffordians.

Proof – It is clear that it suffices to prove that $\Delta^m e^{\lambda x}$ and $\Delta^m x^n$ are monogenics. By the lemma, taking the real part of $e^{\lambda z}$, where $z \in \mathbb{C}$, one has :

$$\Delta^{m+1} e^{\lambda x_0} \cos(\lambda |\vec{x}|) = 0.$$

We will obtain a holomorphic Cliffordian function taking $D^* e^{\lambda x_0} \cos(\lambda |\vec{x}|)$. Let us compute this :

$$\begin{aligned} D^* e^{\lambda x_0} \cos(\lambda |\vec{x}|) &= \lambda e^{\lambda x_0} \cos(\lambda |\vec{x}|) - \lambda e^{\lambda x_0} \sin(\lambda |\vec{x}|) D^*(|\vec{x}|) \\ &= \lambda e^{\lambda x_0} \left[\cos(\lambda |\vec{x}|) - \sin(\lambda |\vec{x}|) \frac{D^*(|\vec{x}|^2)}{2|\vec{x}|} \right] = \\ &= \lambda e^{\lambda x_0} \left[\cos(\lambda |\vec{x}|) + \frac{\vec{x}}{|\vec{x}|} \sin(\lambda |\vec{x}|) \right] = \lambda e^{\lambda x_0} e^{\lambda \vec{x}} = \lambda e^{\lambda x}. \end{aligned}$$

It follows immediately that all the terms of the expansion of $e^{\lambda x}$ are holomorphic Cliffordian, and in particular x^n for $n \in \mathbb{N}$. ■

Remark.- When f is holomorphic Cliffordian, then the same is true for $\frac{\partial}{\partial x_j} f$, $j = 0, \dots, 2m + 1$. Indeed : all

$$D\Delta^m\left(\frac{\partial}{\partial x_j} f\right) = \frac{\partial}{\partial x_j}(D\Delta^m f) = 0.$$

More generally, let us denote by D^α the operator of derivation :

$$D^\alpha = \frac{\partial^{\alpha_0 + \alpha_1 + \dots + \alpha_{2m-1}}}{\partial x_0^{\alpha_0} \partial x_1^{\alpha_1} \dots \partial x_{2m+1}^{\alpha_{2m+1}}}.$$

Where $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_{2m+1}) \in \mathbb{N}^{2m+2}$ is a multiindice, then if f is holomorphic Cliffordian, then $D^\alpha f$ est also holomorphic Cliffordian.

See [2] and [4].

5. Polynomial solutions of $D\Delta^m f = 0$

Now, we know that all integer powers of x are monomials which are solutions of the equation

$$D\Delta^m(x^n) = 0, \quad n \in \mathbb{N}.$$

Let us find all possible "monomials". For this purpose, set $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_{2m+1})$ with $\alpha_i \in \mathbb{N}$ and $|\alpha| = \sum_{i=0}^{2m+1} \alpha_i$. Consider the set $\{e_\nu\} = \{e_0, \dots, e_0, e_1, \dots, e_1, \dots, e_{2m+1}, \dots, e_{2m+1}\}$ where e_0 is written α_0 times, $e_i : \alpha_i$ times and $e_{2m+1} : \alpha_{2m+1}$ times. Then set :

$$(6) \quad P_\alpha(x) = \sum_{\mathfrak{S}} \prod_{\nu=1}^{|\alpha|-1} (e_{\sigma(\nu)} x) e_{\sigma(|\alpha|)},$$

the sum being expanded over all distinguishable elements σ of the permutation group \mathfrak{S} of the set $\{e_\nu\}$.

The function $P_\alpha(x)$, as a function of x , is a polynomial of degree $|\alpha| - 1$. A straightforward calculation carried on P_α shows that P_α is equal, up to a rational constant, to $D^\alpha(x^{2|\alpha|-1})$. It follows then that $P_\alpha(x)$ is a holomorphic Cliffordian function.

As an illustration, let us compute $P_{(0,1,1,0)}(x)$, $P_{(1,1,0,0)}(x)$ and $P_{(2,0,0,0)}(x)$ in the case when $m = 1$. Following our notations, we have $|\alpha| = 2$ and

$$P_{(0,1,1,0)}(x) = e_1 x e_2 + e_2 x e_1$$

$$P_{(1,1,0,0)}(x) = e_0 x e_1 + e_1 x e_0$$

$$P_{(2,0,0,0)}(x) = e_0 x e_0.$$

Now, as the first polynomial is concerned, let us calculate

$$\begin{aligned} \frac{\partial^2}{\partial x_1 \partial x_2}(x^3) &= \frac{\partial}{\partial x_1} (e_2 x^2 + x e_2 x + x^2 e_2) = \\ &= (e_2 e_1 x + e_2 x e_1) + (e_1 e_2 x + x e_2 e_1) + (e_1 x e_2 + x e_1 e_2) \\ &= e_1 x e_2 + e_2 x e_1 = P_{(0,1,1,0)}(x) \end{aligned}$$

For the second one :

$$\frac{\partial^2}{\partial x_0 \partial x_1}(x^3) = \frac{\partial}{\partial x_1}(3x^2) = 3(e_1 x + x e_1) = 3(e_0 x e_1 + e_1 x e_0) = 3 P_{(1,1,0,0)}(x).$$

Finally :

$$\frac{\partial^2}{\partial x_0^2}(x^3) = 6x = 6 e_0 x e_0 = 6 P_{(2,0,0,0)}(x).$$

The general formula is :

$$D^\alpha(x^{2|\alpha|-1}) = \begin{cases} P_\alpha(x), & \text{if } \alpha_0 = 0 \\ \alpha_0! C_{2|\alpha|-1}^{\alpha_0} P_\alpha(x), & \text{if } \alpha_0 \neq 0. \end{cases}$$

Later, we will be able to prove that the polynomials $P_\alpha(x)$ form a basis of the space of polynomial solution of the equation $D\Delta^m P = 0$.

Remark : the polynomials $P_\alpha(x)$ are left and right holomorphic Cliffordian.

Put

$$\begin{aligned} \lambda &= \sum_{i=0}^{2m+1} \lambda_i e_i, \quad \lambda_i \in \mathbb{R} \\ \lambda_\alpha &= \prod_{i=1}^{2m+1} \lambda_i^{\alpha_i} \end{aligned}$$

then, the following formal series gives the generating function :

$$(1 - \lambda x)^{-1} \lambda = \sum_{\alpha} P_\alpha(x) \lambda_\alpha.$$

It is convenient, for certain computations, to modify slightly these polynomials a little bit :

Let $\vec{\alpha} = (\alpha_1, \dots, \alpha_{2m+1})$, $\alpha_j \in \mathbb{N}$ $P_{\vec{\alpha}}^n(x) = \frac{1}{|\vec{\alpha}|!} D^{\vec{\alpha}} x^{n+|\vec{\alpha}|}$
then $P_{\vec{\alpha}}^n$ is of degree n

$$|\alpha|! P_{\vec{\alpha}}^{|\alpha|-1}(x) = \frac{(2|\alpha| - \alpha_0 - 1)!}{(2|\alpha| - 1)!} P_{\alpha}(x)$$

$$\frac{\partial}{\partial x_0} P_{\vec{\alpha}}^n(x) = n P_{\vec{\alpha}}^{n-1}(x)$$

$$\frac{\partial}{\partial x_k} P_{\vec{\alpha}}^n(x) = P_{(\alpha_1, \dots, \alpha_k+1, \dots, \alpha_{2m+1})}^{n-1}(x).$$

6. The Cauchy kernel of holomorphic Cliffordian functions

Following Brackx, Delanghe and Sommen [1], recall that there exists a Cauchy kernel connected with the theory of monogenic functions. In our situation, when we study functions of the type :

$$f : S \oplus V \rightarrow \mathbb{R}_{0,2m+1},$$

the related Cauchy kernel is :

$$(7) \quad E(x) = \frac{1}{\omega_m} \frac{x^*}{|x|^{2m+2}}, \quad x \in S \oplus V \setminus \{0\},$$

where $\omega_m = 2\pi^{m+1} \frac{1}{\Gamma(m+1)}$ is the area of the unit sphere in \mathbb{R}^{2m+2} .

Recall also that $E(x)$ is a monogenic function with singularity at the origin, i.e :

$$DE(x) = \delta \quad \text{for } x \in S \oplus V$$

where δ is the Dirac measure.

Let $\omega(y) = dy_0 \wedge \dots \wedge dy_{2m+1}$ and $\gamma(y) = \sum_{i=0}^{2m+1} (-1)^i e_i dy_0 \wedge \dots \wedge \widehat{dy}_i \wedge \dots \wedge dy_{2m+1}$.

Then, we have :

Theorem.- [Integral representation formula (general case)] [1].
If $f \in \mathcal{C}^1(U, \mathbb{R}_{0,2m+1})$, then :

$$\int_{\partial\Omega} E(y-x)\gamma(y)f(y) - \int_{\Omega} E(y-x)Df(y)\omega(y) = \begin{cases} f(x), & x \in \overset{\circ}{\Omega} \\ 0, & x \notin \Omega, \end{cases}$$

where Ω is an oriented compact differentiable variety of dimension $2m+2$ with boundary $\partial\Omega$ and $\Omega \subset U$.

From this theorem follows the following integral representation formula for monogenic functions called also the Cauchy representation formula [1] :

Theorem.- If f is monogenic in U and if $\Omega \subset U$

$$\int_{\partial\Omega} E(y-x)\gamma(y)f(y) = \begin{cases} f(x), & x \in \overset{\circ}{\Omega} \\ 0, & x \notin \Omega \end{cases}$$

It is natural to have an integral representation formula of this type because the Cauchy operator D is of order 1. In our situation, the operator $D\Delta^m$ which gives the holomorphic Cliffordian functions is of order $2m+1$ and the corresponding integral formula would be much more complicated.

But, the first step to obtain such a formula, is to exhibit an analogous of the Cauchy kernel.

Remember that the fundamental solution of the iterated Laplacian, i.e. the function $h : S \oplus V \setminus \{0\} \rightarrow \mathbb{R}$ verifying the equation $\Delta^{m+1}h(x) = 0$ for $x \in S \oplus V \setminus \{0\}$, is in fact well-known : that is

$$h(x) = \ell n |x|, \quad x \in S \oplus V \setminus \{0\}.$$

Recall briefly the idea : using spherical coordinates, i.e. introducing $\rho = |x|$, the radial form of the Laplacian is

$$\Delta_{\rho} = \frac{d^2}{d\rho^2} + \frac{2m+1}{\rho} \frac{d}{d\rho}.$$

Calculating the iterated Laplacian, one get, for $k \in \mathbb{N}$:

$$\Delta_{\rho}^k \ell n \rho = (-1)^{k+1} 2^{k-1} (k-1)! (2m)(2m-2) \cdots (2m-2k+2) \frac{1}{\rho^{2k}}$$

and, thus, when $k = m+1$, one has outside the singularity :

$$\Delta_{\rho}^{m+1} \ell n \rho = 0.$$

Similarly as in the complex case when we know that $\ell n \sqrt{x^2 + y^2}$ is the fundamental solution of the Laplace equation and when we write it

as $\frac{1}{2} \ln (z\bar{z})$, here also we will resort to the relation $xx^* = |x|^2$ for $x = x_0 + \vec{x} \in S \oplus V$ and the final conclusion of our first step is :

The fundamental solution of the iterated Laplacian Δ^{m+1} is $h(x) = \frac{1}{2} \ln (xx^*)$.

Now, according to the property (ii) of §3. $h(x)$ being a polyharmonic function of order $m+1$, then $D^*(\frac{1}{2}\ln (xx^*))$ will be a holomorphic Cliffordian function on $S \oplus V \setminus \{0\}$. But

$$D^* \left(\frac{1}{2} \ln (xx^*) \right) = \frac{1}{2} \frac{D^*(|x|^2)}{|x|^2} = \frac{x^*}{|x|^2} = x^{-1}.$$

In this way, we have found the first holomorphic Cliffordian function with singularity at the origin.

Again, according to the remark of §2. since x^{-1} is holomorphic Cliffordian on $S \oplus V \setminus \{0\}$, then $\Delta^m(x^{-1})$ should be monogenic on the same set. Let us compute

$$\Delta^m(x^{-1}) = \Delta^m D^* \ln \rho = D^* \Delta^m \ln \rho,$$

where we have noted $\rho = |x| = (xx^*)^{\frac{1}{2}}$.

Now explicitly,

$$\begin{aligned} \Delta^m(x^{-1}) &= D^*(-1)^{m+1} 2^{2m-1} (m-1)! (2m)(2m-2) \cdots 2 \cdot \frac{1}{\rho^{2m}} = \\ &= (-1)^{m+1} 2^{2m-1} (m-1)! m! D^* \left(\frac{1}{\rho^{2m}} \right) = \\ &= (-1)^m 2^{2m-1} (m!)^2 \frac{1}{(|x|^2)^{m+1}} D^*(|x|^2) = \\ &= (-1)^m 2^{2m} (m!)^2 \frac{x^*}{|x|^{2m+2}} = \\ &= (-1)^m 2^{2m} (m!)^2 \omega_m E(x). \end{aligned}$$

Thus, we get :

$$\frac{(-1)^m (m+1)}{2^{2m+1} m! \pi^{m+1}} \Delta^m(x^{-1}) = E(x).$$

It becomes natural to introduce a new kernel :

$$N(x) = \varepsilon_m x^{-1},$$

where $\varepsilon_m = (-1)^m \frac{m+1}{2^{2m+1} m! \pi^{m+1}}$.

Remember the basic properties of the kernel $N(x)$:

(i) It is related to the Cauchy kernel of the monogenic functions E by :

$$\Delta^m N(x) = E(x), \quad x \in S \oplus V \setminus \{0\}.$$

(ii) N is holomorphic Cliffordian on $S \oplus V \setminus \{0\}$ because :

$$D\Delta^m N(x) = DE(x) = \delta.$$

7. Integral representation formula for holomorphic Cliffordian functions

Let $f : S \oplus V \rightarrow R_{0,2m+1}$ be a function of class \mathcal{C}^{2m+1} and B be the unit ball in \mathbb{R}^{2m+2} . According to [1], for $x \in \overset{\circ}{B}$, we have

$$f(x) = \int_{\partial B} E(y-x)\gamma(y)f(y) - \int_B E(y-x)Df(y)\omega(y).$$

Substitute $\Delta^m N$ on the place of E , one has :

$$f(x) = \int_{\partial B} \Delta^m N(y-x)\gamma(y)f(y) - \int_B \Delta^m N(y-x)Df(y)\omega(y).$$

Making use of the Green's formula :

$$\int_{\Omega} u\Delta v = \int_{\Omega} v\Delta u + \int_{\partial\Omega} u \frac{\partial v}{\partial n} - \int_{\partial\Omega} v \frac{\partial u}{\partial n}$$

applied on the second integral with $u = Df$ and $v = \Delta^{m-1}N$, we will deduce :

$$\begin{aligned} f(x) &= \int_{\partial B} \Delta^m N(y-x)\gamma(y)f(y) - \int_B \Delta^{m-1}N(y-x)\Delta Df(y)\omega(y) \\ &\quad - \int_{\partial B} \left(\frac{\partial}{\partial n} \Delta^{m-1}N(y-x) \right) Df(y) d\sigma_y + \int_{\partial B} (\Delta^{m-1}N(y-x)) \frac{\partial}{\partial n} Df(y) d\sigma_y. \end{aligned}$$

Iterating the process of applying the Green's formula on the second integral of the preceding formula with $u = D\Delta f$ and $v = \Delta^{m-2}N$, we will deduce a sum of six integrals as follows :

$$\begin{aligned} f(x) &= \int_{\partial B} (\Delta^m N)\gamma f - \int_B (\Delta^{m-2}N)D\Delta^2 f - \\ &\quad - \int_{\partial B} \left(\frac{\partial}{\partial n} \Delta^{m-2}N \right) D\Delta f + \int_{\partial B} (\Delta^{m-2}N) \frac{\partial}{\partial n} D\Delta f \\ &\quad - \int_{\partial B} \left(\frac{\partial}{\partial n} \Delta^{m-1}N \right) Df + \int_{\partial B} (\Delta^{m-1}N) \frac{\partial}{\partial n} Df. \end{aligned}$$

So, applying m times the Green's formula, we have :

$$\begin{aligned}
f(x) &= \int_{\partial B} (\Delta^m N(y-x)) \gamma(y) f(y) \\
&\quad - \sum_{k=1}^m \int_{\partial B} \left(\frac{\partial}{\partial n} \Delta^{m-k} N(y-x) \right) D \Delta^{k-1} f(y) d\sigma_y \\
&\quad + \sum_{k=1}^m \int_{\partial B} (\Delta^{m-k} N(y-x)) \frac{\partial}{\partial n} D \Delta^{k-1} f(y) d\sigma_y \\
&\quad - \int_B N(y-x) D \Delta^m f(y) \omega(y).
\end{aligned}$$

This would be the general integral representation formula for functions $f : S \oplus V \rightarrow \mathbb{R}_{0,2m+1}$.

The Cauchy integral formula for holomorphic Cliffordian functions will be obtained erasing the last integral because in that case $D \Delta^m f = 0$.

Remark that the obtained Cauchy integral formula involves $2m+1$ integrals on ∂B . That means that, for holomorphic Cliffordian function, one can reconstitute the values of f in a point of the interior of B knowing the values on ∂B of f , $D \Delta^{k-1} f$ and $\frac{\partial}{\partial n} D \Delta^{k-1} f$, with $k = 1, \dots, m$.

Remark also, that when $m = 0$, i.e. the case of holomorphic functions, we have :

$$N(z) = \frac{1}{2\pi} \cdot \frac{1}{z}, \quad E(z) = \frac{1}{2\pi} \frac{\bar{z}}{|z|^2}.$$

8. Taylor expansion of a holomorphic Cliffordian function

Here we will imitate the well-know process for the obtention of a Taylor formula for holomorphic functions starting with the Cauchy formula and developping the Cauchy kernel. Our Cauchy kernel is :

$$N(y-x) = \varepsilon_m (y-x)^{-1}.$$

In order to developp $(y-x)^{-1}$, let us proceed as follows :

$$\begin{aligned}
(y-x)^{-1} &= \left(y(1-y^{-1}x) \right)^{-1} = (1-y^{-1}x)^{-1} y^{-1} = \\
&= y^{-1} + y^{-1}xy^{-1} + y^{-1}xy^{-1}xy^{-1} + \dots + (y^{-1}x)^n y^{-1} + \dots
\end{aligned}$$

In view of $yy^* = |y|^2$, we have $y^{-1} = \frac{y^*}{|y|^2}$, and thus :

$$(y - x)^{-1} = \sum_{n=0}^{\infty} \frac{(y^*x)^n y^*}{|y|^{2n+2}}.$$

Let have a look at the second term of this developpement :

$$\begin{aligned} y^*xy^* &= (y_0 - \vec{y})x(y_0 - \vec{y}) = \\ &= (e_0xe_0)y_0^2 + \sum_{j=1}^{2m+1} (e_0xe_j)y_0(-y_j) + \sum_{k=1}^{2m+1} (e_kxe_0)(-y_k)y_0 + \sum_{j,k=1}^{2m+1} (e_jxe_k)y_jy_k. \end{aligned}$$

It is not difficult to observe that the polynomials $P_\alpha(x)$ appear again and one can write :

$$y^*xy^* = \sum_{|\alpha|=2} P_\alpha(x)Y^\alpha,$$

where we have made use of the notation :

$$Y^\alpha = y_0^{\alpha_0}(-y_1)^{\alpha_1} \cdots (-y_{2m+1})^{\alpha_{2m+1}}.$$

A straightforward calculation gives finally :

$$(y - x)^{-1} = \sum_{k=1}^{\infty} \frac{1}{|y|^{2k}} \sum_{|\alpha|=k} P_\alpha(x)Y^\alpha$$

or more concisely :

$$(y - x)^{-1} = \sum_{|\alpha|=1}^{\infty} P_\alpha(x) \frac{Y^\alpha}{|y|^{2|\alpha|}}.$$

In order to obtain the Taylor series of f , take the Cauchy integral formula and substitute the expansion of $N(y - x)$.

Observe that $\Delta_x^m N(y - x) = \Delta_y^m N(y - x)$, so that in the first integral of the Cauchy formula, we have :

$$\begin{aligned} \int_{\partial B} \Delta^m N(y - x) \gamma(y) f(y) &= \int_{\partial B} \Delta_y^m \left(\varepsilon_m \sum_{|\alpha|=1}^{\infty} P_\alpha(x) \frac{Y^\alpha}{|y|^{2|\alpha|}} \right) \gamma(y) f(y) = \\ &= \varepsilon_m \sum_{|\alpha|=1}^{\infty} P_\alpha(x) \int_{\partial B} \left(\Delta_y^m \frac{Y^\alpha}{|y|^{2|\alpha|}} \right) \gamma(y) f(y) = \sum_{|\alpha|=1}^{\infty} P_\alpha(x) A_\alpha^{(0)}, \end{aligned}$$

where the $A_\alpha^{(0)}$ are in $\mathbb{R}_{0,2m+1}$ and are given by :

$$A_\alpha^{(0)} = \varepsilon_m \int_{\partial B} \left(\Delta_y^m \frac{Y^\alpha}{|y|^{2|\alpha|}} \right) \gamma(y) f(y).$$

Similarly, as the other integrals in the Cauchy formula are concerned, we have :

$$\frac{\partial}{\partial n} \Delta^\ell N(y-x) = \frac{\partial}{\partial n_y} \Delta_y^\ell N(y-x)$$

which allows to deduce finally :

$$f(x) = \sum_{|\alpha|=1}^{\infty} P_\alpha(x) C_\alpha,$$

where the coefficients $C_\alpha \in \mathbb{R}_{0,2m+1}$, and more precisely :

$$C_\alpha = A_\alpha^{(0)} + A_\alpha^{(1)} + \dots + A_\alpha^{(2m)}$$

with :

$$A_\alpha^{(j)} = \varepsilon_m \int_{\partial S} \left(\frac{\partial}{\partial n_y} \Delta_y^{m-j} \frac{Y^\alpha}{|y|^{2|\alpha|}} \right) D \Delta^{j-1} f(y) d\sigma_y, \quad j = 1, \dots, m$$

and

$$A_\alpha^{(\ell+m)} = \varepsilon_m \int_{\partial S} \left(\Delta_y^{m-\ell} \frac{Y^\alpha}{|y|^{2|\alpha|}} \right) \frac{\partial}{\partial n} D \Delta^{\ell-1} f(y) d\sigma_y, \quad \ell = 1, \dots, m.$$

At the end of this paragraph let us prove that the polynomials P_α span the space of polynomial solutions of $D \Delta^m f = 0$. Indeed, according to the Taylor expansion if $P(x)$ is an arbitrary polynomial, we have :

$$P(x) = \sum_{|\alpha|=1}^{\infty} P_\alpha(x) C_\alpha$$

as a holomorphic Cliffordian function. But P is a polynomial, so that the sum is finite :

$$P(x) = \sum_{|\alpha|=1}^d P_\alpha(x) C_\alpha$$

and this shows that P is a linear (right) combination of the P_α .

Let Q be any polynomial of degree smaller or equal to $2m$, then Q is holomorphic Cliffordian and

$$Q(x) = \sum_{|\alpha|=1}^{2m+1} P_\alpha(x) C_\alpha.$$

9. Laurent series

Consider a function which is holomorphic Cliffordian on a punctured neighborhood of the origin, say, for example on $B \setminus \{0\}$, where B is the unit ball in $S \oplus V$.

Let Γ_1 and Γ_2 be two balls, centered at the origin, with radii r_1 and r_2 , respectively, and such that $0 < r_1 < r_2 < 1$. One can apply the Cauchy representation formula on the region, which is limited by Γ_1 and Γ_2 , namely on $\Gamma_2 \setminus \Gamma_1$. Those integrals, taken on $\partial\Gamma_2$, will give us, as in the previous paragraph, the regular part of the Laurent series. Because of the sense of the integration, we have now to integrate on $\partial\Gamma_1$ those terms of the representation formula, which contain $N(x-y)$ and its derivatives.

In this way, one needs to develop $(x-y)^{-1}$. So :

$$\begin{aligned} (x-y)^{-1} &= (x(1-x^{-1}y))^{-1} = (1-x^{-1}y)^{-1}x^{-1} = \\ &= x^{-1} + x^{-1}yx^{-1} + x^{-1}yx^{-1}yx^{-1} + \dots \\ &\quad \dots + (x^{-1}y)^k x^{-1} + \dots \\ &= x^{-1} + \sum_{i=0}^{2m+1} (x^{-1}e_i x^{-1})y_i + \\ &\quad + \sum_{0 \leq i_1, i_2 \leq 2m+1} (x^{-1}e_{i_1} x^{-1}e_{i_2} x^{-1} + x^{-1}e_{i_2} x^{-1}e_{i_1} x^{-1})y_{i_1}y_{i_2} + \dots \end{aligned}$$

Remark that the rational functions appearing in the last development are of negative powers on x , resp. : -1, -2, -3,

Using a similar manner of notation as in the case of the polynomials $P_\alpha(x)$, we set $\beta = (\beta_0, \beta_1, \dots, \beta_{2m+1})$, with $\beta_i \in \mathbb{N}$ and $|\beta| = \sum_{i=0}^{2m+1} \beta_i$. Consider again the set $\{e_\nu\}$, where e_0 is written β_0 times, e_1, β_1 times, etc ... and e_{2m+1}, β_{2m+1} times. Set now :

$$S_\beta(x) = \sum_{\mathfrak{S}} \prod_{\nu=1}^{|\beta|} (x^{-1}e_{\sigma(\nu)})x^{-1},$$

the sum being expanded over all distinguishable elements σ of the permutation group \mathfrak{S} .

$S_\beta(x)$ is left and right holomorphic Cliffordian.

We recognize easily $S_{(1,0,0,0)}(x) = x^{-1}e_0x^{-1}$, $S_{(0,1,0,0)}(x) = x^{-1}e_1x^{-1}$, $S_{(0,1,1,0)}(x) = x^{-1}e_1x^{-1}e_2x^{-1} + x^{-1}e_2x^{-1}e_1x^{-1}$ in the special case when $m = 1$. Remark also that $S_0(x) = x^{-1}$ and that the power of x^{-1} in S_β is exactly $|\beta| + 1$.

Thus, we get :

$$N(x - y) = \varepsilon_m \sum_{|\beta|=0}^{\infty} S_{\beta}(x)y^{\beta},$$

where $y^{\beta} = (y_0)^{\beta_0}(y_1)^{\beta_1} \dots (y_{2m+1})^{\beta_{2m+1}}$.

In the same way as in paragraph 8, one deduces the following Laurent series for a holomorphic Cliffordian function $f : B \setminus \{0\} \rightarrow \mathbb{R}_{0,2m+1}$, $B \subset S \oplus V$: for each $x \in B \setminus \{0\}$, we have

$$f(x) = \sum_{|\beta|=0}^{\infty} S_{\beta}(x)D_{\beta} + \sum_{|\alpha|=1}^{\infty} P_{\alpha}(x)C_{\alpha},$$

where C_{α} and D_{β} belong to $\mathbb{R}_{0,2m+1}$.

The first sum is the analogous of the singular part of a Laurent expansion for a holomorphic function, while the second sum represents the analogous of its regular part.

Here, we centered our expansions at the origin. Of course, they remain valid in neighborhoods of every point $a \in S \oplus V$. If $f : B \setminus \{a\} \rightarrow \mathbb{R}_{0,2m+1}$, is a holomorphic Cliffordian function, where B is a ball centered at a , then for every $x \in B \setminus \{a\}$, one has :

$$f(x) = \sum_{|\beta|=0}^{\infty} S_{\beta}(x - a)D_{\beta} + \sum_{|\alpha|=1}^{\infty} P_{\alpha}(x - a)C_{\alpha},$$

with $C_{\alpha}, D_{\beta} \in \mathbb{R}_{0,2m+1}$.

Remark : the rational functions $S_{\beta}(x)$ are left and right holomorphic Cliffordian.

The present paper is a detailed exposition of part of the results announced in [8]. However, some modifications were brought, especially concerning the multiplicative constants appearing in the definitions of the polynomials $P_{\alpha}(x)$ and the rational functions $S_{\beta}(x)$.

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