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JACOBI ELLIPTIC CLIFFORDIAN FUNCTIONS

by

GUY LAVILLE* and IVAN RAMADANOFF†

The well-known Jacobi elliptic functions $sn(z)$, $cn(z)$, $dn(z)$ are defined in higher dimensional spaces by the following method. Consider the Clifford algebra of the antieclidean vector space of dimension $2m + 1$. Let x be the identity mapping on the space of scalars + vectors. The holomorphic Cliffordian functions may be viewed roughly as generated by the powers of x , namely x^n , their derivatives, their sums, their limits (cf : z^n for classical holomorphic functions). In that context it is possible to define the same type of functions as Jacobi's.

Keywords : Clifford analysis, Elliptic functions, Jacobi functions, Holomorphic Cliffordian functions.

AMS Classifications : 30G35, 33E05.

Introduction

The theory of elliptic functions, i.e. holomorphic periodic functions of one complex variable is well-known. The theory of periodic functions in higher dimensional spaces, (be they real, vector or Clifford-valued) has a long history. The Dirichlet problem in a box was the motivation of the works of P. Appell [1], [2], [3], A. Dixon [6]. After a long drowsiness Fueter [8] made some studies in the context of his theory of regular quaternion-valued functions. More recently J. Ryan [15] and S. Krausshar [9] worked with Clifford-valued functions. Here we are looking at the well-known Jacobi elliptic functions $sn(z)$, $cn(z)$, $dn(z)$ in higher dimensional spaces, in the framework of what we think to be the natural context : holomorphic Cliffordian functions. The main tool is the fundamental ζ_N functions introduced in [12].

§1. Prerequisites

In this first paragraph, we will recall the notion of holomorphic Cliffordian function, introduced in [10], [11], and also what we could call an elliptic

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Cliffordian function, [12]. Some basic properties of the Cliffordian analogous of the ζ Weierstrass function will be remembered and some ingredients as well, which we will make use further.

Let $\mathbb{R}_{0,2m+1}$ be the Clifford algebra of the real vector space V of dimension $2m+1$, provided with a quadratic form of negative signature and $m \in \mathbb{N}$. Denote by S the set of the scalars in $\mathbb{R}_{0,2m+1}$, which can be identified to \mathbb{R} . Let $\{e_i\}$, $i = 1, 2, \dots, 2m+1$, be an orthonormal basis of V and set $e_0 = 1$. Thus, in the algebra $\mathbb{R}_{0,2m+1}$, the calculus rules will be generated by $e_i e_j + e_j e_i = -2\delta_{ij}$ for $0 \leq i, j \leq 2m+1$, where δ_{ij} is the Kronecker symbol.

A point $x = (x_0, x_1, \dots, x_{2m+1})$ of \mathbb{R}^{2m+2} could be considered as an element of $S \oplus V$ and be written as $x = x_0 + \vec{x}$, where x_0 means its scalar part and $\vec{x} = \sum_{i=1}^{2m+1} e_i x_i$ its vector part.

Let Ω be an open set of $S \oplus V$. A function $f : \Omega \rightarrow \mathbb{R}_{0,2m+1}$ is said to be (left) holomorphic Cliffordian in Ω if and only if :

$$D \Delta^m f(x) = 0,$$

for each x of Ω . Here, Δ^m means the m times iterated Laplacian Δ and D is the well-known operator :

$$D = \sum_{i=0}^{2m+1} e_i \frac{\partial}{\partial x_i}$$

lying on the basis of the theory of (left) monogenic functions ([4], see also [7]).

In [11], the foundations of a theory of holomorphic Cliffordian functions were achieved, constructing a corresponding Cauchy kernel, obtaining a Cauchy integral representation formula allowing to derive a similar to the Taylor expansion series. Perhaps the most significant phenomenon in this theory against the theory of monogenic functions is the fact that the function $x \mapsto x^n$, $n \in \mathbb{N}$ is a holomorphic Cliffordian one. Moreover, the function $x \mapsto x^{-1}$ is also holomorphic Cliffordian on $S \oplus V \setminus \{0\}$, so that, restricting us only to pointwise singularities, there is no major difficulties to obtain a similar to the Laurent expansion series for meromorphic Cliffordian functions, [12].

Take now an integer N in $\{1, 2, \dots, 2m+2\}$ and let $\omega_\alpha \in S \oplus V$ for $\alpha = 1, 2, \dots, N$. Suppose always the paravectors $\omega_1, \dots, \omega_N$ linearly

independent in $S \oplus V$. For convenience the ω_α will play the role of half periods. So, a function $f : S \oplus V \rightarrow \mathbb{R}_{0,2m+1}$ is said to be N -periodic if

$$f(x + 2\omega_\alpha) = f(x)$$

for every $x \in S \oplus V$ and $\alpha = 1, 2, \dots, N$. Further, let us call the set $2\mathbb{Z}^N\omega = \{2k\omega, k \in \mathbb{Z}^N\}$ a lattice. Here we will make use of the notations : $\omega = (\omega_1, \dots, \omega_N)$, for a N -uple of paravectors, $k = (k_1, \dots, k_N)$ for a multiindex, $k_\alpha \in \mathbb{Z}$, and $k\omega = \sum_{\alpha=1}^N k_\alpha \omega_\alpha$. Obviously, for a N -periodic function, we have :

$$f(x + 2k\omega) = f(x).$$

Recall a general theorem of elliptic Cliffordian functions, i.e. meromorphic and N -periodic :

the theorem for the principal parts.- *If f_1 and f_2 are two elliptic Cliffordian functions with the same pointwise poles and the same principal parts of their Laurent expansions on the neighborhoods of their poles, then they differ just up to an additive constant, [12].*

Now recall the definition of the Weierstrass ζ_N functions. First, we need to rearrange the lattice $2\mathbb{Z}^N\omega$ in a countable set : $\{w_p\}_{p=0}^\infty$, where $w_0 = (0, 0, \dots, 0)$. Then set :

$$\zeta_N(x) = x^{-1} + \sum_{p=1}^\infty \left\{ (x - w_p)^{-1} + \sum_{\mu=0}^{N-1} (w_p^{-1}x)^\mu w_p^{-1} \right\}.$$

In such a way we have a function $\zeta_N : S \oplus V \setminus 2\mathbb{Z}^N\omega \rightarrow \mathbb{R}_{0,2m+1}$ for which one can show it is a holomorphic Cliffordian one and that ζ_N possesses simple poles at the vertices of the lattice. Moreover, one has also :

$$\zeta_N(x) = x^{-1} - \sum_{k \geq [\frac{N}{2}]} \sum_{p=1}^\infty (w_p^{-1}x)^{2k+1} w_p^{-1},$$

from which it follows that ζ_N is an odd function.

The function ζ_N itself is not N -periodic, but it satisfies a property of quasi-periodicity (i.e. up to a holomorphic Cliffordian polynomial). More precisely :

$$\zeta_N(x + 2\omega) - \zeta_N(x) = 2 \sum_{p=0}^{[\frac{N+1}{2}]-1} \frac{((x + \omega) | \nabla_y)^{2p}}{(2p)!} \zeta_N(y) \Big|_{y=\omega}$$

or equivalently :

$$\zeta_N(x + \omega) - \zeta_N(x - \omega) = 2 \sum_{p=0}^{\lfloor \frac{N+1}{2} \rfloor - 1} \frac{(x | \nabla_y)^{2p}}{(2p)!} \zeta_N(y) \Big|_{y=\omega}.$$

Let us make some comments. Here we have made use of the notation : $(x | \nabla_y)^{2p} \zeta_N(y) \Big|_{y=\omega}$. That means we start with the usual scalar product in \mathbb{R}^{2m+2} of the two vectors $x = (x_0, x_1, \dots, x_{2m+1})$ and the gradient $\nabla_y = \left(\frac{\partial}{\partial y_0}, \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_{2m+1}} \right)$ which is applied to the function ζ_N as a function of the variables $(y_0, y_1, \dots, y_{2m+1})$. Then $(x | \nabla_y)^{2p}$ means an iteration $2p$ times of $(x | \nabla_y)$ and the final result is obtained substituting $y = \omega$.

In the particular case when $N = 2m + 2$ which will be the most natural case appearing later, we have :

$$\zeta_{2m+2}(x + 2\omega) - \zeta_{2m+2}(x) = 2 \sum_{p=0}^m \frac{((x + \omega) | \nabla_y)^{2p}}{(2p)!} \zeta_N(y) \Big|_{y=\omega}$$

Note the right-hand side is a polynomial on x of degree $2m$ which will be denoted by $p_{2m}(x ; \omega)$.

Finally, let us write down the Laurent expansion of ζ_N in a neighborhood of the origin. This can be done using the formula :

$$(x | \nabla_w)^n (w^{-1}) \Big|_{w=w_p} = (-1)^n n! (w_p^{-1} | x)^n w_p^{-1},$$

where $w, x, w_p \in S \oplus V$ and $p, n \in \mathbb{N}$. So, we get :

$$\zeta_N(x) = x^{-1} - \sum_{n=N}^{\infty} \sum_{p=1}^{\infty} (-1)^n \frac{(x | \nabla_w)^n (w^{-1}) \Big|_{w=w_p}}{n!}.$$

Because of the imparity of ζ_N , there is a more simple formula :

$$(1) \quad \zeta_N(x) = x^{-1} + \sum_{k \geq \lfloor \frac{N}{2} \rfloor} \frac{1}{(2k+1)!} \sum_{p=1}^{\infty} (x | \nabla_w)^{2k+1} (w^{-1}) \Big|_{w=w_p}$$

Just note that in the special case $N = 2m + 2$, the first sum starts from $k = m + 1$.

§2. Translations operators

Introduce the translation operators E_j : for a fixed lattice generated by N paravectors $2\omega_1, \dots, 2\omega_N$ and for an arbitrary function $\varphi : S \oplus V \longrightarrow \mathbb{R}_{0,2m+1}$, set :

$$E_j(\varphi)(x) = \varphi(x + \omega_j), \quad j = 1, 2, \dots, N$$

The composition of the two operators E_i and E_j will be denoted simply as $E_j E_i$. Obviously, $E_j E_i = E_i E_j$. Actually, the set $\{E_j\}$, $j = 1, \dots, N$ generates a commutative algebra of operators. This algebra is isomorphic to the polynomial algebra of N independent variables $\mathbb{P}(X_1, \dots, X_N)$. Remark also the square of any translation operator gives a translation on the whole period : $E_j^2(\varphi)(x) = \varphi(x + 2\omega_j)$.

Suppose now φ is a N -periodic function. We could translate this fact using the language of the translation operators, saying φ is N -periodic if and only if :

$$(I - E_j^2)(\varphi)(x) = 0$$

for $j = 1, 2, \dots, N$.

But the algebraic structure of this set of operators allows us to write :

$$I - E_j^2 = (I - E_j)(I + E_j).$$

Sometime, we could look on a special translation as, for example, $I - E_i E_j$.

In this case note that :

$$(I - E_i E_j)(\varphi)(x) = \varphi(x) - \varphi(x + \omega_i + \omega_j).$$

The same result would be obtained if we use the identity :

$$I - E_i E_j = (I - E_i)(I + E_j) + (I + E_i) - (I + E_j),$$

$1 \leq i, j \leq N$.

The next step is to understand how to write the quasi periodicity of a function as, for example, the function ζ_N . But it is clear that applying the operator $I - E_j^2$ on ζ_N , we will get the opposite of the polynomial giving the right hand side of the quasi periodicity. In the case $N = 2m + 2$, we will have :

$$(I - E_j^2)(\zeta_{2m+2})(x) = - p_{2m}(x ; \omega_j)$$

for $j = 1, 2, \dots, 2m + 2$.

Recapitulate : for a N -periodic function, the operators $I - E_j^2$, $j = 1, 2, \dots, N$, give zero, while for a quasi-periodic function, they generate a polynomial. For ζ_N , the degree of the corresponding polynomial is $2\left(\left[\frac{N+1}{2}\right] - 1\right)$.

The translation operators possess a very beneficial property. Remember that any holomorphic Cliffordian polynomial could be written as a sum of monomials of the type $(\lambda x)^n \lambda$, where λ, x are paravectors, $n \in \mathbb{N}$. A direct observation on $(I - E_j)((\lambda x)^n \lambda)$ shows that the last expression should be a polynomial of degree $n - 1$. So, applying once $I - E_j$ on a polynomial of degree n , one get a polynomial of degree $n - 1$. Obviously, at the end of the chain, one has

$$(I - E_j)(h) = 0,$$

for $h \in S \oplus V$, which is a polynomial of degree 0.

Let us remark that *in order to annihilate a polynomial of degree n , one needs to apply $n + 1$ operators of the type $I - E_j$, but not necessarily with the same j .* For example a polynomial of second degree could be annihilated independently by $(I - E_1)^3$ or $(I - E_1)(I - E_2)(I - E_3)$ etc.

When one looks at the function ζ_{2m+2} , then it is true that :

$$\prod_{j=1}^{2m+1} (I - E_j) (I - E_i^2) (\zeta_{2m+2})(x) = 0,$$

because, reading this line from the right to the left, we have $I - E_i^2$ applied to ζ_{2m+2} which generates a polynomial of degree $2m$, and then $\prod_j (I - E_j)$ annihilates this polynomial. But let us write the same in the opposite order, namely :

$$(I - E_i^2) \prod_{j=1}^{2m+1} (I - E_j) (\zeta_{2m+2})(x) = 0.$$

This can be looked as the authentic periodicity only on $2\omega_i$ of the function $\prod_{j=1}^{2m+1} (I - E_j) (\zeta_{2m+2})(x)$.

In such a way, we dispose with a receipt to construct periodic functions starting by a quasi-periodic.

Till now we will denote for brevity the function ζ_{2m+2} by ζ .

§3. Other ways for getting periodic functions

First consider the case $m = 0$ and $N = 2$. The corresponding ζ_2 function coincides with the classical Weierstrass function in \mathbb{C} . Its quasi-periodicity is realized up to a polynomial of 0 degree :

$$\zeta_2(x + \omega) - \zeta_2(x - \omega) = 2\zeta_2(\omega).$$

As usually, ω is a generic notation for the two periods ω_1 and ω_2 .

Could we construct a $(2\omega_1, 2\omega_2)$ periodic function having two simple poles at α and $-\alpha$ saying, with opposite residues k and $-k$? The answer is yes and the construction is simple. Set :

$$\varphi(x) = k \zeta_2(x - \alpha) - k \zeta_2(x + \alpha).$$

Obviously φ has the required residues and the required poles. Verify φ is a 2ω -periodic function :

$$\begin{aligned} \varphi(x + \omega) &= k \zeta_2(x - \alpha + \omega) - k \zeta_2(x + \alpha + \omega) = k \zeta_2(x - \alpha - \omega) + k 2\zeta_2(\omega) - \\ &- k \zeta_2(x + \alpha - \omega) - k 2\zeta_2(\omega) = \varphi(x - \omega). \end{aligned}$$

Look now at the case $m = 1$ and $N = 4$. Here the quasi periodicity of the corresponding ζ_4 function is guaranteed up to an even polynomial of second degree :

$$\zeta_4(x + \omega) - \zeta_4(x - \omega) = 2\zeta_4(\omega) + (x \mid \nabla_w)^2 \zeta_4(w) \Big|_{w=\omega}.$$

If we want to construct again a 4 periodic function with two simple poles and opposite scalar residues, we need such a method able to destroy the polynomial. One possible way is to use the complexified Clifford algebra $\mathbb{R}_{0,m+1} \otimes \mathbb{C}$, i.e. the complex space $\mathbb{R}_{0,3} \oplus i\mathbb{R}_{0,3}$ and then set :

$$\varphi(x) = k \zeta_4(x - \alpha) - k \zeta_4(x + \alpha) + ik \zeta_4(x - i\alpha) - ik \zeta_4(x + i\alpha).$$

Thus, φ would have the required poles at α and $-\alpha$ with residues k and $-k$, respectively. Those poles belong to $\mathbb{R}_{0,3}$ (in fact in $S \oplus V$). Of course φ inherited also two other poles. A long, but direct computation, carried on $\frac{1}{k} \varphi(x + \omega)$, shows that φ is periodic. The fact that the polynomial disappears is due just to $i^2 = -1$.

So, the complexification method gives the result, i.e. a way to annihilated a polynomial of second degree. There is another way, coming from iteration

processes usual in numerical analysis : if $p(x)$ is a polynomial of degree 2 of the real variable x and $h \in \mathbb{R}$, then :

$$p(x + 3h) - 3p(x + 2h) + 3p(x + h) - p(x) = 0.$$

It is not difficult to generalize the last proposition in the Cliffordian case. The result remains true if $x, h \in S \oplus V$ and p is a holomorphic Cliffordian polynomial of degree 2.

Now, take ζ_4 in $\mathbb{R}_{0,3}$. Set :

$$\varphi(x) = \zeta_4(x) - 3\zeta_4(x + \beta) + 3\zeta_4(x + 2\beta) - \zeta_4(x + 3\beta)$$

with an arbitrary $\beta \in S \oplus V$. Denote by $p(x; \omega)$ the polynomial $2\zeta_4(\omega) + (x | \nabla_\omega)^2 \zeta_4(\omega) |_{w=\omega}$. We are in a position to show that φ is 2ω periodic. It suffices to form $\varphi(x + \omega)$ and to apply the quasi periodicity of ζ_4 :

$$\begin{aligned} \varphi(x + \omega) &= \zeta_4(x + \omega) - 3\zeta_4(x + \beta + \omega) + 3\zeta_4(x + 2\beta + \omega) - \zeta_4(x + 3\beta + \omega) \\ &= \zeta_4(x - \omega) + p(x) - 3\zeta_4(x + \beta - \omega) - \\ &\quad - 3p(x + \beta) + 3\zeta_4(x + 2\beta - \omega) + 3p(x + 2\beta) \\ &\quad - \zeta_4(x + 3\beta - \omega) - p(x + 3\beta) \\ &= \varphi(x - \omega). \end{aligned}$$

Some additional receipts : imagine we dispose with a 4-periodic function φ , when $m = 1$ and $N = 4$. Set the four periods as $2\omega_1, 2\omega_2, 2\omega_3, 2\omega_4$. If we want to obtain a new 4-periodic function with some of the periods unchanged, some of them divided by half, then it suffices to add corresponding translations. For example :

$$\eta(x) = (I + E_1)(\varphi)(x) = \varphi(x) + \varphi(x + \omega_1)$$

would be periodic on $\omega_1, 2\omega_2, 2\omega_3, 2\omega_4$. The proof is obvious. With the same initial situation :

$$\begin{aligned} \theta(x) &= (I + E_2)(I + E_3)(I + E_4)(\varphi)(x) = \varphi(x) + \sum_{i=2}^4 \varphi(x + \omega_i) + \\ &\quad + \sum_{2 \leq i < j \leq 4} \varphi(x + \omega_i + \omega_j) + \varphi(x + \sum_{i=2}^4 \omega_i) \end{aligned}$$

would be periodic on $2\omega_1, \omega_2, \omega_3, \omega_4$.

A last remark : remember that in the last part of §2 we got a method for constructing periodic functions from quasi-periodic in the general frame of $\mathbb{R}_{0,2m+1}$. This would be the tool we will make use systematically.

§4. The Jacobi elliptic Cliffordian functions

The aim of this paragraph is to build in $\mathbb{R}_{0,2m+1}$ a system of functions which could be viewed as the analogous of the Jacobi elliptic functions : sn, cn and dn . First of all, in the complex case, which coincides with our case $m = 0$, $N = 2$, they are three elliptic functions whose general characteristics are : all of them are 2-periodic and they have the same simple poles at the same points. In the traditional notations [16], the two main periods are $4K$ and $4iK'$, with $K, K' \in \mathbb{R}$ related with the evaluation of the elliptic integral under the form of Legendre. Furthermore, they are different because sn is $(4K, 2iK')$ periodic, dn is $(2K, 4iK')$ periodic and, for cn , the periods are submitted to a strange perturbation : they are $(4K, 2K + i2K')$, or equivalently $(2K + i2K', 4iK')$. Note that the three vectors $4K, 2K + i2K'$ and $4iK'$ are \mathbb{R} -dependent.

According to the end of §2, we are in a position to construct periodic functions starting from ζ : the abbreviated notation for ζ_{2m+2} , applying products of (at least) $2m + 1$ operators of the type $I - E_j$. Such a product $\prod_j (I - E_j)$, of length $2m + 1$, independently if the j are equal or different, just belonging to $\{1, 2, \dots, 2m + 2\}$, will be enough for insuring the periodicity of $\prod_j (I - E_j)(\zeta)(x)$ on the whole system of paravectors $2\omega_1, \dots, 2\omega_{2m+2}$.

Recall that concerning the periods of sn and dn , there is a phenomenon consisting in a division by two along the two directions of \mathbb{R}^2 , but recall also we have a receipt allowing us to reduce by a half a given period, (see the end of §3). Thus, set :

DEFINITION 1.- Define :

$$S_i(x) = (I + E_i) \prod_{\substack{j=1 \\ j \neq i}}^{2m+2} (I - E_j)(\zeta)(x)$$

for $i = 1, 2, \dots, 2m + 2$.

We claim that S_i is periodic with periods $2\omega_1, \dots, 2\omega_{i-1}, \omega_i, 2\omega_{i+1}, \dots, 2\omega_{2m+2}$. Let us verify S_i is periodic on $2\omega_k$, $k \neq i$. For this, take

$$(I - E_k^2)(S_i)(x) = (I + E_i) \prod_{j \neq i} (I - E_j)(I - E_k^2)(\zeta)(x)$$

and let us read the last line from the right to the left : $(I - E_k^2)(\zeta)(x) = -p_{2m}(x ; \omega_k)$: a polynomial which is annihilated by the product. In this case $I + E_i$ does not play any role.

However, the role of $I + E_i$ is playing when we say that S_i will be periodic on ω_i . For this, we need to show that :

$$(I - E_i)(S_i)(x) = 0.$$

And so, take :

$$(I - E_i)(S_i)(x) = \prod_{j \neq i} (I - E_j)(I - E_i^2)(\zeta)(x)$$

and this is clearly equal to zero.

At this stage, we dispose with $2m + 2$ analogues of sn and dn . How to find an analogue to cn ? Arguing that the previous functions were built via products of length $2m + 2$, the most natural way is to set :

DEFINITION 2.- Define :

$$C(x) = \prod_{j=1}^{2m+2} (I - E_j)(\zeta)(x).$$

We claim the periods of C are $\{\omega_i + \omega_k\}_{i=1}^{2m+2}$, where k is arbitrarily fixed in $\{1, \dots, 2m + 2\}$. The proof that $2\omega_k$ is a period for C is obvious. Let us show $\omega_i + \omega_k$, $i = 1, \dots, 2m + 2$, $i \neq k$ are periods for C :

$$\begin{aligned} (I - E_i E_k)(C)(x) &= [(I - E_i)(I + E_k) + (I + E_i) - (I + E_k)](C)(x) = \\ &= (I - E_i) \prod_{j \neq k} (I - E_j)(I - E_k^2)(\zeta)(x) + \prod_{j \neq i} (I - E_j)(I - E_i^2)(\zeta)(x) \\ &\quad - \prod_{j \neq k} (I - E_j)(I - E_k^2)(\zeta)(x). \end{aligned}$$

At each line, ζ generates a polynomial of degree $2m$ via $I - E_k^2$ or $I - E_i^2$ and then the polynomial is annihilated by a product of at least $2m + 1$ operators.

Remark there is no need to associate to the fixed k an appropriate function, named C_k . Even if we do this, following the definition of C , one has : $C = C_1 = \dots = C_{2m+2}$.

In such a way, we constructed a set of elliptic Cliffordian functions $\{C, S_1, \dots, S_{2m+2}\}$, whose number $2m + 3$ does not suffer any change. It was clear the set of functions has to be at least $2m + 3$. They can not be more because we want $(2m + 2)$ - periodic functions, so the number of the operators in the product must be unchanged. The only thing theoretically possible is to put more than one operator of the type $I + E_j$ in the product. Look at :

$$F(x) = (I + E_1)(I + E_2) \prod_{j=3}^{2m+2} (I - E_j)(\zeta)(x).$$

The last would not be periodic on $2\omega_1$, even on ω_1 , because it remains an annihilating product of only $2m$ operators which is not sufficient for the destruction of $p_{2m}(x; \omega_1)$.

Finally, the number of $2m + 3$ functions is optimal and the rules of their constructions are rigid.

Let us raise an ambiguity. We remarked that any product $\prod_j (I - E_j)$, of length $2m + 1$, with the j different or equal, annihilates the quasi-periodicity polynomials on each direction of the paravectors belonging to the lattice. When we defined the functions S_i , we made use of products only of different operators. The reason comes from the necessity to obey to the second constraint that all our functions need to have the same poles at the same points. A study of the number and the position of the poles will be done in the next paragraph.

Come back to the case $m = 0, N = 2$ and, of course, $2m + 3$ is 3. Look at the functions C, S_1, S_2 . Actually, we started a beginning of description of the similarity between C, S_1, S_2 and cn, dn and sn , respectively. At this stage, the similarity concerns only the periods. As we said, the problem of the poles will be studied in §5. Anyway, it becomes and will be clear that C, S_1, S_2 are nothing else then $ikcn(z+iK')$, $idn(z+iK')$ and $ksn(z+iK')$, respectively, the last being written in the traditional notations, [16].

§5. General properties of the Jacobi elliptic Cliffordian functions

Come back to the definitions of C, S_1, \dots, S_{2m+2} . If we want to explicit each of them, we have to be patient. Each function is a sum of 2^{2m+2} terms which are : $\zeta(x)$, followed by the sum of $E_j(\zeta)(x) = \zeta(x + \omega_j)$, for $j = 1, \dots, 2m+2$, this sum containing C_{2m+2}^1 terms, then we have to add the $E_j E_k(\zeta)(x) = \zeta(x + \omega_j + \omega_k)$, $1 \leq j < k \leq 2m+2$, whose number is C_{2m+2}^2 and so on, till the last term, which is $E_1 E_2 \dots E_{2m+2}(\zeta)(x) = \zeta(x + \sum_{j=1}^{2m+2} \omega_j)$. In addition, we have also to take into account the corresponding signs. For example :

$$(2) \quad C(x) = \zeta(x) - \sum_{j=1}^{2m+2} \zeta(x + \omega_j) + \sum_{1 \leq i < j \leq 2m+2} \zeta(x + \omega_i + \omega_j) - \sum_{i < j < k} \zeta(x + \omega_i + \omega_j + \omega_k) + \dots + (-1)^{2m+2} \zeta(x + \sum_{j=1}^{2m+2} \omega_j).$$

A first look at (2) shows an enormous set of simple poles organized in groups : the first group contains only 0, the second all the half periods, then they are combined by pairs, etc ... and in the final group we have only the vertex $\sum_{j=1}^{2m+2} \omega_j$. A geometrical description of the set of poles would be better : they are nothing else than all the vertices of a hyperparallelogram spanned over $0, \omega_1, \dots, \omega_{2m+2}$, whose number is 2^{2m+2} .

Another observation could be done : all the residues are +1 or -1 and, at least at this moment, concerning C , one can say that the sum of the residues appearing in the expanded version of the definition is zero. In fact, we have an equal number of +1 and -1.

What will be surprising is that, because of the known periods and due also to other hidden periodicities, we will find later, each Jacobi function will be uniquely determined by just $2m+2$ poles and the knowledge of the respective residues (being among +1 and -1).

Remark that the expanded expression of C can be written as a shorter formula :

$$(3) \quad C(x) = \sum_{k=0}^{2m+2} (-1)^k \sum E_{j_1} \dots E_{j_k}(\zeta)(x),$$

where, in the second sum, we take $1 \leq j_1 < \dots < j_k \leq 2m+2$.

In such a way, one can say that the residue at the pole $\omega_{j_1} + \dots + \omega_{j_k}$ is exactly $(-1)^k$.

Once we have adopted this formalism, it is not difficult to write down expanded expressions of S_i :

$$(4) \quad S_i(x) = \sum_{k=0}^{2m+2} \sum (-1)^{\varepsilon(j_1, \dots, j_k)} E_{j_1} \dots E_{j_k}(\zeta)(x)$$

where :

$$(-1)^{\varepsilon(j_1, \dots, j_k)} = \begin{cases} (-1)^k & \text{if } i \notin \{j_1, \dots, j_k\} \\ (-1)^{k-1} & \text{if } i \in \{j_1, \dots, j_k\} \end{cases}$$

Thus, the residue at the pole $\omega_{j_1} + \dots + \omega_{j_k}$ is $(-1)^k$ if $i \notin \{j_1, \dots, j_k\}$ and $(-1)^{k-1}$ if $i \in \{j_1, \dots, j_k\}$.

As we can see, the signs of the residues of S_i , $i = 1, \dots, 2m+2$, are also well organized. For each function there is an equal number of signs $+$ and signs $-$.

An illustration of the previous calculations in the case $m = 0$, $N = 2$ can be summarized in the following table :

	$C(x)$	$S_1(x)$	$S_2(x)$
First group :	0	+1	+1
Second group :	$\begin{cases} \omega_1 \\ \omega_2 \end{cases}$	$\begin{matrix} -1 \\ -1 \end{matrix}$	$\begin{matrix} +1 \\ -1 \end{matrix}$
Third group :	$\omega_1 + \omega_2$	+1	-1

Here, they are 4 residues. The classical theory of elliptic functions says the functions are of order 2, i.e. the knowledge of only 2 poles with opposite residues is sufficient for determining each function at all. And this is really the case. For S_1 , one needs to know the residues at 0 and ω_2 , those in ω_1 and $\omega_1 + \omega_2$ follow because of the period ω_1 of S_1 . The same is true for S_2 : once we know the residues at 0 and ω_1 , those in ω_2 and $\omega_1 + \omega_2$ will be deduced by periodicity (on ω_2). Concerning C , look at $C(\omega_2) = C(\omega_2 + 2\omega_1) = C(\omega_1)$ and take into account the periodicity $C(\omega_1 + \omega_2) = C(0)$, so only two poles $\{0, \omega_1\}$ are sufficient. Let us agree that in the sets of poles they are “determining” poles and “additional” poles. For C, S_1, S_2 , the corresponding sets of determining poles are $\{0, \omega_1\}$, $\{0, \omega_2\}$ and $\{0, \omega_1\}$, respectively.

Now let us prove that **C is an odd function**. By the expanded formula for C , (2), we see C is a sum of 2^{2m+2} terms of the form $\zeta(x + \omega)$, provided

with their signs, where here ω is a generic notation for a half period or a sum of half periods. When we will take $C(-x)$, they will appear terms of the type $\zeta(-x + \omega)$, which one can transform, because of the imparity and the quasi-periodicity as :

$$\zeta(-x + \omega) = -\zeta(x - \omega) = -\zeta(x + \omega - 2\omega) = -\zeta(x + \omega) + p_{2m}(x ; \omega).$$

And thus we see that, putting $x \mapsto -x$, we have the opposite of the respective term, the sum of which will give $-C(x)$, with a sum of polynomials. Remark also that

$$\zeta(-x + \omega_i + \omega_j) = -\zeta(x + \omega_i + \omega_j) + p_{2m}(x ; \omega_i) + p_{2m}(x ; \omega_j).$$

The remaining term would be :

$$\begin{aligned} & \sum_{j=1}^{2m+2} p_{2m}(x ; \omega_j) - \sum_{i < j} (p_{2m}(x ; \omega_i) + p_{2m}(x ; \omega_j)) + \cdots - \\ & (p_{2m}(x ; \omega_1) + p_{2m}(x ; \omega_2) + \cdots + p_{2m}(x ; \omega_{2m+2})). \end{aligned}$$

In order to prove this expression is zero, introduce an abreviated notation for $p_{2m}(x ; \omega_i)$ as p_i for example. Our result would be derived from the formula :

$$(5) \quad \sum_{j=1}^n p_j - \sum_{i < j} (p_i + p_j) + \sum_{i < j < k} (p_i + p_j + p_k) - \cdots + (-1)^{n-1} \sum_{j=1}^n p_j = 0.$$

Let us prove it. For convenience, put $p = \sum_{i=1}^n p_i$.

LEMMA.- For any p_1, \dots, p_n and p as below :

$$(6) \quad \sum_{1 \leq i_1 < \cdots < i_k \leq n} p_{i_1} + p_{i_2} + \cdots + p_{i_k} = C_{n-1}^{k-1} p.$$

The proof is achieved by a recurrence on $n \geq 2$. For $n = 2$, first, if $k = 1$, $p_1 + p_2 = C_1^0 p$ and if $k = 2$, $p_1 + p_2 = C_1^1 p$. Suppose (6) is satisfied for $n - 1$ and all $k = 1, \dots, n - 1$. Take the sum on the left hand side of (6). It has two type of terms :

$$p_1 + \sum_{2 \leq j_1 < \cdots < j_{k-1} \leq n} p_{j_1} + \cdots + p_{j_{k-1}}$$

which gives by the recurrence hypothesis $C_{n-2}^{k-2} p$. The last terms are of the type :

$$\sum_{2 \leq i_1 < \dots < i_k \leq n} p_{i_1} + \dots + p_{i_k}$$

equal to $C_{n-2}^{k-1} p$. It remains to take into account that $C_{n-2}^{k-2} + C_{n-2}^{k-1} = C_{n-1}^{k-1}$ and thus the lemma is proved.

The sum of the left hand side of (5) is equal to

$$C_{n-1}^0 p - C_{n-1}^1 p + \dots + (-1)^{n-1} C_{n-1}^{n-1} p = p \sum_{\ell=0}^{n-1} (-1)^\ell C_{n-1}^\ell = 0.$$

This ends the proof that the Jacobi function C is odd.

Similar procedures can be applied in order to prove that the functions S_1, \dots, S_{2m+2} are all odd. Because of the signs, the respective formulas of combinatorics are a little bit more complicated.

A direct consequence of the imparity of the Jacobi functions is *the possibility to determine some of their zeroes*. In fact, if φ is a periodic function of period ω which is odd, then $\frac{\omega}{2}$ is a candidate for a zero of φ . Write $\varphi(x + \omega) = \varphi(x)$ and put $x = -\frac{\omega}{2}$. Thus $\varphi(\frac{\omega}{2}) = \varphi(-\frac{\omega}{2}) = -\varphi(\frac{\omega}{2})$.

So, remember the sequences of periods of C, S_1, \dots, S_{2m+2} respectively, and, after elimination of those half-periods which are poles, it remains that :

$$C\left(\frac{\omega_1}{2} + \frac{\omega_j}{2}\right) = 0, \quad j = 2, 3, \dots, 2m+2$$

and

$$S_j\left(\frac{\omega_j}{2}\right) = 0, \quad j = 1, 2, \dots, 2m+2.$$

Moreover, playing with the known periodicities we can increase the lists of zeroes.

Let us prove : S_i becomes zero for $x = \frac{\omega_i}{2}, \omega_j + \frac{\omega_i}{2}, j = 1, 2, \dots, 2m+2, j \neq i$, modulo $\omega_1, \dots, \omega_{2m+2}$. Really, we have shown already that $S_i(\frac{\omega_i}{2}) = 0$. Thanks to the periodicity, we have also : $S_i(x + \omega_i + 2\omega_j) = S_i(x), j \neq i, j = 1, 2, \dots, 2m+2$. Put $x = -(\omega_j + \frac{\omega_i}{2})$ and use the imparity of S_i . Then $S_i(\omega_j + \frac{\omega_i}{2}) = -S_i(\omega_j + \frac{\omega_i}{2})$ and so $S_i(\omega_j + \frac{\omega_i}{2}) = 0$. We said modulo

$\omega_1, \dots, \omega_{2m+2}$. Let us verify : starting from $S_i(\frac{\omega_i}{2}) = 0$, we could add ω_j for $j \neq i$ and ω_i . In the two cases S_i will vanish : in the first case because we already proved it, the second thanks to the periodicity on ω_i . Start now from $S_i(\omega_j + \frac{\omega_i}{2}) = 0$. If we add $\omega_j, j \neq i$, remember S_i is periodic on $2\omega_j$, so :

$$S_i(\omega_j + \frac{\omega_i}{2} + \omega_j) = S_i(\frac{\omega_i}{2}) = 0.$$

Finally, let us add ω_i . But using now the periodicity of S_i on ω_i , we have again zero.

In the same way, we can prove C becomes zero for $x = \frac{1}{2}(\omega_1 + \omega_j)$, $j = 2, \dots, 2m+2$ and for $x = \frac{3}{2}\omega_1 + \frac{\omega_j}{2}$, modulo $\omega_1, \dots, \omega_{2m+2}$.

We do not affirm have found all the zeroes of the Jacobi functions, we just know these are surely zeroes.

Now, let us reach the *problem of the hide periodicities*. Start with the function C . Remember that, by construction, it possesses the following periods : $2\omega_1, \omega_1 + \omega_2, \dots, \omega_1 + \omega_{2m+2}$. Moreover, C is constructed by 2^{2m+2} poles. We claim that only $2m+2$ poles are sufficient for determining C at all and they are $0, \omega_2, \dots, \omega_{2m+2}$, those we called determining poles. Really, look at $C(x + \omega_1) = C(x - \omega_1 + 2\omega_1) = C(x - \omega_1) = C(x - \omega_1 + \omega_1 + \omega_j) = C(x + \omega_j)$ $j = 2, \dots, 2m+2$ and put $x = 0$. The residue of C at ω_1 is determined as being the residue at any point ω_j , and we know it is -1 .

Then, the residues at $\omega_1 + \omega_j$ are the same as the residue at 0 because of the "official" periods $\omega_1 + \omega_j$. Look now at the points $\omega_j + \omega_k$ with $j, k = 2, \dots, 2m+2, j < k$. Take $C(x + \omega_j + \omega_k) = C(x + \omega_j + \omega_k + 2\omega_1) = C(x + \omega_1 + \omega_j + \omega_1 + \omega_k) = C(x)$ and put $x = 0$.

Further, we have : $C(\omega_1 + \omega_j + \omega_k) = C(\omega_k)$, or $C(\omega_j)$, $C(\omega_i + \omega_j + \omega_k) = C(2\omega_1 + \omega_i + \omega_j + \omega_k) = C(\omega_k)$, where $2 \leq i < j < k \leq 2m+2$, and $C(\omega_1 + \omega_i + \omega_j + \omega_k) = C(\omega_j + \omega_k)$. By a chain argument we can deduce all the residues at the vertices of the hyperparallelogram from those in $0, \omega_2, \dots, \omega_{2m+2}$.

Concerning the hide periodicities of S_i , we will mention that by tedious calculations, one can show, in the case $m = 1, N = 4$, that :

$$\begin{cases} S_1(x + \omega_2 + \omega_3) = -S_1(x + \omega_4) \\ S_1(x + \omega_3 + \omega_4) = -S_1(x + \omega_2) \\ S_1(x + \omega_4 + \omega_2) = -S_1(x + \omega_3) \end{cases}$$

and also

$$S_1(x + \omega_2 + \omega_3 + \omega_4) = -S_1(x).$$

For S_2 , one get :

$$\begin{aligned} S_2(x + \omega_j + \omega_k) &= S_2(x), \\ S_2(x + \omega_1 + \omega_j + \omega_k) &= -S_2(x) \end{aligned}$$

for $1 \leq j < k \leq 4$.

Finally, one may say that the sets of determining poles for S_i are : $\{0, \omega_1, \dots, \widehat{\omega}_i, \dots, \omega_{2m+1}\}$ where $\widehat{}$ means we omit the term and $i = 1, \dots, 2m + 2$.

We will end this paragraph with a remark on the sum of the residues of the Jacobi functions, under the condition to take the sum of the residues only at the determining poles of the function. It is easily seeing that, for each Jacobi function, the sum of the residues is $+1 + (-1)(2m + 1) = -2m$.

The study of the $2m + 3$ Jacobi elliptic Cliffordian functions shows that the structure of general elliptic Cliffordian functions seems to be complicated and subtle because even the general theorem on the sum of the residues for classical elliptic functions appears to be a very particular case of the Cliffordian one.

§6. On the Laurent expansions of the Jacobi elliptic Cliffordian functions

Following formula (1) of §1, the Laurent expansion of ζ in a neighborhood of the origin is :

$$(7) \quad \zeta(x) = x^{-1} + \sum_{k \geq m+1} \frac{1}{(2k+1)!} \sum_{p=1}^{\infty} (x | \nabla_w)^{2k+1} (w^{-1}) \Big|_{w=w_p}$$

As in [12], let us resort to a formal writing of (7) considering that by definition :

$$(x | \nabla)^{2k+1} \left(\sum_{p=1}^{\infty} w_p^{-1} \right) := \sum_{p=1}^{\infty} (x | \nabla_w)^{2k+1} (w^{-1}) \Big|_{w=w_p}$$

even in the left hand side the sum $\sum_{p=1}^{\infty} w_p^{-1}$ is obviously not convergent.

Even more, let us introduce the notation W for $\sum_{p=1}^{\infty} w_p^{-1}$. In such a way, we have :

$$(8) \quad \zeta(x) = x^{-1} + \sum_{k \geq m+1} \frac{(x | \nabla)^{2k+1}}{(2k+1)!} (W).$$

As we already remarked in [12], for the complex case, i.e. $m = 0$, $N = 2$, (8) reduces to :

$$\zeta(z) = \frac{1}{z} + \frac{(z | \nabla)^3}{3!} \left(\sum_{p=1}^{\infty} w_p^{-1} \right) + \dots,$$

which is another way to write the well-known Laurent expansion of the Weierstrass ζ function in \mathbb{C} :

$$\zeta(z) = \frac{1}{z} - z^3 \left(\sum_{p=1}^{\infty} \frac{1}{w_p^4} \right) - z^5 \left(\sum_{p=1}^{\infty} \frac{1}{w_p^6} \right) - \dots .$$

Our aim is to get the Laurent expansions of the Jacobi elliptic Cliffordian functions C, S_1, \dots, S_{2m+2} in a neighborhood of the origin. Look at (2), §5 : we see that $C(x) = \zeta(x) + \phi(x)$, where in ϕ we have introduced the sum of $2^{2m+2} - 1$ terms containing translations of ζ , i.e. $\zeta(x + \omega)$.

Consequently, ϕ has no pole at the origin, so ϕ is a holomorphic Cliffordian function in the considered neighborhood. Moreover, $\phi = C - \zeta$ is an odd function, so that $\phi(0) = 0$. Combining the Laurent expansion of ζ and the usual Taylor expansion of ϕ :

$$\phi(x) = \frac{(x | \nabla_w)}{1!} \phi(w) \Big|_{w=0} + \frac{(x | \nabla_w)^3}{3!} \phi(w) \Big|_{w=0} + \dots$$

we can deduce :

$$\begin{aligned} C(x) &= x^{-1} + \frac{(x | \nabla_w)}{1!} \phi(w) \Big|_{w=0} + \frac{(x | \nabla_w)^3}{3!} \phi(w) \Big|_{w=0} + \dots + \\ &\dots + \sum_{k \geq m+1} \frac{1}{(2k+1)!} \left[(x | \nabla)^{2k+1} (W) + (x | \nabla_w)^{2k+1} \phi(w) \Big|_{w=0} \right]. \end{aligned}$$

Obviously, the same procedure can be applied to $S_i(x) = \zeta(x) + \psi_i(x)$, $i = 1, \dots, 2m+2$, in order to deduce the Laurent expansions of S_i .

Remark also the conditions $\phi(0) = \psi_i(0) = 0$, $i = 1, \dots, 2m+2$, lead to a numerous quantity of relations concerning the behavior of ζ in its half periods. For example, $\phi(0) = 0$ means that :

$$\sum_{k=1}^{2m+2} (-1)^{k-1} \sum_{1 \leq i_1 < \dots < i_k \leq 2m+2} \zeta(\omega_{i_1} + \dots + \omega_{i_k}) = 0.$$

Let us see how looks this relation in the case $m = 0$, $N = 2$:

$$\zeta_2(\omega_1) + \zeta_2(\omega_2) = \zeta_2(\omega_1 + \omega_2).$$

That is a right formula which admits a direct proof setting $x = 0$ in :

$$\zeta_2(x + \omega_1 + \omega_2) = \zeta_2(x - \omega_1 - \omega_2) + 2\zeta_2(\omega_1) + 2\zeta_2(\omega_2)$$

and using the fact ζ_2 is odd.

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