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Models with Recoil for Bose-Einstein Condensation and Superradiance

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Abstract

In this paper we consider two models which exhibit *equilibrium* BEC superradiance. They are related to two different types of superradiant scattering observed in recent experiments. The first one corresponds to the amplification of matter-waves due to Raman superradiant scattering from a BE condensate, when the recoiled and the condensed atoms are in different internal states. The main mechanism is stimulated Raman scattering in two-level atoms, which occurs in a superradiant way. Our second model is related to the superradiant Rayleigh scattering from a BE condensate. This again leads to a matter-waves amplification but now with the recoiled atoms in the same state as the atoms in the condensate. Here the recoiling atoms are able to interfere with the condensate at rest to form a matter-wave grating (interference *fringes*) which is observed experimentally.

Keywords: Bose-Einstein Condensation, Raman/Rayleigh Superradiance, Optic Lattice, Matter-Wave Grating
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1 Introduction

This paper is the third in a series about models for equilibrium Bose-Einstein Condensation (BEC) superradiance motivated by the discovery of the Dicke superradiance and BEC matter waves amplification [1]-[5]. In these experiments the condensate is illuminated with a laser beam, the so called *dressing beam*. The BEC atoms then scatter photons from this beam and receive the corresponding recoil momentum producing coherent *four-wave mixing* of light and atoms [5]. The aim of our project is the construction of soluble statistical mechanical models for these phenomena.

In the first paper [6], motivated by the principle of *four-wave mixing* of light and atoms [5], we considered two models with a linear interaction between Bose atoms and photons, one with a global gauge symmetry and another one in which this symmetry is broken. In both cases we provided a rigorous proof for the emergence of a cooperative effect between BEC and superradiance. We proved that there is equilibrium superradiance and also that there is an enhancement of condensation compared with that occurring in the case of the free Bose gas.

In the second paper [7] we formalized the ideas described in [4, 5] by constructing a thermodynamically stable model whose main ingredient is the *two-level* internal states of the Bose condensate atoms. We showed that our model is equivalent to a *bosonized* Dicke maser model. Besides determining its equilibrium states, we computed and analyzed the thermodynamic functions, again finding the existence of a cooperative effect between BEC and superradiance. Here the phase diagram turns out to be more complex due to the two-level atomic structure.

In the present paper we study the effect of *momentum recoil* which was omitted in [6] and [7]. Here we consider two models motivated by two different types of superradiant scattering observed in recent experiments carried out by the MIT group, see e.g. [1]-[3]. Our first model (*Model 1*) corresponds to the Raman superradiant scattering from a cigar-shaped BE condensate considered in [1]. This leads to the amplification of matter waves (recoiled atoms) in the situation when amplified and condensate atoms are in *different* internal states. The main mechanism is stimulated Raman scattering in two-level atoms, which occurs in a way similar to Dicke superradiance [7].

Our second model (*Model 2*) is related to the superradiant Rayleigh scattering from a cigar-shaped BE condensate [2], [3]. This again leads to a matter-wave amplification but now with recoiled atoms in the *same* state as the condensate at rest. This is because the condensate is now illuminated by an off-resonant pump laser beam, so that for a long-pulse the atoms remain in their lower level states. In this case the (*non-Dicke*) superradiance is due to self-stimulated Bragg scattering [3].

From a theoretical point of view both models are interesting as they describe homogeneous systems in which there is *spontaneous breaking of translation invariance*. In the case of the Rayleigh superradiance this means that the phase transition corresponding to BEC is at the same time also a transition into a *matter-wave grating* i.e. a "frozen" spatial density wave structure, see Section 4. The fact that *recoiling* atoms are able to interfere with the condensate *at rest* to form a matter-wave grating (interference *fringes*) has been recently observed experimentally, see [3]-[5], and discussion in [8] and [9].

In the case of the Raman superradiance there is an important difference: the internal atomic states for condensed and recoiled bosons are *orthogonal*. Therefore these bosons are *different* and consequently cannot interfere to produce a matter-wave grating as in the first case. Thus the observed spatial modulation is not in the atomic density of interfering recoiled and condensed bosons, but in the *off-diagonal* coherence and photon condensate producing a one-dimensional (*corrugated*) optical lattice, see discussion in Section 4.

Now let us make the definition of our models more exact. Consider a system of identical bosons of mass m enclosed in a cube $\Lambda \subset \mathbb{R}^\nu$ of volume $V = |\Lambda|$ centered at the origin. We impose periodic boundary conditions so that the momentum dual set is $\Lambda^* = \{2\pi p/V^{1/\nu} | p \in \mathbb{Z}^\nu\}$.

In *Model 1* the bosons have an internal structure which we model by considering them as two-level atoms, the two levels being denoted by $\sigma = \pm$. For momentum k and level σ, $a^*_{k,\sigma}$ and $a_{k,\sigma}$ are the usual boson creation and annihilation operators with $[a_{k,\sigma}, a^*_{k',\sigma'}] = \delta_{k,k'}\delta_{\sigma,\sigma'}$. Let $\epsilon(k) = \|k\|^2/2m$ be the single particle kinetic energy and $N_{k,\sigma} = a^*_{k,\sigma}a_{k,\sigma}$ the operator for the number of particles with momentum k and level σ. Then the total kinetic energy is

$$T_{1,\Lambda} = \sum_{k\in\Lambda^*} \epsilon(k)(N_{k,+} + N_{k,-}) \tag{1.1}$$

and the total number operator is $N_{1,\Lambda} = \sum_{k\in\Lambda^*}(N_{k,+} + N_{k,-})$. We define the Hamiltonian $H_{1,\Lambda}$ for *Model 1* by

$$H_{1,\Lambda} = T_{1,\Lambda} + U_{1,\Lambda} \tag{1.2}$$

where

$$U_{1,\Lambda} = \Omega\, b^*_q b_q + \frac{g}{2\sqrt{V}}(a^*_{q+}a_{0-}b_q + a_{q+}a^*_{0-}b^*_q) + \frac{\lambda}{2V}N^2_{1,\Lambda}, \tag{1.3}$$

$g > 0$ and $\lambda > 0$. Here b_q, b^*_q are the creation and annihilation operators of the photons, which we take as a one-mode boson field with $[b_q, b^*_q] = 1$ and a frequency Ω. g is the coupling constant of the interaction of the bosons with the photon external field which, without loss of generality, we can take to be positive as we can always incorporate the sign of g into b. Finally the λ-term is added in (1.2) to obtain a thermodynamical stable system and to ensure the right thermodynamic behaviour. This is explained in Section 2.

In *Model 2* we consider the situation when the excited atoms have already irradiated photons, i.e. we deal only with de-excited atoms $\sigma = -$. In other words, we neglect the atom excitation and consider only *elastic* atom-photon scattering. This is close to the experimental situation [3]-[5], in which the atoms in the BE condensate are irradiated by *off-resonance* laser beam. Assuming that detuning between the optical fields and the atomic two-level resonance is much larger that the natural line of the atomic transition (superradiant Rayleigh regime [2, 3]) we get that the atoms always remain in their *lower* internal energy state. We can then ignore the internal structure of the atoms and let a^*_k and a_k be the usual boson creation and annihilation operators for momentum k with $[a_k, a^*_{k'}] = \delta_{k,k'}$, $N_k = a^*_k a_k$ the operator for the number of particles with momentum k,

$$T_{2,\Lambda} = \sum_{k\in\Lambda^*} \epsilon(k)N_k \tag{1.4}$$

the total kinetic energy, and $N_{2,\Lambda} = \sum_{k\in\Lambda^*} N_k$ the total number operator. We then define the Hamiltonian $H^{(2)}_\Lambda$ for *Model 2* by

$$H_{2,\Lambda} = T_{2,\Lambda} + U_{2,\Lambda} \tag{1.5}$$

where

$$U_{2,\Lambda} = \Omega\, b_q^* b_q + \frac{g}{2\sqrt{V}}(a_q^* a_0 b_q + a_q a_0^* b_q^*) + \frac{\lambda}{2V} N_{2,\Lambda}^2 \,. \tag{1.6}$$

This paper is structured as follows:
In Section 2 we give a complete rigorous solution of the variational principle for the equilibrium state for *Model 1* and we compute the corresponding pressure as a function of the temperature and the chemical potential. We prove that this model exhibits Raman superradiance.
In Section 3 we study *Model 2* and show that Rayleigh superradiance occurs in this case. The analysis is very similar to that of *Model 1* and therefore we do not repeat it but simply state the results.
In Section 4 we show that in both models there is spontaneous breaking of translation invariance in the equilibrium state. We relate this with the spatial modulation of matter-waves (*matter-wave grating*). We find that in *Model 1* there is no such spatial modulation in spite of the breaking of translation invariance while in *Model 2* this spatial modulation exists. We conclude with several remarks.

We close this introduction with the following comments:
- In our models we *do not* use for effective photon-boson interaction the *four-wave mixing principle*, see [5], [6], [10]. The latter seems to be important for the geometry, when a linearly polarized pump laser beam is incident in a direction perpendicular to the long axis of a cigar-shaped BE condensate, inducing the "45°- recoil pattern" picture [1]-[3]. Instead as in [7], we consider a *minimal* photon-atom interaction only with superradiated photons, cf [11]. This corresponds to superradiance in a "one-dimensional" geometry, when a pump laser beam is collimated and aligned along the long axis of a cigar-shaped BE condensate, see [9], [12].
- In this geometry the superradiant photons and recoiled matter-waves propagate in the same direction as the incident pump laser beam. If one considers it as a classical "source" (see [5]), then we get a *minimal* photon-atom interaction [7] generalized to take into account the effects of recoil. Notice that the further approximation of the BEC operators by c-numbers leads to a bilinear photon-atom interaction studied in [5], [6].
- In this paper we study *equilibrium* BEC superradiance while the experimental situation (as is the case with Dicke superradiance [13]) is more accurately described by non-equilibrium statistical mechanics. However we believe that for the purpose of understanding the quantum coherence interaction between light and the BE condensate our analysis is as instructive and is in the same spirit as the rigorous study of the Dicke model in thermodynamic equilibrium, see e.g. [14]-[16].
- In spite of the simplicity of our exactly soluble *Models 1* and *2* they are able to demonstrate the main features of the BEC superradiance with recoil: the photon-boson condensate *enhancement* with formation of the *light corrugated optical lattice* and the *matter-wave grating*. The corresponding phase diagrams are very similar to those in [7]. However, though the type of behaviour is similar, this is now partially due to the momentum recoil and not entirely to the internal atomic level structure.

2 Model 1

2.1 The effective Hamiltonian

We start with the *stability* of Hamiltonian (1.2). Consider the term $U_{1,\Lambda}$ in (1.3). This gives

$$\begin{aligned} U_{1,\Lambda} &= \Omega\,(b_q^* + \frac{g}{2\Omega\sqrt{V}} a_{q+} a_{0-}^*)(b_q + \frac{g}{2\Omega\sqrt{V}} a_{q+}^* a_{0-}) - \frac{g^2}{4\Omega^2 V} N_{0-}(N_{q+}+1) + \frac{\lambda}{2V} N_{1,\Lambda}^2 \\ &\geq \frac{\lambda}{2V} N_{1,\Lambda}^2 - \frac{g^2}{4\Omega^2 V} N_{0-}(N_{q+}+1). \end{aligned} \tag{2.1}$$

On the basis of the trivial inequality $4ab \leq (a+b)^2$, the last term in the lower bound in (2.1) is dominated by the first term if $\lambda > g^2/8\Omega$, that is if the stabilizing coupling λ is large with respect to the coupling constant g or if the external frequency is large enough. We therefore assume the *stability condition*: $\lambda > g^2/8\Omega$.

Since we want to study the equilibrium properties of the model (1.2) in the grand-canonical ensemble, we shall work with the Hamiltonian

$$H_{1,\Lambda}(\mu) = H_{1,\Lambda} - \mu N_{1,\Lambda} \tag{2.2}$$

where μ is the chemical potential. Since $T_{1,\Lambda}$ and the interaction $U_{1,\Lambda}$ conserve the quasi-momentum, Hamiltonian (1.2) describes a *homogeneous* (translation invariant) system. To see this explicitly, notice that the external laser field possesses a natural quasi-local structure as the Fourier transform of the field operator $b(x)$:

$$b_q = \frac{1}{\sqrt{V}} \int_\Lambda dx\, e^{iq\cdot x} b(x). \tag{2.3}$$

If for $z \in \mathbb{R}^\nu$, we let τ_x be the translation automorphism $(\tau_z b)(x) = b(x+z)$, then since we have periodic boundary conditions, $\tau_z(b_q) = e^{-iq\cdot z} b_q$ and similarly $\tau_z(a_{k,\sigma}) = e^{-ik\cdot z} a_{k,\sigma}$. Therefore, the Hamiltonian (1.2) is translation invariant. Consequently, in the thermodynamic limit, it is natural to look for translation invariant or homogeneous equilibrium states at all inverse temperatures β and all values of the chemical potential μ.

Because the interaction (1.3) is not bilinear or quadratic in the creation and annihilation operators the system cannot be diagonalized by a standard symplectic or Bogoliubov transformation. Therefore at the first glance one is led to conclude that the model is not soluble. However on closer inspection one notices that all the interaction terms contain space averages, namely, either

$$\frac{a_{0-}}{\sqrt{V}} = \frac{1}{V} \int_\Lambda dx\, a_-(x), \tag{2.4}$$

and its adjoint, or

$$\frac{1}{V} \int_\Lambda dx\, a_\sigma^*(x) a_\sigma(x). \tag{2.5}$$

Without going into all the mathematical details it is well-known [17] that space averages tend weakly to a multiple of the identity operator for all space-homogeneous *extremal* or *mixing* states. Moreover as all the methods of characterizing the equilibrium states (e.g. the variational principle, the KMS-condition, the characterization by correlation inequalities etc.

[17]) involve only affine functionals on the states, we can limit ourselves to looking for the extremal or mixing equilibrium states and in so doing we can exploit the above mentioned property for space averages. One way of accomplishing this is by applying the so-called *effective Hamiltonian* method, which is based on the fact that an equilibrium state is not determined by the Hamiltonian but by its *Liouvillian*. The best route to prove the exactness of the effective Hamiltonian method (cf [18]) is to use the characterization of the equilibrium state by means of the *correlation inequalities* [19], [17]:
A state ω is an equilibrium state for $H_{1,\Lambda}(\mu)$ at inverse temperature β, if and only if for all local observables A, it satisfies

$$\lim_{V\to\infty} \beta\omega\left([A^*,[H_{1,\Lambda}(\mu),A]]\right) \geq \omega(A^*A)\ln\frac{\omega(A^*A)}{\omega(AA^*)}. \tag{2.6}$$

Clearly only the *Liouvillian* $[H_{1,\Lambda}(\mu),\cdot]$ of the Hamiltonian enters into these inequalities and therefore we can replace $H_{1,\Lambda}(\mu)$ by a simpler Hamiltonian, the *e*ffective Hamiltonian, which gives in the limiting state ω the same Liouvillian as $H_{1,\Lambda}(\mu)$ and then look for the equilibrium states corresponding to it. Now in our case for an extremal or mixing state ω we define the effective translation invariant Hamiltonian $H^{\rm eff}_{1,\Lambda}(\mu,\eta,\rho)$ such that for all local observables A and B

$$\lim_{V\to\infty} \omega\left(A,[H_{1,\Lambda}(\mu),B]\right) = \lim_{V\to\infty} \omega\left(A,[H^{\rm eff}_{1,\Lambda}(\mu,\eta,\rho),B]\right). \tag{2.7}$$

The significance of the *parameters* η and ρ will become clear below. One can then replace (2.6) by

$$\lim_{V\to\infty} \beta\omega\left([A^*,[H^{\rm eff}_{1,\Lambda}(\mu,\eta,\rho),A]]\right) \geq \omega(A^*A)\ln\frac{\omega(A^*A)}{\omega(AA^*)}. \tag{2.8}$$

We can choose $H^{\rm eff}_{1,\Lambda}(\mu,\eta,\rho)$ so that it can be diagonalized and thus (2.8) can be solved explicitly. For a given chemical potential μ, the inequalities (2.8) can have *more than one* solution. We determine the physical solution by minimizing the free energy density with respect to the set of states or equivalently by maximizing the grand canonical pressure on this set.
Let the effective Hamiltonian be defined by

$$\begin{aligned} H^{\rm eff}_{1,\Lambda}(\mu,\eta,\rho) &= (\lambda\rho-\mu+\epsilon(q))a^*_{q+}a_{q+} + (\lambda\rho-\mu)a^*_{0-}a_{0-} + \frac{g}{2}(\eta a^*_{q+}b_q + \bar\eta a_{q+}b^*_q) \\ &\quad +\Omega\, b^*_q b_q + \frac{g\sqrt{V}}{2}\left(\bar\zeta a_{0-} + \zeta a^*_{0-}\right) + T'_{1,\Lambda} + (\lambda\rho-\mu)N'_{1,\Lambda} \end{aligned} \tag{2.9}$$

where

$$T'_{1,\Lambda} = \sum_{k\in\Lambda^*,\,k\neq q} \epsilon(k)N_{k,+} + \sum_{k\in\Lambda^*,\,k\neq 0} \epsilon(k)N_{k,-}, \tag{2.10}$$

$$N'_{1,\Lambda} = \sum_{k\in\Lambda^*,\,k\neq q} N_{k,+} + \sum_{k\in\Lambda^*,\,k\neq 0} N_{k,-}, \tag{2.11}$$

η and ζ are complex numbers and ρ is a positive real number. Notice that the Hamiltonian (2.9) is translation invariant, but it is not gauge invariant for $\zeta\neq 0$ because of the linear terms in a^*_{0-}, a_{0-} operators. Therefore in this case, (2.9) generates translation invariant states, which are extremal with respect to the gauge group in the zero-minus mode. They are labeled by the arg ζ. One can easily check that (2.7) is satisfied if

$$\eta = \frac{\omega(a_{0-})}{\sqrt{V}}, \qquad \zeta = \frac{\omega(a_{q+}b^*_q)}{V} \qquad \text{and} \qquad \rho = \frac{\omega(N_{1,\Lambda})}{V}, \tag{2.12}$$

where the state ω coincides with the equilibrium state $\langle\cdot\rangle_{H^{\rm eff}_{1,\Lambda}(\mu,\eta,\rho)}$ defined by the effective Hamiltonian $H^{\rm eff}_{1,\Lambda}(\mu,\eta,\rho)$. From (2.7) and (2.12) we then obtain the *self-consistency* equations

$$\eta = \frac{1}{\sqrt{V}}\langle a_{0-}\rangle_{H^{\rm eff}_{1,\Lambda}(\mu,\eta,\rho)}, \qquad \zeta = \frac{1}{V}\langle a_{q+}b^*_q\rangle_{H^{\rm eff}_{1,\Lambda}(\mu,\eta,\rho)}, \qquad \rho = \frac{1}{V}\langle N_{1,\Lambda}\rangle_{H^{\rm eff}_{1,\Lambda}(\mu,\eta,\rho)}. \tag{2.13}$$

The structure of (2.9) implies that the parameter ζ is a function of η and ρ through (2.13). So, we do not need to label the effective Hamiltonian by ζ. The important simplification here is that $H^{\rm eff}_{1,\Lambda}(\mu,\eta,\rho)$ can be diagonalized:

$$\begin{aligned} H^{\rm eff}_{1,\Lambda}(\mu,\eta,\rho) &= E_+(\mu,\eta,\rho)\alpha_1^*\alpha_1 + E_-(\mu,\eta,\rho)\alpha_2^*\alpha_2 \\ &\quad +(\lambda\rho-\mu)\alpha_3^*\alpha_3 + T'_{1,\Lambda} + (\lambda\rho-\mu)N'_{1,\Lambda} + \frac{g^2V|\zeta|^2}{4(\mu-\lambda\rho)}, \end{aligned} \tag{2.14}$$

where

$$\begin{aligned} E_+(\mu,\eta,\rho) &= \frac{1}{2}(\Omega-\mu+\lambda\rho+\epsilon(q)) + \frac{1}{2}\sqrt{(\Omega+\mu-\lambda\rho-\epsilon(q))^2+g^2|\eta|^2}, \\ E_-(\mu,\eta,\rho) &= \frac{1}{2}(\Omega-\mu+\lambda\rho+\epsilon(q)) - \frac{1}{2}\sqrt{(\Omega+\mu-\lambda\rho-\epsilon(q))^2+g^2|\eta|^2}, \end{aligned} \tag{2.15}$$

$$\alpha_1 = a_{q+}\cos\theta + b_q\sin\theta, \qquad \alpha_2 = a_{q+}\sin\theta - b_q\cos\theta, \qquad \alpha_3 = a_{0-} + \frac{g\sqrt{V}\zeta}{2(\lambda\rho-\mu)}, \tag{2.16}$$

and

$$\tan 2\theta = -\frac{g|\eta|}{\Omega+\mu-\lambda\rho-\epsilon(q)}. \tag{2.17}$$

Note that the correlation inequalities (2.8) (see [19]) imply that

$$\lim_{V\to\infty}\omega\left(A^*,[H_{1,\Lambda}(\mu),A]\right) \geq 0 \tag{2.18}$$

for all observables A. Applying (2.18) with $A = a^*_{0+}$, one gets the condition $\lambda\rho-\mu \geq 0$. Similarly, one obtains the condition $\lambda\rho+\epsilon(q)-\mu \geq 0$ by applying (2.18) to $A = a^*_{q+}$. We also have that $E_+(\mu,\eta,\rho) \geq E_-(\mu,\eta,\rho)$ and $E_-(\mu,\eta,\rho) = 0$ when $|\eta|^2 = 4\Omega(\lambda\rho+\epsilon(q)-\mu)/g^2$ and then $E_+(\mu,\eta,\rho) = \Omega-\mu+\lambda\rho+\epsilon(q)$. Thus we have the constraint:

$$|\eta|^2 \leq 4\Omega(\lambda\rho+\epsilon(q)-\mu)/g^2 \tag{2.19}$$

We shall need the above information to make sense of the thermodynamic functions below. Of course the parameters η, ρ and consequently $E_\pm$ are V dependent but for simplicity we do not indicate this dependence explicitly.

We can foresee that for some values of μ there will be Bose-Einstein condensation in the mode $\{0-\}$. We know that in this case the gauge invariant, homogeneous states are not extremal within the class of translation invariant equilibrium states [20]. Therefore to ensure that the states that we shall obtain are extremal we add to the Hamiltonian a gauge breaking term

$$-\frac{g\sqrt{V}}{2}\left(\overline{h}a_{0-} + ha^*_{0-}\right), \tag{2.20}$$

and then let $h \to 0$ after the thermodynamic limit. The corresponding effective Hamiltonian is then

$$H^{\text{eff}}_{1,\Lambda}(\mu,\eta,\rho;h) := H^{\text{eff}}_{1,\Lambda}(\mu,\eta,\rho) - \frac{g\sqrt{V}}{2}\left(\overline{h}a_{0-} + ha^*_{0-}\right) \tag{2.21}$$

with $h \in \mathbb{C}$. The equations corresponding to (2.13) now become

$$\eta = \frac{1}{\sqrt{V}}\langle a_{0-}\rangle_{H^{\text{eff}}_{1,\Lambda}(\mu,\eta,\rho;h)}, \qquad \zeta = \frac{1}{V}\langle a_{q+}b^*_q\rangle_{H^{\text{eff}}_{1,\Lambda}(\mu,\eta,\rho;h)}, \qquad \rho = \frac{1}{V}\langle N_{1,\Lambda}\rangle_{H^{\text{eff}}_{1,\Lambda}(\mu,\eta,\rho;h)}. \tag{2.22}$$

These consistency equations can be made explicit by using the above diagonalization:

$$\eta = \frac{g}{2(\mu-\lambda\rho)}(\zeta - h), \tag{2.23}$$

$$\zeta = \frac{1}{2}\frac{g\,\eta}{V(E_+ - E_-)}\left\{\frac{1}{e^{\beta E_+}-1} - \frac{1}{e^{\beta E_-}-1}\right\} \tag{2.24}$$

and

$$\begin{aligned}\rho \;=\; & |\eta|^2 + \frac{1}{V}\frac{1}{e^{-\beta(\mu-\lambda\rho)}-1} + \frac{1}{2V}\left\{\frac{1}{e^{\beta E_+}-1} + \frac{1}{e^{\beta E_-}-1}\right\} \\ & - \frac{(\mu-\lambda\rho-\epsilon(q)+\Omega)}{2V(E_+-E_-)}\left\{\frac{1}{e^{\beta E_+}-1} - \frac{1}{e^{\beta E_-}-1}\right\} \\ & + \frac{1}{V}\sum_{k\in\Lambda^*,\,k\neq 0}\frac{1}{e^{\beta(\epsilon(k)-\mu+\lambda\rho)}-1} + \frac{1}{V}\sum_{k\in\Lambda^*,\,k\neq 0,\,q}\frac{1}{e^{\beta(\epsilon(k)-\mu+\lambda\rho)}-1}.\end{aligned} \tag{2.25}$$

Combining (2.23) and (2.24) we obtain the equation:

$$\eta = \frac{g^2\,\eta}{4(\mu-\lambda\rho)V(E_+-E_-)}\left\{\frac{1}{e^{\beta E_+}-1} - \frac{1}{e^{\beta E_-}-1}\right\} - \frac{gh}{2(\mu-\lambda\rho)}. \tag{2.26}$$

It is now clear that the equilibrium states are determined by the limiting form of the consistency equations (2.23) - (2.26). We solve these equations and obtain the corresponding pressure so that we can determine the equilibrium state, when there are several solutions for a particular chemical potential.

We shall need the following definitions:

$$\varepsilon_0(\mu) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} d^3k \frac{\epsilon(k)-\mu}{e^{\beta(\epsilon(k)-\mu)}-1}, \tag{2.27}$$

$$\rho_0(\mu) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} d^3k \frac{1}{e^{\beta(\epsilon(k)-\mu)}-1} \tag{2.28}$$

and

$$p_0(\mu) = -\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} d^3k \ln(1-e^{-\beta(\epsilon(k)-\mu)}), \tag{2.29}$$

that is the grand-canonical energy density, the particle density and the pressure for the *free Bose-gas* for $\mu \leq 0$. Let

$$s_0(\mu) = \beta(\varepsilon_0(\mu) + p_0(\mu)), \tag{2.30}$$

and note that $s_0(\mu)$ is an increasing function of μ. We shall denote the free Bose-gas *critical* density by ρ_c, i.e. $\rho_c := \rho_0(0)$. Recall that ρ_c is infinite for $\nu < 3$ and finite for $\nu \geq 3$.

2.2 Solution of consistency equations

Notice first that equations (2.23) and (2.24) imply

$$\arg\zeta = \arg\eta = \arg h = \varphi\,, \tag{2.31}$$

i.e. one can consider the corresponding parameters in the consistency equations (2.23) - (2.26) to be real and non-negative.

Remark 2.1 *By virtue of the upper bound (2.19) and equation (2.26) we get that* $\delta := \lambda\rho - \mu > 0$ *for any volume* V *as soon as* $h \neq 0$. *By the same reasoning one gets from (2.26) that* $\lim_{V\to\infty}\inf E_- > 0$ *(Case A), or at most* $\lim_{V\to\infty} E_- = 0$ *in such a way that* $\lim_{V\to\infty} VE_-$ *is finite (Case B).*

We start our analysis of the solution of the consistency equations from small densities (small chemical potentials), when there is no condensates, passing then to higher values. So, later on we distinguish the two possibilities indicated in Remark 2.1:

$$Case\ A: \qquad \lim_{V\to\infty} E_- > 0. \tag{2.32}$$

Then by Remark 2.1 the consistency equations (2.23) - (2.26) in the thermodynamic limit yield:

$$\eta = \frac{gh}{2\delta}\ ,\quad \zeta = 0\,, \tag{2.33}$$

and the equation for the particle density (2.25) takes the form:

$$\mu = 2\lambda\rho_0(-\delta) - \delta + \frac{\lambda g^2|h|^2}{4\delta^2}. \tag{2.34}$$

We also have the limiting expectations

$$\lim_{V\to\infty}\frac{1}{V}\langle a^*_{0-}a_{0-}\rangle_{H^{\rm eff}_{1,\Lambda}(\mu,\eta,\rho;h)} = |\eta|^2 = \frac{g^2|h|^2}{4\delta^2} \tag{2.35}$$

and (by virtue of (2.32))

$$\lim_{V\to\infty}\frac{1}{V}\langle a^*_{q+}a_{q+}\rangle_{H^{\rm eff}_{1,\Lambda}(\mu,\eta,\rho;h)} = \lim_{V\to\infty}\frac{1}{V}\langle b^*_q b_q\rangle_{H^{\rm eff}_{1,\Lambda}(\mu,\eta,\rho;h)} = 0. \tag{2.36}$$

Let us now examine equation (2.34). For $h \neq 0$ it is solvable for any μ and we shall denote its unique solution by $\delta(\mu,h) = \lambda\rho(\mu,h) - \mu$.

Solution 1: Suppose that $\mu \leq \mu_c := 2\lambda\rho_c \equiv 2\lambda\rho_0(0)$. Since $|h|/\delta \to 0$ as $h \to 0$, then $\delta(\mu,h) \to \delta(\mu)$, where $\delta(\mu)$ is the unique solution of equation:

$$\mu = 2\lambda\rho_0(-\delta) - \delta. \tag{2.37}$$

Then we see from (2.35) and (2.36) that in this case in the thermodynamic limit there is *no condensation* in the $\{0-\}$ and other modes:

$$\lim_{V\to\infty}\frac{1}{V}\langle a^*_{0-}a_{0-}\rangle_{H^{\rm eff}_{1,\Lambda}(\mu,0,\rho)} = \lim_{V\to\infty}\frac{1}{V}\langle a^*_{q+}a_{q+}\rangle_{H^{\rm eff}_{1,\Lambda}(\mu,0,\rho)} = \lim_{V\to\infty}\frac{1}{V}\langle b^*_q b_q\rangle_{H^{\rm eff}_{1,\Lambda}(\mu,0,\rho)} = 0. \tag{2.38}$$

After letting $h \to 0$, the energy density is given by

$$\begin{aligned}\lim_{V\to\infty}\frac{1}{V}\langle H_{1,\Lambda}(\mu)\rangle_{H^{\rm eff}_{1,\Lambda}(\mu,0,\rho)} &= 2\varepsilon_0(-\delta(\mu)) - 2(\delta(\mu)+\mu)\rho_0(-\delta(\mu)) + \frac{1}{2}\lambda(\delta(\mu)+\mu)^2 \\ &= 2\varepsilon_0(-\delta(\mu)) - \frac{1}{2}\frac{(\delta(\mu)+\mu)^2}{\lambda} \end{aligned} \tag{2.39}$$

and the entropy density is equal to

$$s(\mu) = 2s_0(-\delta(\mu)). \tag{2.40}$$

Since the grand-canonical pressure is given by

$$p(\mu) = \frac{1}{\beta}s(\mu) - \lim_{V\to\infty}\frac{1}{V}\langle H_{1,\Lambda}(\mu)\rangle_{H^{\rm eff}_{1,\Lambda}(\mu,\eta,\rho)}, \tag{2.41}$$

then

$$p(\mu) = 2p_0(-\delta(\mu)) + \frac{1}{2}\frac{(\delta(\mu)+\mu)^2}{\lambda}. \tag{2.42}$$

Solution 2: Now suppose that $\mu > \mu_c = 2\lambda\rho_c$. Then to verify equation (2.34) in the limit $h \to 0$ the solution must converge to zero: $\delta(\mu, h) \to 0$, in such a way that

$$\frac{\lambda g^2|h|^2}{4\delta^2(\mu,h)} \to \mu - 2\lambda\rho_c. \tag{2.43}$$

Therefore it follows from (2.34) and (2.35) that

$$|\eta|^2 = \lim_{V\to\infty}\frac{1}{V}\langle a^*_{0-}a_{0-}\rangle_{H^{\rm eff}_{1,\Lambda}(\mu,\eta,\rho)} = \frac{\mu}{\lambda} - 2\rho_c \tag{2.44}$$

and (again by (2.32)) the limit (2.36) gives

$$\lim_{V\to\infty}\frac{1}{V}\langle a^*_{q+}a_{q+}\rangle_{H^{\rm eff}_{1,\Lambda}(\mu,\eta,\rho)} = \lim_{V\to\infty}\frac{1}{V}\langle b^*_q b_q\rangle_{H^{\rm eff}_{1,\Lambda}(\mu,\eta,\rho)} = 0, \tag{2.45}$$

i.e., there is *no condensation* in the $q \neq 0$ modes and the laser boson field.

In this case the energy density is given by:

$$\lim_{V\to\infty}\frac{1}{V}\langle H_{1,\Lambda}(\mu)\rangle_{H^{\rm eff}_{1,\Lambda}(\mu,\eta,\rho)} = 2\varepsilon_0(0) - \frac{\mu^2}{2\lambda} \tag{2.46}$$

and the entropy density has the form:

$$s(\mu) = 2s_0(0) = 2\beta(\varepsilon_0(0) + p_0(0)). \tag{2.47}$$

Thus for the pressure one gets:

$$p(\mu) = 2p_0(0) + \frac{\mu^2}{2\lambda}. \tag{2.48}$$

Notice that the bound (2.19) and (2.44) imply the upper limit on chemical potential

$$\mu \leq \mu_c + \frac{4\Omega\lambda\epsilon(q)}{g^2} \tag{2.49}$$

for which Solution 2 applies.

This means that for the higher densities or chemical potentials:

$$\mu > \mu_c + \frac{4\Omega\lambda\epsilon(q)}{g^2}, \tag{2.50}$$

to satisfy equation (2.25) we have to consider

$$Case\ B: \qquad \lim_{V\to\infty} E_- = 0. \tag{2.51}$$

Then (see (2.19) and Remark 2.1) in the thermodynamic limit: $|\eta|^2 \to 4\Omega(\delta+\epsilon(q))/g^2$. In fact, to obtain a finite limit in (2.26) the corresponding large-volume asymptotic should to be

$$|\eta|^2 \approx \frac{4\Omega}{g^2}\left(\delta + \epsilon(q) - \frac{1}{\beta V\tau}\right) \tag{2.52}$$

for some $\tau > 0$. This implies that

$$E_+ \to \Omega + \epsilon(q) - \delta, \qquad E_- \approx \frac{\Omega}{\beta V\tau(\Omega+\epsilon(q)-\delta)} \tag{2.53}$$

and (2.26) becomes in the limit:

$$\eta\left(1 - \frac{g^2\tau}{4\delta\Omega}\right) = \frac{gh}{2\delta}. \tag{2.54}$$

The last equation gives

$$\tau = \frac{4\delta\Omega}{g^2} - \frac{2h\Omega}{g\eta}. \tag{2.55}$$

Taking the limit $V \to \infty$ in (2.25) we get

$$\mu = \frac{4\lambda\Omega(\delta+\epsilon(q))}{g^2} + \frac{4\lambda\delta\Omega}{g^2} - \frac{2\lambda h\Omega}{g\eta} + 2\lambda\rho_0(-\delta) - \delta. \tag{2.56}$$

We can also check by using the diagonalization that in this case

$$\lim_{V\to\infty}\frac{1}{V}\langle a_{0-}^* a_{0-}\rangle_{H^{\rm eff}_{1,\Lambda}(\mu,\eta,\rho;h)} = |\eta|^2 = \frac{4\Omega(\delta+\epsilon(q))}{g^2}, \tag{2.57}$$

$$\lim_{V\to\infty}\frac{1}{V}\langle a_{q+}^* a_{q+}\rangle_{H^{\rm eff}_{1,\Lambda}(\mu,\eta,\rho;h)} = \tau = \frac{4\delta\Omega}{g^2} - \frac{2h\Omega}{g\eta}, \tag{2.58}$$

$$\lim_{V\to\infty}\frac{1}{V}\langle b_q^* b_q\rangle_{H^{\rm eff}_{1,\Lambda}}(\mu,\eta,\rho;h) = \frac{g^2|\eta|^2\tau}{4\Omega^2} = (\delta+\epsilon(q))\left(\frac{4\delta}{g^2} - \frac{2h}{g\eta}\right). \tag{2.59}$$

We note here that if we take $\delta = 0$ in (2.56), then for $h \to 0$ this expression gives the limiting value of the chemical potential (2.49) for Solution 2.

Solution 3: Let $\mu > \mu_c + 4\Omega\lambda\epsilon(q)/g^2$, see (2.50). Now we take Case B and let $h \to 0$. Then by (2.57), (2.58) and (2.59) we obtain a *simultaneous* condensation of the excited/non-excited bosons and the laser photons in the q-mode:

$$\lim_{V\to\infty}\frac{1}{V}\langle a_{0-}^* a_{0-}\rangle_{H^{\rm eff}_{1,\Lambda}(\mu,\eta,\rho)} = |\eta|^2 = \frac{4\Omega(\delta+\epsilon(q))}{g^2}, \tag{2.60}$$

$$\lim_{V\to\infty}\frac{1}{V}\langle a^*_{q+}a_{q+}\rangle_{H^{\rm eff}_{1,\Lambda}(\mu,\eta,\rho)}=\tau=\frac{4\delta\Omega}{g^2}, \tag{2.61}$$

$$\lim_{V\to\infty}\frac{1}{V}\langle b^*_{q}b_{q}\rangle_{H^{\rm eff}_{1,\Lambda}(\mu,\eta,\rho)}=\frac{4(\delta+\epsilon(q))\delta}{g^2}. \tag{2.62}$$

Equation (2.56) becomes:

$$\mu=\frac{4\lambda\Omega(\delta+\epsilon(q))}{g^2}+\frac{4\lambda\delta\Omega}{g^2}+2\lambda\rho_0(-\delta)-\delta. \tag{2.63}$$

Using the diagonalization of (2.9) one computes also

$$\lim_{V\to\infty}\frac{1}{V}\langle a^*_{q+}b_{q}\rangle_{H^{\rm eff}_{1,\Lambda}(\mu,\eta,\rho)}=\frac{4\delta\sqrt{\Omega(\delta+\epsilon(q))}}{g^2}. \tag{2.64}$$

In this case the energy density is given by:

$$\begin{aligned}\lim_{V\to\infty}\frac{1}{V}\langle H_{1,\Lambda}(\mu)\rangle_{H^{\rm eff}_{1,\Lambda}(\mu,\eta,\rho)} &= (\epsilon(q)-\mu)\frac{4\delta\Omega}{g^2}-\mu\frac{4\Omega(\delta+\epsilon(q))}{g^2}-\frac{8\Omega\delta(\delta+\epsilon(q))}{g^2}\\ &\quad+\Omega\frac{4(\delta+\epsilon(q))\delta}{g^2}+2\varepsilon_0(-\delta)-2(\delta+\mu)\rho_0(-\delta)+\frac{1}{2}\frac{(\delta+\mu)^2}{\lambda}\\ &= \frac{4\Omega(\delta+\epsilon(q))\delta}{g^2}+2\varepsilon_0(-\delta)-\frac{1}{2}\frac{(\delta+\mu)^2}{\lambda}.\end{aligned} \tag{2.65}$$

The entropy density is again given by

$$s(\mu)=2s_0(\mu-\lambda\rho) \tag{2.66}$$

and the pressure becomes

$$p(\mu)=2p_0(-\delta)+\frac{1}{2}\frac{(\delta+\mu)^2}{\lambda}-\frac{4\Omega(\delta+\epsilon(q))\delta}{g^2}. \tag{2.67}$$

Notice that only $|\eta|$ is determined and not the phase (2.31) of η , that is that we can only determine the state up to a gauge transformation in the $\sigma=-$ fields.

The fact that here we have condensation in a state with non-zero momentum is extremely significant and is related to the *spontaneous breaking* of translation invariance. We shall examine this important aspect in Section 4.

The Solutions 1, 2 and 3 represent possible equilibrium states. For a given value of the chemical potential μ, two or even three of these solutions may be possible, see Figure 1 below. To distinguish between them we have to compare the corresponding pressures to determine which is maximum. The analysis, which is given in the next subsection, involves a detailed study of the pressure. We find that the situation is as described below.

Let $\kappa=8\Omega\lambda/g^2-1$ and $\alpha=\epsilon(q)(\kappa+1)/2$. From the condition for thermodynamic stability we know that $\kappa>0$. In this notation **Solution 2** applies for $\mu_c\le\mu\le\mu_c+\alpha$. Let δ_0 be the unique value of $\delta\in[0,\infty)$ such that $2\lambda\rho_0'(-\delta)=\kappa$ and let $\mu_0=2\lambda\rho_0(-\delta_0)+\kappa\delta_0$. Note that $\mu_0<2\lambda\rho_c$.
The case when $\mu_0+\alpha\ge 2\lambda\rho_c$ is easy. In this situation **Solution 1** applies for $\mu\le 2\lambda\rho_c$

and there exists $\mu_1 > \mu_0 + \alpha$ (see definition after (2.80)) such that **Solution 2** applies for $2\lambda\rho_c < \mu < \mu_1$ and **Solution 3** for $\mu \geq \mu_1$.

When $\mu_0 + \alpha < 2\lambda\rho_c$ the situation is more subtle. In Subsection 2.3 we shall show that there exists $\mu_1 > \mu_0 + \alpha$ (2.80), such that **Solution 3** applies for $\mu \geq \mu_1$. However we are not able to decide on which side of $2\lambda\rho_c$, the point μ_1 lies. If $\mu_1 > 2\lambda\rho_c$ the situation is as in the previous subcase, while if $\mu_0 + \alpha < \mu_1 < 2\lambda\rho_c$ the *intermediate phase* where **Solution 2** obtains is *eliminated*. This the situation is similar to [7], where one has $\alpha = 0$.

Note that for $\nu < 3$, **Solution 1** applies when $\mu < \mu_1$ and **Solution 3** when $\mu \geq \mu_1$.

2.3 The Pressure for Model 1

This subsection is devoted to a detailed study of the pressure for Model 1 as a function of the chemical potential μ.

Recall that δ is the limiting value of $\lambda\rho - \mu$, $\kappa = 8\Omega\lambda/g^2 - 1$ and $\alpha = \epsilon(q)(\kappa + 1)/2$. From above we have the following classification:

Solution 1: Here $\mu \leq \mu_c$. The density equation

$$\mu = 2\lambda\rho_0(-\delta) - \delta \tag{2.68}$$

has a unique solution in δ, denoted by $\delta_1(\mu)$ (previously denoted by $\delta(\mu)$). Let

$$p_1(\delta, \mu) := 2p_0(-\delta) + \frac{(\delta + \mu)^2}{2\lambda}. \tag{2.69}$$

Then

$$p(\mu) = p_1(\delta_1(\mu), \mu). \tag{2.70}$$

Solution 2: Here $\mu > \mu_c$, $\delta = 0$ and the pressure is given by

$$p(\mu) = p_2(\mu) := 2p_0(0) + \frac{\mu^2}{2\lambda}. \tag{2.71}$$

Solution 3: The equation (2.63) can be re-written as

$$\mu = 2\lambda\rho_0(-\delta) + \kappa\delta + \alpha. \tag{2.72}$$

Recall that δ_0 is the unique value of $\delta \in [0, \infty)$ such that $2\lambda\rho_0'(-\delta) = \kappa$, $\mu_0 = 2\lambda\rho_0(-\delta_0) + \kappa\delta_0$ and that $\mu_0 < 2\lambda\rho_c$.

Then for $\mu < \mu_0 + \alpha$, equation (2.72) has *no* solutions. For $\mu_0 + \alpha \leq \mu \leq 2\lambda\rho_c + \alpha$ this equation has *two* solutions: $\tilde{\delta}_3(\mu)$ and $\delta_3(\mu)$, where $\tilde{\delta}_3(\mu) < \delta_3(\mu)$ if $\mu \neq \mu_0 + \alpha$, and $\tilde{\delta}_3(\mu_0 + \alpha) = \delta_3(\mu_0 + \alpha)$. Finally for $\mu > 2\lambda\rho_c + \alpha$ it has a *unique* solution $\delta_3(\mu)$. Let

$$p_3(\delta, \mu) := 2p_0(-\delta) + \frac{\{(\delta + \mu)^2 - (\kappa + 1)\delta^2 - 2\alpha\delta\}}{2\lambda}. \tag{2.73}$$

Then

$$\frac{dp_3(\tilde{\delta}_3(\mu), \mu)}{d\mu} = \frac{\tilde{\delta}_3(\mu) + \mu}{\lambda} < \frac{\delta_3(\mu) + \mu}{\lambda} = \frac{dp_3(\delta_3(\mu), \mu)}{d\mu} \tag{2.74}$$

for $\mu \neq \mu_0 + \alpha$. Since $p_3(\tilde{\delta}_3(\mu_0+\alpha), \mu_0+\alpha) = p_3(\delta_3(\mu_0+\alpha), \mu_0+\alpha)$,

$$p_3(\tilde{\delta}_3(\mu), \mu) < p_3(\delta_3(\mu), \mu) \tag{2.75}$$

for $\mu_0 + \alpha < \mu \leq 2\lambda\rho_c + \alpha$. Therefore

$$p(\mu) = p_3(\delta_3(\mu), \mu) \tag{2.76}$$

for all $\mu \geq \mu_0 + \alpha$.

Note that $\tilde{\delta}_3(2\lambda\rho_c + \alpha) = 0$ so that

$$p_3(\tilde{\delta}_3(2\lambda\rho_c+\alpha), 2\lambda\rho_c+\alpha) = p_1(\delta_1(2\lambda\rho_c), 2\lambda\rho_c) = p_2(2\lambda\rho_c) = 2p_0(0) + 2\lambda\rho_c^2. \tag{2.77}$$

therefore

$$p_3(\delta_3(\mu_0+\alpha), \mu_0+\alpha) = p_3(\tilde{\delta}_3(\mu_0+\alpha), \mu_0+\alpha) < 2p_0(0) + 2\lambda\rho_c^2. \tag{2.78}$$

Also for large μ, $p_3(\delta_3(\mu), \mu) \approx (\mu^2/2\lambda)((\kappa+1)/\kappa)$ while $p_2(\mu) \approx (\mu^2/2\lambda)$, so that $p_3(\delta_3(\mu), \mu) > p_2(\mu)$ eventually. We remark finally that the slope of $p_3(\delta_3(\mu), \mu)$ is greater than that of $p_2(\mu)$,

$$\frac{dp_3(\delta_3(\mu), \mu)}{d\mu} = \frac{\delta_3(\mu) + \mu}{\lambda} > \frac{\mu}{\lambda} = \frac{dp_2(\mu)}{d\mu}, \tag{2.79}$$

so that the corresponding curves intersect at most once.

The case $\alpha = 0$, i.e. $\epsilon(q = 0) = 0$, has been examined in [7].

For the case $\alpha > 0$ we have two *subcases*:
The subcase $\mu_0 + \alpha \geq 2\lambda\rho_c$ is easy. In this situation Solution 1 applies for $\mu \leq 2\lambda\rho_c$. From (2.78) we see that

$$p_3(\delta_3(\mu_0+\alpha), \mu_0+\alpha) < 2p_0(0) + 2\lambda\rho_c^2 < p_2(\mu_0+\alpha \tag{2.80}$$

and therefore from the behaviour for large μ we can deduce that there exists $\mu_1 > \mu_0 + \alpha$ such that Solution 2 applies for $2\lambda\rho_c < \mu < \mu_1$ and Solution 3 for $\mu \geq \mu_1$.

The subcase $\mu_0 + \alpha < 2\lambda\rho_c$ is more complicated. In Figure 1 we have drawn $y = 2\lambda\rho_0(-\delta) - \delta$ and $y = 2\lambda\rho_0(-\delta) + \kappa\delta + \alpha$ for this subcase. We know that

$$p_3(\tilde{\delta}_3(2\lambda\rho_c), 2\lambda\rho_c) < p_3(\tilde{\delta}_3(2\lambda\rho_c+\alpha), 2\lambda\rho_c+\alpha) = p_1(\delta_1(2\lambda\rho_c), 2\lambda\rho_c). \tag{2.81}$$

Therefore since the slope of $p_3(\tilde{\delta}_3(\mu), \mu)$ is greater than the slope of $p_1(\delta_1(\mu), \mu)$ for $\mu_0 + \alpha < \mu < 2\lambda\rho_c$, (see Figure 1):

$$\frac{dp_3(\tilde{\delta}_3(\mu), \mu)}{d\mu} = \frac{\tilde{\delta}_3(\mu) + \mu}{\lambda} > \frac{\delta_1(\mu) + \mu}{\lambda} = \frac{dp_1(\delta_1(\mu), \mu)}{d\mu}, \tag{2.82}$$

we can conclude that

$$p_3(\delta_3(\mu_0+\alpha), \mu_0+\alpha) = p_3(\tilde{\delta}_3(\mu_0+\alpha), \mu_0+\alpha) < p_1(\delta_1(\mu_0+\alpha), \mu_0+\alpha). \tag{2.83}$$

We also know by the arguments above that there exists $\mu_1 > \mu_0 + \alpha$ such that Solution 3 applies for for $\mu \geq \mu_1$. However we do know on which side of $2\lambda\rho_c$, the point μ_1 lies. If $\mu_1 > 2\lambda\rho_c$ the situation is as in the previous subcase while if $\mu_0 + \alpha < \mu_1 < 2\lambda\rho_c$ the intermediate phase where Solution 2 obtains is eliminated.

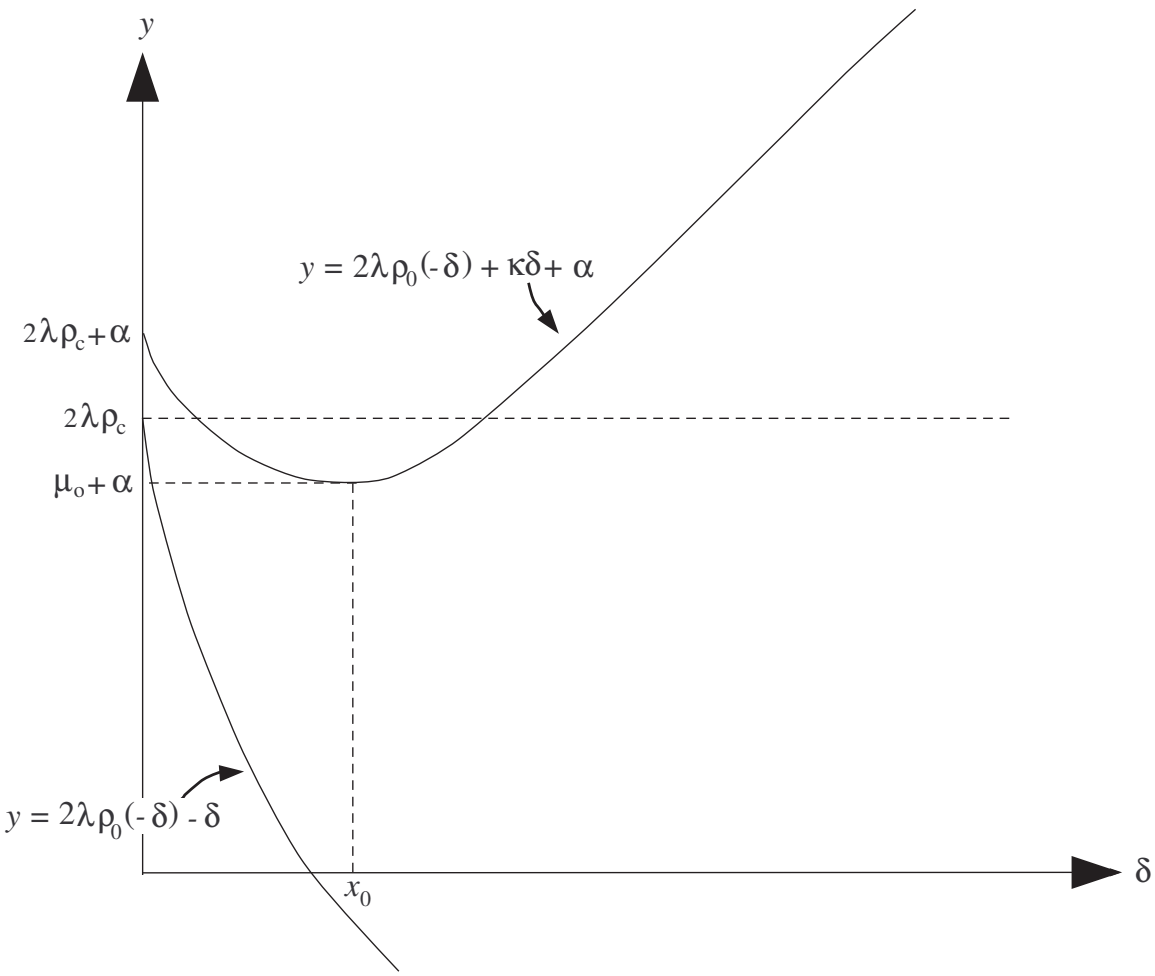

Figure 1: The density equation for $h = 0$ and $\nu \geq 3$

3 Model 2

As we said in the introduction the analysis for this model is very similar to that of *Model 1*. Therefore we briefly summarize the results without repeating the details. For *Model 2* the effective Hamiltonian is

$$\begin{aligned} H^{\text{eff}}_{2,\Lambda}(\mu,\eta,\rho) &= (\lambda\rho - \mu + \epsilon(q))a^*_q a_q + (\lambda\rho - \mu)a^*_0 a_0 + \frac{g}{2}(\eta a^*_q b_q + \bar{\eta} a_q b^*_q) \\ &\quad + \Omega\, b^*_q b_q + \frac{g\sqrt{V}}{2}\left(\zeta a_0 + \bar{\zeta} a^*_0\right) + T'_{2,\Lambda} + (\lambda\rho - \mu)N'_{2,\Lambda} \end{aligned} \tag{3.1}$$

where

$$T'_{2,\Lambda} = \sum_{k\in\Lambda^*,\, k\neq 0\, k\neq q} \epsilon(k) N_k, \tag{3.2}$$

$$N'_{2,\Lambda} = \sum_{k\in\Lambda^*,\, k\neq 0\, k\neq q} \epsilon(k) N_k. \tag{3.3}$$

The parameters η, ζ and ρ satisfy the *self-consistency* equations:

$$\eta = \frac{1}{\sqrt{V}}\langle a_0\rangle_{H^{\text{eff}}_{2,\Lambda}(\mu,\eta,\rho)}, \qquad \zeta = \frac{1}{V}\langle a^*_q b_q\rangle_{H^{\text{eff}}_{2,\Lambda}(\mu,\eta,\rho)}, \qquad \rho = \frac{1}{V}\langle N_{2,\Lambda}\rangle_{H^{\text{eff}}_{2,\Lambda}(\mu,\eta,\rho)}. \tag{3.4}$$

Using the external sources (2.20) and the same treatment as for the *Model 1* in Section 2.2, we again obtain three cases:

Solution 1: $\mu \leq \lambda\rho_c \equiv \mu_c$. In this case there is *no condensation*:

$$\lim_{V\to\infty}\frac{1}{V}\langle a_0^* a_0\rangle_{H^{\rm eff}_{2,\Lambda}(\mu,\eta,\rho)} = \lim_{V\to\infty}\frac{1}{V}\langle a_q^* a_q\rangle_{H^{\rm eff}_{2,\Lambda}(\mu,\eta,\rho)} = \lim_{V\to\infty}\frac{1}{V}\langle b_q^* b_q\rangle_{H^{\rm eff}_{2,\Lambda}(\mu,\eta,\rho)} = 0, \tag{3.5}$$

the density equation is

$$\mu = \lambda\rho_0(-\delta) - \delta \tag{3.6}$$

and the pressure is

$$p(\mu) = p_0(-\delta) + \frac{1}{2}\frac{(\delta+\mu)^2}{\lambda}. \tag{3.7}$$

Solution 2: Let $\mu_c < \mu \leq \mu_c + 4\Omega\lambda\epsilon(q)/g^2$. Then $\delta(\mu,h) = \lim_{h\to 0}(\lambda\rho(\mu,h) - \mu) = 0$.

$$|\eta|^2 = \lim_{V\to\infty}\frac{1}{V}\langle a_0^* a_0\rangle_{H^{\rm eff}_{2,\Lambda}(\mu,\eta,\rho)} = \frac{\mu}{\lambda} - \rho_c. \tag{3.8}$$

There is condensation in the $k=0$ mode but there is *no condensation* in the $k=q$ mode and of the photon laser field:

$$\lim_{V\to\infty}\frac{1}{V}\langle a_q^* a_q\rangle_{H^{\rm eff}_{2,\Lambda}(\mu,\eta,\rho)} = \lim_{V\to\infty}\frac{1}{V}\langle b_q^* b_q\rangle_{H^{\rm eff}_{2,\Lambda}(\mu,\eta,\rho)} = 0. \tag{3.9}$$

The pressure density is given by

$$p(\mu) = p_0(0) + \frac{\mu^2}{2\lambda}. \tag{3.10}$$

Solution 3: Let $\mu > \mu_c + 4\Omega\lambda\epsilon(q)/g^2$, see (2.50). Then there is simultaneous condensation of the zero-mode and the q-mode bosons as well as the laser q-mode photons:

$$\lim_{V\to\infty}\frac{1}{V}\langle a_0^* a_0\rangle_{H^{\rm eff}_{2,\Lambda}(\mu,\eta,\rho)} = \frac{4\Omega(\delta+\epsilon(q))}{g^2}, \tag{3.11}$$

$$\lim_{V\to\infty}\frac{1}{V}\langle a_q^* a_q\rangle_{H^{\rm eff}_{2,\Lambda}(\mu,\eta,\rho)} = \frac{4\delta\Omega}{g^2}, \tag{3.12}$$

$$\lim_{V\to\infty}\frac{1}{V}\langle b_q^* b_q\rangle_{H^{\rm eff}_{2,\Lambda}(\mu,\eta,\rho)} = \frac{4(\delta+\epsilon(q))\delta}{g^2}, \tag{3.13}$$

$$\lim_{V\to\infty}\frac{1}{V}\langle a_q^* b_q\rangle_{H^{\rm eff}_{2,\Lambda}(\mu,\eta,\rho)} = \frac{4\delta\sqrt{\Omega(\delta+\epsilon(q))}}{g^2}. \tag{3.14}$$

The density equation is

$$\mu = \frac{8\lambda\Omega}{g^2}(\delta+\epsilon(q)/2) + \rho_0(-\delta) - \delta \tag{3.15}$$

and pressure is

$$p(\mu) = p_0(-\delta) + \frac{1}{2}\frac{(\delta+\mu)^2}{\lambda} - \frac{4\Omega(\delta+\epsilon(q))\delta}{g^2}. \tag{3.16}$$

Note that relations between the values of μ in the three cases above are exactly the same as for *Model 1* apart from the fact that $2\rho_0$ is now replaced by ρ_0 and $2\rho_c$ by ρ_c. To see this one has to compare the kinetic energy operators (1.1) and (1.4).

4 Spontaneous Breaking of Translation Invariance and Matter-Wave Grating

The recently observed phenomenon of *periodic spatial variation in the boson-density* is responsible for the light and matter-wave *amplification* in superradiant condensation, see [2]-[4], [12]. This so called *matter-wave grating* is produced by the interference of two different macroscopically occupied momentum states: the first corresponds to a macroscopic number of *recoiled* bosons and the second to *residual* BE condensate at rest. Clearly this cannot happen in the translation invariant states and so it must be due to a spontaneous breaking of this invariance.

Let us consider the situation in Solution 3 for *Model 1*. For simplicity we shall take $q = (2\pi/\gamma)\mathbf{e}_1$, with $\mathbf{e}_1 = (1, 0, \ldots, 0) \in \mathbb{R}^\nu$ and $\gamma > 0$ and we shall denote the limit Gibbs state for the effective Hamiltonian by ω:

$$\omega(\,\cdot\,) = \lim_{V\to\infty} \langle\,\cdot\,\rangle_{H^{\rm eff}_{1,\Lambda}(\mu,\eta,\rho)}\,. \tag{4.1}$$

We know that the existence of condensation in the zero-mode of the $\sigma = -$ bosons implies that the extremal states have broken gauge symmetry in the corresponding fields in this mode. As was remarked earlier this is indicated here by the fact that η is not zero.

It has been shown in [20] that condensation in the zero-mode implies that the gauge invariant spatially homogeneous equilibrium states are not extremal but can be decomposed into a convex combination of extremal space homogeneous equilibrium states with broken gauge symmetry (spontaneous braking of gauge symmetry).

From (2.61), (2.62) and (2.64) we see that in Solution 3

$$\lim_{V\to\infty}\left|\frac{1}{V}\omega(a^*_{q+}b_q)\right|^2 = \lim_{V\to\infty}\frac{1}{V}\omega(a^*_{q+}a_{q+})\lim_{V\to\infty}\frac{1}{V}\omega(b^*_q b_q). \tag{4.2}$$

This strongly suggests a similar decomposition when there is condensation in the q-mode. In fact when there is condensation in a mode $q \neq 0$ one can argue, along the same lines as in [20], that the spatially homogeneous equilibrium states when considered within the equilibrium states which are periodic in the $\mathbf{e}_1$ direction with period γ, are not extremal. They can be decomposed into a convex combination of extremal periodic equilibrium states which are not spatially homogeneous:

$$\omega = \frac{1}{\gamma}\int_0^\gamma dx\ \ \omega_x \tag{4.3}$$

where

$$\omega_x \circ \tau_{\gamma\mathbf{e}_1} = \omega_x \tag{4.4}$$

and

$$\omega_y \circ \tau_{x\mathbf{e}_1} = \omega_{(x+y)\mathrm{mod}\ \gamma}. \tag{4.5}$$

Therefore in this model we have spontaneous breaking of translation invariance. The rigorous and explicit construction of the states ω_x involves mathematical details which are outside of the scope of the present paper and is carried out in [21].

Let $\Lambda_1 = \{x | x \in \Lambda, 0 < x < \gamma\}$ and let $V_1 = |\Lambda_1|$. Then we can write, for example,

$$\lim_{V\to\infty} a^\#_{q+}/\sqrt{V} = \lim_{N\to\infty}\frac{1}{2N}\sum_{n=-N}^{N}\frac{1}{V_1}\int_{n\mathbf{e}_1+\Lambda_1} e^{\pm iqx}a^\#_+(x)dx. \tag{4.6}$$

Therefore in the representation corresponding to each of the extremal states ω_x, $a^{\#}_{q+}/\sqrt{V}$ converges weakly to a complex number which by (4.4) and (4.5) is equal to

$$e^{\mp iqx} \lim_{V\to\infty} \frac{1}{\sqrt{V}}\, \omega_0(a^{\#}_{q+}), \tag{4.7}$$

where $\omega_0 = \omega_{x=0}$. It then follows from (2.61), with $\delta = \lim_{V\to\infty}(\lambda\rho - \mu)$, that

$$\lim_{V\to\infty} \frac{1}{V}\omega_0(a^*_{q+}a_{q+}) = \left|\lim_{V\to\infty} \frac{1}{\sqrt{V}}\, \omega_0(a_{q+})\right|^2 = \frac{4\delta\Omega}{g^2}. \tag{4.8}$$

Similarly $a^{\#}_{0-}/\sqrt{V}$ and $b^{\#}/\sqrt{V}$ converge weakly to a complex numbers and from (2.60) and (2.62) respectively we obtain

$$\lim_{V\to\infty} \frac{1}{V}\omega_0(a^*_{0-}a_{0-}) = \left|\lim_{V\to\infty} \frac{1}{\sqrt{V}}\, \omega_0(a_{0-})\right|^2 = \frac{4\Omega(\delta + \epsilon(q))}{g^2} \tag{4.9}$$

and

$$\lim_{V\to\infty} \frac{1}{V}\omega_0(b^*_q b_q) = \left|\lim_{V\to\infty} \frac{1}{\sqrt{V}}\, \omega_0(b_q)\right|^2 = \frac{4(\delta + \epsilon(q))\delta}{g^2}. \tag{4.10}$$

The weak convergence of $a^{\#}_{q+}/\sqrt{V}$ and $a^{\#}_{0-}/\sqrt{V}$ to complex numbers also implies that

$$\lim_{V\to\infty} \frac{1}{V}\omega_0(a^*_{q,+}a_{k,+}) = \lim_{V\to\infty} \frac{1}{\sqrt{V}}\omega_0(a^*_{q,+}) \lim_{V\to\infty} \frac{1}{\sqrt{V}}\omega_0(a_{k,+}) = 0 \tag{4.11}$$

for $k \neq q$ and

$$\lim_{V\to\infty} \frac{1}{V}\omega_0(a^*_{0-}a_{k,-}) = \lim_{V\to\infty} \frac{1}{\sqrt{V}}\omega_0(a^*_{0,-}) \lim_{V\to\infty} \frac{1}{\sqrt{V}}\omega_0(a_{k,-}) = 0 \tag{4.12}$$

for $k \neq 0$.

An alternative strategy to the one we have developed above would be to use the traditional method of introducing source terms in the Hamiltonian (1.2) to break both the gauge and translation symmetries and let:

$$H_{1,\Lambda}(\xi) := H_{1,\Lambda} - \sqrt{V}(\xi\, b^*_q + \overline{\xi}\, b_q)\,. \tag{4.13}$$

Then the effective Hamiltonian becomes

$$\begin{aligned} H^{\text{eff}}_{1,\Lambda}(\mu,\rho;\zeta,\xi) \;=\; & (\lambda\rho - \mu + \epsilon(q))a^*_{q+}a_{q+} + (\lambda\rho - \mu)a^*_{0-}a_{0-} \\ & + \frac{g}{2}\left\{(\zeta + \frac{\xi}{\Omega})a^*_{q+}a_{0-} + (\overline{\zeta} + \frac{\overline{\xi}}{\Omega})a^*_{0-}a_{q+}\right\} + \Omega\, \hat{b}^*_q\hat{b}_q \\ & + V\Omega|\zeta|^2 - V\frac{|\xi|^2}{\Omega} + T'_{1,\Lambda} + (\lambda\rho - \mu)N'_{1,\Lambda} - \frac{1}{2}V\rho^2\,, \end{aligned} \tag{4.14}$$

where

$$\hat{b}_q = b_q - \sqrt{V}(\zeta + \frac{\xi}{\Omega}) \tag{4.15}$$

and we can carry out the procedure of Section 2.1 to obtain the results equivalent to (4.8)-(4.12).

We now want to examine the possibility of interference between two different macroscopically occupied momentum states in *Model 1* in the periodic states. Without loss of generality we can restrict ourselves to the state ω_0. In this state the mean local particle density for the $\sigma = +$ bosons is

$$\rho_+(x) = \lim_{V\to\infty} \frac{1}{V} \sum_{k\in\Lambda^*} \sum_{p\in\Lambda^*} e^{i(k-p)x} \omega_0(a^*_{k,+} a_{p,+}) = \rho + \lim_{V\to\infty} \frac{1}{V} \sum_{k\in\Lambda^*} \sum_{p\in\Lambda^* \, p\neq k} e^{i(k-p)x} \omega_0(a^*_{k,+} a_{p,+}). \tag{4.16}$$

We know that condensation occurs only in the q-mode for the $\sigma = +$ bosons and in the $q = 0$ - mode for the $\sigma = -$ bosons and therefore only the terms containing q survive in the integral sum (4.16) in the thermodynamic limit:

$$\rho_+(x) = \rho + \lim_{V\to\infty} \frac{1}{V} \sum_{k\in\Lambda^* \, k\neq q} 2\,\mathfrak{Re}\left\{ e^{i(k-q)x} \omega_0(a^*_{k,+} a_{q,+}) \right\} \tag{4.17}$$

From the above discussion, in particular from (4.11), we see that $\rho_+(x) = \rho$. Similarly $\rho_-(x) = \rho$. Thus in spite of the fact that the state ω_0 is not space homogeneous, the total particle density is constant and equal to 2ρ. This means that in *Model 1* we get no particle density space variation even in the presence of the *light corrugated lattice of condensed photons*, see (4.10).

Let us now turn our attention to the corresponding situation for *Model 2* in Solution 3. The decomposition into periodic states still stands and again we have spontaneous breaking of gauge symmetry. In the representation corresponding to each of the extremal states, $a^\#_q/\sqrt{V}$, $a^\#_0/\sqrt{V}$ and $b^\#/\sqrt{V}$ all converge weakly to a complex numbers and

$$\lim_{V\to\infty} \frac{1}{V} \omega_0(a^*_q a_q) = \left| \lim_{V\to\infty} \frac{1}{\sqrt{V}} \omega_0(a_q) \right|^2 = \frac{4\delta\Omega}{g^2}, \tag{4.18}$$

$$\lim_{V\to\infty} \frac{1}{V} \omega_0(a^*_0 a_0) = \left| \lim_{V\to\infty} \frac{1}{\sqrt{V}} \omega_0(a_0) \right|^2 = |\eta|^2 = \frac{4\Omega(\delta + \epsilon(q))}{g^2}, \tag{4.19}$$

$$\lim_{V\to\infty} \frac{1}{V} \omega_0(b^*_q b_q) = \left| \lim_{V\to\infty} \frac{1}{\sqrt{V}} \omega_0(b_q) \right|^2 = \frac{4(\delta + \epsilon(q))\delta}{g^2}. \tag{4.20}$$

We have again

$$\rho(x) = \lim_{V\to\infty} \frac{1}{V} \sum_{k\in\Lambda^*} \sum_{p\in\Lambda^*} e^{i(k-p)x} \omega_0(a^*_k a_p) = \rho + \lim_{V\to\infty} \frac{1}{V} \sum_{k\in\Lambda^*} \sum_{p\in\Lambda^* \, p\neq k} e^{i(k-p)x} \omega_0(a^*_k a_p). \tag{4.21}$$

The important difference here is that in this model the *same* boson atoms may condense in two states and therefore

$$\begin{aligned} \rho(x) \;=\; & \rho + \lim_{V\to\infty} \frac{1}{V} 2\,\mathfrak{Re}\left\{ e^{-iqx} \omega_0(a^*_0 a_q) \right\} \\ & + \lim_{V\to\infty} \frac{1}{V} \sum_{k\in\Lambda^* \, k\neq 0,q} 2\,\mathfrak{Re}\left\{ e^{i(k-q)x} \omega_0(a^*_k a_q) \right\} \\ & + \lim_{V\to\infty} \frac{1}{V} \sum_{k\in\Lambda^* \, k\neq 0,q} 2\,\mathfrak{Re}\left\{ e^{ikx} \omega_0(a^*_k a_0) \right\}. \end{aligned} \tag{4.22}$$

The last two sums in (4.22) vanish in the thermodynamic limit by the same argument as for *Model 1*. However

$$\lim_{V\to\infty}\frac{1}{V}\omega_0(a_q^*a_0)=\lim_{V\to\infty}\frac{1}{\sqrt{V}}\omega_0(a_q^*)\lim_{V\to\infty}\frac{1}{\sqrt{V}}\omega_0(a_0)\equiv C\neq 0. \tag{4.23}$$

Therefore, the bosons contained in the two condensates may interfere and by virtue of (4.22) and (4.23) this gives the matter-wave grating formed by the quantum interference of the two coherent states with different momenta:

$$\rho(x)=\rho+(Ce^{iqx}+\overline{C}e^{-iqx}). \tag{4.24}$$

Notice that by (4.23) and by (4.24) there is no matter-wave grating in the Solution 2, when one of the condensates (in this case the *q-condensate*) is empty, see (2.45).

5 Concluding Remarks

We conclude this paper with few remarks concerning the importance of the *matter-wave grating* for the amplification of light and matter waves observed in recent experiments.
It is clear that the absence of the matter-wave grating in *Model 1* and its presence in *Model 2* provides a physical distinction between Raman and Rayleigh superradiance. Note first that matter-wave amplification differs from light amplification in one important aspect: a matter-wave amplifier has to possess a *reservoir* of atoms. In *Models 1* and *2* this is the BE condensate. In both models the superradiant scattering transfers atoms from the condensate at rest to a recoil mode.
The *gain* mechanism for the Raman amplifier is superradiant Raman scattering in a two-level atoms, transferring bosons from the condensate into the recoil state [1]
The Rayleigh amplifier is in a sense even more effective. Since now the atoms in a recoil state interfere with the BE condensate at rest, the system exhibits a space *matter-wave grating* and the quantum-mechanical amplitude of transfer into the recoil state is proportional to the product of the boson occupation numbers $n_0(n_q+1)$ for the wave-vectors $k=0,q$. Each time the momentum imparted by photon scattering is absorbed by the matter-wave grating via the coherent transfer of an atom from the condensate into the recoil mode. Thus, the variance of the grating grows, since the quantum amplitude for scattered atom to be transferred into a recoiled state is increasing [2]-[4], [12]. At the same time the dressing laser beam prepares from the BE condensate a gain medium able to amplify the light. The matter-wave grating diffracts the dressing beam into the path of the probe light resulting in the amplification of the latter [5].

In the case of equilibrium BEC superradiance the amplification of the light and the matter waves manifests itself in *Models 1* and *2* as a *mutual enhancement* of the BEC and the photons condensations, see Solutions 3 in Sections 2 and 3. Note that the corresponding formulæ for condensation densities for *Model 1* (2.60)-(2.62) and for *Model 2* (3.11), (3.12), (3.13) are *identical*. The same is true for the *boson-photon correlations* (*entanglements*) between recoiled bosons and photons, see (2.64), (3.14), as well as between photons and the BE condensate at rest:

$$\lim_{V\to\infty}\frac{1}{V}\langle a_{0-}^*b_q\rangle_{H^{\rm eff}_{1,\Lambda}(\mu,\eta,\rho)}=\frac{1}{V}\langle a_0^*b_q\rangle_{H^{\rm eff}_{2,\Lambda}(\mu,\eta,\rho)}=\frac{4\Omega(\lambda\rho+\epsilon(q)-\mu)\sqrt{\lambda\rho-\mu}}{g^2}, \tag{5.1}$$

and for the *off-diagonal coherence* between recoiled atoms and the condensate at rest:

$$\lim_{V\to\infty}\frac{1}{V}\langle a_{0-}a^*_{q+}\rangle_{H^{\rm eff}_{1,\Lambda}(\mu,\eta,\rho)}=\frac{1}{V}\langle a_0 a^*_q\rangle_{H^{\rm eff}_{2,\Lambda}(\mu,\eta,\rho)}=\frac{4\Omega\sqrt{(\lambda\rho+\epsilon(q)-\mu)(\lambda\rho-\mu)}}{g^2}. \tag{5.2}$$

As we have shown above, the *difference* between *Models 1* and *2* becomes visible only on the level of the wave-grating or spatial modulation of the *particle density* (4.24).

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