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# Note on the generalized Hansen and Laplace coefficients

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**Abstract.** Recently, Breiter *et al.*(2004) report the computation of Hansen coefficients  $X_k^{\gamma,m}$  for non integer values of  $\gamma$ . In fact, the Hansen coefficients are closely related to the Laplace  $b_s^{(m)}$ , and generalized Laplace coefficients  $b_{s,r}^{(m)}$  (Laskar and Robutel, 1995) that do not require  $s, r$  to be integers. In particular, the coefficients  $X_0^{\gamma,m}$  have very simple expressions in terms of the usual Laplace coefficients  $b_{\gamma+2}^{(m)}$ , and all their properties derive easily from the known properties of the Laplace coefficients.

**Keywords:** Keplerian motion, Hansen coefficients, Laplace coefficients, analytical methods

## 1. Introduction

The aim of this note, mostly didactical, is to clarify some simple relations between the Hansen coefficients, and the Laplace, and generalized Laplace coefficients. Once these relations are explicated, the results quoted by Breiter *et al.*(2004) become simple translations of known results on the Laplace coefficients.

## 2. Hansen and Laplace coefficients

The Hansen coefficients (Hansen, 1855) are defined as the Fourier coefficients  $X_k^{\gamma,m}$  of the series

$$\left(\frac{r}{a}\right)^\gamma e^{imv} = \sum_{k=-\infty}^{+\infty} X_k^{\gamma,m} e^{ikM} \quad (1)$$

where  $v, M$  are the true and mean anomaly,  $r, a$  the radial distance and semi-major axis. The transformation  $v \rightarrow -v$  transforms  $M$  in  $-M$ . Thus  $X_k^{\gamma,m}$  is real and  $X_{-k}^{\gamma,-m} = X_k^{\gamma,m}$ . We have

$$X_k^{\gamma,m} = \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{r}{a}\right)^\gamma e^{imv} e^{-ikM} dM. \quad (2)$$

In particular, for  $k = 0$ ,

$$X_0^{\gamma,m} = \frac{1}{\sqrt{1-e^2}} \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{r}{a}\right)^{\gamma+2} e^{imv} dv. \quad (3)$$

Hansen (1855) uses the expressions in term of the true anomaly  $v$

$$\frac{r}{a} = \frac{1-e^2}{1+e\cos v} = \frac{(1-e^2)(1+\beta^2)}{(1+\beta\xi)(1+\beta\xi^{-1})} \quad (4)$$



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where  $\xi = e^{iv}$ , and  $\beta = (1 - \sqrt{1 - e^2})/e$ . We thus obtain immediately the expansion of  $(r/a)^{\gamma+2}$  in Laurent series of  $\xi$

$$\left(\frac{r}{a}\right)^{\gamma+2} = (1 - e^2)^{\gamma+2} (1 + \beta^2)^{\gamma+2} \frac{1}{2} \sum_{k=-\infty}^{+\infty} b_{\gamma+2}^{(k)}(-\beta) \xi^k \quad (5)$$

where  $b_s^k(\alpha)$  are the classical Laplace coefficients defined as the coefficients of the Laurent series

$$(1 - \alpha z)^{-s} (1 - \alpha z^{-1})^{-s} = \frac{1}{2} \sum_{k=-\infty}^{+\infty} b_s^{(k)}(\alpha) z^k, \quad (6)$$

with  $b_s^{(-k)}(\alpha) = b_s^{(k)}(\alpha)$ , and for  $k \geq 0$ ,

$$b_s^{(k)}(\alpha) = \frac{(s)_k}{k!} \alpha^k F(s, s+k, k+1; \alpha^2), \quad (7)$$

where  $(s)_0 = 1$ ,  $(s)_k = s(s+1) \cdots (s+k-1)$  for  $k \geq 1$ . Thus

$$X_0^{\gamma, m} = (1 - e^2)^{\gamma+3/2} (1 + \beta^2)^{\gamma+2} \frac{1}{2} b_{\gamma+2}^{(m)}(-\beta). \quad (8)$$

Any property of the Laplace coefficients can thus be trivially translated into a property on the Hansen coefficients  $X_0^{\gamma, m}$ . In particular, in 1785, Laplace demonstrated the most useful relations

$$\begin{aligned} b_{s+1}^{(j)}(\alpha) &= \frac{(s+j)}{s} \frac{(1+\alpha^2)}{(1-\alpha^2)^2} b_s^{(j)}(\alpha) - \frac{2(j-s+1)}{s} \frac{\alpha}{(1-\alpha^2)^2} b_s^{(j+1)}(\alpha) \\ b_{s+1}^{(j+1)}(\alpha) &= \frac{j}{j-s} \left(\alpha + \frac{1}{\alpha}\right) b_{s+1}^{(j)}(\alpha) - \frac{j+s}{j-s} b_{s+1}^{(j-1)}(\alpha) \end{aligned} \quad (9)$$

that translate immediately as

$$\begin{aligned} X_0^{\gamma, m} &= \frac{\gamma+1+m}{\gamma+1} X_0^{\gamma-1, m} + \frac{m-\gamma}{\gamma+1} e X_0^{\gamma-1, m+1} \\ X_0^{\gamma, m+1} &= -\frac{2}{e} \frac{m}{m-\gamma-1} X_0^{\gamma, m} - \frac{m+\gamma+1}{m-\gamma-1} X_0^{\gamma, m-1}. \end{aligned} \quad (10)$$

As with the Laplace coefficients, these relations then allows to express all coefficients with respect to the two first ones  $X_0^{\gamma, 0}$  and  $X_0^{\gamma, 1}$ . Brieter *et al.* (2004) treat as a special case  $\gamma = (2n+1)/2$ . This is precisely the case of the expansion of the Newtonian potential in Laplace coefficients. In fact, the recurrence formulas of Laplace (10), allows to express all coefficients not with 4 initial coefficients, as quoted in Breiter *et al.*, 2004, but from only two of them (Laplace, 1785), namely

$$X_0^{-3/2, 0} = (1 + \beta^2)^{1/2} \frac{1}{2} b_{1/2}^{(0)}(\beta); \quad X_0^{-3/2, 1} = -(1 + \beta^2)^{1/2} \frac{1}{2} b_{1/2}^{(1)}(\beta), \quad (11)$$

with the expressions in term of the elliptic integrals of first and second kind  $K(\beta), E(\beta)$  (Tisserand, 1889)

$$b_{1/2}^{(0)}(\beta) = \frac{4}{\pi}K(\beta) ; \quad b_{1/2}^{(1)}(\beta) = \frac{4}{\pi\beta}(K(\beta) - E(\beta)) . \quad (12)$$

More generally,  $X_0^{\gamma,m}$  can be expressed with respect to  $X_0^{\gamma-[\gamma],0}$  and  $X_0^{\gamma-[\gamma],1}$ , where  $[\gamma]$  denotes the integer part of  $\gamma$ .

### 3. Hansen coefficients for $k \in \mathbb{Z}$

The expressions of the Hansen coefficients for  $k \neq 0$  are obtained in a similar way using expansions of (2) in eccentric anomaly  $E$  as

$$X_k^{\gamma,m} = \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{r}{a}\right)^{\gamma+1} e^{imv} e^{ike \sin E} e^{-ikE} dE . \quad (13)$$

With

$$\frac{r}{a} = \frac{1}{1+\beta^2}(1-\beta\eta)(1-\beta\eta^{-1}); \quad e^{iv} = \eta \frac{(1-\beta\eta^{-1})}{(1-\beta\eta)} , \quad (14)$$

we have

$$\left(\frac{r}{a}\right)^{\gamma+1} e^{imv} = \frac{\eta^m}{2(1+\beta^2)^{\gamma+1}} \sum_{l=-\infty}^{+\infty} b_{-\gamma-1+m, -\gamma-1-m}^{(l)}(\beta) \eta^l \quad (15)$$

where  $b_{s,r}^{(k)}$  are the generalized Laplace coefficients defined in (Laskar and Robutel, 1995) as the coefficients of the Laurent series

$$(1-\alpha z)^{-s}(1-\alpha z^{-1})^{-r} = \frac{1}{2} \sum_{k=-\infty}^{+\infty} b_{s,r}^{(k)}(\alpha) z^k . \quad (16)$$

We have here  $b_{s,r}^{(-k)} = b_{r,s}^{(k)}$ , and for  $k \geq 0$

$$b_{s,r}^{(k)}(\alpha) = \frac{\binom{s}{k}}{k!} \alpha^k F(r, s+k, k+1; \alpha^2) . \quad (17)$$

The classical expansion in Bessel functions

$$e^{ike \sin E} = \sum_{n=-\infty}^{+\infty} J_n(ke) \eta^n , \quad (18)$$

allows then the computation of the Hansen coefficients  $X_k^{\gamma,m}$  in term of Bessel functions and Laplace coefficients as

$$X_k^{\gamma,m} = \frac{1}{(1+\beta^2)^{\gamma+1}} \sum_{n=-\infty}^{+\infty} b_{-\gamma-1+m, -\gamma-1-m}^{(k-n-m)}(\beta) J_n(ke) . \quad (19)$$

#### 4. Conclusion

The computation of Hansen (1855), or the later computation of Hill (1875) and Tisserand (1889), are very similar to the present presentation, the only novelty here being the use of Laplace, and generalized Laplace coefficients to express the Hansen coefficients in a simple form, that explicits even more the fact that nowhere in the original demonstration of Hansen is requested the fact that  $\gamma$  is an integer. This is particularly visible for the computation of  $X_0^{\gamma,m}$  that are very simple expressions of the usual Laplace coefficients  $b_{\gamma+2}^{(m)}$ .

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