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Structure Formation by Modulational Interaction between Lower-Hybrid Waves and Dispersive Alfvén Waves

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We consider the interaction between finite amplitude lower hybrid (LH) waves and dispersive Alfvén (DA) waves. For this purpose, we derive a set of three-dimensional equations governing nonlinear coupled LH and DA waves that are propagating obliquely to an external magnetic field. The interaction between the two wave modes is due to the Reynolds' stress of the LH waves and the density perturbation associated with the DA wave. Additional terms due to self-nonlinearity of the DA wave are included in order to describe large amplitude DA waves. The governing equations are analyzed for three-wave decay and modulational interaction. We present nonlinear dispersion relations describing the instability of a finite amplitude pump LH wave. In addition, we present a numerical solution of the three-dimensional equations.

I. INTRODUCTION

The modulational interaction between lower-hybrid (LH) waves and dispersive Alfvén (DA) is considered. The DA waves are important in many plasma phenomena in space and laboratory plasmas, see Ref. [1] for a comprehensive review. In the present treatment we derive and analyze a set of three-dimensional equations governing nonlinear coupled LH and DA waves that are propagating obliquely to an external magnetic field. The LH waves are considered in the electrostatic approximation and the present treatment describes waves on the resonance cone waves as well as pressure driven oscillations. The DA wave accompany a quasi-neutral density perturbation and a sheared magnetic field which modulates the LH waves. Further, the ponderomotive force associated with LH waves facilitates interaction between the DA and LH wave. Recently, the problem has attracted attention [2, 3, 4] and it has been shown that nonlinear structures can be formed as a result of the interaction [2, 3]. In addition to the coupling mechanisms presented in Ref. [4] the present set of equations includes the scalar nonlinearity considered in Ref. [3]. We derive a nonlinear dispersion relation which describes the nonlinear generation of DA waves by a large amplitude LH pump wave. The dispersion relation generalizes previous results [2] to include effects originating from the parallel electron kinetics. Further, we present numerical solutions of the three-dimensional equations.

The paper is organized as follows: The basic equations for low and medium β plasma are presented in Section II. In addition, some basic properties of LH wave and DA wave are briefly reviewed. In section III we derive a nonlinear dispersion relations describing excitation of DA waves by a pump LH wave. Analytical results as well as numerical calculations of growth rates are presented. In section IV we present a numerical solution of the three dimensional equations. The paper is concluded by a summary in section V.

II. BASIC EQUATIONS

Before considering the equation governing the nonlinear evolution we shall review some basic properties of LH wave and DA wave. LH wave are electrostatic polarized waves in a magnetized plasma with frequencies much lower than the electron cyclotron frequency ω_{ce} but much larger than the ion cyclotron frequency ω_{ci} , i.e., $\omega_{ci} \ll \omega \ll \omega_{ce}$ where ω is the wave frequency. In the linear approximation the LH wave is characterized by the approximate dispersion relation

$$\omega^2 = \omega_{LH}^2 \left(1 + \rho_T^2 k^2 + \frac{m_i}{m_e} \frac{k_z^2}{k^2} \right), \quad (1)$$

where $\omega_{LH} = \omega_{pi}/(1 + \omega_{pe}^2/\omega_{ce}^2)^{1/2}$ is the LH frequency, k_{\perp} and k_z is the perpendicular and parallel wave number, respectively. In the electrostatic cold plasma approximation the mode belongs to the resonance cone. However, for large k_{\perp} finite Larmor radius effects must be included, giving rise additional perpendicular dispersion. The strength of the finite Larmor radius effects is described by the thermal dispersion length ρ_T in the second term in Eq. (1). For sufficiently large wavelengths, i.e., $\lambda_e k \sim 1$ where $\lambda_e = c/\omega_{pe}$ is the electron skin depth, the LH wave is connected to the electromagnetic R whistler/fast magnetosonic wave mode. This transition is not described by Eq. (1) but it is straight forward to include lowest order electromagnetic effects due to finite electron current parallel to the magnetic field. However, the present treatment of LH wave is restricted to the electrostatic limit $\lambda_e k \gg 1$ in which Eq. (1) provides an appropriate description. The group velocity of the LH wave is

$$\mathbf{v}_g \approx \omega_{LH} \left(\rho_T^2 k_{\perp}^2 - \frac{m_i}{m_e} \frac{k_z^2}{k_{\perp}^2} \right) \frac{\mathbf{k}_{\perp}}{k_{\perp}^2} + \omega_{LH} \frac{m_i}{m_e} \frac{k_z}{k_{\perp}^2} \hat{\mathbf{z}}. \quad (2)$$

The wave is a backward wave in the perpendicular direction when k_{\perp} is sufficiently small, i.e., when $\rho_T^2 k_{\perp}^4 < m_i/m_e k_z^2$.

In a low- β ($\beta \ll 1$) plasma the DA wave is characterized by the dispersion relation

$$\omega^2 = \frac{k_z^2 v_A^2}{1 + k_\perp^2 \lambda_e^2 + k_z^2 \lambda_i^2} (1 + k_\perp^2 \rho_s^2) \quad (3)$$

where $\lambda_i = c/\omega_{pi}$ is the ion skin depth, $\rho_s = (T_e/m_i)^{1/2}/\omega_{ci}$ is the ion gyro radius at the electron temperature T_e . The term $\rho_s^2 k_\perp^2$ originates from the parallel electron kinetics and the $k_z^2 \lambda_i^2$ term is a correction due to finite ω/ω_{ci} . In a medium β ($m_e/m_i \ll \beta \ll 1$) with $k_\perp \lambda_e \ll k_\perp \rho_s \sim 1$ and $k_z \lambda_i \ll 1$ Eq. (3) describes the kinetic Alfvén wave (KAW) with

$$\omega = k_z v_A (1 + k_\perp^2 \rho_s^2)^{1/2}. \quad (4)$$

For an extremely low- β plasma ($\beta \ll m_e/m_i \Leftrightarrow \rho_s \ll \lambda_e$) with $\lambda_i k_z \ll 1$ Eq. (3) describes the dispersive inertial Alfvén wave (DIAW) and the modified convective cell mode. The DIAW is characterized by $k_\perp \rho_s \ll k_\perp \lambda_e \sim 1$ for which the dispersion relation is reduced to

$$\omega = \frac{k_z v_A}{(1 + k_\perp^2 \lambda_e^2)^{1/2}}. \quad (5)$$

For sufficiently large k_\perp , i.e., $k_\perp \lambda_e \gg 1$, the finite electron pressure becomes important and the $k_\perp \rho_s$ term in Eq. (3) must be retained. This electrostatic limit of the DA wave is known as the modified convective cell mode and is described by the approximate dispersion relation

$$\omega = \frac{k_z}{k_\perp} (\omega_{ce} \omega_{ci})^{1/2} (1 + \rho_s^2 k_\perp^2)^{1/2}. \quad (6)$$

Both the DIAW and the convective cell mode are propagating backward in the perpendicular direction.

A. LH wave equation

Here we will derive a equation governing the evolution of the LH wave. The LH wave are coupled to the DA wave through the low frequency fluctuations in density and magnetic field. In addition, the the low frequency fluid motion associated with the DA wave convects the LH perturbations and gives rises to additional nonlinear frequency shifts. The high frequency electric field \mathbf{E}_L which is associated with the LH wave is assumed to be electrostatic, i.e., $\mathbf{E}_L = -\nabla \phi_L$ where ϕ_L is the electrostatic potential. The LH frequency regime is characterized by $\omega_{ci} \ll \omega \ll \omega_{ce}$. Thus, the electron motion perpendicular to \mathbf{B}_0 can be evaluated in the drift approximation while the ions can be considered as unmagnetized. The perpendicular electron velocity $\mathbf{v}_{e\perp}$ is given by

$$\mathbf{v}_{e\perp} \approx \frac{c}{B_0} \hat{\mathbf{z}} \times \nabla_\perp \phi_L + \frac{c}{B_0 \omega_{ce}} \frac{\partial}{\partial t} \nabla_\perp \phi_L - v_{ez} \frac{\hat{\mathbf{z}} \times \nabla A_z}{B_0} - \frac{1}{\omega_{ce}} \hat{\mathbf{z}} \times [(\mathbf{v}_{e\perp} \cdot \nabla) \mathbf{u}_{e\perp} + (\mathbf{u}_{e\perp} \cdot \nabla) \mathbf{v}_{e\perp}], \quad (7)$$

where the first two terms are linear and represents the $\mathbf{E} \times \mathbf{B}$ drift and the polarization drift, respectively. The DA wave accompany a sheared magnetic field $\mathbf{B}_\perp = \nabla A_z \times \hat{\mathbf{z}}$ which gives rise to the third term. The last term is due to convection.

The electron velocity associated with the DA wave, $\mathbf{u}_{e\perp}$, can be evaluated in the lowest order drift approximation, we have that $\mathbf{u}_{e\perp} \approx c/B_0 \hat{\mathbf{z}} \times \nabla_\perp \phi_A$ where ϕ_A is the electrostatic potential associated with the DA wave. The parallel component of the electron velocity is governed by the approximate equation

$$\partial_t v_{ez} = e/m_e \partial_z \phi_L. \quad (8)$$

In the LH frequency domain the ions can be regarded as unmagnetized as $\omega \gg \omega_{ci}$ and the ion velocity is governed by the approximate equation

$$\partial_t \mathbf{v}_i = -e/m_i \nabla \phi_L. \quad (9)$$

Equations (7)–(9) can be used to calculate the high frequency current density $\mathbf{J} = en_0(1+\eta)(\mathbf{v}_i - \mathbf{v}_e) + e(n_i \mathbf{u}_i - n_e \mathbf{u}_e)$ where n_0 is the background plasma density, n_e (n_i) is the high frequency electron (ion) density fluctuation, and $\eta = (c/B_0 \omega_{ci}) \nabla_\perp^2 \phi_A$ is the quasi-neutral density perturbation associated with the DA wave. By substituting Eqs. (7)–(9) into the charge density conservation equation $\partial_t \rho + \nabla \cdot \mathbf{J} = 0$ and the Poisson equation $\nabla^2 \phi_L = -4\pi\rho$, we obtain the wave equation for LH waves

$$\begin{aligned} \mathcal{L}_L \phi_L &= -\omega_{LH}^2 \nabla_\perp \cdot (\eta \nabla_\perp \phi_L) - \omega_{pe}^2 \partial_z (\eta \partial_z \phi_L) \\ &\quad - \frac{\omega_{pe}^2}{\omega_{ce}} \partial_t (\nabla \phi_L \times \nabla \eta) \cdot \hat{\mathbf{z}} - \frac{c}{B_0} \partial_t (\hat{\mathbf{z}} \times \nabla \phi_A \cdot \nabla \nabla_\perp^2 \phi_L) \\ &\quad - \frac{c}{B_0} \frac{\omega_{pe}^2}{\omega_{ce}^2} \partial_t \nabla \cdot (\hat{\mathbf{z}} \times \nabla \phi_A \cdot \nabla \nabla_\perp \phi_L + \hat{\mathbf{z}} \times \nabla \phi_L \cdot \nabla \nabla_\perp \phi_A) \\ &\quad - \frac{e}{m_e} \frac{\omega_{pi}^2}{\omega_{ci}^2} \partial_t \nabla_\perp \cdot [(\partial_t \nabla_\perp \phi_A) \partial_t^{-2} \nabla^2 \phi_L] \\ &\quad + \frac{\omega_{pe}^2}{B_0} \partial_t [\hat{\mathbf{z}} \times \nabla A_z \cdot \nabla \partial_t^{-1} \partial_z \phi_L]. \quad (10) \end{aligned}$$

The operator \mathcal{L}_L describes the linear dispersion of the LH waves and is given by

$$\mathcal{L}_L = \partial_t^2 \nabla_\perp^2 (1 + \omega_{pe}^2/\omega_{ce}^2) + \omega_{pe}^2 \partial_z^2 + \omega_{pi}^2 \nabla^2 + \partial_t^2 [a_1 \nabla_\perp^4 + a_2 \nabla_\perp^2 \partial_z^2 + a_3 \partial_z^4]. \quad (11)$$

In addition to the cold plasma effects described by Eqs. (7)–(9) we have also included corrections due to finite pressure in Eq. (10). The thermal dispersion is described by the coefficients a_1 , a_2 , and a_3 . In the linear approximation we recover the dispersion relation given in Eq (1).

B. DA wave equation

The DA wave accompany a sheared magnetic field \mathbf{B}_\perp as well as a parallel electric field E_z . The electric and magnetic field of the DA wave are given by

$\mathbf{E}_A = -\nabla\phi_A - \hat{\mathbf{z}}c^{-1}\partial_t A_z$ and $\mathbf{B}_\perp = \nabla A_z \times \hat{\mathbf{z}}$, respectively. As the frequency of the considered DA wave is much smaller than ω_{ci} both the electron and ion velocity can be calculated in the drift approximation. The perpendicular electron velocity associated with the DA wave is

$$\mathbf{u}_{e\perp} \approx \frac{c}{B_0}\hat{\mathbf{z}} \times \nabla_\perp \phi_A - \rho_s^2 \frac{c}{B_0}\hat{\mathbf{z}} \times \nabla \nabla_\perp^2 \phi_A - u_{ez} \frac{\hat{\mathbf{z}} \times \nabla A_z}{B_0} - \frac{1}{\omega_{ce}} \hat{\mathbf{z}} \times \langle \mathbf{v}_e \cdot \nabla \mathbf{v}_{e\perp} \rangle, \quad (12)$$

where u_{ez} is the parallel electron velocity and the angular bracket denotes the ensemble average over the LH wave period. The first two terms are linear and are due to the $\mathbf{E} \times \mathbf{B}$ drift and the diamagnetic drift, respectively. The third term is a self nonlinearity due to field line bending associated with the electromagnetic DA wave. The last term arises from the perpendicular Reynolds' stress due to the high frequency LH wave. The electron polarization drift has been neglected in Eq. (12) as the contribution of the ion polarization drift to the current is a factor m_i/m_e times larger. The perpendicular ion velocity is

$$\mathbf{u}_{i\perp} \approx \frac{c}{B_0}\hat{\mathbf{z}} \times \nabla_\perp \phi_A - \frac{c}{B_0\omega_{ci}} \left(\frac{\partial}{\partial t} + \frac{c}{B_0}\hat{\mathbf{z}} \times \nabla_\perp \phi_A \cdot \nabla_\perp \right) \nabla_\perp \phi_A - \frac{1}{\omega_{ci}^2} \left(\frac{\partial}{\partial t} - \omega_{ci}\hat{\mathbf{z}} \times \right) \langle \mathbf{v}_i \cdot \nabla \mathbf{v}_{i\perp} \rangle, \quad (13)$$

where we have assumed that $|\hat{\mathbf{z}} \times \nabla_\perp \phi_A \cdot \nabla_\perp| \gg (B_0/c)v_{iz}\partial/\partial z$. The parallel (to $\hat{\mathbf{z}}$) electron velocity is governed by the equation,

$$\left(\frac{\partial}{\partial t} + \frac{c}{B_0}\hat{\mathbf{z}} \times \nabla_\perp \phi_A \cdot \nabla_\perp \right) u_{ez} = \frac{e}{m_e c} \left[c \frac{\partial \phi_A}{\partial z} + \left(\frac{\partial}{\partial t} + \frac{c}{B_0}\hat{\mathbf{z}} \times \nabla \phi_A \cdot \nabla \right) A_z - \rho_s^2 c \left(\frac{\partial}{\partial z} - \frac{1}{B_0}\hat{\mathbf{z}} \times \nabla A_z \right) \nabla_\perp^2 \phi_A \right] - \langle \mathbf{v}_{e\perp} \cdot \nabla_\perp v_{ez} \rangle - \frac{1}{2} \langle \partial_z v_{ez}^2 \rangle. \quad (14)$$

where the self nonlinearity in the left hand side is due to convection and the nonlinearities in the right hand side is due to the nonlinear $\mathbf{E} \times \mathbf{B}$ drift. Further, a finite parallel electron pressure term has been included in the electron equation. The non linear terms in the right hand side of Eq. (14) can be evaluated using the linearized electron velocity associated with the LH wave. As the time scales of the LH wave and the DA wave are well separated the LH wave potential can be represented as $\phi_L = \bar{\phi}_L \exp(-i\omega_{LH}t) + \text{c.c.}$ where $\bar{\phi}_L = \bar{\phi}_L(\mathbf{r}, t)$ is a temporally slowly varying envelope function. With this representation of ϕ_L the parallel Reynolds' stress in Eq. (14)

can be written as

$$\langle \mathbf{v}_{e\perp} \cdot \nabla_\perp v_{ez} \rangle + \frac{1}{2} \langle \partial_z v_{ez}^2 \rangle \approx i \frac{c}{B_0\omega_L} \frac{e}{m_e} \frac{\partial}{\partial z} [(\nabla_\perp \bar{\phi}_L^* \times \nabla_\perp \bar{\phi}_L) \cdot \hat{\mathbf{z}}] - \frac{e^2}{m_e^2 \omega_{ce}^2} \frac{\partial}{\partial z} |\nabla_\perp \bar{\phi}_L|^2 + \frac{e^2}{m_e^2 \omega_L^2} \frac{\partial}{\partial z} |\partial_z \bar{\phi}_L|^2, \quad (15)$$

where the first term is due to the $\mathbf{E} \times \mathbf{B}$ drift, the second is due to the electron polarization current, and the last term is due to nonlinear parallel electron convection. Similarly, the perpendicular Reynolds' stress in Eq. (13) can be evaluated in terms of ϕ_L .

The parallel ion current is insignificantly small in comparison to the electron current. Substituting Eqs. (14) and (15) into the parallel component of the Ampères's law, i.e., $\nabla_\perp^2 A_z \approx 4\pi en_0 u_{ez}/c$, gives

$$d_t (1 - \lambda_e^2 \nabla_\perp^2) A_z + c \partial_z \phi_A - c \rho_s^2 d_z \nabla_\perp^2 \phi_A = i \frac{c^2}{B_0\omega_L} \frac{\partial}{\partial z} [(\nabla_\perp \bar{\phi}_L^* \times \nabla_\perp \bar{\phi}_L) \cdot \hat{\mathbf{z}}] - \frac{ec}{m_e \omega_{ce}^2} \frac{\partial}{\partial z} |\nabla_\perp \bar{\phi}_L|^2 + \frac{ec}{m_e \omega_L^2} \frac{\partial}{\partial z} |\partial_z \bar{\phi}_L|^2, \quad (16)$$

where the differential operators $d_t = \partial_t + (c/B_0)\hat{\mathbf{z}} \times \nabla \phi_A \cdot \nabla$, $d_z = \partial_z - B_0^{-1}\hat{\mathbf{z}} \times \nabla A_z \cdot \nabla$ includes the self nonlinearities. The continuity equation in conjunction with the quasi neutrality condition and the parallel component of the Ampères's law gives

$$d_t \nabla_\perp^2 \phi_A + \frac{v_A^2}{c} d_z \nabla_\perp^2 A_z = - \frac{e}{m_i \omega_L^2} \partial_t \nabla_\perp^2 |\nabla \bar{\phi}_L|^2 + \frac{cm_e}{B_0 m_i} (\hat{\mathbf{z}} \times \nabla \bar{\phi}_L) \cdot \nabla \nabla_\perp^2 \bar{\phi}_L^* + \text{c.c.} \quad (17)$$

Equations (10), (16), and (17) are the desired equations for investigating parametric excitation of DA waves by large amplitude LH waves. This set of equations can be analyzed analytically as well as numerically in order to investigate three-wave decay and modulational interaction. In addition, the self nonlinearities in the DA equations allows studies of interaction between LH waves and Alfvénic vortices.

III. NONLINEAR DISPERSION RELATION

In this section we will consider the stability of a pump LH wave in a homogeneous plasma. For small amplitudes of the DA wave the self nonlinearities can be neglected, i.e., $d_t \approx \partial_t$ and $d_z \approx \partial_z$. Equations (16) and (17) can be combined to eliminate A_z and to obtain one single equation governing ϕ_A . In the resulting equation the nonlinearities originating from the right hand side of (17) are proportional to ∂_t while the nonlinearities originating from (16) does not depend on temporal derivatives.

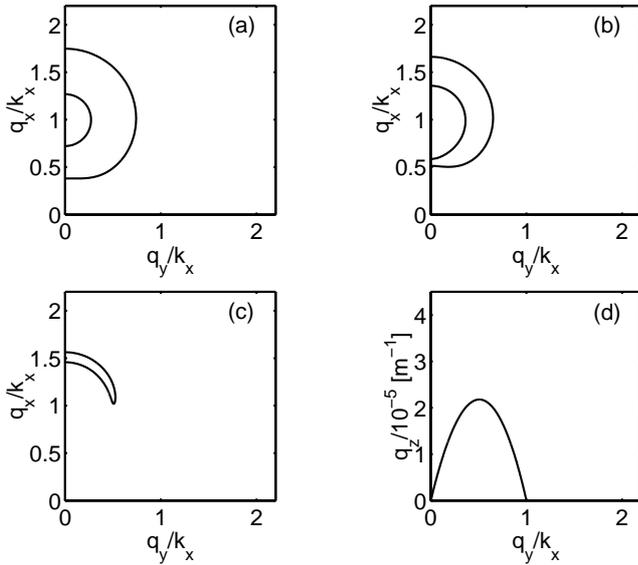


FIG. 1: Projections of the resonance surface $\omega_{\mathbf{k}} = \omega_{\mathbf{k}-\mathbf{q}} + \Omega_A$. Panel (a)–(c) shows the projection of the resonance surface in the q_x – q_y plane for $q_z = 1.7 \times 10^{-5} \text{ m}^{-1}$, $q_z = 2.0 \times 10^{-5} \text{ m}^{-1}$, and $q_z = 2.2 \times 10^{-5} \text{ m}^{-1}$, respectively. Panel (d) shows the projection on the q_y – q_z plane for $q_x = k_x$.

Thus, for sufficiently slow variations the main nonlinearities arises from the parallel Reynold's stress. Further, if the length scales of the DA and LH wave is not extremely well separated [3] the first term in the right hand side of Eq. (16) is dominating the coupling scenario. We obtain a simplified wave equation governing the DA wave,

$$\mathcal{L}_A \phi_A = -i \frac{cv_A^2}{B_0 \omega_{LH}} \partial_z^2 (\nabla_{\perp} \bar{\phi}_L^* \times \nabla_{\perp} \bar{\phi}_L) \cdot \hat{z}, \quad (18)$$

where $\mathcal{L}_A = \partial_t^2 (1 - \lambda_e^2 \nabla_{\perp}^2) - v_A^2 \partial_z^2 (1 - \rho_s^2 \nabla_{\perp}^2)$. In the linear approximation we immediately obtain the DAW dispersion relation Eq. (3) from Eq. (18). Similarly, for $k_{\perp} L \ll \omega_{LH} / \omega_{ci}$ and $k_{\perp} L \ll \omega_{pi}^2 / \omega_{LH} \omega_{ci}$, where L is the length scale of the DA perturbation, Eq. (10) can be reduced to

$$\mathcal{L}_L \phi_L = -\frac{\omega_{pe}^2}{B_0 \omega_{ce}} \partial_t (\nabla \phi_L \times \nabla \eta) \cdot \hat{z}. \quad (19)$$

To investigate the parametric instability of a pump LH we write electrostatic potential of the excited DAW as $\phi_A = a \exp[i(\mathbf{q} \cdot \mathbf{x} - \Omega t)] + \text{c.c.}$ and decompose the LH wave as

$$\begin{aligned} \phi_L = & \phi_0 e^{i(\mathbf{k} \cdot \mathbf{x} - \omega_{\mathbf{k}} t)} + \phi_+ e^{i[(\mathbf{k} + \mathbf{q}) \cdot \mathbf{x} - (\omega_{\mathbf{k}} + \Omega) t]} \\ & + \phi_- e^{i[(\mathbf{k} - \mathbf{q}) \cdot \mathbf{x} - (\omega_{\mathbf{k}} - \Omega) t]} + \text{c.c.}, \quad (20) \end{aligned}$$

where $\omega_{\mathbf{k}}$ is the frequency of the LH pump wave and ϕ_0 (ϕ_{\pm}) is the electrostatic potential of the LH wave pump (sidebands). The amplitudes of the up shifted and down

shifted satellites can be calculated from Eq. (19), we have that

$$D_+ \phi_+ = i \omega_L \frac{c}{B_0} \frac{\omega_{pi}^2}{\omega_{ci}^2} q_{\perp}^2 (\mathbf{k}_{\perp} \times \mathbf{q}_{\perp})_z a \phi_0, \quad (21)$$

$$D_- \phi_-^* = i \omega_L \frac{c}{B_0} \frac{\omega_{pi}^2}{\omega_{ci}^2} q_{\perp}^2 (\mathbf{k}_{\perp} \times \mathbf{q}_{\perp})_z a \phi_0^*, \quad (22)$$

where D_{\pm} is the Fourier transform of the operator \mathcal{L}_L evaluated in $\mathbf{k} \pm \mathbf{q}$, $\omega_{\mathbf{k}} \pm \Omega$. Approximately,

$$D_{\pm} \approx -2\omega_{LH} \left(1 + \frac{\omega_{pe}^2}{\omega_{ce}^2} \right) (\mathbf{k}_{\perp} \pm \mathbf{q}_{\perp})^2 [\mp \Omega + \omega_{\mathbf{k} \pm \mathbf{q}} - \omega_{\mathbf{k}}], \quad (23)$$

where $\omega_{\mathbf{k} \pm \mathbf{q}}$ are the frequencies of the side bands. The amplitude of the DA wave is given by

$$D_A a = i \frac{c}{B_0 \omega_L} v_A^2 q_z^2 (\mathbf{k}_{\perp} \times \mathbf{q}_{\perp})_z (\phi_+ \phi_0^* + \phi_-^* \phi_0), \quad (24)$$

where $D_A = -\Omega^2 (1 + \lambda_e^2 q_{\perp}^2) + v_A^2 q_z^2 (1 + \rho_s^2 q_{\perp}^2)$. By combining Eqs. (21), (22), and (24) we obtain a nonlinear dispersion relation of the form

$$\begin{aligned} 1 + \frac{\omega_{LH}}{8} \frac{\Omega_A^2}{\Omega^2 - \Omega_A^2} \frac{\lambda_e^2 q_{\perp}^2}{1 + \rho_s^2 q_{\perp}^2} \frac{E_0^2}{E_{TH}^2} \frac{(\mathbf{k} \times \mathbf{q})_z^2}{k^2 q^2} \\ \times \left[\frac{\alpha_+}{-\Omega + \omega_{\mathbf{k} + \mathbf{q}} - \omega_{\mathbf{k}}} + \frac{\alpha_-}{\Omega + \omega_{\mathbf{k} - \mathbf{q}} - \omega_{\mathbf{k}}} \right] = 0, \quad (25) \end{aligned}$$

where $E_{TH} = \omega_{ce} m_e B_0 / (\omega_{pe} m_i)$ is a characteristic field strength of the interaction and $\alpha_{\pm} = q^2 / (\mathbf{k} \pm \mathbf{q})^2$. Equation (25) generalizes the dispersion relation obtained in Ref. [2] by including finite electron temperature effects. In the limit $\rho_s q_{\perp} \approx 0$ we recover the previous result. The finite T_e effect included in Eq. (25) is necessary in order to describe excitation of KAW in a medium β plasma. Furthermore, this correction is also important in the short wave limit $q_{\perp} \lambda_e \gg 1$ in extremely low β plasma where Eq. (25) describes excitation of convective cell modes.

The excitation of DA waves via a modulational instability of the pump LH wave is investigated by considering the limit $k \gg q \sim \lambda_e^{-1}$ and $\Omega \ll \Omega_A$. Equation (25) is reduced to the approximate dispersion relation

$$(\Omega - \mathbf{v}_g \cdot \mathbf{q})^2 = \delta^2 \left[1 - \frac{\omega_{LH}}{4\delta} \frac{\lambda_e^2 q_{\perp}^4}{k^2} \frac{E_0^2}{E_{TH}^2} \sin^2 \alpha \right], \quad (26)$$

where \mathbf{v}_g is the group velocity of the pump wave,

$$\begin{aligned} \delta \approx & \frac{\omega_{LH}}{2} \left[\rho_T^2 k^2 + \frac{m_i}{m_e} \frac{k_z^2}{k^2} (4 \cos^2 \alpha - 1) \right] \frac{q^2}{k^2} \\ & - 2\omega_{LH} \frac{m_i}{m_e} \frac{k_z q_z k q \cos \alpha}{k^4} + \frac{\omega_{LH}}{2} \frac{m_i}{m_e} \frac{q_z^2}{k^2} \quad (27) \end{aligned}$$

is a small frequency shift arising from the nonlinear interaction, and $\cos \alpha = \mathbf{k}_{\perp} \cdot \mathbf{q}_{\perp} / (k_{\perp} q_{\perp})$. As δ is quadratic in q_i it follows from the solution of Eq. (26) that the pump wave is unstable when $E_0 > E_{\text{mod}} \propto q^{-1}$. Thus, the

modulational instability can not, according to this condition, excite long wave length DA wave as the threshold goes as q^{-1} .

For $q \sim k \gg \lambda_e^{-1}$ Eq. (25) describes three-wave decay where the LH pump decays into a down shifted LH sideband and a modified convective cell wave. For small amplitudes the up shifted sideband is not strongly excited and we can omit the the first term in the square bracket in Eq. (25). We obtain

$$(\Omega^2 - \Omega_A^2)(\Omega + \omega_{\mathbf{k}-\mathbf{q}} - \omega_{\mathbf{k}}) + \frac{\omega_{LH}}{8} \frac{\lambda_e^2 q_{\perp}^2}{1 + \rho_s^2 q_{\perp}^2} \frac{E_0^2}{E_{TH}^2} \frac{(\mathbf{k} \times \mathbf{q})_z^2}{k^2 q^2} \frac{\Omega_A^2 q^2}{(\mathbf{k} - \mathbf{q})^2} = 0, \quad (28)$$

which describes a three-wave decay instability near the resonance surface $\omega_{\mathbf{k}} = \omega_{\mathbf{k}-\mathbf{q}} + \Omega_A$. The growth rate is

$$\gamma = \left[\frac{v_A q_z \omega_{LH}}{8} \frac{\lambda_e q_{\perp}}{\sqrt{1 + \rho_s^2 q_{\perp}^2}} \frac{E_0^2}{E_{TH}^2} \sin^2 \alpha \frac{q^2}{(\mathbf{k} - \mathbf{q})^2} \right]^{1/2}. \quad (29)$$

The decay spectrum has a fairly complicated structure as the negative perpendicular dispersion of the LH wave allows several intersections between the surfaces $\omega_{\mathbf{k}} = \omega_{\mathbf{k}-\mathbf{q}} + \Omega$ and $\Omega = \Omega_A$. The main features of the resonance surface is illustrated in Fig. 1 for $\mathbf{k} = k_x \hat{\mathbf{x}}$ with $k_x = 2\pi/20 \text{ m}^{-1}$. The plasma parameters used in Fig. 1 are $n_0 = 900 \text{ cm}^3$, $B_0 = 25 \text{ } \mu\text{T}$, $m_i/m_e = 1600$, $T_e = 3000 \text{ K}$, and $T_i = 2400 \text{ K}$. The considered plasma parameters are appropriate for the Earth's upper ionosphere. The plasma is an extremely low β plasma with $\beta \approx 1.6 \times 10^{-6}$, $\lambda_e \approx 180 \text{ m}$, $\lambda_i = c/\omega_{pi} \approx 30 \text{ km}$, and $\rho_s \approx 8 \text{ m}$. Figure 2 shows the growth rate of the three-wave decay for a moderate amplitude ($E_0 = 0.5 \text{ mV/m}$) pump LH wave. The growth rate $\gamma = \text{Im}(\Omega)$ was obtained by solving the nonlinear dispersion relation Eq. (25) numerically. At this amplitude the instability occurs close to the resonance surfaces shown in Fig. 1. The multiple local maximas are attributed to the negative perpendicular LH dispersion. Two local maximas in γ is seen in panel (a). For larger q_z the regions of instability are merging. The maximum γ is at $q_x = k_x$, $q_y = k_x/2$, and $q_z \approx 2 \times 10^{-5} \text{ m}^{-1}$.

For larger amplitudes of the pump wave both the up shifted and down shifted sidebands are excited. For $\mathbf{q}_{\perp} = q_y \hat{\mathbf{y}} \perp \mathbf{k} = k_x \hat{\mathbf{x}}$ Eq. (25) can be solved analytically, we have that

$$\Omega^2 = \frac{\Omega_A^2 + \delta_{\pm}^2}{2} \pm \left[\left(\frac{\Omega_A^2 - \delta_{\pm}^2}{2} \right)^2 + \Omega_A^2 \Gamma \right]^{1/2}, \quad (30)$$

where

$$\Gamma = \frac{\omega_{LH}^2}{8} \frac{\lambda_e^2 q_{\perp}^2}{1 + \rho_s^2 q_{\perp}^2} \frac{E_0^2}{E_{TH}^2} \frac{q^2}{k^2 + q^2} \left(\rho_T^2 q_{\perp}^2 + \frac{m_i}{m_e} \frac{q_z^2}{q_{\perp}^2} \right), \quad (31)$$

and $\delta_{\pm} = \omega_{LH}/2(\rho_T^2 q_{\perp}^2 + m_i/m_e q_z^2/q_{\perp}^2)$. As seen from the solution in Eq. (30), for sufficiently large E_0 there

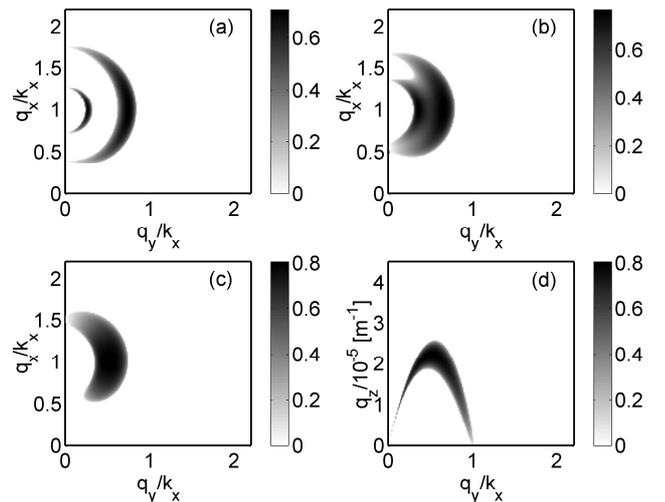


FIG. 2: Three-wave decay spectrum obtained from Eq. (25) for a moderate amplitude ($E_0 = 0.5 \text{ mV/m}$) pump LH wave. The amplitude of the growth rate is color coded. Panel (a)–(c) shows the decay spectrum for $q_z = \text{const.}$, where $q_z = 1.7 \times 10^{-5} \text{ m}^{-1}$, $q_z = 2.0 \times 10^{-5} \text{ m}^{-1}$, and $q_z = 2.2 \times 10^{-5} \text{ m}^{-1}$, respectively. Panel (d) shows the decay spectrum for $q_x = k_x$.

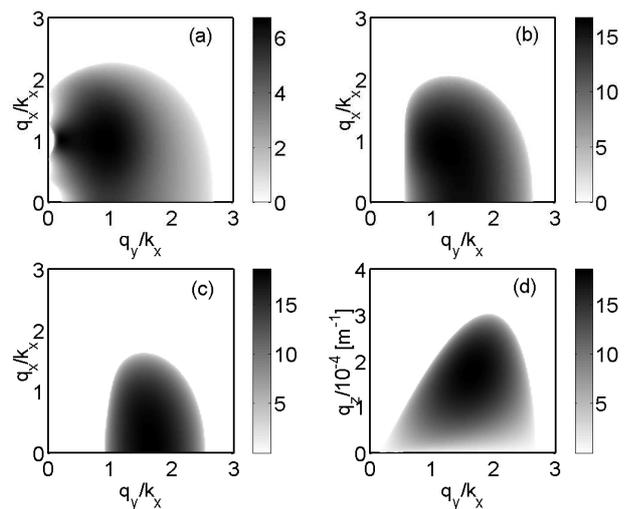


FIG. 3: Decay spectrum obtained from Eq. (25) for a large amplitude ($E_0 = 6 \text{ mV/m}$) pump LH wave. The amplitude of the growth rate is color coded. Panel (a)–(c) shows the decay spectrum for $q_z = \text{const.}$, where $q_z = 1.7 \times 10^{-5} \text{ m}^{-1}$, $q_z = 1.0 \times 10^{-4} \text{ m}^{-1}$, and $q_z = 1.8 \times 10^{-4} \text{ m}^{-1}$, respectively. Panel (d) shows the decay spectrum for $q_x = 0$.

is a purely growing instability for a limited range of q_y . The instability can be characterized as a modified decay instability with $|\Omega| > \Omega_A$. Figure 3 shows the growth rate of the instability for a large amplitude ($E_0 = 6 \text{ mV/m}$) pump LH wave. The growth rate $\gamma = \text{Im}(\Omega)$ was obtained by solving the nonlinear dispersion relation Eq. (25) numerically. The purely growing

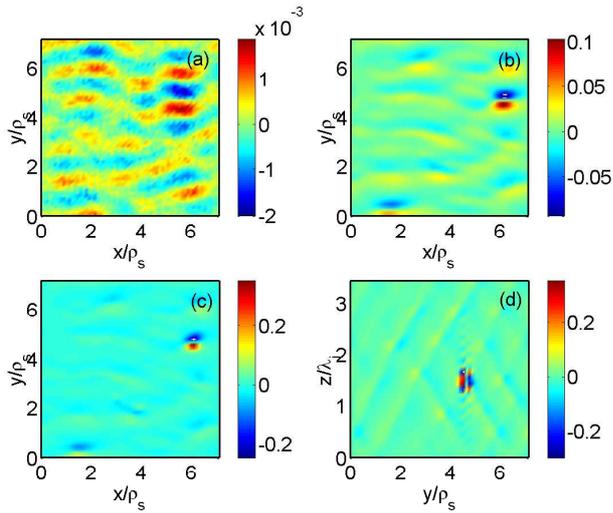


FIG. 4: Numerical solution of Eqs. (18) and (32). Panel (a), (b), and (c) shows $\eta/100$ for $z = \text{const.}$ at $\omega_{LH}t = 2.4 \times 10^3$, $\omega_{LH}t = 4.2 \times 10^3$, and $\omega_{LH}t = 4.4 \times 10^3$, respectively. Panel (d) shows the same quantity for $x/\rho_s = 6$.

instability described by Eq. (30) is illustrated in panel (d).

IV. NUMERICAL SOLUTION

We have solved the simplified set of equations (18) and (19) numerically. As the time scale of the the LH and DA waves are well separated we can use the WKB representation $\phi_L = \bar{\phi}_L(\mathbf{x}, t) \exp(-i\omega_{LH}t) + \text{c.c.}$. With this representation Eq. (19) can be reduced to an equation describing the envelop function $\bar{\phi}_L$. We have that

$$\bar{\mathcal{L}}_L \bar{\phi}_L = i \frac{\omega_{LH}}{\omega_{ci}} (\nabla \phi_L \times \nabla \eta)_z, \quad (32)$$

where $\bar{\mathcal{L}}_L = -2i/\omega_{LH} \partial_t \nabla^2 - \rho_T^2 \nabla_{\perp}^4 + m_i/m_e \partial_z^2$. A pseudo-spectral method was used to approximate the spatial derivatives and the solution was advanced in time using a standard fourth order Runge-Kutta method. A monochromatic LH wave with $\mathbf{k} = 2\pi/20 \hat{\mathbf{x}} \text{ m}^{-1}$ and amplitude E_0 was given as initial condition. For the DA wave, low amplitude noise was given as initial condition. By considering the two cases $E_0 = 0.5 \text{ mV/m}$ and

$E_0 = 6 \text{ mV/m}$ we have investigated the nonlinear evolution of the three-wave decay instability and the modified decay instability.

Low amplitude LH pump $E_0 = 0.5 \text{ mV/m}$: The numerical solution exhibits the three-wave decay instability illustrated in Fig. 1. The growth rate of the fastest growing mode is in agreement with the growth rate shown in Fig. 1. Due to depletion of the pump wave the instability cease to grow at $\Omega_{At} \approx 30$ ($t \approx 6 \text{ s}$). No further instabilities were observed and the simulation was interrupted at $\Omega_{At} \approx 60$.

Large amplitude LH pump $E_0 = 6 \text{ mV/m}$: The initial stage is dominated by the modified decay instability and the growth rate of the fastest growing mode agrees with the growth rate shown in Fig. 2. Figure 3 shows the result of a simulation. Panel (a)–(c) shows the time evolution of η in the x – y plane for a fix z . It is clearly seen in panel (a) that DA waves with $\mathbf{q}_{\perp} \perp \mathbf{k} \parallel \hat{\mathbf{x}}$ are excited by the purely growing instability. At $\omega_{LH}t = 4.2 \times 10^3$ the amplitude of η has grown to $\sim 0.1\%$ and regions with relatively large η and large electric field has been formed, see panel (b) and (c). Panel (d) shows the structure of the density fluctuation along \mathbf{B}_0 in the y – z plane for $x/\rho_s = 6$. For larger t the amplitude of the density fluctuations continue to grow and the spatial size decreases.

V. SUMMARY

In the present paper we have considered the nonlinear interaction between LH and DA waves. We have derived a set of equations describing parametric interaction between the two wave modes. The governing equation has been analyzed analytically and numerically. It is demonstrated by solving the governing equations numerically that small scaled structures, i.e., smaller than the wave length of the pump wave, can be generated.

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- [1] K. Stasiewicz, P. Bellan, C. Chaston, C. Kletzing, R. Lysak, J. Maggs, O. Pokhotelov, C. Seyler, P. Shukla, L. Stenflo, et al., *Space Sci. Rev.* **92**, 423 (2000).
 [2] V. D. Shapiro, *Phys. Rev. Lett.* **81**, 3415 (1998).
 [3] V. D. S. Ücer, *Phys. Lett. A* **328**, 196 (2004).

- [4] P. K. Shukla and L. Stenflo, *Nonlinear Phenomena Involving Dispersive Alfvén Waves* (Springer Verlag, Berlin, 1999), pp. 1–30.