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A Dicke Type Model for Equilibrium BEC Superradiance

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Abstract

We study the effect of electromagnetic radiation on the condensate of a Bose gas. In an earlier paper we considered the problem for two simple models showing the cooperative effect between Bose-Einstein condensation and superradiance. In this paper we formalise the model suggested by Ketterle et al in which the Bose condensate particles have a two level structure. We present a soluble microscopic Dicke type model describing a thermodynamically stable system. We find the equilibrium states of the system and compute the thermodynamic functions giving explicit formulæ expressing the cooperative effect between Bose-Einstein condensation and superradiance.

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1 Introduction

The present paper is motivated by the recent experiments exhibiting a special coherent interaction between matter and light, which has been nicknamed "four-wave mixing" [1]. In these experiments boson atoms with an internal structure, condensed in a trap, are irradiated with light produced by an external laser beam. The structure of the atoms is usually represented by considering them as having two levels [2, 3]. A system of two level-atoms interacting with light is very reminiscent of the Dicke model [4]. Moreover an important feature of this model, namely superradiance has been observed in these experiments, where it is found that there is an enhancement of both Bose-Einstein Condensation (BEC) and light radiation (superradiance) due to the interaction.

Recently various models [1] - [3] for this BEC-superradiance coupling were constructed and discussed in order to describe both equilibrium and non-equilibrium superradiance by condensed atoms. It is interesting to note that as early as 1978 Girardeau [5] had already anticipated this phenomenon in the context of superfluid helium and had discussed the possible impact of the equilibrium superradiance on the thermodynamic properties of the latter. In our recent letter [6] we have considered two simple systems by which we modelled the coherent behaviour of the BEC atoms irradiated by a laser beam, showing rigorously that a weakened form of the "four-wave mixing" interaction enhances the superradiance and BEC as proposed by Ketterle et al [1] - [3].

The aim of the present paper is to consider a model which takes explicitly into account the *internal structure* of the boson atoms. In fact we assume that our bosons have an internal two-level structure of the type described by $SU(2)$-spin symmetry. Therefore the one-particle wave functions are of the form $\psi \otimes s$ where $\psi \in L^2(\mathbb{R}^\nu)$ describing the spacial localization and $s \in \mathbb{C}^2$ describing the internal (spin) state. Only the condensate particles, i.e., the particles in the ground state are supposed to interact with the external field, and therefore only the ground state boson particles are given a different ground state energy parameterized by a separation level parameter ε. If ε is put equal to zero, it is as if we have just two different types of boson particles. The interaction turns out to be a second quantized version of the well known Dicke maser model. In our model we suppose that the *recoil* of the particles is negligible. The model is in fact a realization of the physical mechanism explained in [2]. For our model we study the equilibrium states in the infinite volume limit (thermodynamic limit) and compute the corresponding thermodynamic functions. We examine the presence of cooperation between the BEC condensate and superradiance as a function of the separation level parameter ε. The existence of this phenomenon confirms the results obtained in [6] for a simpler model. It can be seen explicitly from the expressions for the occupation densities for the bosons and photons. Our results predict that with conventional BEC one obtains the same phenomenon of BEC-superradiance cooperation as is observed for trap experiments.

We note that experimentally one can observe the photon recoil effect which, on light atoms, can be non-negligible [1]. However in the present paper we consider the case when the photon momentum is very small so that the recoil effect can still be neglected. In a later publication we shall study another model in which the influence of recoil is included.

2 The model and its equilibrium states

We consider a system of *two types* of bosons of mass m enclosed in a cubic box Λ in ν dimensions ($\Lambda \subset \mathbb{R}^\nu$) with volume V, centered at the origin.

As usual let $\Lambda^* = \{2\pi k/V^{1/\nu} | k \in \mathbb{Z}^\nu\}$ be the dual space of Λ used to formulate the model with periodic boundary conditions. For $k \in \Lambda^*$, $\sigma = \pm$, $a^*_{k,\sigma}$ and $a_{k,\sigma}$ are the usual boson creation and annihilation operators of the two types of bosons satisfying the commutation relations:

$$[a_{k,\sigma}, a^*_{k',\sigma'}] = \delta_{k,k'}\delta_{\sigma,\sigma'}. \tag{2.1}$$

The kinetic energy of the system is given by

$$T_\Lambda = \sum_{\sigma=\pm} \sum_{k\in\Lambda^*,\, k\neq 0} \epsilon(k) a^*_{k,\sigma} a_{k,\sigma} + \varepsilon(a^*_{0,+}a_{0,+} - a^*_{0,-}a_{0,-}), \tag{2.2}$$

where $\varepsilon \geq 0$ and $\epsilon(k) = \|k\|^2/2m$. Note that the two $k = 0$ mode bosons (the ground state for non-interacting bosons) have a supplementary internal energy, a spin-state energy, making the internal structure of the bosons explicit. On the other hand the excited bosons $k \neq 0$ are not distinguished by their internal energy, but it is straightforward to make them also distinguished. The reader will able to see that that our arguments cover also the situation, when the single particle boson spectrum is presented by *two* branches: $\epsilon_\sigma(k) := \epsilon(k) + \sigma\,\varepsilon$ for two internal states os bosons.

We represent the external one mode laser field by a single mode boson field with creation and annihilation operators b, b^* satisfying $[b, b^*] = 1$. As we indicated in the introduction here we consider the case, when the photon momentum is very small so that the recoil effect is *negligible*. In this approximation we can take $k = 0$ and then

$$b = \frac{1}{\sqrt{V}} \int_\Lambda dx\, b(x), \tag{2.3}$$

where $b(x), x \in \mathbb{R}^\nu$ stands for the local (annihilation) photon field. As suggested in the introduction we define our model Hamiltonian as

$$H_\Lambda = T_\Lambda + U_\Lambda \tag{2.4}$$

where

$$U_\Lambda = \frac{g}{2\sqrt{V}}(a^*_{0+}a_{0-}b + a_{0+}a^*_{0-}b^*) + \Omega\, b^*b + \frac{\lambda}{2V}N_\Lambda^2 \tag{2.5}$$

and

$$N_\Lambda = \sum_{k\in\Lambda^*} N_k\,,\ \ N_k = (N_{k,+} + N_{k,-})\,,\ \ N_{k,\sigma} = a^*_{k,\sigma}a_{k,\sigma}$$

are respectively total boson number operator, the k-boson number operator and the boson number operator for momentum k and type σ.

Furthermore $\Omega > 0$ is the laser frequency and g is the coupling constant of the interaction between the bosons and the external field. Note that without loss of generality we can take g to be positive as we can always incorporate the argument of g into b by a gauge transformation.

Notice in (2.5) the presence of the mean-field repulsive particle interaction with a positive coupling constant $\lambda > 0$. This term is essential in order to obtain a model describing a thermodynamically stable system, i.e. ensuring the right thermodynamic behaviour. Indeed one can check by considering the interaction U_Λ in (2.5), that

$$\begin{aligned} U_\Lambda &= \Omega\,(b^* + \frac{g}{2\Omega\sqrt{V}} a_{0+}a_{0-}^*)(b + \frac{g}{2\Omega\sqrt{V}} a_{0+}^* a_{0-}) - \frac{g^2}{4\Omega V} N_{0-}(N_{0+}+1) + \frac{\lambda}{2V} N_\Lambda^2 \\ &\geq \frac{\lambda}{2V} N_\Lambda^2 - \frac{g^2}{4\Omega V} N_{0-}(N_{0+}+1). \end{aligned} \tag{2.6}$$

On the basis of the trivial inequality $4ab \leq (a+b)^2$, the lower bound of (2.6) is bounded from below, if $\lambda > g^2/8\Omega$, that is if the stabilizing repulsive interaction coupling constant λ is large with respect to the coupling constant g or if the laser frequency Ω is large enough. Therefore we assume that $\lambda > g^2/8\Omega$ is satisfied for the model (2.4). The reader will see all along in the explicit analysis of the model below, the importance of this stabilizing condition. We note that, so far, neither the coherent recoil model, nor the "four-wave mixing" model nor Girardeau's model are thermodynamically stable, although Girardeau in [5] has stressed the importance of this stabilization. The models in [6] are stable because of the linearity of the interaction.

In the present paper we study the equilibrium states of the model (2.4) in the grand-canonical ensemble and therefore we shall work with the Hamiltonian

$$H_\Lambda(\mu) = H_\Lambda - \mu N_\Lambda \tag{2.7}$$

where μ is the chemical potential. Specifically our objective is to identify the infinite volume equilibrium states corresponding to the Hamiltonian (2.7) for a system of three different types of bosons. One way of achieving this goal is through the basic variational principle of statistical mechanics. Before starting to do this we prefer to reformulate the model with the purpose of showing that our model (2.4) is nothing but a second quantized bosonic form of the Dicke model and hence it realizes the ideas proposed in [2] and [3].

We have a system of atoms with internal states $\sigma = \pm$. Dicke regarded the two-level atom as a spin-1/2 system. This is what we shall also do and therefore we start from a two-dimensional representation of the Pauli matrices generating the Lie algebra of $SU(2)$, given by

$$\sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.8}$$

and the basis vectors $\{e_+ = (1,0),\, e_- = (0,1)\}$ of $\mathbb{C}^2$ diagonalizing σ^3.

The one-particle space of bosons is $\mathcal{H} = L^2(\mathbb{R}^\nu) \otimes \mathbb{C}^2$. Let $f \otimes s$ be an element of $\mathcal{H}$, then $a^*(f \otimes s)$ is the creation operator of a boson particle with state vector $f \otimes s$. In particular one can make the following identifications.

$$a_{k,\pm}^* = a^*(f_k \otimes e_\pm) \tag{2.9}$$

where for $k \in \Lambda^*$, f_k is the plane wave function

$$f_k(x) = \frac{1}{\sqrt{V}}\, e^{ik\cdot x}, \quad x \in \mathbb{R}^\nu. \tag{2.10}$$

In particular we have

$$a^*_{0,\pm} = a^*(f_0 \otimes e_\pm). \tag{2.11}$$

For any $\phi \in \mathcal{H}$, the creation and annihilation operators $a^*(\phi)$ and $a(\phi)$ are linearly defined on arbitrary n-particle subspaces of Fock space $\mathcal{F}(\mathcal{H})$:

$$a^*(\phi)\mathbf{sym}(\phi_1 \otimes \phi_2 \otimes \ldots \otimes \phi_n) = (n+1)^{\frac{1}{2}}\mathbf{sym}(\phi \otimes \phi_1 \otimes \ldots \otimes \phi_n) \tag{2.12}$$

and

$$a(\phi)\mathbf{sym}(\phi_1 \otimes \phi_2 \otimes \ldots \otimes \phi_n) = n^{-\frac{1}{2}} \sum_{r=1}^{n} \langle \phi, \phi_r \rangle_{\mathcal{H}}\ \mathbf{sym}(\phi_1 \otimes \ldots \otimes \widehat{\phi_r} \otimes \ldots \otimes \phi_n), \tag{2.13}$$

where **sym** denotes symmetrization, $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ is the scalar product in $\mathcal{H}$, and $\widehat{\phi_r}$ means that ϕ_r is omitted.

Applying these definitions for $a^{\#}_{0,\pm}$ on the n-particle $k=0$ mode states and using the identity $\sigma^+ s = \langle e_-, s \rangle_{\mathbb{C}^2}\ e_+$ we obtain

$$\begin{aligned} a^*_{0+}a_{0-}\ &\mathbf{sym}((f_0 \otimes s_1) \otimes (f_0 \otimes s_2) \otimes \ldots \otimes (f_0 \otimes s_n)) \\ &= \sum_{r=1}^{n} \sigma^+_r \mathbf{sym}((f_0 \otimes s_1) \otimes (f_0 \otimes s_2) \otimes \ldots \otimes (f_0 \otimes s_n)) \end{aligned} \tag{2.14}$$

where

$$\sigma^+_r(f_0 \otimes s_1) \otimes (f_0 \otimes s_2) \otimes \ldots \otimes (f_0 \otimes s_n) = (f_0 \otimes s_1) \otimes (f_0 \otimes s_2) \otimes \ldots \otimes (f_0 \otimes \sigma^+ s_r) \otimes \ldots \otimes (f_0 \otimes s_n). \tag{2.15}$$

The $k=0$ mode kinetic energy term can be treated similarly. Hence, on the n-particle $k=0$ mode states the sum of the $k=0$ kinetic-energy term (2.2) and the interaction term with the laser field (2.5) takes the form

$$\varepsilon \sum_{i=1}^{n} \sigma^3_i + \frac{g}{2\sqrt{V}} \sum_{i=1}^{n} (\sigma^+_i b + \sigma^-_i b^*) \tag{2.16}$$

which *coincides* with the Dicke maser model. This proves that the model (2.4) (or (2.7)) realizes the suggestions of [3], namely that it is nothing but a second quantized bosonized form of the Dicke maser model.

So far we have discussed the structure of our model. The rest of the section is devoted to the technical preparation of the basic variational principle of statistical mechanics applied to our model (2.7).

The variational principle states that if $\mathcal{S}$ is the set of the extremal translation invariant states and f is the free energy density defined on $\mathcal{S}$ by

$$f(\omega) = \lim_{V \to \infty} \omega(H_\Lambda(\mu)/V) - (1/\beta)S(\omega) \tag{2.17}$$

where $S(\omega)$ is the entropy density of the state ω, then a state $\omega_\beta \in \mathcal{S}$ satisfying

$$f(\omega_\beta) = \inf_{\omega \in \mathcal{S}} f(\omega) \tag{2.18}$$

is an equilibrium state of (2.7) at inverse temperature β.

The Hamiltonian (2.7) is not quadratic in the creation and annihilation operators, and therefore cannot be diagonalised by a standard symplectic or Bogoliubov transformation and thus, on this basis, one is tempted to conclude at first sight that the model is not soluble. However on closer inspection we find that we can write (2.5) in the form

$$\begin{aligned}\frac{U_\Lambda}{V} &= \frac{g}{2}\left\{\left(\frac{a_{0+}^*}{\sqrt{V}}\right)\left(\frac{a_{0-}}{\sqrt{V}}\right)\left(\frac{b}{\sqrt{V}}\right)+\left(\frac{a_{0+}}{\sqrt{V}}\right)\left(\frac{a_{0-}^*}{\sqrt{V}}\right)\left(\frac{b^*}{\sqrt{V}}\right)\right\}\\ &\qquad +\Omega\left(\frac{b^*}{\sqrt{V}}\right)\left(\frac{b}{\sqrt{V}}\right)+\frac{\lambda}{2}\left(\frac{N_\Lambda}{V}\right)^2,\end{aligned} \tag{2.19}$$

so that all the terms are *space averages*. We have

$$\frac{a_{0\pm}}{\sqrt{V}}=\frac{1}{V}\int_\Lambda dx\, a_\pm(x),\quad \frac{a_{0\pm}^*}{\sqrt{V}}=\frac{1}{V}\int_\Lambda dx\, a_\pm^*(x)\quad\text{and}\quad \frac{N_\Lambda}{V}=\sum_{\sigma=\pm}\frac{1}{V}\int_\Lambda dx\, a_\sigma^*(x)\, a(x)_\sigma. \tag{2.20}$$

and by virtue of (2.3), $b^*/\sqrt{V}$ and $b/\sqrt{V}$ are clearly also *space averages*. Without going into all the mathematical details, the reason why space averages are such a simplifying feature is that they tend *weakly* to a multiples of the identity operator [7]. For example if ω is a space homogeneous extremal (mixing) state then for all local observables, A and B one has

$$\begin{aligned}\lim_{V\to\infty}\omega\left(A\,\frac{1}{V}\int_\Lambda dx\, a^*(x)\, a(x)B\right) &= \omega\left(A\,B\right)\lim_{V\to\infty}\omega\left(\frac{1}{V}\int_\Lambda dx\, a_\sigma^*(x)\, a_\sigma(x)\right)\\ &= \omega\left(A\,B\right)\omega\left(a_\sigma^*(0)\, a_\sigma(0)\right),\end{aligned} \tag{2.21}$$

so that N_Λ/V tends weakly to $\sum_{\sigma=\pm}\omega\left(a^*(0)_\sigma\, a_\sigma(0)\right)$. Similarly

$$\lim_{V\to\infty}\frac{a_{0\pm}}{\sqrt{V}}=\omega(a_\pm(0)),\qquad\text{and}\qquad \lim_{V\to\infty}\frac{b}{\sqrt{V}}=\omega(b(0)). \tag{2.22}$$

Thus if $\omega\in\mathcal{S}$, then the contribution of the term (2.5) to the energy density in (2.17) yields

$$\begin{aligned}\lim_{V\to\infty}\frac{\omega(U_\Lambda)}{V} &= \frac{g}{2}\left\{\omega(a_+^*(0))\omega(a_-(0))\omega(b(0))+\omega(a_+(0))\omega(a_-^*(0))\omega(b^*(0))\right\}\\ &\qquad +\Omega\left|\omega(b(0))\right|^2+\frac{\lambda}{2}\left(\sum_{\sigma=\pm}\omega\left(a_\sigma^*(0)\, a_\sigma(0)\right)\right)^2,\end{aligned} \tag{2.23}$$

The result follows readily from (2.21) with A and B a multiple of the identity. We can therefore conclude that in the study of the equilibrium states of (2.4) or (2.7), we can limit ourselves to searching for solutions ω which are product states on the tensor product *canonical commutation relations* algebra (CCR) of the three different kinds of particles, namely on

$$\mathcal{A}:=\mathcal{A}_+\otimes\mathcal{A}_-\otimes\mathcal{B}, \tag{2.24}$$

where $\mathcal{A}_\pm$ is the C* algebra generated by the Weyl operators:

$$W_\pm(f):=\exp\left\{i\,\frac{a_\pm^*(f)+a_\pm(f)}{\sqrt{2}}\right\},$$

for all $f \in L^2(\mathbb{R}^\nu) \cap L^1(\mathbb{R}^\nu)$, and $\mathcal{B}$ by the Weyl operators:

$$W_b(f) := \exp\left\{ i \, \frac{b^*(f) + b(f)}{\sqrt{2}} \right\} .$$

The above discussion makes it clear that we find the equilibrium states of our model amongst the states which are determined completely by their one-point and two-point functions, that is, among the set of extremal space invariant *quasi-free* states [7] on the respective CCR-algebras. This is a consequence of the fact that if $\omega \in \mathcal{S}$, the set of states on $\mathcal{A}$, and $\tilde{\omega} \in \mathcal{S}$ is a quasi-free state with the same one-point and two-point functions as ω, then it follows by Klein's inequality [8] that

$$S(\tilde{\omega}) \geq S(\omega). \tag{2.25}$$

Therefore since our energy density involves only the one-point and two-point functions, if $\mathcal{S}_{QF}$ is the set of quasi-free state on $\mathcal{A}$, then

$$\inf_{\omega \in \mathcal{S}} f(\omega) \geq \inf_{\omega \in \mathcal{S}_{QF}} f(\omega) \tag{2.26}$$

and consequently

$$f(\omega_\beta) = \inf_{\omega \in \mathcal{S}_{QF}} f(\omega). \tag{2.27}$$

We denote the set of *quasi-free states* on $\mathcal{A}_\sigma$ by ω_σ determined by the constants α_σ and the non-negative operators A_σ on $L^2(\mathbb{R}^\nu)$ and satisfying

$$\omega_\sigma(W_\sigma(f)) = \exp\left(i\sqrt{2}\, \mathfrak{Re}\, (\alpha_\sigma \langle 1, f\rangle) - \frac{1}{4}\|f\|^2 - \frac{1}{2}\langle f, A_\sigma f\rangle \right) \tag{2.28}$$

for all $f \in L^2(\mathbb{R}^\nu) \cap L^1(\mathbb{R}^\nu)$, see [7].

Note that the states ω_σ are completely determined by the *one-point function*

$$\omega_\sigma(a_\sigma(f)) = \bar{\alpha}_\sigma \langle f, 1\rangle \tag{2.29}$$

and the *two-point function*

$$\omega_\sigma(a_\sigma^*(f) a_\sigma(g)) = \langle g,\, A_\sigma f\rangle + |\alpha_\sigma|^2 \langle 1, f\rangle \langle g, 1\rangle \tag{2.30}$$

for all $f,\, g \in L^2(\mathbb{R}^\nu) \cap L^1(\mathbb{R}^\nu)$.

On $\mathcal{B}$ we consider the extremal invariant state, which is determined by one constant α_b, see [7].

$$\omega_b(W_b(f)) = \exp\left(i\sqrt{2}\, \mathfrak{Re}\, (\alpha_b \langle 1, f\rangle) - \frac{1}{4}\|f\|^2 \right) \tag{2.31}$$

Its one- and two-point functions are

$$\omega_b(b(f)) = \bar{\alpha}_b \, \langle f, 1\rangle \tag{2.32}$$

and

$$\omega_b(b^*(f) b(g)) = |\alpha_b|^2 \, \langle 1, f\rangle \, \langle g, 1\rangle \, . \tag{2.33}$$

Note that this one-mode *coherent* state depends only on the $k = 0$ mode. It is possible also to consider a more general quasi-free states of the form (2.31) to take into account

other photons modes. However since only the $k = 0$ mode interacts with the bosons, this is unnecessary.

Thus the candidates for the equilibrium states are among the set, $\mathcal{S}_P$, of products of quasi-free states, i.e. states of the form

$$\omega = \omega_+ \otimes \omega_- \otimes \omega_b. \tag{2.34}$$

They are completely parameterized by the set of parameters: $\alpha_\pm,\ \alpha_b \in \mathbb{C}$ and the integral operators $A_\pm$ on $L^2(\mathbb{R}^\nu)$:

$$(A_\pm f)(x) = \int_{\mathbb{R}^\nu} A_\pm(x-y) f(y) d^\nu y, \quad x \in \mathbb{R}^\nu \tag{2.35}$$

If $\hat{A}_\pm$ is the Fourier transform of $A_\pm$, then $\hat{A}_\pm(k) \geq 0$, expressing the positivity of the states $\omega_\pm$.

The variational principle (2.18) is now reduced to

$$f(\omega_\beta) = \inf_{\omega \in \mathcal{S}_P} f(\omega). \tag{2.36}$$

The entropy density for states in $\mathcal{S}_P$ is explicitly given by, see [9]:

$$S(\omega) = S(\omega_+) + S(\omega_-), \tag{2.37}$$

where $S(\omega_b) = 0$ because only *one* photon mode is taken into account. Here

$$S(\omega_\pm) = \int_{\mathbb{R}^\nu} \left\{ \left(1 + \hat{A}_\pm(k)\right) \ln\left(1 + \hat{A}_\pm(k)\right) - \hat{A}_\pm(k) \ln \hat{A}_\pm(k) \right\} \frac{d^\nu k}{(2\pi)^\nu}. \tag{2.38}$$

A straightforward computation yields:

$$\begin{aligned} \lim_{V\to\infty} \omega(H_\Lambda(\mu)/V) &= -(\mu-\varepsilon)|\alpha_+|^2 - (\mu+\varepsilon)|\alpha_-|^2 \\ &+ \int_{\mathbb{R}^\nu} (\epsilon(k) - \mu) \left(\hat{A}_+(k) + \hat{A}_-(k)\right) \frac{d^\nu k}{(2\pi)^\nu} \\ &+ \frac{g}{2} \left(\bar{\alpha}_+ \alpha_- \bar{\alpha}_b + \alpha_+ \bar{\alpha}_- \alpha_b\right) + \Omega\, |\alpha_b|^2 \\ &+ \frac{\lambda}{2} \left\{ \int_{\mathbb{R}^\nu} \left(\hat{A}_+(k) + \hat{A}_-(k)\right) \frac{d^\nu k}{(2\pi)^\nu} + |\alpha_+|^2 + |\alpha_-|^2 \right\}^2. \end{aligned} \tag{2.39}$$

Note that the pressure $P(\mu)$ of the system (2.7), as a function of the chemical potential μ, is related to the grand-canonical free-energy density by

$$P(\mu) = -f(\omega_\beta) = -\inf_{\mathcal{S}} f(\omega). \tag{2.40}$$

3 Variational Solutions

In this section we give a systematic derivation of the equilibrium states for our model as well as explicit expression for the corresponding grand-canonical pressure. To this end we solve the variational principal (2.36), and we start by substituting (2.37)-(2.39) into (2.36) to

obtain an expression for the functional $f(\omega)$ in terms of the variational parameters $\alpha_\pm,\ \alpha_b \in \mathbb{C}$ and $\hat{A}_\pm(k)$:

We find that there are *two critical* chemical potentials $\mu_c^{(1)}(\varepsilon)$ and $\mu_c^{(2)}(\varepsilon)$, $\mu_c^{(1)}(\varepsilon) \leq \mu_c^{(2)}(\varepsilon)$.

For $\mu < \mu_c^{(1)}(\varepsilon)$, the two $\sigma = \pm$ Bose gases behave like two mean field Bose gases with no BEC and they do not interact with the external b-boson laser field, in which there is no condensation either.

For $\mu > \mu_c^{(2)}(\varepsilon)$, there is BEC for the two $\sigma = \pm$ Bose gases and for the external boson laser field (superradiance).

When $\mu_c^{(1)}(\varepsilon) < \mu_c^{(2)}(\varepsilon)$, for $\mu_c^{(1)}(\varepsilon) < \mu < \mu_c^{(2)}(\varepsilon)$ there is BEC only for the $\sigma = -$ Bose gas.

First we remark that we can take $\alpha_\pm,\ \alpha_b$ real after a suitable gauge transformation on the boson creation and annihilation operators $a_{0\pm}$ and b, see (2.29) and (2.32). Note that the squares of these parameters are in fact the condensate densities of the corresponding boson modes. For notational convenience we introduce the particle density for an arbitrary quasi-free state ω of the form $\omega_+ \otimes \omega_- \otimes \omega_b$,

$$\rho := \int_{\mathbb{R}^\nu} \left(\hat{A}_+(k) + \hat{A}_-(k)\right) \frac{d^\nu k}{(2\pi)^\nu} + |\alpha_+|^2 + |\alpha_-|^2 = \lim_{V\to\infty} \frac{\omega\left(N_\Lambda\right)}{V}, \tag{3.1}$$

that is, ρ is the density of $\sigma = \pm$ particles, excluding the b-particles. We get the following Euler-Lagrange equations for the variational principle (2.36):
Take

$$\alpha_+,\ \alpha_-,\ \alpha_b \in \mathbb{R}. \tag{3.2}$$

(i) Differentiation of $f(\omega)$ with respect to α_+ gives:

$$2\left(\lambda\rho + \varepsilon - \mu\right)\alpha_+ + g\alpha_-\alpha_b = 0, \tag{3.3}$$

(ii) differentiation with respect to α_-:

$$2\left(\lambda\rho - \varepsilon - \mu\right)\alpha_- + g\alpha_+\alpha_b = 0, \tag{3.4}$$

(iii) differentiation with respect to α_b,

$$2\Omega\alpha_b + g\alpha_+\alpha_- = 0. \tag{3.5}$$

(iv) and finally differentiating with respect to $\hat{A}_+$ and $\hat{A}_-$ yields:

$$\hat{A}_+(k) = \hat{A}_-(k) = \frac{1}{e^{\beta(\epsilon(k)-\mu+\lambda\rho)} - 1}. \tag{3.6}$$

Note that the last equation implies that $\lambda\rho - \mu \geq 0$, since the $\hat{A}_\pm(k)$ are *positive*. Moreover, the correlation inequality (see e.g. [10])

$$\omega\left([A^*, [H_\Lambda(\mu), A]]\right) \geq 0 \tag{3.7}$$

for all observables A, applied here with $A = a_{0-}^*$, implies that $\lambda\rho - \mu \geq \varepsilon \geq 0$. Substituting (3.6) into (3.1) we get

$$\rho = |\alpha_+|^2 + |\alpha_-|^2 + 2\rho_0(\mu - \lambda\rho) \tag{3.8}$$

where

$$\rho_0(\mu) := \int_{\mathbb{R}^\nu} \frac{1}{e^{\beta(\epsilon(k)-\mu)}-1} \frac{d^\nu k}{(2\pi)^\nu} \tag{3.9}$$

is the density of the *free* Bose gas at chemical potential μ. Recall that $\rho_0(\mu<0)<\infty$ and that $\rho_0(\mu=0)<\infty$ for $\nu>2$.

Solving (3.3), (3.4) and (3.5) we have to distinguish *three* cases:

Case 1: $\alpha_+ = \alpha_- = \alpha_b = 0$.
Substituting zero for α_+ and α_- into (3.8) we get the standard equation for density of the *mean-field interacting* bosons

$$\rho = 2\rho_0(\mu - \lambda\rho), \tag{3.10}$$

see e.g. [11]. By virtue of the stability condition $\lambda\rho - \mu \geq \varepsilon \geq 0$ we see that this equation has no solution for $\mu > \mu_1(\varepsilon) := 2\lambda\rho_0(-\varepsilon) - \varepsilon$, while if $\mu \leq \mu_1(\varepsilon)$ it has a unique solution $\rho = \rho_1(\mu)$. (See Figure 1, where $x = \lambda\rho - \mu$ so that $x \geq \varepsilon$ and (3.10) becomes $\mu = 2\lambda\rho_0(-x) - x$). Putting this value of ρ into (3.6), we determine $\hat{A}_\pm$. Substituting these and $\alpha_+ = \alpha_- = \alpha_b = 0$ in the expressions (2.31), (2.33) and (2.33), for $\omega_\pm$ and ω_b respectively, we find a solution, $\omega^{(1)}_{\beta,\mu}$, of the Euler-Lagrange equations for the variational principle (2.36) for $\mu \leq \mu_1(\varepsilon)$. From (3.8) we are able to compute the free energy density for the state $\omega^{(1)}_{\beta,\mu}$:

$$f(\omega^{(1)}_{\beta,\mu}) = -2p_0(\mu - \lambda\rho_1(\mu)) - \frac{1}{2}\lambda\rho_1^2(\mu) \tag{3.11}$$

where $p_0(\mu)$ is the pressure of the free Bose gas:

$$p_0(\mu) := -\frac{1}{\beta}\int_{\mathbb{R}^\nu} \ln\left(1 - e^{-\beta(\epsilon(k)-\mu)}\right) \frac{d^\nu k}{(2\pi)^\nu}. \tag{3.12}$$

Case 2: α_+, α_- and α_b are non-zero.
We obtain from (3.3), (3.4) and (3.5) that

$$\alpha_+ = \frac{2\sqrt{\Omega\,(\lambda\rho - \varepsilon - \mu)}}{g}, \quad \alpha_- = \frac{2\sqrt{\Omega\,(\lambda\rho + \varepsilon - \mu)}}{g}, \quad \alpha_b = -\frac{2\sqrt{(\lambda\rho-\mu)^2 - \varepsilon^2}}{g}. \tag{3.13}$$

From these we see that in this case BEC is indeed present. Again substituting these values for α_+, α_- into (3.8) we get

$$\rho = \frac{8\Omega}{g^2}(\lambda\rho - \mu) + 2\rho_0(\mu - \lambda\rho). \tag{3.14}$$

Note that the first term corresponds to the condensate density. Let $\eta := (8\Omega\lambda/g^2 - 1)$. From the thermodynamic stability condition for (Section 2) we know that $\eta > 0$. Then equation (3.14) has a *unique* solution $\rho = \rho_2(\mu)$ for $\mu > \mu_2(\varepsilon) := 2\lambda\rho_0(-\varepsilon) + \eta\varepsilon$. Substituting this value of ρ into (3.13) and (3.6) we obtain all the parameters α_+, α_-, α_b and $\hat{A}_\pm$ and consequently we get another solution, $\omega^{(2)}_{\beta,\mu}$, of the Euler-Lagrange equations.

The free energy density for the state $\omega^{(2)}_{\beta,\mu}$ can again be computed:

$$f(\omega^{(2)}_{\beta,\mu}) = -2p_0(\mu - \lambda\rho_2(\mu)) - \frac{1}{2}\lambda\rho_2^2(\mu) + \frac{4\Omega}{g^2}(\lambda\rho_2(\mu) - \mu)^2 - \frac{4\Omega\varepsilon^2}{g^2}. \tag{3.15}$$

Denote by x_0 the unique solution of equation $2\lambda\rho_0'(-x) = \eta$ corresponding to the minimum of the function $2\lambda\rho_0(-x)+\eta x$, and let $\mu_0 = 2\lambda\rho_0(-x_0)+\eta x_0$. For $\mu_0 < \mu \leq \mu_2(\varepsilon)$ the equation (3.14) has two solutions $\rho = \rho_2(\mu)$ and $\rho = \tilde{\rho}_2(\mu)$, $\rho_2(\mu) > \tilde{\rho}_2(\mu)$. The corresponding states $\omega^{(2)}_{\beta,\mu}$ and $\tilde{\omega}^{(2)}_{\beta,\mu}$ can be found as above. The free energy density for the state $\omega^{(2)}_{\beta,\mu}$ is as in (3.15) and for $\tilde{\omega}^{(2)}_{\beta,\mu}$ it is the same with $\rho_2(\mu)$ replaced by $\tilde{\rho}_2(\mu)$.

Case 3: $\alpha_- \neq 0$ and $\alpha_+ = \alpha_b = 0$.
From (3.3), (3.4) and (3.5) one can see that this is possible only if

$$\rho = \frac{\mu+\varepsilon}{\lambda}, \tag{3.16}$$

corresponding to the *boundary* $x = \varepsilon$ of the stability domain, see Figure 1. The equation (3.8) then requires that $\mu > \mu_1(\varepsilon)$ and gives

$$\alpha_+ = \sqrt{\frac{\mu+\varepsilon}{\lambda} - 2\rho_0(-\varepsilon)}. \tag{3.17}$$

This case corresponds to yet another solution of the Euler-Lagrange equations, $\omega^{(3)}_{\beta,\mu}$, whose free energy density is given by:

$$f(\omega^{(3)}_{\beta,\mu}) = -2p_0(-\varepsilon) - \frac{(\mu+\varepsilon)^2}{2\lambda}. \tag{3.18}$$

We see from above that for certain values of μ there are several solutions of the Euler-Lagrange equations. Since these equations determine only the stationary points of the free energy functional, if there is more than one such point, in order to obtain the equilibrium state for a fixed μ we have to decide which of the solutions, has the *lowest* grand-canonical free-energy density.

To proceed with explicit analysis of solutions of the Euler-Lagrange equations it is easier to work with the variable $x = \lambda\rho - \mu$ rather than ρ. Also in the grand-canonical ensemble it is more usual to use the pressure instead of the free energy density. These allow to find the grand-canonical pressure as a function of its natural variable, the chemical potential. In terms of x and η the equations (3.10), (3.14)and (3.16) become:

$$2\lambda\rho_0(-x) - x = \mu \quad \text{for} \quad \mu \leq \mu_1(\varepsilon), \tag{3.19}$$

$$2\lambda\rho_0(-x) + \eta x = \mu \quad \text{for} \quad \mu \geq \mu_0 \tag{3.20}$$

and

$$x = \varepsilon \quad \text{for} \quad \mu > \mu_1(\varepsilon). \tag{3.21}$$

We consider first the case $\varepsilon = 0$. Then $\mu_1(0) = \mu_2(0) = 2\lambda\rho_c$, where $\rho_c := \rho_0(0)$. So, in this case the *lower* critical dimensionality the same as for the free (or mean-field) Bose-gas: $\nu = 2$. The equations (3.19), (3.20) and (3.21) become :

$$2\lambda\rho_0(-x) - x = \mu \quad \text{for} \quad \mu \leq 2\lambda\rho_c, \tag{3.22}$$

$$2\lambda\rho_0(-x) + \eta x = \mu \quad \text{for} \quad \mu \geq \mu_0 \tag{3.23}$$

and

$$x = 0 \quad \text{for} \quad \mu > 2\lambda\rho_c. \tag{3.24}$$

In Figure 1 we have drawn $y = 2\lambda\rho_0(-x) - x$ and $y = 2\lambda\rho_0(-x) + \eta x$. Recall that x_0 is the unique solution of $2\lambda\rho_0'(-x) = \eta$ and $\mu_0 = 2\lambda\rho_0(-x_0) + \eta x_0$. It is easy to see that:

1. For $\mu < \mu_0$, (3.22) has a unique solution $x_1(\mu)$ while (3.23) does not have a solution.
2. In the region $\mu_0 < \mu < 2\lambda\rho_c$, (3.22) has a unique solution $x_1(\mu)$ while (3.23) has two solutions $x_2(\mu)$ and $\tilde{x}_2(\mu)$, $x_2(\mu) > \tilde{x}_2(\mu)$.
3. Finally for $\mu > 2\lambda\rho_c$, (3.22) has no solution while (3.23) has a unique solutions $x_2(\mu)$ and we also have to consider the solution (3.24), $x = 0$.

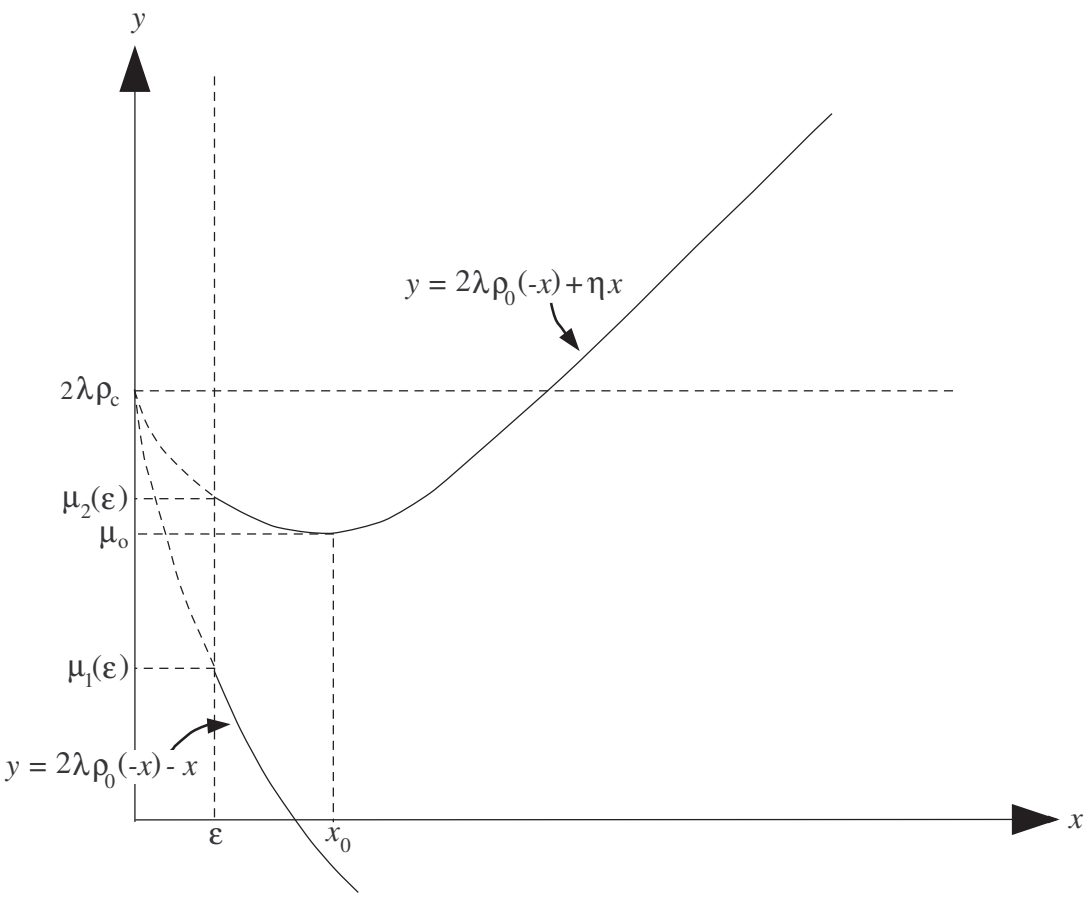

Figure 1: Solution of the density equation

Let

$$P_1(x, \mu) = 2p_0(-x) + \frac{(x+\mu)^2}{2\lambda}. \tag{3.25}$$

and

$$P_2(x, \mu) = 2p_0(-x) + \frac{\{(x+\mu)^2 - (\eta+1)x^2\}}{2\lambda}, \tag{3.26}$$

Then the situation is as follows:

1. For $\mu < \mu_0$, the solution of the variational problem (2.36) is $\omega^{(1)}_{\beta,\mu}$ and the corresponding pressure $P(\mu) := -f(\omega^{(1)}_{\beta,\mu}) = P_1(x_1(\mu), \mu)$.

2. For $\mu_0 < \mu < 2\lambda\rho_c$, the solution of the variational problem is the state out of $\omega^{(1)}_{\beta,\mu}$, $\omega^{(2)}_{\beta,\mu}$ and $\tilde{\omega}^{(2)}_{\beta,\mu}$ which minimizes the free energy density or equivalently maximizes the pressure. The pressures for these states are $P_1(x_1(\mu), \mu)$, $P_2(x_2(\mu), \mu)$ and $P_2(\tilde{x}_2(\mu), \mu)$ respectively.

3. For $\mu > 2\lambda\rho_c$, the two candidates for the solution of the variational problem (2.36) are $\omega^{(2)}_{\beta,\mu}$ and $\omega^{(3)}_{\beta,\mu}$. The pressures for these states are $P_2(x_2(\mu), \mu)$ and $P_3(\mu) := -f(\omega^{(3)}_{\beta,\mu}) = 2p_0(0) + \mu^2/2\lambda$.

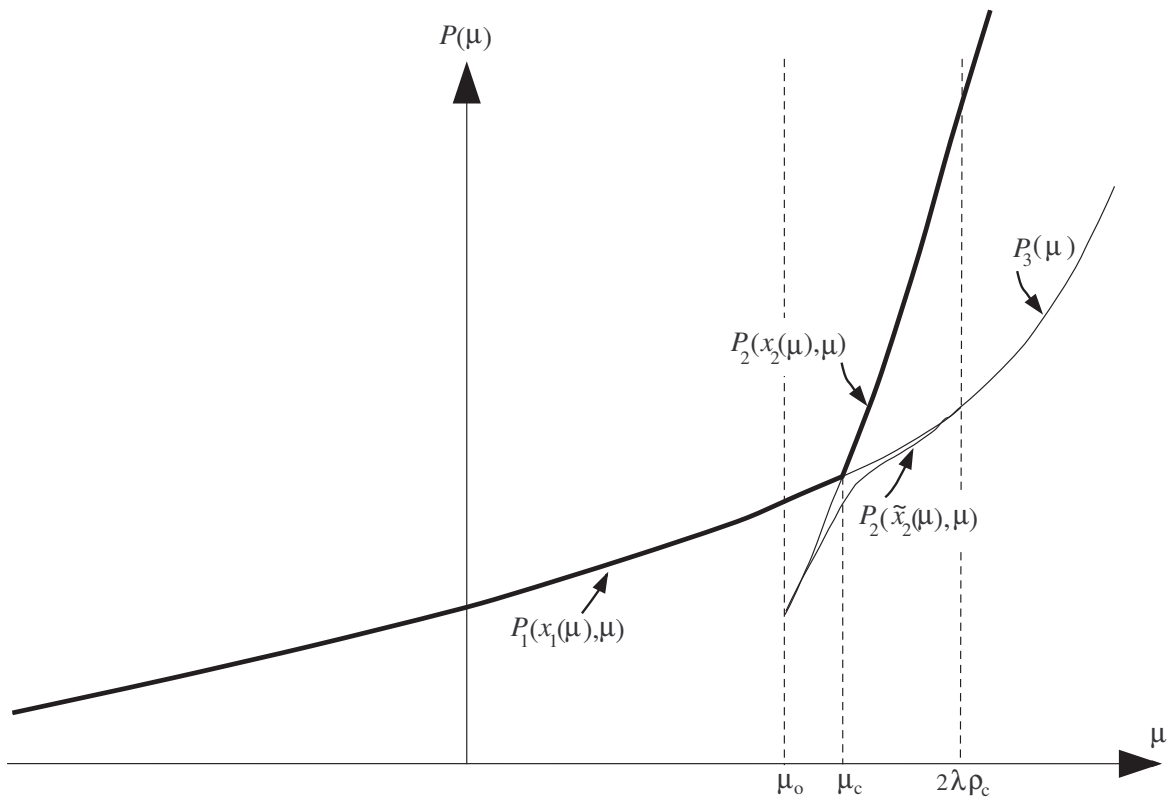

Figure 2: The pressure

In Figure 2 we have sketched $P_1(x_1(\mu), \mu)$, $P_2(x_2(\mu), \mu)$, $P_2(\tilde{x}_2(\mu), \mu)$ and $P_3(\mu_0)$. One can check that $P_1(x_1(\mu), \mu)$ and $P_2(x_2(\mu), \mu)$ are convex in μ. One also has

$$\frac{dP_2(x_2(\mu), \mu)}{d\mu} = \frac{x_2(\mu) + \mu}{\lambda}, \quad \frac{dP_2(\tilde{x}_2(\mu), \mu)}{d\mu} = \frac{\tilde{x}_2(\mu) + \mu}{\lambda} \tag{3.27}$$

and

$$\frac{dP_1(x_1(\mu), \mu)}{d\mu} = \frac{x_1(\mu) + \mu}{\lambda}. \tag{3.28}$$

Therefore since for $\mu_0 < \mu < 2\lambda\rho_c$, $x_2(\mu) > \tilde{x}_2(\mu) > x_1(\mu)$, in this interval we have

$$\frac{dP_2(x_2(\mu), \mu)}{d\mu} > \frac{dP_2(\tilde{x}_2(\mu), \mu)}{d\mu} > \frac{dP_1(x_1(\mu), \mu)}{d\mu}. \tag{3.29}$$

As $P_2(x_2(\mu_0),\mu_0) = P_2(\tilde{x}_2(\mu_0),\mu_0)$, it follows from (3.29) that $P_2(x_2(\mu),\mu) > P_2(\tilde{x}_2(\mu),\mu)$ for $\mu_0 < \mu < 2\lambda\rho_c$. Now

$$P_1(x_1(2\lambda\rho_c), 2\lambda\rho_c) = P_2(\tilde{x}_2(2\lambda\rho_c), 2\lambda\rho_c) = 2p_0(0) + 2\lambda\rho_c^2 \tag{3.30}$$

and consequently $P_2(x_2(2\lambda\rho_c), 2\lambda\rho_c) > P_1(x_1(2\lambda\rho_c), 2\lambda\rho_c)$. Also if $P_2(\tilde{x}_2(\mu_0),\mu_0)$ were greater than $P_1(x_1(\mu_0),\mu_0)$, then (3.29) would imply that $P_2(\tilde{x}_2(2\lambda\rho_c), 2\lambda\rho_c) > P_1(x_1(2\lambda\rho_c), 2\lambda\rho_c)$ contradicting (3.30). Thus we must have

$$P_2(x_2(\mu_0),\mu_0) = P_2(\tilde{x}_2(\mu_0),\mu_0) < P_1(x_1(\mu_0),\mu_0). \tag{3.31}$$

Therefore there exists a unique μ_c satisfying $\mu_0 < \mu_c < 2\lambda\rho_c$ such that $P_2(x_2(\mu_c),\mu_c) = P_1(x_1(\mu_c),\mu_c)$.
Finally we consider $\mu > 2\lambda\rho_c$. We have $P_3(2\lambda\rho_c) = P_1(x_2(2\lambda\rho_c), 2\lambda\rho_c) < P_1(x_2(2\lambda\rho_c), 2\lambda\rho_c)$ and

$$\frac{dP_3(\mu)}{d\mu} = \frac{\mu}{\lambda} < \frac{(x_2(\mu)+\mu)}{\lambda} = \frac{dP_2(x_2(\mu),\mu)}{d\mu}. \tag{3.32}$$

Therefore $P_2(x_2(\mu),\mu) > P_3(\mu)$ for $\mu > 2\lambda\rho_c$.

Summarizing: There exists a unique critical chemical potential μ_c such that

1. For $\mu < \mu_c$, the solution of the variational problem (2.36) is $\omega^{(1)}_{\beta,\mu}$. For $\omega^{(1)}_{\beta,\mu}$, $\alpha_+ = \alpha_- = \alpha_b = 0$, i.e. the two $\pm$ Bose gases behave like two mean field Bose gases with no BEC and do not interact with the external b-bosons which do not condense either. The corresponding pressure is $P(\mu) = P_1(x_1(\mu),\mu)$.

2. For $\mu > \mu_c$, the solution of the variational problem is $\omega^{(2)}_{\beta,\mu}$. For this state

$$\alpha_+ = \alpha_- = \frac{2\sqrt{\Omega\, x_2(\mu)}}{g} \quad \text{and} \quad \alpha_b = -\frac{2x_2(\mu)}{g}, \tag{3.33}$$

   i.e. there is BEC for the two $\pm$ Bose gases and for the external bosons laser field (superradiance). Moreover the condensation of the $\pm$ bosons is enhanced by the presence of the laser field (b-bosons), known as the equilibrium BEC superradiance [6]. The pressure for the system is $P(\mu) = P_2(x_2(\mu),\mu)$.

We now return to the case $\varepsilon > 0$. We have to redefine P_2 and P_3 but P_1 remains unchanged:

$$P_2(x,\mu) = 2p_0(-x) + \frac{\{(x+\mu)^2 - (\eta+1)x^2\}}{2\lambda} + \frac{(\eta+1)\varepsilon^2}{2\lambda} \tag{3.34}$$

and

$$P_3(\mu) = 2p_0(-\varepsilon) + \frac{(\mu+\varepsilon)^2}{2\lambda}. \tag{3.35}$$

Note that

$$P_1(x_1(\mu_1(\varepsilon)),\mu_1(\varepsilon)) = P_3(\mu_1(\varepsilon)) \tag{3.36}$$

and

$$\begin{aligned} P_2(\tilde{x}_2(\mu_2(\varepsilon)),\mu_2(\varepsilon)) &= P_3(\mu_2(\varepsilon)) \text{ for } \varepsilon < x_0 \\ P_2(x_2(\mu_2(\varepsilon)),\mu_2(\varepsilon)) &= P_3(\mu_2(\varepsilon)) \text{ for } \varepsilon > x_0. \end{aligned} \tag{3.37}$$

Also by again considering the derivatives

$$P_2(x_2(\mu), \mu) > P_3(\mu) \tag{3.38}$$

for $\mu > \mu_2(\varepsilon)$ and as before $P_2(x_2(\mu), \mu) > P_2(\tilde{x}_2(\mu), \mu)$ for $\mu_0 < \mu < 2\lambda\rho_0(-\varepsilon)$ in the region where it applies.

(a) The simplest case to consider is when $\varepsilon > x_0$. In this case for $\mu < \mu_1(\varepsilon)$ only (3.19) has a solution $x_1(\mu)$, for $\mu_1(\varepsilon) < \mu < \mu_2(\varepsilon)$ only (3.21) is satisfied i.e. $x = \varepsilon$ and for $\mu > \mu_2(\varepsilon)$ only (3.20) has a solution $x_2(\mu)$. Thus the states are $\omega_\beta^{(1)}$, $\omega_\beta^{(3)}$ and $\omega_\beta^{(2)}$ as μ increases. This means that as we increase μ the system goes from no BEC, to BEC for the $\sigma = -$ bosons only, to BEC for both species and superradiance.

(b) When $\varepsilon < x_0$ we have to consider two cases, $\mu_1(\varepsilon) < \mu_0 < \mu_2(\varepsilon)$ and $\mu_0 < \mu_1(\varepsilon)$.

In the first case we can use the same arguments as for $\varepsilon = 0$ to show that $P_3(\mu_0) > P_2(x_2(\mu_0), \mu_0)$ and $P_3(\mu_2(\varepsilon)) < P_2(x_2(\mu_2(\varepsilon)), \mu_2(\varepsilon))$. This implies that there *exists* $\mu_c(\varepsilon)$ between μ_0 and $\mu_2(\varepsilon)$ such that $P_3(\mu_c(\varepsilon)) = P_2(x_2(\mu_c(\varepsilon)), \mu_c(\varepsilon))$. Thus at $\mu_c(\varepsilon)$ the state changes from $\omega_\beta^{(3)}$ to $\omega_\beta^{(2)}$. This means that the situation is the same as for $\varepsilon > x_0$ except that the changes of state occur at $\mu_1(\varepsilon)$ and at $\mu_c(\varepsilon)$.

For $\varepsilon < x_0$ and $\mu_0 < \mu_1(\varepsilon)$ the same argument applies. However we did not determine on which side of $\mu_1(\varepsilon)$, the value of $\mu_c(\varepsilon)$ lies. Thus we know that there is no BEC for $\mu < \mu_0$ and there is BEC for both types of bosons for $\mu > \mu_c(\varepsilon)$, but we do not know if the intermediate phase with BEC for $-$ bosons only is present.

4 Conclusion: Equilibrium BEC Superradiance

The above results may be summarized as follows:

There exist *two critical* chemical potentials $\mu_c^{(1)}(\varepsilon)$ and $\mu_c^{(2)}(\varepsilon)$, $\mu_c^{(1)}(\varepsilon) \leq \mu_c^{(2)}(\varepsilon)$.

For $\mu < \mu_c^{(1)}(\varepsilon)$, the solution of the variational problem (2.36) is $\omega_{\beta,\mu}^{(1)}$. For the state $\omega_{\beta,\mu}^{(1)}$,

$$\alpha_+ = \alpha_- = \alpha_b = 0, \tag{4.1}$$

i.e. the two $\sigma = \pm$ Bose gases behave like two mean field Bose gases with no BEC and they do not interact with the external b-boson laser field, in which there is no condensation either.

For $\mu > \mu_c^{(2)}(\varepsilon)$, the solution of the variational problem is $\omega_{\beta,\mu}^{(2)}$. By virtue of (3.13) for this state we have

$$0 < \alpha_+ \leq \alpha_- \quad \text{and} \quad \alpha_b \neq 0, \tag{4.2}$$

i.e. there is BEC for the two $\sigma = \pm$ Bose gases and for the external boson laser field (superradiance). Moreover, for $\varepsilon > 0$ the condensation of the $\sigma = \pm$ bosons is *enhanced* by the presence of this laser field: one gets it even for dimensions $\nu = 1, 2$, because $\rho_0(-\varepsilon) < \infty$ for $\nu \geq 1$. We interpret this quantum state as that of equilibrium BEC superradiance [6].

When $\mu_c^{(1)}(\varepsilon) < \mu_c^{(2)}(\varepsilon)$, for $\mu_c^{(1)}(\varepsilon) < \mu < \mu_c^{(2)}(\varepsilon)$, the solution of the variational problem is $\omega_{\beta,\mu}^{(3)}$. For this state we have

$$\alpha_- \neq 0 \quad \text{and} \quad \alpha_+ = \alpha_b = 0, \tag{4.3}$$

i.e. there is BEC only for the $\sigma = -$ Bose gas.

(a) The simplest case to consider is when $\varepsilon > x_0$. In this case $\mu_c^{(1)}(\varepsilon) = \mu_1(\varepsilon)$ and $\mu_c^{(2)}(\varepsilon) = \mu_2(\varepsilon)$. Thus the states are $\omega_\beta^{(1)}$, $\omega_\beta^{(3)}$ and $\omega_\beta^{(2)}$ as μ increases. This means that as we increase μ we observe three stages: the system goes from *no* BEC, to BEC for *only* the $\sigma = -$ bosons, and then to BEC for both $\sigma = \pm$ boson species and for the laser field (superradiance).

(b) When $0 < \varepsilon < x_0$ we have to consider *two* subcases: $\mu_1(\varepsilon) < \mu_0 < \mu_2(\varepsilon)$ and $\mu_0 < \mu_1(\varepsilon)$.

In the first subcase $\mu_1(\varepsilon) = \mu_c^{(1)}(\varepsilon) < \mu_c^{(2)}(\varepsilon) < \mu_2(\varepsilon)$. Otherwise the situation is as in (a).

For $\mu_0 < \mu_1(\varepsilon)$, $\mu_1(\varepsilon) < \mu_c^{(2)}(\varepsilon) < \mu_2(\varepsilon)$ but we did not determine if $\mu_c^{(1)}(\varepsilon) < \mu_c^{(2)}(\varepsilon)$. Thus we do not know if the *intermediate phase* with BEC for only $\sigma = -$ bosons is present.

(c) If $\varepsilon = 0$, then $\mu_c^{(1)}(\varepsilon) = \mu_c^{(2)}(\varepsilon)$ and the *intermediate phase* with BEC for only $\sigma = -$ bosons is not present.

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FIGURE CAPTIONS

Figure 1: Solution of the density equation

Figure 2: The pressure