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HURWITZ ACTION ON TUPLES OF EUCLIDEAN REFLECTIONS

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This note was prompted by the reading of [4], which purports to show that if an $n$-tuple of Euclidean reflections has a finite orbit under the Hurwitz action of the braid group, then the generated group is finite. I noticed that the proof given is fatally flawed; however, using the argument of Vinberg given in [3], I found a short (hopefully correct) proof which at the same time considerably simplifies the computational argument given in [4]. This is what I expound below. I first recall all the necessary notation and assumptions, expounding some facts in slightly more generality than necessary.

0.1. Hurwitz action.

Definition. Given a group $G$, we call Hurwitz action the action of the $n$-strand braid group $B_n$ on $G^n$ given by

$$\sigma_i(s_1, \ldots, s_n) = (s_1, \ldots, s_{i-1}, s_i s_{i+1}^{-1}, s_i, s_{i+2}, \ldots, s_n).$$

The inverse is given by $\sigma_i^{-1}(s_1, \ldots, s_n) = (s_1, \ldots, s_{i-1}, s_i^{-1} s_{i+1}, s_i, s_{i+2}, \ldots, s_n)$. Here $a b$ is $b^{-1} a b$ and $b a$ is $b a b^{-1}$.

This action preserves the product of the $n$-tuple. We need to repeat some remarks in [3]. By decreasing induction on $i$ one sees that $\sigma_1 \ldots \sigma_n(s_1, \ldots, s_n) = (s_1, \ldots, s_{i-1}, s_n, s_i^n, \ldots, s_{n-1}^n)$. In particular if $\gamma = \sigma_1 \ldots \sigma_{n-1}$ we get $\gamma(s_1, \ldots, s_n) = (s_n, s_1, \ldots, s_{n-1})^{s_n}$ whence, if $c = s_1 \ldots s_n$, we get that $\gamma^n(s_1, \ldots, s_n) = (s_1, \ldots, s_n)^c$.

We also deduce that given any subsequence $(i_1, \ldots, i_k)$ of $(1, \ldots, n)$, there exists an element of the Hurwitz orbit of $(s_1, \ldots, s_n)$ which begins by $(s_{i_1}, \ldots, s_{i_k})$.

Assume now that the Hurwitz orbit of $(s_1, \ldots, s_n)$ is finite. Then some power of $\gamma$ fixes $(s_1, \ldots, s_n)$, thus some power of $c$ is central in the subgroup generated by the $s_i$. Similarly, by looking at the action of $\sigma_1 \ldots \sigma_{n-1}$ on an element of the orbit beginning by $(s_{i_1}, \ldots, s_{i_k})$ we get that for any subsequence $(i_1, \ldots, i_k)$ of $(1, \ldots, n)$ there exists a power of $s_{i_1} \ldots s_{i_k}$ central in the subgroup generated by $(s_{i_1}, \ldots, s_{i_k})$.

0.2. Reflections. Let $V$ be a vector space on some subfield $K$ of $\mathbb{C}$. We call complex reflection a finite order element $s \in \text{GL}(V)$ whose fixed points are a hyperplane. If $\zeta$ (a root of unity) is the unique non-trivial eigenvalue of $s$, the action of $s$ can be written $s(x) = x - \tilde{r}(x)r$ where $r \in V$ and $\tilde{r}$ is an element of the dual of $V$ satisfying

$$(a^b)^{-1} = a^{-1} b a^{-1} b^{-1} a b.$$
$\hat{r}(r) = 1 - \zeta$. These elements are unique up to multiplying $r$ by a scalar and $\hat{r}$ by the inverse scalar. We say that $r$ (resp. $\hat{r}$) is a root (resp. coroot) associated to $s$.

0.3. Cartan Matrix. If $(s_1, \ldots, s_n)$ is a tuple of complex reflections and if $r_i, \hat{r}_i$ are corresponding roots and coroots, we call Cartan matrix the matrix $C = \{c_i(r_j)\}_{i,j}$. This matrix is unique up to conjugating by a diagonal matrix. Conversely, a class modulo the action of diagonal matrices of Cartan matrices is an invariant of the $\text{GL}(V)$-conjugacy class of the tuple. It determines this class if it is invertible and $n = \dim V$. Indeed, this implies that the $r_i$ form a basis of $V$; and in this basis the matrix $s_i$ differs from the identity matrix only on the $i$-th line, where the opposed of the $i$-th line of $C$ has been added; thus $C$ determines the $s_i$.

If $C$ can be chosen Hermitian (resp. symmetric), such a choice is then unique up to conjugating by a diagonal matrix of norm 1 elements of $K$ (resp. of signs).

If $C$ is Hermitian (which implies that the $s_i$ are of order 2), then the sesquilinear form given by $\hat{C}$ is invariant by the $s_i$ (if the $s_i$ are not of order 2, but the matrix obtained by replacing all elements on the diagonal of $\hat{C}$ by 2's is Hermitian, then the latter matrix defines a sesquilinear form invariant by the $s_i$).

0.4. Coxeter element. We keep the notation as above and we assume that the $r_i$ form a basis of $V$. We recall a result of [2] on the “Coxeter” element $c = s_1 \ldots s_2$. If we write $C = U + V$ where $U$ is upper triangular unipotent and where $V$ is lower triangular (with diagonal terms $-\zeta_i$, thus $V$ is also unipotent when $s_i$ are of order 2), then the matrix of $c$ in the $r_i$ basis is $-U^{-1}V$ (to see this write it as $U s_1 \ldots s_n = -V$ and look at partial products in the left-hand side starting from the left). As $U$ is of determinant 1, we deduce that $\chi(c) = \det(xI + U^{-1}V) = \det(xU + V)$ where $\chi(c)$ denotes the characteristic polynomial; in particular $\det(C) = \chi(c)_{|x=1}$; one also gets that the fix-point set of $c$ is the kernel of $C$, equal to the intersection of the reflecting hyperplanes.

0.5. The main theorem. The next theorem implies the statement given in [4]. (1.1] considers Euclidean reflections with the $r_i$ linearly independent; if the $r_i$ are chosen of the same length this implies that $C$ is symmetric, and as $C$ is then the Gram matrix of the $r_i$, it is invertible:

Theorem. Let $(s_1, \ldots, s_n)$ be a tuple of reflections in $\text{GL}(\mathbb{R}^n)$ which have an associated Cartan matrix symmetric and invertible. Assume in addition that the Hurwitz orbit of the tuple is finite. Then the group generated by the $s_i$ is finite.

Proof. In the next paragraph, we just need that $(s_1, \ldots, s_n)$ is a tuple of complex reflections with a finite Hurwitz orbit and with the $r_i$ a basis of $V$.

A straightforward computation shows that an element of $GL(V)$ commutes to the $s_i$ if and only if it acts as a scalar on the subspaces generated by $\{r_i\}_{i \in I}$ where $I$ is a block of $C$ (i.e., a connected component of the graph with vertices $\{1, \ldots, n\}$ and edges $(i,j)$ for each pair such that either $C_{i,j}$ or $C_{j,i}$ is not zero). The finiteness of the Hurwitz orbit implies that for any subsequence $(i_1, \ldots, i_k)$ of $(1, \ldots, n)$, there exists a power of $s_{i_1} \ldots s_{i_k}$ which commutes to $s_{i_1}, \ldots, s_{i_k}$. This power acts thus as a scalar on each subspace generated by the $r_{i_j}$ in a block of the submatrix of $C$ determined by $(i_1, \ldots, i_k)$. As the determinant of each $s_{i_j}$ on this subspace is a root of unity, the scalar must be a root of unity. Thus, the restriction of each $s_{i_1} \ldots s_{i_k}$ to the subspace $< r_{i_1}, \ldots, r_{i_k} >$ generated by the $r_{i_j}$ is of finite order.
We use from now on all the assumptions of the theorem. Thus the \( s_i \) are order 2 elements of \( O(C) \), the orthogonal group of the quadratic form defined by \( C \).

Also, \( \chi(c) \) is a polynomial with real coefficients. As \( c \) is of finite order, any real root of \( \chi(c) \) is 1 or \(-1\). This implies that \( \chi(c) \big|_{x=1} \) is a nonnegative real number, and thus \( \det C \) also. The same holds for any principal minor of \( C \), since such a minor is \( \chi(c') \big|_{x=1} \) where \( c' \) is the restriction of some \( s_{i_1} \ldots s_{i_k} \) to \( < r_{i_1}, \ldots, r_{i_k} > \).

The quadratic form defined by \( C \) is thus positive, and as \( \det C \neq 0 \) it is positive definite (cf. [1, §7, exercice 2]).

We now digress about the Cartan matrix of two reflections \( s_1 \) et \( s_2 \). Such a matrix is of the form \( \begin{pmatrix} 2 & a \\ b & 2 \end{pmatrix} \). If \( a = 0 \) and \( b \neq 0 \) or \( a \neq 0 \) and \( b = 0 \) then \( s_1s_2 \) is of infinite order. Otherwise, the number \( ab \) is a complete invariant of the conjugacy class of \( (s_1, s_2) \) restricted to \( < r_1, r_2 > \), and \( s_1s_2 \) restricted to this subspace is of finite order \( m \) if and only if there exists \( k \) prime to \( m \) such that \( ab = 4 \cos^2 k\pi/m \).

Since \( C \) is symmetric and since the restriction of \( s_is_j \) to \( < r_i, r_j > \) is of finite order, there exists prime integer pairs \( (k_{i,j}, m_{i,j}) \) such that \( C_{i,j} = \pm 2 \cos k_{i,j} \pi/m_{i,j} \). If \( K \) is the cyclotomic subfield containing the \( \text{lcm}(2m_{i,j}) \)-th roots of unity, and if \( O \) is the ring of integers of \( K \), we get that all coefficients of \( C \) lie in \( O \). It follows, if \( G \) is the group generated by the \( s_i \), that in the \( r_i \) basis we have \( G \subset \text{GL}(O^n) \).

We now apply Vinberg’s argument as in [3, 1.4.2]. Let \( \sigma \in \text{Gal}(K/Q) \). Then \( \sigma(C) \) is again positive definite: all arguments used to prove that \( C \) is positive definite still apply for \( \sigma(C) \): it is real, symmetric, invertible and the Hurwitz orbit of \( (\sigma(s_1), \ldots, \sigma(s_n)) \) is still finite. Since \( G \subset O(C) \), which is compact, the entries of the elements of \( G \) in the \( r_i \) basis are of bounded norm. Since \( O(\sigma(C)) \) is also compact for any \( \sigma \in \text{Gal}(K/Q) \), we get that entries of elements of \( G \) are elements of \( O \) all of whose complex conjugates have a bounded norm. There is a finite number of such elements, so \( G \) is finite. \( \square \)

References