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HURWITZ ACTION ON TUPLES OF EUCLIDEAN REFLECTIONS

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This note was prompted by the reading of [3], which purports to show that if an n-tuple of Euclidean reflections has a finite orbit under the Hurwitz action of the braid group, then the generated group is finite. I noticed that the proof given is fatally flawed \(^1\); however, using the argument of Vinberg given in [3], I found a short (hopefully correct) proof which at the same time considerably simplifies the computational argument given in [3]. This is what I expound below. I first recall all the necessary notation and assumptions, expounding some facts in slightly more generality than necessary.

0.1. Hurwitz action.

Definition. Given a group G, we call Hurwitz action the action of the n-strand braid group \(B_n\) on \(G^n\) given by

\[
\sigma_i(s_1, \ldots, s_n) = (s_1, \ldots, s_{i-1}, s_{i+1}, s_i^{s_i+1}, s_{i+2}, \ldots, s_n).
\]

The inverse is given by \(\sigma_i^{-1}(s_1, \ldots, s_n) = (s_1, \ldots, s_{i-1}, s_i, s_{i+1}, \ldots, s_n).\) Here \(a^b\) is \(b^{-1}ab\) and \(b^a\) is \(b^{-1}ab^{-1}a.\)

This action preserves the product of the n-tuple. We need to repeat some remarks in [3]. By decreasing induction on \(i\) one sees that \(\sigma_1 \cdots \sigma_n(s_1, \ldots, s_n) = (s_1, \ldots, s_{i-1}, s_i, s_i^{s_i}, \ldots, s_n^{s_{n-1}}).\) In particular if \(\gamma = \sigma_1 \cdots \sigma_{n-1}\) we get \(\gamma(s_1, \ldots, s_n) = (s_n, s_1, \ldots, s_{n-1})^{s_n}\) whence, if \(c = s_1 \cdots s_n\), we get that \(\gamma^n(s_1, \ldots, s_n) = (s_1, \ldots, s_n)^c.\)

We also deduce that given any subsequence \((i_1, \ldots, i_k)\) of \((1, \ldots, n)\), there exists an element of the Hurwitz orbit of \((s_1, \ldots, s_n)\) which begins by \((s_{i_1}, \ldots, s_{i_k}).\)

Assume now that the Hurwitz orbit of \((s_1, \ldots, s_n)\) is finite. Then some power of \(\gamma\) fixes \((s_1, \ldots, s_n),\) thus some power of \(c\) is central in the subgroup generated by the \(s_i.\) Similarly, by looking at the action of \(\sigma_1 \cdots \sigma_{n-1}\) on an element of the orbit beginning by \((s_{i_1}, \ldots, s_{i_k})\) we get that for any subsequence \((i_1, \ldots, i_k)\) of \((1, \ldots, n)\) there exists a power of \(s_{i_1} \cdots s_{i_k}\) central in the subgroup generated by \((s_{i_1}, \ldots, s_{i_k}).\)

0.2. Reflections. Let \(V\) be a vector space on some subfield \(K\) of \(\mathbb{C}.\) We call complex reflection a finite order element \(s \in \text{GL}(V)\) whose fixed points are a hyperplane. If \(\zeta\) (a root of unity) is the unique non-trivial eigenvalue of \(s,\) the action of \(s\) can be written \(s(x) = x - \bar{r}(x)r\) where \(r \in V\) and \(\bar{r}\) is an element of the dual of \(V\) satisfying

\[\zeta^r(x) = x - \bar{r}(x)r\]

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\(^1\)The problem is in proposition 2.3, which is essential to the main theorem (1.1) of the paper. The argument given there is basically that if a Coxeter group has a reflection representation where the image of the Coxeter element is of finite order, then the image of that representation is finite. However this is false: the Cartan matrix \((-1/2 -1 -2)\) where \(P^2 + l = \sqrt{2}\) defines Euclidean reflections which give a representation of an infinite rank 3 Coxeter group, such that the image of the Coxeter group is infinite but the image of the Coxeter element is of order 8 (personal communication of F.Zara).
\( \hat{r}(r) = 1 - \zeta \). These elements are unique up to multiplying \( r \) by a scalar and \( \hat{r} \) by the inverse scalar. We say that \( r \) (resp. \( \hat{r} \)) is a root (resp. coroot) associated to \( s \).

0.3. **Cartan Matrix.** If \( (s_1, \ldots, s_n) \) is a tuple of complex reflections and if \( r_i, \hat{r}_i \) are corresponding roots and coroots, we call **Cartan matrix** the matrix \( C = \{ \tilde{c}(r_j) \}_{i,j} \). This matrix is unique up to conjugating by a diagonal matrix. Conversely, a class modulo the action of diagonal matrices of Cartan matrices is an invariant of the \( GL(V) \)-conjugacy class of the tuple. It determines this class if it is invertible and \( n = \dim V \). Indeed, this implies that the \( r_i \) form a basis of \( V \); and in this basis the matrix \( s_i \) differs from the identity matrix only on the \( i \)-th line, where the opposite of the \( i \)-th line of \( C \) has been added; thus \( C \) determines the \( s_i \).

If \( C \) can be chosen Hermitian (resp. symmetric), such a choice is then unique up to conjugating by a diagonal matrix of norm 1 elements of \( K \) (resp. of signs).

If \( C \) is Hermitian (which implies that the \( s_i \) are of order 2), then the sesquilinear form given by \( ^tC \) is invariant by the \( s_i \) (if the \( s_i \) are not of order 2, but the matrix obtained by replacing all elements on the diagonal of \( ^tC \) by 2’s is Hermitian, then the latter matrix defines a sesquilinear form invariant by the \( s_i \)).

0.4. **Coxeter element.** We keep the notation as above and we assume that the \( r_i \) form a basis of \( V \). We recall a result of [8] on the “Coxeter” element \( c = s_1 \cdots s_n \). If we write \( C = U + V \) where \( U \) is upper triangular unipotent and where \( V \) is lower triangular (with diagonal terms \(-\zeta_i\), thus \( V \) is also unipotent when \( s_i \) are of order 2), then the matrix of \( c \) in the \( r_i \) basis is \(-U^{-1}V\) (to see this write it as \( U_{s_1} \cdots s_{n} = -V \) and look at partial products in the left-hand side starting from the left). As \( U \) is of determinant 1, we deduce that \( \chi(c) = \det(xI + U^{-1}V) = \det(xU + V) \) where \( \chi(c) \) denotes the characteristic polynomial; in particular \( \det(C) = \chi(c) |_{x=1} \); one also gets that the fix-point set of \( c \) is the kernel of \( C \), equal to the intersection of the reflecting hyperplanes.

0.5. **The main theorem.** The next theorem implies the statement given in [8] ([8, 1.1] considers Euclidean reflections with the \( r_i \) linearly independent; if the \( r_i \) are chosen of the same length this implies that \( C \) is symmetric, and as \( C \) is then the Gram matrix of the \( r_i \) it is invertible):

**Theorem.** Let \( (s_1, \ldots, s_n) \) be a tuple of reflections in \( GL(\mathbb{R}^n) \) which have an associated Cartan matrix symmetric and invertible. Assume in addition that the Hurwitz orbit of the tuple is finite. Then the group generated by the \( s_i \) is finite.

**Proof.** In the next paragraph, we just need that \( (s_1, \ldots, s_n) \) is a tuple of complex reflections with a finite Hurwitz orbit and with the \( r_i \) a basis of \( V \).

A straightforward computation shows that an element of \( GL(V) \) commutes to the \( s_i \) if and only if it acts as a scalar on the subspaces generated by \( \{ r_i \}_{i \in I} \) where \( I \) is a block of \( C \) (i.e., a connected component of the graph with vertices \( \{ 1, \ldots, n \} \) and edges \( (i,j) \) for each pair such that either \( C_{i,j} \) or \( C_{j,i} \) is not zero). The finiteness of the Hurwitz orbit implies that for any subsequence \( (i_1, \ldots, i_k) \) of \( \{ 1, \ldots, n \} \), there exists a power of \( s_{i_1} \cdots s_{i_k} \) which commutes to \( s_{i_1}, \ldots, s_{i_k} \). This power acts thus as a scalar on each subspace generated by the \( r_{i_1} \) in a block of the submatrix of \( C \) determined by \( (i_1, \ldots, i_k) \). As the determinant of each \( s_{i_1} \) on this subspace is a root of unity, the scalar must be a root of unity. Thus, the restriction of each \( s_{i_1} \cdots s_{i_k} \) to the subspace \( < r_{i_1}, \ldots, r_{i_k} > \) generated by the \( r_{i_j} \) is of finite order.
We use from now on all the assumptions of the theorem. Thus the $s_i$ are order 2 elements of $O(C)$, the orthogonal group of the quadratic form defined by $C$.

Also, $\chi(c)$ is a polynomial with real coefficients. As $c$ is of finite order, any real root of $\chi(c)$ is 1 or $-1$. This implies that $\chi(c) \mid_{x=1}$ is a nonnegative real number, and thus $\det C$ also. The same holds for any principal minor of $C$, since such a minor is $\chi(c') \mid_{x=1}$ where $c'$ is the restriction of some $s_{i_1} \ldots s_{i_k}$ to $< r_{i_1}, \ldots, r_{i_k} >$. The quadratic form defined by $C$ is thus positive, and as $\det C \neq 0$ it is positive definite (cf. [1, §7, exercice 2]).

We now digress about the Cartan matrix of two reflections $s_1$ et $s_2$. Such a matrix is of the form $\begin{pmatrix} 2 & a \\ b & 2 \end{pmatrix}$. If $a = 0$ and $b \neq 0$ or $a \neq 0$ and $b = 0$ then $s_1 s_2$ is of infinite order. Otherwise, the number $ab$ is a complete invariant of the conjugacy class of $(s_1, s_2)$ restricted to $< r_1, r_2 >$, and $s_1 s_2$ restricted to this subspace is of finite order $m$ if and only if there exists $k$ prime to $m$ such that $ab = 4 \cos^2 k\pi/m$.

Since $C$ is symmetric and since the restriction of $s_i s_j$ to $< r_i, r_j >$ is of finite order, there exists prime integer pairs $(k_{i,j}, m_{i,j})$ such that $C_{i,j} = \pm 2 \cos k_{i,j} \pi/m_{i,j}$. If $K$ is the cyclotomic subfield containing the lcm$(2m_{i,j})$-th roots of unity, and if $\mathcal{O}$ is the ring of integers of $K$, we get that all coefficients of $C$ lie in $\mathcal{O}$. It follows, if $G$ is the group generated by the $s_i$, that in the $r_i$ basis we have $G \subset GL(\mathcal{O}_n)$.

We now apply Vinberg’s argument as in [3, 1.4.2]. Let $\sigma \in \text{Gal}(K/Q)$. Then $\sigma(C)$ is again positive definite: all arguments used to prove that $C$ is positive definite still apply for $\sigma(C)$: it is real, symmetric, invertible and the Hurwitz orbit of $(\sigma(s_1), \ldots, \sigma(s_n))$ is still finite. Since $G \subset O(C)$, which is compact, the entries of the elements of $G$ in the $r_i$ basis are of bounded norm. Since $O(\sigma(C))$ is also compact for any $\sigma \in \text{Gal}(K/Q)$, we get that entries of elements of $G$ are elements of $\mathcal{O}$ all of whose complex conjugates have a bounded norm. There is a finite number of such elements, so $G$ is finite. 

\[\Box\]

References