A duality between q-multiplicities in tensor products and q-multiplicities of weights for the root systems B, C or D

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Abstract

Starting from Jacobi-Trudi's type determinental expressions for the Schur functions of types B, C and D, we define a natural q-analogue of the multiplicity $[V(\lambda) : M(\mu)]$ when $M(\mu)$ is a tensor product of row or column shaped modules defined by μ. We prove that these q-multiplicities are equal to certain Kostka-Foulkes polynomials related to the root systems C or D. Finally we express the corresponding multiplicities in terms of Kostka numbers

1 Introduction

Given two partitions λ and μ of length n, the Kostka number $K_{\lambda,\mu}^{A_n}$ is equal to the dimension of the weight space μ in the finite dimensional irreducible sl_{n+1}-module $V(\lambda)$ of highest weight λ. The Schur duality is a classical result in representation theory establishing that $K_{\lambda,\mu}^{A_n}$ is also equal to the multiplicity of $V(\lambda)$ in the tensor products

$$V(\mu_1\Lambda_1) \otimes \cdots \otimes V(\mu_n\Lambda_1) \text{ and } V(\Lambda_{\mu'_1}) \otimes \cdots \otimes V(\Lambda_{\mu'_m})$$

where $\mu' = (\mu'_1, ..., \mu'_m)$ is the conjugate partition of μ and the Λ_i's $i = 1, ..., n-1$ are the fundamental weights of sl_{n+1}. Another way to define $K_{\lambda,\mu}^{A_n}$ is to use the Jacobi-Trudi identity which gives a determinantal expression of the Schur function $s_\mu = \mathrm{char}(V(\mu))$ in terms of the characters $h_k = \mathrm{char}(V(k\Lambda_1))$ of the k-th symmetric power representation. This formula can be rewritten

$$s_\mu = \prod_{1 \le i < j \le n} (1 - R_{i,j}) h_\mu \tag{1}$$

where $h_\mu = h_{\mu_1} \cdots \cdot h_{\mu_n}$ and the $R_{i,j}$ are the raising operators (see 3.2). Then one can prove that it makes sense to write

$$h_\mu = \prod_{1 \le i < j \le n} (1 - R_{i,j})^{-1} s_\mu \tag{2}$$

which gives the decomposition of h_μ on the basis of Schur functions. From this decomposition we derive the following expression for $K_{\lambda,\mu}^{A_n}$:

$$K_{\lambda,\mu}^{A_n} = \sum_{\sigma \in \mathcal{S}_n} (-1)^{l(\sigma)} \mathcal{P}^{A_n}(\sigma(\lambda + \rho) - (\mu + \rho)) \tag{3}$$

where $\mathcal{S}_n$ is the symmetric group of order n and $\mathcal{P}^{A_n}$ the ordinary Kostant's partition function defined from the equality:

$$\prod_{\alpha \text{ positive root}} \frac{1}{(1 - x^\alpha)} = \sum_\beta \mathcal{P}^{A_n}(\beta) x^\beta$$

with β running on the set of nonnegative integral combinations of positive roots of sl_n.
There exists a q-analogue $K_{\lambda,\mu}^{A_n}(q)$ of $K_{\lambda,\mu}^{A_n}$ obtained by replacing the ordinary Kostant's partition function $\mathcal{P}^{A_n}$ by its q-analogue $\mathcal{P}_q^{A_n}$ satisfying

$$\prod_{\alpha \text{ positive root}} \frac{1}{(1-qx^{\alpha})} = \sum_{\beta} \mathcal{P}_q^{A_n}(\beta)x^{\beta}.$$

So we have

$$K_{\lambda,\mu}^{A_n}(q) = \sum_{\sigma\in\mathcal{S}_n} (-1)^{l(\sigma)} \mathcal{P}_q^{A_n}(\sigma(\lambda+\rho)-(\mu+\rho)) \tag{4}$$

which is a polynomial in q with nonnegative integer coefficients [8], [9]. In [11], Nakayashiki and Yamada have shown that $K_{\lambda,\mu}^{A_n}(q)$ can also be computed from the combinatorial R matrix corresponding to Kashiwara's crystals associated to some $U_q(\widehat{sl_n})$-modules.

For $g = so_{2n+1}, sp_{2n}$ or so_{2n} there also exist expressions similar to (3) for the multiplicities $K_{\lambda,\mu}^{g}$ of the weight μ in the finite dimensional irreducible module $V(\lambda)$ but a so simple duality as for sl_n does not exist although it is possible to obtain certain duality results between multiplicities of weights and tensor product multiplicities of representations by using duals pairs of algebraic groups ([5]). This implies that the quantifications of weight multiplicities and tensor product multiplicities can not coincide for the root systems B_n, C_n and D_n. The Kostka-Foulkes polynomials $K_{\lambda,\mu}^{g}(q)$ are the q-analogues of $K_{\lambda,\mu}^{g}$ defined as in (4) by quantifying the partition function corresponding to the root system associated to g (see 2.2). In [13], Hatayama, Kuniba, Okado and Takagi have introduced for type C_n a quantification $X_{\lambda,\mu}^{C_n}(q)$ of the multiplicity of $V(\lambda)$ in the tensor product

$$W(\mu_1\Lambda_1)\otimes\cdots\otimes W(\mu_n\Lambda_1)$$

where for any $i = 1, ..., n$,

$$W(\mu_i\Lambda_1) = V(\mu_i\Lambda_1)\oplus V((\mu_i-2)\Lambda_1)\oplus\cdots\oplus V((\mu_i \text{mod} 2)\Lambda_1).$$

This quantification is based on the determination of the combinatorial R matrix of some $U_q'(\widehat{g})$-crystals in the spirit of [11]. Note that there also exist q-multiplicities for the sp_2-module $V(\lambda)$ in a tensor product

$$V(\Lambda_1)^{\otimes k}\otimes V(\Lambda_2)^{\otimes l}$$

where k, l are positive integers obtained by Yamada [17].

In this paper we first use Jacobi-Trudi's type determinantal expressions for the Schur functions associated to g to introduce q-analogues of the multiplicity of $V(\lambda)$ in the tensor products

$$\begin{array}{l}\text{(i)}: \mathfrak{h}(\mu) = V(\mu_1\Lambda_1)\otimes\cdots\otimes V(\mu_n\Lambda_1), \mathfrak{H}(\mu) = W(\mu_1\Lambda_1)\otimes\cdots\otimes W(\mu_n\Lambda_1)\\ \text{(ii)}: \mathfrak{e}(\mu) = V(\Lambda_{\mu_1'})\otimes\cdots\otimes V(\Lambda_{\mu_m'}), \mathfrak{E}(\mu) = W(\Lambda_{\mu_1'})\otimes\cdots\otimes W(\Lambda_{\mu_m'}) \text{ with } n\geq|\mu|\end{array}$$

where

$$\left\{\begin{array}{c} W(\mu_i\Lambda_1) = V(\mu_i\Lambda_1)\oplus V((\mu_i-2)\Lambda_1)\oplus\cdots\oplus V((\mu_i \text{mod} 2)\Lambda_1) \\ W(\Lambda_k) = V(\Lambda_k)\oplus V(\Lambda_{k-2})\oplus\cdots\oplus V(\Lambda_{k \bmod 2}) \end{array}\right. .$$

With the condition $n\geq|\mu|$ for (ii), these multiplicities are independent of the Lie algebra g of type B_n, C_n or D_n considered. When $q=1$, we recover a remarkable property already used by Koike and Terada in [6]. Next we prove that these q-multiplicities are in fact equal to Kostka-Foulkes polynomials associated to the root systems of types C and D. It is possible to extend the definition (4) of the Kostka-Foulkes polynomials associated to the root system A_n by replacing μ by $\gamma \in \mathbb{N}^n$ where γ is not a partition. In this case $K_{\lambda,\gamma}^{A_n}(q)$ may have nonnegative coefficients but $K_{\lambda,\gamma}^{A_n}(1)$ is equal to the dimension of the weight space γ in $V(\lambda)$. Now if we extend (4) by replacing λ by $\xi \in \mathbb{N}^n$, the polynomial $K_{\xi,\mu}^{A_n}(q)$ is equal up to a sign to a Kostka-Foulkes polynomial $K_{\nu,\mu}^{A_n}(q)$ where ν is a partition. We obtained two expressions of the q-multiplicities defined above respectively in terms of the polynomials $K_{\lambda,\gamma}^{A_n}(q)$ and $K_{\xi,\mu}^{A_n}(q)$. By specializing at $q=1$, this yields expressions of the corresponding multiplicities in terms of Kostka numbers.

In section 1 we recall the background on the root systems B_n, C_n and D_n and the corresponding Kostka-Foulkes polynomials. We review in section 2 the determinantal identities for Schur functions that we need in the sequel and we introduce the formalism suggested in [1] to prove the expressions of Schur functions in terms of raising and lowering operators implicitly contain in [15]. Thank to this formalism we are able to obtain expressions for multiplicities similar to (3). We quantify these multiplicities to obtain the desired q-analogues in section 3. We prove in Section 4 two duality theorems between our q-analogues and certain Kostka-Foulkes polynomials of types C and D. Finally we establish formulas expressing the associated multiplicities in terms of Kostka numbers.

Notation: In the sequel we frequently define similar objects for the root systems B_n C_n and D_n. When they are related to type B_n (resp. C_n, D_n), we implicitly attach to them the label B (resp. the labels C, D). To avoid cumbersome repetitions, we sometimes omit the labels B, C and D when our definitions or statements are identical for the three root systems.

Note: *While writing this work, I have been informed that Shimozono and Zabrocki [16] have introduced independently and by using creating operators essentially the same tensor power multiplicities. Thanks to this formalism they recover in particular Jacobi-Trudi's type determinantal expressions of the Schur functions associated to the root systems B, C and D which constitute the starting point of this article.*

2 Background on the root systems B_n, C_n and D_n

2.1 Convention for the positive roots

Consider an integer $n \geq 1$. The weight lattice for the root system C_n (resp. B_n and D_n) can be identified with $P_{C_n} = \mathbb{Z}^n$ (resp. $P_{B_n} = P_{D_n} \left(\frac{\mathbb{Z}}{2}\right)^n$) equipped with the orthonormal basis ε_i, $i = 1, ..., n$. We take for the simple roots

$$\left\{ \begin{array}{l} \alpha_n^{B_n} = \varepsilon_n \text{ and } \alpha_i^{B_n} = \varepsilon_i - \varepsilon_{i+1},\ i = 1, ..., n-1 \text{ for the root system } B_n \\ \alpha_n^{C_n} = 2\varepsilon_n \text{ and } \alpha_i^{C_n} = \varepsilon_i - \varepsilon_{i+1},\ i = 1, ..., n-1 \text{ for the root system } C_n \\ \alpha_n^{D_n} = \varepsilon_n + \varepsilon_{n-1} \text{ and } \alpha_i^{D_n} = \varepsilon_i - \varepsilon_{i+1},\ i = 1, ..., n-1 \text{ for the root system } D_n \end{array} \right. . \tag{5}$$

Then the set of positive roots are

$$\left\{ \begin{array}{l} R_{B_n}^+ = \{\varepsilon_i - \varepsilon_j, \varepsilon_i + \varepsilon_j \text{ with } 1 \leq i < j \leq n\} \cup \{\varepsilon_i \text{ with } 1 \leq i \leq n\} \text{ for the root system } B_n \\ R_{C_n}^+ = \{\varepsilon_i - \varepsilon_j, \varepsilon_i + \varepsilon_j \text{ with } 1 \leq i < j \leq n\} \cup \{2\varepsilon_i \text{ with } 1 \leq i \leq n\} \text{ for the root system } C_n \\ R_{D_n}^+ = \{\varepsilon_i - \varepsilon_j, \varepsilon_i + \varepsilon_j \text{ with } 1 \leq i < j \leq n\} \text{ for the root system } D_n \end{array} \right. .$$

Denote respectively by $P_{B_n}^+, P_{C_n}^+$ and $P_{D_n}^+$ the sets of dominant weights of so_{2n+1}, sp_{2n} and so_{2n}.

Let $\lambda = (\lambda_1, ..., \lambda_n)$ be a partition with n parts. We will classically identify λ with the dominant weight $\sum_{i=1}^{n} \lambda_i \varepsilon_i$. Note that there exists dominant weights associated to the orthogonal root systems whose coordinates on the basis ε_i, $i = 1, ..., n$ are not positive integers (hence which can not be regarded as a partition). For each root system of type B_n, C_n or D_n, the set of weights having nonnegative integer coordinates on the basis $\varepsilon_1, ..., \varepsilon_n$ can be identify with the set π_n^+ of partitions of length n. For any partition λ, the weights of the finite dimensional so_{2n+1}, sp_{2n} or so_{2n}-module of highest weight λ are all in $\pi_n = \mathbb{Z}^n$. For any $\alpha \in \pi_n$ we write $|\alpha| = \alpha_1 + \cdots + \alpha_n$.
The conjugate partition of the partition λ is denoted λ' as usual. Consider λ, μ two partitions of length n and set $m = \max(\lambda_1, \mu_1)$. Then by adding to λ' and μ' the required numbers of parts 0 we will consider them as partitions of length m.

The Weyl group $W_{B_n} = W_{C_n}$ of so_{2n+1} and sp_{2n} is identified to the sub-group of the permutation group of the set $\{\overline{n}, ..., \overline{2}, \overline{1}, 1, 2, ..., n\}$ generated by $s_i = (i, i+1)(\overline{i}, \overline{i+1})$, $i = 1, ..., n-1$ and $s_n = (n, \overline{n})$ where for $a \neq b$ (a, b) is the simple transposition which switches a and b. We denote by l_B the length function corresponding to the set of generators s_i, $i = 1, ...n$.
The Weyl group W_{D_n} of so_{2n} is identified to the sub group of W_{B_n} generated by $s_i = (i, i+1)(\overline{i}, \overline{i+1})$, $i = 1, ..., n-1$ and $s_n' = (n, \overline{n-1})(n-1, \overline{n})$. We denote by l_D the length function corresponding to the set of generators s_n' and s_i, $i = 1, ...n-1$.
Note that $W_{D_n} \subset W_{B_n}$ and any $w \in W_{B_n}$ verifies $w(\overline{i}) = \overline{w(i)}$ for $i \in \{1, ..., n\}$. The action of w on $\beta = (\beta_1, ..., \beta_n) \in P_n$ is given by

$$w \cdot (\beta_1, ..., \beta_n) = (\beta_1^w, ..., \beta_n^w)$$

where $\beta_i^w = \beta_{w(i)}$ if $\sigma(i) \in \{1, ..., n\}$ and $\beta_i^w = -\beta_{w(\overline{i})}$ otherwise.
The half sums ρ_{B_n}, ρ_{C_n} and ρ_{D_n} of the positive roots associated to each root system B_n, C_n and D_n verify:

$$\rho_{B_n} = (n - \frac{1}{2}, n - \frac{3}{2}, ..., \frac{1}{2}), \rho_{C_n} = (n, n-1, ..., 1) \text{ and } \rho_{B_n} = (n-1, n-2, ..., 0).$$

In the sequel we identify the symmetric group $\mathcal{S}_n$ with the sub group of W_{B_n} or W_{D_n} generated by the s_i's, $i = 1, ..., n-1$.

2.2 Schur functions and Kostka-Foulkes polynomials

We now briefly review the notions of Schur functions and Kostka-Foulkes polynomials associated to the roots systems B_n, C_n and D_n and refer the reader to [12] for more details. For any weight $\beta = (\beta_1, ..., \beta_n) \in \pi_n$ we set $x^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n}$ where $x_1, ..., x_n$ are fixed indeterminates. We set

$$a_\beta^{B_n} = \sum_{w \in W_{B_n}} (-1)^{l(\sigma)} (w \cdot x^\beta)$$

where $w \cdot x^\mu = x^{w(\mu)}$. The Schur function $s_\beta^{B_n}$ is defined as in [12] by

$$s_\beta^{B_n} = \frac{a_{\beta + \rho_{B_n}}^{B_n}}{a_{\rho_{B_n}}^{B}}.$$

When $\nu \in \pi_n^+$, $s_\nu^{B_n}$ is the Weyl character of $V(\nu)$ the finite dimensional irreducible module with highest weight ν. For any $w \in W_{B_n}$, the dot action of w on $\beta \in \pi_n$ is defined by

$$w \circ \beta = w \cdot (\beta + \rho_{B_n}) - \rho_{B_n}.$$

We have the following straightening law for the Schur functions. For any $\beta \in \pi_n$, $s_\beta^{B_n} = 0$ or there exists a unique $\nu \in \pi_n^+$ such that $s_\beta^{B_n} = (-1)^{l(w)} s_\nu^{B_n}$ with $w \in W_{B_n}$ and $\nu = w \circ \beta$. Set $\mathbb{K} = \mathbb{Z}[q, q^{-1}]$ and write $\mathbb{K}[\pi_n]$ for the $\mathbb{K}$-module generated by the x^β, $\beta \in \pi_n$. Set $\mathcal{C}_{B_n} = \mathbb{K}[\pi_n]^{W_{B_n}} = \{f \in \mathbb{K}[\pi_n],$ $w \cdot f = f$ for any $w \in W_{B_n}\}$. Then $\{s_\nu^{B_n}\}, \nu \in \pi_n^+$ is a basis of $\mathbb{K}[\pi_n]^{W_{B_n}}$.
We define $s_\beta^{C_n}$ and $s_\beta^{D_n}$ belonging to $\mathcal{C}_{C_n} = \mathcal{C}_{B_n}$ and $\mathcal{C}_{D_n}$ in the same way and we obtain similarly that $\{s_\nu^{C_n}, \nu \in \pi_n^+\}$ and $\{s_\nu^{D_n}, \nu \in \pi_n^+\}$ are respectively bases of $\mathcal{C}_{C_n}$ and $\mathcal{C}_{D_n}$.

The q-analogue $\mathcal{P}_q^{B_n}$ of Kostant's partition function corresponding to the root system B_n is defined by the equality

$$\prod_{\alpha \in R_{B_n}^+} \frac{1}{1 - q x^\alpha} = \sum_{\beta \in \pi_n} \mathcal{P}_q^{B_n}(\beta) x^\beta.$$

Note that $\mathcal{P}_q^{B_n}(\beta) = 0$ if β is not a linear combination of positive roots of $R_{B_n}^+$ with nonnegative coefficients. We write similarly $\mathcal{P}_q^{C_n}$ and $\mathcal{P}_q^{D_n}$ for the q-partition functions associated respectively to the root systems C_n and D_n. Given λ and μ two partitions of length n, the Kostka-Foulkes polynomials of types B_n, C_n and D_n are then respectively defined by

$$K_{\lambda,\mu}^{B_n}(q) = \sum_{\sigma \in W_{B_n}} (-1)^{l(\sigma)} \mathcal{P}_q^{B_n}(\sigma(\lambda + \rho_{B_n}) - (\mu + \rho_{B_n})),$$
$$K_{\lambda,\mu}^{C_n}(q) = \sum_{\sigma \in W_{C_n}} (-1)^{l(\sigma)} \mathcal{P}_q^{C_n}(\sigma(\lambda + \rho_{C_n}) - (\mu + \rho_{C_n})),$$
$$K_{\lambda,\mu}^{D_n}(q) = \sum_{\sigma \in W_{D_n}} (-1)^{l(\sigma)} \mathcal{P}_q^{D_n}(\sigma(\lambda + \rho_{D_n}) - (\mu + \rho_{D_n})).$$

Remarks:
(i) : We have $K_{\lambda,\mu}(q) = 0$ when $|\lambda| < |\mu|$.
(ii) : When $|\lambda| = |\mu|$, $K_{\lambda,\mu}^{B_n}(q) = K_{\lambda,\mu}^{C_n}(q) = K_{\lambda,\mu}^{D_n}(q) = K_{\lambda,\mu}^{A_{n-1}}(q)$ that is, the Kostka-Foulkes polynomials associated to the root systems B_n, C_n and D_n are Kostka-Foulkes polynomials associated to the root system A_{n-1}.

3 Determinantal identities and multiplicities of representations

3.1 Determinantal identities for Schur functions

Consider $k \in \mathbb{Z}$. When k is a nonnegative integer, write $(k)_n = (k, 0, ..., 0)$ for the partition of length n with a unique non-zero part equal to k. Then set

$$h_k^{B_n} = s_{(k)_n}^{B_n}, h_k^{C_n} = s_{(k)_n}^{C_n}, h_k^{D_n} = s_{(k)_n}^{D_n}$$

and

$$H_k^{B_n} = h_k^{B_n} + h_{k-2}^{B_n} + \cdots + h_{k \bmod 2}^{B_n}, H_k^{C_n} = h_k^{C_n} + h_{k-2}^{C_n} + \cdots + h_{k \bmod 2}^{B_n},$$
$$H_k^{D_n} = h_k^{D_n} + h_{k-2}^{D_n} + \cdots + h_{k \bmod 2}^{D_n}.$$

When k is a negative integer we set $h_k^{B_n}=h_k^{C_n}=h_k^{D_n}=0$ and $H_k^{B_n}=H_k^{C_n}=H_k^{D_n}=0$.
For any $\alpha=(\alpha_1,...,\alpha_n)\in\mathbb{Z}^n$ define

$$u_\alpha^{B_n}=\det\begin{pmatrix} h_{\alpha_1}^{B_n} & h_{\alpha_1+1}^{B_n}+h_{\alpha_1-1}^{B_n} & \cdots\cdots\cdots & h_{\alpha_1+n-1}^{B_n}+h_{\alpha_1-n+1}^{B_n} \\ h_{\alpha_2-1}^{B_n} & h_{\alpha_2}^{B_n}+h_{\alpha_2-2}^{B_n} & \cdots\cdots\cdots & h_{\alpha_2+n-2}^{B_n}+h_{\alpha_2-n}^{B_n} \\ . & . & \cdots\cdots\cdots & . \\ . & . & \cdots\cdots\cdots & . \\ h_{\alpha_n-n+1}^{B_n} & h_{\alpha_n-n+2}^{B_n}+h_{\alpha_n-n}^{B_n} & \cdots\cdots\cdots & h_{\alpha_n}^{B_n}+h_{\alpha_n-2n+2}^{B_n} \end{pmatrix}. \quad (6)$$

By using the equalities $h_k^{B_n}=H_k^{B_n}-H_{k-2}^{B_n}$ and simple computations on determinants we have also

$$u_\alpha^{B_n}=\det\begin{pmatrix} H_{\alpha_1}^{B_n}-H_{\alpha_1-2}^{B_n} & H_{\alpha_1+1}^{B_n}-H_{\alpha_1-1}^{B_n} & \cdots\cdots\cdots & H_{\alpha_1+n-1}^{B_n}-H_{\alpha_1-n-1}^{B_n} \\ H_{\alpha_2-1}^{B_n}-H_{\alpha_2-3}^{B_n} & H_{\alpha_2}^{B_n}-H_{\alpha_2-4}^{B_n} & \cdots\cdots\cdots & H_{\alpha_2+n-2}^{B_n}-H_{\alpha_2-n-2}^{B_n} \\ . & . & \cdots\cdots\cdots & . \\ . & . & \cdots\cdots\cdots & . \\ H_{\alpha_n-n+1}^{B_n}-H_{\alpha_n-n-1}^{B_n} & H_{\alpha_n-n+2}^{B_n}-H_{\alpha_n-n-2}^{B_n} & \cdots\cdots\cdots & H_{\alpha_n}^{B_n}-H_{\alpha_n-2n-2}^{B_n} \end{pmatrix}. \quad (7)$$

We define $u_\alpha^{C_n}$ and $u_\alpha^{D_n}$similarly by replacing $h_k^{B_n}$ respectively by $h_k^{C_n}$ and $h_k^{D_n}$.

Consider p and n two integers such that $n\geq 1$. When p is nonnegative and $n\geq p$, write $(1^p)_n=(1,...,1,0,...,0)$ for the partition of length n having p non zero parts equal to 1. We set

$$\begin{cases} e_p^{B_n}=s_{(1^p)_n}^{B_n}, e_p^{C_n}=s_{(1^p)_n}^{C_n}, e_p^{D_n}=s_{(1^p)_n}^{D_n} \text{ if } 0\leq p\leq n \\ e_p^{B_n}=e_{2p-n}^{B_n}, e_p^{C_n}=e_{2p-n}^{C_n}, e_k^{D_n}=e_{2p-n}^{D_n} \text{ if } n+1\leq p\leq 2n \\ e_p^{B_n}=e_p^{C_n}=e_p^{D_n}=0 \text{ otherwise} \end{cases}$$

and

$$E_k^{B_n}=e_k^{B_n}+e_{k-2}^{B_n}+\cdots+e_{k \bmod 2}^{B_n}, E_k^{C_n}=e_k^{C_n}+e_{k-2}^{C_n}+\cdots+e_{k \bmod 2}^{B_n},$$
$$E_k^{D_n}=e_k^{D_n}+e_{k-2}^{D_n}+\cdots+e_{k \bmod 2}^{D_n}.$$

For any $\beta=(\beta_1,...,\beta_n)\in\mathbb{Z}^n$ define

$$v_\beta^{B_n}=\det\begin{pmatrix} e_{\beta_1}^{B_n} & e_{\beta_1+1}^{B_n}+e_{\beta_1-1}^{B_n} & \cdots\cdots\cdots & e_{\beta_1+n-1}^{B_n}+e_{\beta_1-n+1}^{B_n} \\ e_{\beta_2-1}^{B_n} & e_{\beta_2}^{B_n}+e_{\beta_2-2}^{B_n} & \cdots\cdots\cdots & e_{\beta_2+n-2}^{B_n}+e_{\beta_2-n}^{B_n} \\ . & . & \cdots\cdots\cdots & . \\ . & . & \cdots\cdots\cdots & . \\ e_{\beta_n-n+1}^{B_n} & e_{\beta_n-n+2}^{B_n}+e_{\beta_n-n}^{B_n} & \cdots\cdots\cdots & e_{\beta_n}^{B_n}+e_{\beta_n-2n+2}^{B_n} \end{pmatrix}.$$

By using the equalities $e_k^{B_n}=E_k^{B_n}-E_{k-2}^{B_n}$ and simple computations on determinants we have also

$$v_\beta^{B_n}=\det\begin{pmatrix} E_{\beta_1}^{B_n}-E_{\beta_1-2}^{B_n} & E_{\beta_1+1}^{B_n}-E_{\beta_1-1}^{B_n} & \cdots\cdots\cdots & E_{\beta_1+n-1}^{B_n}-E_{\beta_1-n-1}^{B_n} \\ E_{\beta_2-1}^{B_n}-E_{\beta_2-3}^{B_n} & E_{\beta_2}^{B_n}-E_{\beta_2-4}^{B_n} & \cdots\cdots\cdots & E_{\beta_2+n-2}^{B_n}-E_{\beta_2-n-2}^{B_n} \\ . & . & \cdots\cdots\cdots & . \\ . & . & \cdots\cdots\cdots & . \\ E_{\beta_n-n+1}^{B_n}-E_{\beta_n-n-1}^{B_n} & E_{\beta_n-n+2}^{B_n}-E_{\beta_n-n-2}^{B_n} & \cdots\cdots\cdots & E_{\beta_n}^{B_n}-E_{\beta_n-2n-2}^{B_n} \end{pmatrix}$$

The determinants $v_\beta^{C_n}, v_\beta^{D_n}$ are defined similarly.

Proposition 3.1.1 *(see[3]) Consider λ a partition of length n and suppose that $\lambda' = (\lambda'_1, ..., \lambda'_m)$ is a partition of length m. Then $u_\lambda = s_\lambda$ and $v_{\lambda'} = s_\lambda$.*

Lemma 3.1.2 *(straightening law for u_α and v_β)*
Consider $\alpha \in \pi_n$ then

$$u_\alpha = \begin{cases} (-1)^{l(\sigma)} u_\lambda \text{ if there exists } \sigma \in \mathcal{S}_n \text{ and } \lambda \in \pi_n^+ \text{ such that } \sigma \circ \alpha = \lambda \\ 0 \text{ otherwise} \end{cases}.$$

Consider $\beta \in \pi_m$ then

$$v_\beta = \begin{cases} (-1)^{l(\sigma)} v_\nu \text{ if there exists } \sigma \in \mathcal{S}_m \text{ and } \nu \in \pi_m^+ \text{ such that } \sigma \circ \alpha = \nu \\ 0 \text{ otherwise} \end{cases}.$$

Proof. By commuting the rows i and $i+1$ in the determinant (7) we see that $u_{s_i \circ \alpha} = -u_\alpha$. This implies that $u_{\sigma \circ \alpha} = (-1)^{l(\sigma)} u_\alpha$ for any $\sigma \in \mathcal{S}_n$. Then it follows from the definition of the dot action that $u_\alpha = 0$ or there exists $\gamma \in \pi_n$ and $\sigma \in \mathcal{S}_n$ such that $\gamma_1 \geq \cdots \geq \gamma_n$ and $\gamma = \sigma \circ \alpha$. In this last case we have $u_\alpha = (-1)^{l(\sigma)} u_\gamma$. Now if there exists a negative γ_i, $u_\gamma = 0$ since all the H_k which appear in the lowest row of (7) are equal to 0. Thus γ is a partition. The proof is similar for v_β. ■

3.2 Determinantal identities in terms of raising and lowering operators

Let $\mathcal{L}_n = \mathbb{K}[[x_1, x_1^{-1}, ..., x_n, x_n^{-1}]]$ be the ring of formal series in the indeterminates $x_1, x_1^{-1}, ..., x_n, x_n^{-1}$. We consider the two following determinants

$$\delta_n(\alpha) = \det \begin{pmatrix} x_1^{\alpha_1} & x_1^{\alpha_1+1} + x_1^{\alpha_1-1} & \cdots\cdots\cdots & x_1^{\alpha_1+n-1} + x_1^{\alpha_1-n+1} \\ x_2^{\alpha_2} & x_2^{\alpha_2} + x_2^{\alpha_2-2} & \cdots\cdots\cdots & x_2^{\alpha_2+n-2} + x_2^{\alpha_2-n} \\ \cdot & \cdot & \cdots\cdots\cdots & \cdot \\ \cdot & \cdot & \cdots\cdots\cdots & \cdot \\ x_n^{\alpha_n-n+1} & x_n^{\alpha_n-n+2} + x_n^{\alpha_n-n} & \cdots\cdots\cdots & x_n^{\alpha_n} + x_n^{\alpha_n-2n+2} \end{pmatrix} \text{ and}$$

$$\Delta_n(\alpha) = \det \begin{pmatrix} x_1^{\alpha_1} - x_1^{\alpha_1-2} & x_1^{\alpha_1+1} - x_1^{\alpha_1-1} & \cdots\cdots\cdots & x_1^{\alpha_1+n-1} - x_1^{\alpha_1-n-1} \\ x_2^{\alpha_2-1} - x_2^{\alpha_2-3} & x_2^{\alpha_2} - x_2^{\alpha_2-4} & \cdots\cdots\cdots & x_2^{\alpha_2+n-2} - x_2^{\alpha_2-n-2} \\ \cdot & \cdot & \cdots\cdots\cdots & \cdot \\ \cdot & \cdot & \cdots\cdots\cdots & \cdot \\ x_n^{\alpha_n-n+1} + x_n^{\alpha_n-n-1} & x_n^{\alpha_n-n+2} - x_n^{\alpha_n-n} & \cdots\cdots\cdots & x_n^{\alpha_n} - x_n^{\alpha_n-2n-2} \end{pmatrix}$$

From a simple computation we derive the equalities:

$$\delta_n(\alpha) = \prod_{1 \leq i < j \leq n} (1 - \frac{x_i}{x_j}) \prod_{1 \leq r < s \leq n} (1 - \frac{1}{x_i x_j}) x^\alpha \text{ and } \Delta_n(\alpha) = \prod_{1 \leq i < j \leq n} (1 - \frac{x_i}{x_j}) \prod_{1 \leq r \leq s \leq n} (1 - \frac{1}{x_i x_j}) x^\alpha. \quad (8)$$

We set $h_\alpha = h_{\alpha_1} \cdots h_{\alpha_n}, H_\alpha = H_{\alpha_1} \cdots H_{\alpha_n}, e_\alpha = e_{\alpha_1} \cdots e_{\alpha_n}$ and $E_\alpha = E_{\alpha_1} \cdots E_{\alpha_n}$.
Remarks
(i) : For any partition μ of length n, h_μ is the character of $\mathfrak{h}(\mu) = V(\mu_1 \Lambda_1) \otimes \cdots \otimes V(\mu_n \Lambda_1)$ and H_μ is the character of $\mathfrak{H}(\mu) = W(\mu_1 \Lambda_1) \otimes \cdots \otimes W(\mu_n \Lambda_1)$ where for any $k \in \mathbb{N}$, $W(k_1) = V(k\Lambda_1) \oplus V((k-2)\Lambda_1) \oplus \cdots \oplus V((k \bmod 2)\Lambda_1)$.
(ii) : For any partition μ of length n such that μ' is of length m, $e_{\mu'}$ is the character of $\mathfrak{e}(\mu) = V(\Lambda_{\mu'_1}) \otimes \cdots \otimes V(\Lambda_{\mu'_m})$ and $E_{\mu'}$ is the character of $\mathfrak{E}(\mu) = W(\Lambda_{\mu'_1}) \otimes \cdots \otimes W(\Lambda_{\mu'_m})$ where for any $k \in \mathbb{N}$ with $k \leq n$, $W(\Lambda_k) = V(\Lambda_k) \oplus V(\Lambda_{k-2}) \oplus \cdots \oplus V(\Lambda_{k \bmod 2})$.

For the root system B_n we introduce six linear maps $\mathrm{h}_{B_n}, \mathrm{H}_{B_n}, \mathrm{u}_{B_n}$ and $\mathrm{e}_{B_n}, \mathrm{E}_{B_n}, \mathrm{v}_{B_n}$ as follows:

$$\left\{ \begin{array}{c} \mathrm{h}_{B_n} : \mathcal{L}_n \to \mathcal{C}_{B_n} \\ x^\alpha \mapsto h_\alpha^{B_n} \end{array} \right., \left\{ \begin{array}{c} \mathrm{H}_{B_n} : \mathcal{L}_n \to \mathcal{C}_{B_n} \\ x^\alpha \mapsto H_\alpha^{B_n} \end{array} \right., \left\{ \begin{array}{c} \mathrm{u}_{B_n} : \mathcal{L}_n \to \mathcal{C}_{B_n} \\ x^\alpha \mapsto u_\alpha^{B_n} \end{array} \right. \text{ and}$$

$$\left\{ \begin{array}{c} \mathrm{e}_{B_n} : \mathcal{L}_n \to \mathcal{C}_{B_n} \\ x^\alpha \mapsto e_\alpha^{B_n} \end{array} \right., \left\{ \begin{array}{c} \mathrm{E}_{B_n} : \mathcal{L}_n \to \mathcal{C}_{B_n} \\ x^\alpha \mapsto E_\alpha^{B_n} \end{array} \right., \left\{ \begin{array}{c} \mathrm{v}_{B_n} : \mathcal{L}_n \to \mathcal{C}_{B_n} \\ x^\alpha \mapsto v_\alpha^{B_n} \end{array} \right. .$$

Note that these maps are not ring homomorphisms. For the roots systems C_n and D_n we define respectively the maps $\mathrm{h}_{C_n}, \mathrm{H}_{C_n}, \mathrm{u}_{C_n}, \mathrm{e}_{C_n}, \mathrm{E}_{C_n}, \mathrm{v}_{C_n}$ and $\mathrm{h}_{D_n}, \mathrm{H}_{D_n}, \mathrm{u}_{D_n}, \mathrm{e}_{D_n}, \mathrm{E}_{D_n}, \mathrm{v}_{D_n}$ similarly.
Let ω_n and Ω_n be the endomorphisms of $\mathcal{L}_n$ corresponding respectively to the multiplication by

$$\phi_n = \prod_{1\le i<j\le n}(1-\frac{x_i}{x_j})\prod_{1\le r<s\le n}(1-\frac{1}{x_ix_j}) \text{ and } \Phi_n = \prod_{1\le i<j\le n}(1-\frac{x_i}{x_j})\prod_{1\le r\le s\le n}(1-\frac{1}{x_ix_j}).$$

Since ϕ_n^{-1} and Φ_n^{-1} belong to $\mathcal{L}_n$, ω_n and Ω_n are the automorphisms of $\mathcal{L}_n$ corresponding to the multiplication by ϕ_n^{-1} and Φ_n^{-1}.

Proposition 3.2.1 *We have*

1. $\mathrm{u}_n = \mathrm{h}_n \cdot \omega_n$ *and* $\mathrm{u}_n = \mathrm{H}_n \cdot \Omega_n$,
2. $\mathrm{v}_n = \mathrm{e}_n \cdot \omega_n$ *and* $\mathrm{v}_n = \mathrm{E}_n \cdot \Omega_n$.

Proof. 1 : We have seen that h_n is not a ring-homomorphism. Nevertheless we have by definition of the h_α

$$\mathrm{h}_n(x^\alpha) = \mathrm{h}_n(x_1^{\alpha_1})\cdots \mathrm{h}_n(x_n^{\alpha_n}) = h_{\alpha_1}\cdots h_{\alpha_n}.$$

More generally if $P_1, ..., P_n$ are polynomials respectively in the indeterminates $x_1, ..., x_n$, we have

$$\mathrm{h}_n(P_1(x_1)\cdots P_n(x_n)) = \mathrm{h}_n(P_1(x_1))\cdots \mathrm{h}_n(P_n(x_n))$$

by linearity of h_n. We can write

$$\delta_n(\alpha) = \sum_{\sigma\in\mathcal{S}_n}(-1)^{l(\sigma)}x_{\sigma(1)}^{\alpha_1-\sigma(1)+1}(x_{\sigma(2)}^{\alpha_2-\sigma(2)+2}+x_{\sigma(2)}^{\alpha_2-\sigma(2)})\cdots(x_{\sigma(n)}^{\alpha_n-\sigma(n)+n}+x_{\sigma(n)}^{\alpha_n-\sigma(n)-n+2})$$

and by the previous argument

$$\mathrm{h}_n(\delta_n(\alpha)) = \sum_{\sigma\in\mathcal{S}_n}(-1)^{l(\sigma)}h_{\alpha_1-\sigma(1)+1}(h_{\alpha_2-\sigma(2)+2}+h_{\alpha_2-\sigma(2)})\cdots(h_{\alpha_n-\sigma(n)+n}+h_{\alpha_n-\sigma(n)-n+2}) = u_\alpha$$

where the last equality follows from (6). By (8) we have $\delta_n(\alpha) = \omega_n(x^\alpha)$. Thus by applying h_n to this equality we obtain $\mathrm{h}_n(\omega_n(x^\alpha)) = u_\alpha = \mathrm{u}_n(x^\alpha)$. Hence $\mathrm{u}_n = \mathrm{h}_n\cdot\omega_n$. We derive the equality $\mathrm{u}_n = \mathrm{H}_n\cdot\Omega_n$ in a similar way starting from

$$\Delta_n(\alpha) = \sum_{\sigma\in\mathcal{S}_n}(-1)^{l(\sigma)}(x_{\sigma(1)}^{\alpha_1-\sigma(1)+1}+x_{\sigma(1)}^{\alpha_2-\sigma(1)-1})\cdots(x_{\sigma(n)}^{\alpha_n-\sigma(n)+n}+x_{\sigma(n)}^{\alpha_n-\sigma(n)-n}).$$

2 : The arguments are the same than in 1 once replacing the characters h and H respectively by the characters e and E. ■

Consider $\alpha = (\alpha_1, ..., \alpha_n) \in \pi_n$ and two integers i, j such that $1 \le i \le j \le n$. The raising operator $R_{i,j}$ and the lowering operator $L_{i,j}$ are respectively defined on π_n by $R_{i,j}(\alpha) = \alpha + \varepsilon_i - \varepsilon_j$ and $L_{i,j}(\alpha) = \alpha - \varepsilon_i - \varepsilon_j$. From the previous lemma we obtain:

Corollary 3.2.2 *For any partition $\mu = (\mu_1, ..., \mu_n)$ we have*

$$s_\mu = \left(\prod_{1 \le i < j \le n} (1 - R_{i,j}) \prod_{1 \le r < s \le n} (1 - L_{r,s}) \right) h_\mu, \; s_\mu = \left(\prod_{1 \le i < j \le n} (1 - R_{i,j}) \prod_{1 \le r \le s \le n} (1 - L_{r,s}) \right) H_\mu,$$

$$s_\mu = \left(\prod_{1 \le i < j \le m} (1 - R_{i,j}) \prod_{1 \le r < s \le m} (1 - L_{r,s}) \right) e_{\mu'}, \; s_\mu = \left(\prod_{1 \le i < j \le m} (1 - R_{i,j}) \prod_{1 \le r \le s \le m} (1 - L_{r,s}) \right) E_{\mu'}$$

where $\mu' = (\mu'_1, ..., \mu'_m)$ is the conjugate partition of μ.

Proof. Let us write

$$\phi_n = \prod_{1 \le i < j \le n} (1 - \frac{x_i}{x_j}) \prod_{1 \le r < s \le n} (1 - \frac{1}{x_i x_j}) = \sum_{\alpha \in \pi_n} a(\alpha) x^\alpha.$$

Then by 1 of Proposition 3.2.1, we have for any $\mu \in \pi_n^+$,

$$\mathrm{u}_n(x^\mu) = \mathrm{h}_n \left(\sum_{\alpha \in \pi_n} a(\alpha) x^{\alpha + \mu} \right) = \sum_{\alpha \in \pi_n} a(\alpha) h_{\alpha + \mu} = u_\mu = s_\mu$$

where the last equality follows from Proposition 3.1.1. This is exactly equivalent to

$$s_\mu = \left(\prod_{1 \le i < j \le n} (1 - R_{i,j}) \prod_{1 \le r < s \le n} (1 - L_{r,s}) \right) h_\mu.$$

The arguments are essentially the same for the other equalities. ■

3.3 Expressions for the multiplicities of representations

Write

$$\phi_n^{-1} = \sum_{\alpha \in \pi_n} f(\alpha) x^\alpha \text{ and } \Phi_n^{-1} = \sum_{\alpha \in \pi_n} F(\alpha) x^\alpha.$$

From Lemma 3.2.1 we deduce that $\mathrm{h}_n = \mathrm{u}_n \circ \omega_n^{-1}$ and $\mathrm{H}_n = \mathrm{u}_n \circ \Omega_n^{-1}$. By applying these identities to x^μ where μ is a partition of length n with μ' of length m we obtain as in Corollary 3.2.2

$$h_\mu = \left(\prod_{1 \le i < j \le n} \frac{1}{1 - R_{i,j}} \prod_{1 \le r < s \le n} \frac{1}{1 - L_{r,s}} \right) s_\mu, \; H_\mu = \left(\prod_{1 \le i < j \le n} \frac{1}{1 - R_{i,j}} \prod_{1 \le r \le s \le n} \frac{1}{1 - L_{r,s}} \right) s_\mu,$$

$$e_{\mu'} = \left(\prod_{1 \le i < j \le m} \frac{1}{1 - R_{i,j}} \prod_{1 \le r < s \le m} \frac{1}{1 - L_{r,s}} \right) s_\mu \text{ and } E_{\mu'} = \left(\prod_{1 \le i < j \le m} \frac{1}{1 - R_{i,j}} \prod_{1 \le r \le s \le m} \frac{1}{1 - L_{r,s}} \right) s_\mu.$$

These relations must be understood as a short way to write

$$h_\mu = \sum_{\alpha \in \pi_n} f(\alpha) u_{\mu + \alpha}, \; H_\mu = \sum_{\alpha \in \pi_n} F(\alpha) u_{\mu + \alpha},$$

$$e_{\mu'} = \sum_{\beta \in \pi_m} f(\alpha) v_{\mu' + \beta} \text{ and } E_{\mu'} = \sum_{\beta \in \pi_m} F(\alpha) v_{\mu' + \beta}.$$

For any positive integer n write $\rho_l = (n, n-1, ..., 1)$.

Proposition 3.3.1 *Consider a partition μ of length n such that μ' has length m. Then for the three roots systems B_n, C_n and D_n we have:*

$$\text{(i)} : \begin{cases} h_\mu = \sum_{\lambda \in \pi_n^+} \sum_{\sigma \in \mathcal{S}_n} (-1)^{l(\sigma)} f(\sigma(\lambda + \rho_n) - \mu - \rho_n) u_\lambda \\ H_\mu = \sum_{\lambda \in \pi_n^+} \sum_{\sigma \in \mathcal{S}_n} (-1)^{l(\sigma)} F(\sigma(\lambda + \rho_n) - \mu - \rho_n) u_\lambda \end{cases},$$

$$\text{(ii)} : \begin{cases} e_{\mu'} = \sum_{\nu \in \pi_m^+} \sum_{\sigma \in \mathcal{S}_m} (-1)^{l(\sigma)} f(\sigma(\nu + \rho_m) - \mu' - \rho_m) v_\nu \\ E_{\mu'} = \sum_{\nu \in \pi_m^+} \sum_{\sigma \in \mathcal{S}_m} (-1)^{l(\sigma)} F(\sigma(\nu + \rho_m) - \mu' - \rho_m) v_\nu \end{cases}.$$

Proof. (i) : Note first that the above relations do not depend on the root system considered. Indeed for any nonnegative integer n, we have $\rho_{B_m} = \rho_n - (\frac{1}{2}, ..., \frac{1}{2}), \rho_{C_n} = \rho_n$ and $\rho_{D_m} = \rho_n - (1, ..., 1)$. Thus $\sigma(\lambda + \rho_{B_n}) - \mu - \rho_{B_n} = \sigma(\lambda + \rho_{C_n}) - \mu - \rho_{C_n} = \sigma(\lambda + \rho_{D_n}) - \mu - \rho_{D_n} = \sigma(\lambda + \rho_n) - \mu - \rho_n$. We have

$$h_\mu = \sum_{\alpha \in \pi_n} f(\alpha) u_{\mu + \alpha}.$$

From Lemma 3.1.2 we deduce that for any $\alpha \in \pi_n$ we have $u_{\mu+\alpha} = 0$ or there exits a partition λ such that $\mu + \alpha = \sigma(\lambda + \rho_n) - \rho_n$ and $u_{\mu+\alpha} = (-1)^{l(\sigma)} u_\lambda$. By setting $\alpha = \sigma(\lambda + \rho_n) - \mu - \rho_n$ in the above sum we obtain $h_\mu = \sum_{\lambda \in \pi_n} \sum_{\sigma \in \mathcal{S}_n} (-1)^{l(\sigma)} f(\sigma(\lambda + \rho_n) - \mu - \rho_n) u_\lambda$. The arguments are similar for the other assertions. ■

From relations (i) and by using the fact that $u_\lambda = s_\lambda$ for any partition λ of length n, we derive the equalities

$$h_\mu = \sum_{\lambda \in \pi_n} u_{\lambda,\mu} s_\lambda \text{ and } H_\mu = \sum_{\lambda \in \pi_n} U_{\lambda,\mu} s_\lambda$$

where

$$u_{\lambda,\mu} = \sum_{\sigma \in \mathcal{S}_n} (-1)^{l(\sigma)} f(\sigma(\lambda + \rho_n) - \mu - \rho_n) \text{ and } U_{\lambda,\mu} = \sum_{\sigma \in \mathcal{S}_n} (-1)^{l(\sigma)} F(\sigma(\lambda + \rho_n) - \mu - \rho_n) \quad (9)$$

are respectively the multiplicities of $V(\lambda)$ in $\mathfrak{h}(\mu)$ and $\mathfrak{H}(\mu)$. Note that $u_{\lambda,\mu} = 0$ and $U_{\lambda,\mu} = 0$ unless $|\mu| \geq |\lambda|$.

For the relations (ii) the situation is more complicated since the partitions ν obtained by applying straightening laws to the $v_{\mu'+\beta}$ yields polynomials v_ν where $\nu \in \pi_m^+$ is a partition of length m so can not be necessarily regarded as the conjugate partition of a partition $\lambda \in \pi_n^+$. The straightening law of Lemma 3.1.2 implies that $|\nu| = |\mu'|$. Since $|\mu| = |\mu'|$, this problem disappear if we suppose $n \geq |\mu|$ for we will have $\nu_1 \leq |\nu| \leq n$ and thus $\nu' \in \pi_n^+$. We can then set $\nu = \lambda'$ with $\lambda \in \pi_n$ and obtain

$$e_{\mu'} = \sum_{\lambda \in \pi_n} v_{\lambda,\mu} s_\lambda \text{ and } E_{\mu'} = \sum_{\lambda \in \pi_n} V_{\lambda,\mu} s_\lambda.$$

We deduce that

$$v_{\lambda,\mu} = u_{\lambda',\mu'} = \sum_{\sigma \in \mathcal{S}_m} (-1)^{l(\sigma)} f(\sigma(\lambda' + \rho_m) - \mu' - \rho_m) \quad (10)$$

$$V_{\lambda,\mu} = U_{\lambda',\mu'} = \sum_{\sigma \in \mathcal{S}_m} (-1)^{l(\sigma)} F(\sigma(\lambda' + \rho_m) - \mu' - \rho_m) \quad (11)$$

are respectively the multiplicities of $V(\lambda)$ in the tensor products $\mathfrak{e}(\mu)$ and $\mathfrak{E}(\mu)$ when $n \geq |\mu|$.

4 Quantification of the multiplicities

4.1 The functions f_q and F_q

Set

$$\phi_n(q) = \prod_{1\leq i<j\leq n}(1-q\frac{x_i}{x_j})\prod_{1\leq r<s\leq n}(1-\frac{q}{x_ix_j}) \text{ and } \Phi_n(q) = \prod_{1\leq i<j\leq n}(1-q\frac{x_i}{x_j})\prod_{1\leq r\leq s\leq n}(1-\frac{q}{x_ix_j}).$$

The functions f_q and F_q are obtained by considering the formal series expansions of $\phi_n^{-1}(q)$ and $\Phi_n^{-1}(q)$. Namely we have

$$\phi_n^{-1}(q) = \sum_{\alpha\in\pi_n} f_q(\alpha)x^\alpha \text{ and } \Phi_n^{-1}(q) = \sum_{\alpha\in\pi_n} F_q(\alpha)x^\alpha. \tag{12}$$

4.2 Some q-analogues of multiplicities of $V(\lambda)$ in $\mathfrak{h}(\mu)$, $\mathfrak{H}(\mu)$, $\mathfrak{e}(\mu)$ or $\mathfrak{E}(\mu)$

Given λ and μ two partitions of length n, let $c_{\lambda,\mu}(q)$ and $C_{\lambda,\mu}(q)$ be the two polynomials defined by

$$u_{\lambda,\mu}(q) = \sum_{\sigma\in\mathcal{S}_n}(-1)^{l(\sigma)}f_q(\sigma(\lambda+\rho_n)-\mu-\rho_n) \text{ and } U_{\lambda,\mu}(q) = \sum_{\sigma\in\mathcal{S}_n}(-1)^{l(\sigma)}F_q(\sigma(\lambda+\rho_n)-\mu-\rho_n).$$

Then from the equalities (9), (10) and (11) we obtain:

Proposition 4.2.1 *Let λ and μ be two partitions of length n. Then*

1. *$u_{\lambda,\mu}(q)$ and $U_{\lambda,\mu}(q)$ are q-analogues of the multiplicity of the representation $V(\lambda)$ in $\mathfrak{h}(\mu)$ and $\mathfrak{H}(\mu)$,*
2. *$v_{\lambda,\mu}(q) = u_{\lambda,'\mu'}(q)$ and $V_{\lambda,\mu}(q) = U_{\lambda',\mu'}(q)$ are q-analogues of the multiplicity of the representation $V(\lambda)$ in $\mathfrak{e}(\mu)$ and $\mathfrak{E}(\mu)$ when the condition $n \geq |\mu|$ is satisfied.*

The following example is obtained from the explicit computation of the function f_q when $n = 2$.

Example 4.2.2 *Consider μ a partition of length 2 and set $\mathcal{E}_\mu = \{\lambda\in\pi_2^+, \lambda = (\mu_1+r-s, \mu_2-r-s)$, $s\in\{0,...,\mu_2\}$, $r\in\{0,...,\mu_2-s\}\}$. Then for any partition λ of length 2 we have:*

$$u_{\lambda,\mu}(q) = \begin{cases} q^{\mu_1-\lambda_1} & \text{if } \lambda\in\mathcal{E}_\mu \\ 0 & \text{otherwise} \end{cases}.$$

Remarks
(i) : It follows from the definition of the q-functions f_q and F_q that $u_{\lambda,\mu}(q) = U_{\lambda,\mu}(q) = 0$ if $|\lambda| > |\mu|$.
(ii) : It is not trivial from the very definitions that $u_{\lambda,\mu}(q)$ and $U_{\lambda,\mu}(q)$ are polynomials in q with nonnegative integer coefficients. This property will be proved in Section 5 as a corollary of Theorem 5.1.5.

5 The duality theorems

5.1 A duality theorem for the q-multiplicities in $\mathfrak{h}(\mu)$ and $\mathfrak{H}(\mu)$

For any nonnegative integer n, set $\kappa_n = (1, ..., 1) \in \pi_n$.

Lemma 5.1.1 *Consider λ, μ two partitions of length n such that $|\lambda| \geq |\mu|$. Let k be any integer such that $k \geq \frac{|\lambda|-|\mu|}{2}$. Then we have*

$$K_{\lambda+k\kappa_n,\mu+k\kappa_n}(q) = \sum_{\sigma \in S_n} (-1)^{l(\sigma)} \mathcal{P}_q(\sigma(\lambda+\rho_n) - (\mu+\rho_n)) \tag{13}$$

where the sum is indexed by the elements of the symmetric group S_n.

Proof. Since $\mathcal{P}_q(\alpha) = 0$ if α is not a linear combination of positive roots with nonnegative coefficients, we have $\mathcal{P}_q(\alpha) = 0$ for any $\alpha \in \pi_n$ such that $|\alpha| < 0$. Consider $\delta = (\delta_1, ..., \delta_n) \in \pi_n$ and $w \in W_n$. Write $w(\delta) = (\delta_1^w, ..., \delta_n^w)$ and denote by $E_{w,\delta} = \{i_1, ..., i_p\}$ the set of the indices i_k such that δ_{i_k} and $\delta_{i_k}^w$ have opposite signs. Define the sum $S_{w,\delta} = \sum_{i_k \in E_{w,\delta}} \delta_{i_k}$. Then $|w(\delta)| = |\delta| - 2S_{w,\delta}$. Now consider k a nonnegative integer and set $\delta = (\lambda + \rho_n + k\kappa_n)$. We have $|w(\lambda + \rho_n + k\kappa_n)| = |(\lambda + \rho_n + k\kappa_n)| - 2S_{w,\delta}$. But $S_{w,\delta} = S_{w,\lambda+\rho_n} + kp$. Thus we obtain

$$\begin{aligned}|w(\lambda + \rho_n + k\kappa_n) - (\mu + \rho_n + k\kappa_n)| = |(\lambda + \rho_n + k\kappa_n)| - 2S_{w,\lambda+\rho_n} - |(\mu + \rho_n + k\kappa_n)| - 2kp = \\ |\lambda| - |\mu| - 2S_{w,\lambda+\rho_n} - 2kp.\end{aligned}$$

When $w \notin \mathcal{S}_n$, we have $p \geq 1$ and $S_{w,\lambda+\rho_n} \geq 1$ since the coordinates of $\lambda + \rho_n$ are all positive. Hence $|w(\lambda + \rho_n + k\kappa_n) - (\mu + \rho_n + k\kappa_n)| < |\lambda| - |\mu| - 2k$ and is negative as soon as $k \geq \frac{|\lambda|-|\mu|}{2}$. For such an integer k the sum defining $K_{\lambda+k\kappa_n,\mu+k\kappa_n}(q)$ normally running on W_n can be restricted to (13) and we obtain

$$K_{\lambda+k\kappa_n,\mu+k\kappa_n}(q) = \sum_{\sigma \in S_n} (-1)^{l(\sigma)} \mathcal{P}_q(\sigma(\lambda + \rho_n + k\kappa_n) - (\mu + \rho_n + k\kappa_n)).$$

Since $\sigma \in \mathcal{S}_n$, we have $\sigma(k\kappa_n) = k\kappa_n$. Thus

$$K_{\lambda+k\kappa_n,\mu+k\kappa_n}(q) = \sum_{\sigma \in S_n} (-1)^{l(\sigma)} \mathcal{P}_q(\sigma(\lambda+\rho_n) - (\mu+\rho_n)).$$

■

We define the involution I on π_n by $I(\alpha_1, ..., \alpha_n) = (-\alpha_n, ..., -\alpha_1)$ for any $\alpha = (\alpha_1, ..., \alpha_n) \in \pi_n$.

Lemma 5.1.2 *For any $\alpha = (\alpha_1, ..., \alpha_n) \in \pi_n$ we have*

$$f_q(\alpha) = \mathcal{P}_q^{D_n}(I(\alpha)) \text{ and } F_q(\alpha) = \mathcal{P}_q^{C_n}(I(\alpha))$$

where $\mathcal{P}_q^{B_n}$ and $\mathcal{P}_q^{D_n}$ are the q-partition functions associated respectively to the root systems B_n and D_n.

Proof. By abuse of notation we also denote by I the ring automorphism of $\mathcal{L}_n$ defined by $I(x^\alpha) = x^{I(\alpha)}$. The image of the root systems C_n and D_n by I are respectively

$$\left\{ \begin{array}{l} \{\varepsilon_i - \varepsilon_j, -\varepsilon_i - \varepsilon_j \text{ with } 1 \leq i < j \leq n\} \cup \{-2\varepsilon_i \text{ with } 1 \leq i \leq n\} \text{ for the root system } C_n \\ \{\varepsilon_i - \varepsilon_j, -\varepsilon_i - \varepsilon_j \text{ with } 1 \leq i < j \leq n\} \text{ for the root system } D_n \end{array} \right. . \tag{14}$$

By applying I to the equality

$$\prod_{\alpha \in R^+_{C_n}} \frac{1}{1 - qx^\alpha} = \sum_{\beta \in \pi_n} \mathcal{P}_q^{C_n}(\beta) x^\beta$$

we obtain

$$\prod_{1 \leq i < j \leq n} \frac{1}{(1 - q\frac{x_i}{x_j})} \prod_{1 \leq r \leq s \leq n} \frac{1}{(1 - \frac{q}{x_r x_s})} = \sum_{\beta \in \pi_n} \mathcal{P}_q^{C_n}(\beta) x^{I(\beta)}.$$

Set $\alpha = I(\beta)$. The equality becomes

$$\Phi_n^{-1}(q) = \sum_{\alpha \in \pi_n} \mathcal{P}_q^{C_n}(I(\alpha)) x^\alpha$$

and from the definition (see 12) of the function F_q, we obtain $\mathcal{P}_q^{C_n}(I(\alpha)) = F_q(\alpha)$. The assertion with f_q is proved in the same way by considering the root system D_n. ■

Given $\sigma \in \mathcal{S}_n$, denote by σ^* the permutation defined by

$$\sigma^*(k) = \sigma(n - k + 1).$$

For any $i \in \{1, ..., n-1\}$, we have $s_i^* = s_{n-i}$. The following Lemma is straightforward:

Lemma 5.1.3 *The map $\sigma \to \sigma^*$ is an involution of the group $\mathcal{S}_n$. Moreover we have $\sigma(I(\beta)) = I(\sigma^*(\beta))$ and $l(\sigma) = l(\sigma^*)$ for any $\beta \in \pi_n, \sigma \in \mathcal{S}_n$.*

Lemma 5.1.4 *Let λ, μ two partitions of length n and $\sigma \in \mathcal{S}_n$. Then*

$$(-1)^{l(\sigma)} f_q(\sigma(\lambda + \rho_n) - (\mu + \rho_n)) = (-1)^{l(\sigma^*)} \mathcal{P}_q^{D_n}(\sigma^*(I(\lambda) + \rho_n) - (I(\mu) + \rho_n)) \text{ and}$$
$$(-1)^{l(\sigma)} F_q(\sigma(\lambda + \rho_n) - (\mu + \rho)) = (-1)^{l(\sigma^*)} \mathcal{P}_q^{C_n}(\sigma^*(I(\lambda) + \rho_n) - (I(\mu) + \rho_n)).$$

Proof. Since $l(\sigma) = l(\sigma^*)$, it suffices to prove the equalities

$$f_q(\sigma(\lambda + \rho_n) - (\mu + \rho_n)) = \mathcal{P}_q^{D_n}(\sigma^*(I(\lambda) + \rho_n) - (I(\mu) + \rho_n)) \text{ and}$$
$$F_q(\sigma(\lambda + \rho_n) - (\mu + \rho_n)) = \mathcal{P}_q^{C_n}(\sigma^*(I(\lambda) + \rho_n) - (I(\mu) + \rho_n)).$$

Set $P = \mathcal{P}_q^{C_n}(\sigma^*(I(\lambda) + \rho_n) - (I(\mu) + \rho_n))$. From the above Lemma we deduce

$$P = \mathcal{P}_q^{C_n}(I(\sigma(\lambda) + \sigma^*(\rho_n) - I(\mu) - \rho_n).$$

Now an immediate computation shows that $\sigma^*(\rho_n) - \rho_n = I(\sigma(\rho_n) - \rho_n)$. Thus we derive

$$P = \mathcal{P}_q^{C_n}(I(\sigma(\lambda + \rho_n) - \mu - \rho_n)) = F_q(\sigma(\lambda + \rho_n) - \mu - \rho_n)$$

where the last equality follows from Lemma 5.1.2.
We obtain the equality $f_q(\sigma(\lambda + \rho_n) - (\mu + \rho_n)) = \mathcal{P}_q^{D_n}(\sigma(I(\lambda) + \rho_n) - (I(\mu) + \rho_n))$ in a similar way.
■

Theorem 5.1.5 *Consider λ, μ two partitions of length n and set $m = \max(\lambda_1, \mu_1)$. Let k be any integer such that $k \geq \frac{|\mu|-|\lambda|}{2}$. Then $\widehat{\lambda} = (m-\lambda_n, ..., m-\lambda_1)$ and $\widehat{\mu} = (m-\mu_n, ..., m-\mu_1)$ are partitions of length n and*

$$\begin{cases} u_{\lambda,\mu}(q) = K^{D_n}_{\widehat{\lambda}+k\kappa_n, \widehat{\mu}+k\kappa_n}(q) \\ U_{\lambda,\mu}(q) = K^{C_n}_{\widehat{\lambda}+k\kappa_n, \widehat{\mu}+k\kappa_n}(q) \end{cases}$$

Proof. First $\widehat{\lambda}$ and $\widehat{\mu}$ are clearly partitions of length n since $m = \max(\lambda_1, \mu_1)$. It follows from the definition of $U_{\lambda,\mu}(q)$ and the above lemma that

$$U_{\lambda,\mu}(q) = \sum_{\sigma \in \mathcal{S}_n} (-1)^{l(\sigma)} F_q(\sigma(\lambda + \rho_n) - \mu - \rho_n) = \sum_{\sigma^* \in \mathcal{S}_n} (-1)^{l(\sigma^*)} \mathcal{P}^{C_n}_q(\sigma^*(I(\lambda) + \rho_n)) - (I(\mu) + \rho_n)).$$

Then by Lemma 5.1.3 we obtain

$$U_{\lambda,\mu}(q) = \sum_{\sigma \in \mathcal{S}_n} (-1)^{l(\sigma)} \mathcal{P}^{C_n}_q(\sigma(I(\lambda) + \rho_n)) - (I(\mu) + \rho_n)).$$

We have $\sigma(I(\lambda) + \rho_n + m\kappa_n) = \sigma(I(\lambda) + \rho_n) + m\kappa_n$ since $\sigma \in \mathcal{S}_n$. So we can write

$$U_{\lambda,\mu}(q) = \sum_{\sigma \in \mathcal{S}_n} (-1)^{l(\sigma)} \mathcal{P}^{C_n}_q(\sigma(I(\lambda) + m\kappa_n + \rho_n)) - (I(\mu) + m\kappa_n + \rho_n)).$$

Since $\widehat{\lambda} = I(\lambda) + m\kappa_n$ and $\widehat{\mu} = I(\mu) + m\kappa_n$ we derive

$$U_{\lambda,\mu}(q) = \sum_{\sigma \in \mathcal{S}_n} (-1)^{l(\sigma)} \mathcal{P}^{C_n}_q(\sigma(\widehat{\lambda} + \rho_n) - (\widehat{\mu} + \rho_n)) = K^{C_n}_{\widehat{\lambda}+k\kappa_n, \widehat{\mu}+k\kappa_n}(q)$$

by Lemma 5.1.1.
We obtain similarly the equality $u_{\lambda,\mu}(q) = K^{D_n}_{\widehat{\lambda}+k\kappa_n, \widehat{\mu}+k\kappa_n}(q)$ by replacing $\mathcal{P}^{C_n}_q$ by $\mathcal{P}^{D_n}_q$. ■

Example 5.1.6 *Consider $\mu = (4, 2, 1)$ and $\lambda = (2, 1, 0)$. We have $m = 4$, $\widehat{\mu} = (3, 2, 0)$ and $\widehat{\lambda} = (4, 3, 2)$. We choose $k = 2$. Then we obtain the equalities*

$$\begin{cases} u_{\lambda,\mu}(q) = K^{D_n}_{(6,5,4),(5,4,2)}(q) = q^3 + q^2 \\ U_{\lambda,\mu}(q) = K^{C_n}_{(6,5,4),(5,4,2))}(q) = q^5 + 2q^4 + 3q^3 + 2q^2 \end{cases}.$$

By using the fact that the Kostka-Foulkes polynomials have nonnegative integer coefficients [9] we obtain the following corollary.

Corollary 5.1.7 *The polynomials $u_{\lambda,\mu}(q)$ and $U_{\lambda,\mu}(q)$ have nonnegative integers coefficients.*

We also recover a property of the Kostka-Foulkes polynomials associated to the root system A_{n-1} proved in [8].

Corollary 5.1.8 *Consider λ, μ two partitions of length n such that $|\lambda| = |\mu|$ and set $m = \max(\lambda_1, \mu_1)$. Then the Kostka-Foulkes polynomials associated to the root system A_{n-1} verifies*

$$K^{A_{n-1}}_{\lambda,\mu}(q) = K^{A_{n-1}}_{\widehat{\lambda},\widehat{\mu}}(q)$$

where $\widehat{\lambda} = (m - \lambda_n, ..., m - \lambda_1)$ and $\widehat{\mu} = (m - \mu_n, ..., m - \mu_1)$.

Proof. Suppose that β is a linear combination of $I(R^+_{C_n})$ with nonnegative coefficients such that $|\beta| = 0$. Then β is necessarily a linear combination of the roots $\varepsilon_i - \varepsilon_j, 1 \leq i < j \leq n$ with nonnegative coefficients (see (14)) that is, a linear combination with nonnegative coefficients of the positive roots associated to the root system A_{n-1}. This implies that

$$f_q(\beta) = F_q(\beta) = \mathcal{P}_q^{A_{n-1}}(\beta)$$

where $\mathcal{P}_q^{A_{n-1}}$ is the q-partition function associated to the root system A_{n-1}. For any $\sigma \in \mathcal{S}_n$, we have $|\sigma(\lambda+\rho_n) - (\mu+\rho_n)| = 0$ since $|\lambda| = |\mu|$. Thus

$$f_q(\sigma(\lambda+\rho_n) - (\mu+\rho_n)) = F_q(\sigma(\lambda+\rho_n) - (\mu+\rho_n)) = \mathcal{P}_q^{A_{n-1}}(\sigma(\lambda+\rho_n) - (\mu+\rho_n))$$

and the multiplicities $u_{\lambda,\mu}(q)$ and $U_{\lambda,\mu}(q)$ coincide with the Kostka-Foulkes polynomial $K^{A_{n-1}}_{\lambda,\mu}(q)$ when $|\lambda| = |\mu|$. Moreover by applying Theorem 5.1.5 with $|\lambda| = |\mu|$ and $k = 0$, we obtain $U_{\lambda,\mu}(q) = K^{C_n}_{\widehat{\lambda},\widehat{\mu}}(q) = K^{A_{n-1}}_{\widehat{\lambda},\widehat{\mu}}(q)$ where the last equality is due to the fact that the Kostka-Foulkes polynomials of types B_n, C_n or D_n are Kostka-Foulkes polynomials associated to the root system A_{n-1} when $|\lambda| = |\mu|$. So we derive the equality $K^{A_{n-1}}_{\lambda,\mu}(q) = K^{A_{n-1}}_{\widehat{\lambda},\widehat{\mu}}(q)$. ■

We have seen that $U_{\lambda,\mu}(q)$ can be regarded as a q-analogue of the multiplicity of the representation $V(\lambda)$ in $\mathfrak{H}^{C_n}(\mu)$. In [13], Hatayama, Kuniba, Okado and Takagi have introduced another quantification $X_{\lambda,\mu}(q)$ of this multiplicity based on the determination of the combinatorial R matrix of the $U_q'(C_n^{(1)})$-crystals B_k. Considered as the crystal graph of the $U_q(C_n)$-module M_k, B_k can be identify with

$$B(k\Lambda_1) \oplus B((k-2)\Lambda_1) \oplus \cdots \oplus B(k \bmod 2\Lambda_1)$$

where for any $i \in \{k, k-2, ..., k \bmod 2\}$, $B(\,k\Lambda_1)$ is the crystal graph of the irreducible finite dimensional $U_q(C_n)$-module of highest weight $k\Lambda_1$. Note that the character of M_k is equal to $H_k^{C_n}$.
Recall that the combinatorial R-matrix associated to crystals B_k is equivalent to the description of the crystal graph isomorphisms

$$\begin{cases} B_l \otimes B_k \overset{\simeq}{\to} B_k \otimes B_l \\ b_1 \otimes b_2 \longmapsto b_2' \otimes b_1' \end{cases}$$

together with the energy function H on $B_l \otimes B_k$. The multiplicity of $V(\lambda)$ in $\mathfrak{H}^{C_n}(\mu)$ is then equal to the number of highest weight vertices of weight λ in the crystal $B_\mu = B_{\mu_1} \otimes \cdots \otimes B_{\mu_n}$. Then $X_{\lambda,\mu}(q)$ is defined by

$$X_{\lambda,\mu}(q) = \sum_{b \in E_\lambda} q^{\sum_{0 \leq i < j \leq n} H(b_i \otimes b_j^{(i+1)})}$$

where E_λ is the set of highest weight vertices $b = b_1 \otimes \cdots \otimes b_n$ in B_μ of highest weight λ, $b_j^{(i)}$ is determined by the crystal isomorphism

$$\begin{array}{c} B_{\mu_i} \otimes B_{\mu_{i+1}} \otimes B_{\mu_{i+2}} \otimes \cdots \otimes B_{\mu_j} \to B_{\mu_i} \otimes B_{\mu_j} \otimes B_{\mu_{i+1}} \cdots \otimes B_{\mu_{j-1}} \\ b_i \otimes b_{i+1} \otimes \cdots \otimes b_j \to b_j^{(i)} \otimes b_i' \otimes \cdots \otimes b_{j-1}' \end{array}$$

and for any $j = 1, ..., n$, $H(b_0 \otimes b_j^{(1)})$ depends only on $b_j^{(1)}$.
Many computations suggest the following conjecture

Conjecture 5.1.9 *For any partition λ and μ of length n with $|\mu| \geq |\lambda|$*

$$U_{\lambda,\mu}(q) = q^{|\mu|-|\lambda|} X_{\lambda,\mu}(q).$$

Note that the conjecture is in particular true for all the examples given in the tables of [13].

5.2 A duality theorem for the q-multiplicities in $\mathfrak{e}(\mu)$ and $\mathfrak{E}(\mu)$

Consider λ,μ two partitions of length l such that $l\geq|\mu|\geq|\lambda|$. Write $n=\max(\lambda_1,\mu_1)$. Then by adding to λ' and μ' the required numbers of parts 0 we can consider them as partitions of length n. Set $m=\max(\lambda'_1,\mu'_1)$. We define the partitions $\widetilde{\lambda}$ and $\widetilde{\mu}$ belonging to π_n by $\widetilde{\lambda}=(m-\lambda'_n,...,m-\lambda'_1)$ and $\widetilde{\mu}=(m-\mu'_n,...,m-\mu'_1)$.

Theorem 5.2.1 *With the above notations, we have for any integer* $k\geq\frac{|\mu|-|\lambda|}{2}$

$$\left\{\begin{array}{l}\text{(i)}: v_{\lambda,\mu}(q)=K^{D_n}_{\widetilde{\lambda}+k\kappa_n,\widetilde{\mu}+k\kappa_n}(q)\\ \text{(ii)}: V_{\lambda,\mu}(q)=K^{C_n}_{\widetilde{\lambda}+k\kappa_n,\widetilde{\mu}+k\kappa_n}(q)\end{array}\right. .$$

Proof. Since $l\geq|\mu|$, we have by Proposition 4.2.1 the equality $v_{\lambda,\mu}(q)=u_{\lambda',\mu'}(q)$. Moreover we have $m\geq\max(\lambda'_1,\mu'_1)$ and $k\geq\frac{|\mu'|-|\lambda'|}{2}$ for $|\lambda'|=|\lambda|$ and $|\mu'|=|\mu|$. Hence by applying Theorem 5.1.5 we obtain $v_{\lambda,\mu}(q)=K^{D_n}_{\widehat{\lambda'}+k\kappa_n,\widehat{\mu'}+k\kappa_n}(q)$ where $\widehat{\lambda'}=(m-\lambda'_n,...,m-\lambda'_1)=\widetilde{\lambda}$ and $\widehat{\mu'}=(m-\mu'_n,...,m-\mu'_1)=\widetilde{\mu}$. So (i) is proved. We obtain (ii) similarly. ■

Example 5.2.2 *For* $\lambda=(2,1,0,0,0)$ *and* $\mu=(2,2,1,0,0)$ *we have* $l=5,n=2$. *Moreover* $\lambda'=(2,1)$, $\mu'=(3,2)$ *and* $m=3$. *So* $\widetilde{\lambda}=(2,1)$ *and* $\widetilde{\mu}=(1,0)$. *Hence for* $k=1$

$$\left\{\begin{array}{l}\text{(i)}: v_{\lambda,\mu}(q)=K^{D_n}_{(3,2),(2,1)}(q)=q\\ \text{(ii)}: V_{\lambda,\mu}(q)=K^{C_n}_{(3,2),(2,1)}(q)=q^2+q\end{array}\right.$$

Remark When λ,μ are considered as weights associated to the root system C_l, the above theorem is essentially the quantification of a duality result explicited by Foulle [2] from results of [5] for the dual pair $(Sp(2l),Sp(2n))$.

6 Identities for the q-multiplicities $U_{\lambda,\mu}(q)$ and $u_{\lambda,\mu}(q)$

6.1 A relations between q-partition functions

Consider a nonnegative integer k and define the finite sets

$$\left\{\begin{array}{l}\mathcal{C}^n_k=\{\beta\in\pi_n,\beta=\sum_{1\leq r\leq s\leq n}e_{r,s}(\varepsilon_r+\varepsilon_s)\text{ with }e_{r,s}\geq0\text{ and }|\beta|=2k\}\\ \mathcal{D}^n_k=\{\beta\in\pi_n,\beta=\sum_{1\leq r<s\leq n}e_{r,s}(\varepsilon_r+\varepsilon_s)\text{ with }e_{r,s}\geq0\text{ and }|\beta|=2k\}\end{array}\right. .$$

Note that each $\beta\in\mathcal{C}^n_k$ (resp. $\beta\in\mathcal{D}^n_k$) verifies $|\beta|=2\sum_{1\leq r\leq s\leq n}e_{r,s}$ (resp. $|\beta|=2\sum_{1\leq r<s\leq n}e_{r,s}$). This implies that

$$\prod_{1\leq r\leq s\leq n}\frac{1}{(1-\dfrac{q}{x_rx_s})}=\sum_{k\geq0}\sum_{\beta\in\mathcal{C}^n_k}c^{C_n}_\beta q^kx^\beta\text{ and }\prod_{1\leq r<s\leq n}\frac{1}{(1-\dfrac{q}{x_rx_s})}=\sum_{k\geq0}\sum_{\beta\in\mathcal{C}^n_k}c^{D_n}_\beta q^kx^\beta$$

where $c^{C_n}_\beta$ (resp. $c^{D_n}_\beta$) is the number of ways to decompose β as $\beta=\sum_{1\leq r\leq s\leq n}e_{r,s}(\varepsilon_r+\varepsilon_s)$ (resp. $\beta=\sum_{1\leq r<s\leq n}e_{r,s}(\varepsilon_r+\varepsilon_s)$) with $e_{r,s}\geq0$.

Lemma 6.1.1 *For any* $\beta\in\pi_n$ *with* $|\beta|=2k\geq0$, *we have*

$$F_q(\beta)=\sum_{\delta\in\mathcal{C}^n_k}c^{C_n}_\delta q^k\mathcal{P}^{A_n}_q(\beta+\delta)\text{ and }f_q(\beta)=\sum_{\delta\in\mathcal{D}^n_k}c^{D_n}_\delta q^k\mathcal{P}^{A_n}_q(\beta+\delta).$$

Proof. We have:

$$\prod_{1\le i<j\le n}\frac{1}{(1-q\frac{x_i}{x_j})}\prod_{1\le r\le s\le n}\frac{1}{(1-\frac{q}{x_rx_s})}=\sum_{\eta\in\pi_n}\sum_{\delta\in\pi_n}c_\delta^{C_n}q^{|\delta|/2}\mathcal{P}_q^{A_n}(\eta)x^{\eta-\delta}$$

which implies the equality $F_q(\beta)=\sum_{\eta-\delta=\beta}c_\delta^{C_n}q^{|\delta|/2}\mathcal{P}_q^{A_n}(\eta)$. Since $\mathcal{P}_q^{A_n}(\eta)=0$ when $|\eta|\neq 0$, we can suppose $|\eta|=0$ and $|\delta|=|\beta|$ in the previous sum. Then $\delta\in\mathcal{C}_k^n$ and the result follows immediately. The proof for $f_q(\beta)$ is similar. ■

6.2 Expressions of the multiplicities $u_{\lambda,\mu}$ and $U_{\lambda,\mu}$ in terms of Kostka numbers

Suppose that ξ and γ belong to π_n. Then we can define the polynomial

$$K_{\xi,\gamma}^{A_{n-1}}(q)=\sum_{\sigma\in\mathcal{S}_n}(-1)^{l(\sigma)}\mathcal{P}_q^{A_{n-1}}(\sigma(\xi+\rho_n)-(\gamma+\rho_n)).$$

Note that the coefficients of $K_{\xi,\gamma}^{A_{n-1}}(q)$ may be negative. When $\xi=\lambda$ is a partition, $K_{\lambda,\gamma}^{A_{n-1}}=K_{\lambda,\gamma}^{A_{n-1}}(1)$ is equal to the dimension of the weight space of weight γ in $V(\lambda)$. When $\gamma=\mu$ is a partition, we have

$$\begin{cases} K_{\xi,\mu}^{A_{n-1}}(q)=(-1)^{l(\tau)}K_{\nu,\mu}^{A_{n-1}}(q) \text{ if } \xi=\tau\circ(\nu) \text{ with } \tau\in\mathcal{S}_n \text{ and } \nu \text{ a partition}\\ 0 \text{ otherwise}\end{cases}$$

Proposition 6.2.1 *Consider λ,μ two partitions of length n such that $k=|\mu|-|\lambda|\geq 0$. Then*

$$u_{\lambda,\mu}(q)=\sum_{\delta\in\mathcal{D}_k^n}c_\delta^{D_n}q^{\frac{|\mu|-|\lambda|}{2}}K_{\lambda,\mu-\delta}^{A_{n-1}}(q)=\sum_{\delta\in\mathcal{D}_k^n}c_\delta^{D_n}q^{\frac{|\mu|-|\lambda|}{2}}K_{\lambda+\delta,\mu}^{A_{n-1}}(q) \text{ and}$$
$$U_{\lambda,\mu}(q)=\sum_{\delta\in\mathcal{C}_k^n}c_\delta^{C_n}q^{\frac{|\mu|-|\lambda|}{2}}K_{\lambda,\mu-\delta}^{A_{n-1}}(q)=\sum_{\delta\in\mathcal{C}_k^n}c_\delta^{C_n}q^{\frac{|\mu|-|\lambda|}{2}}K_{\lambda+\delta,\mu}^{A_{n-1}}(q).$$

Proof. By definition we have

$$U_{\lambda,\mu}(q)=\sum_{\sigma\in\mathcal{S}_n}(-1)^{l(\sigma)}F_q(\sigma(\lambda+\rho_n)-(\mu+\rho_n)).$$

Hence from the above lemma we derive

$$U_{\lambda,\mu}(q)=\sum_{\delta\in\mathcal{C}_k^n}c_\delta^{C_n}q^{|\delta|/2}\sum_{\sigma\in\mathcal{S}_n}(-1)^{l(\sigma)}\mathcal{P}_q^{A_{n-1}}(\sigma(\lambda+\rho_n)-(\mu-\delta+\rho_n)) \tag{15}$$

which yields the first desired equality since $K_{\lambda,\mu-\delta}^{A_{n-1}}(q)=\sum_{\sigma\in\mathcal{S}_n}(-1)^{l(\sigma)}\mathcal{P}_q^{A_{n-1}}(\sigma(\lambda+\rho_n)-(\mu-\delta+\rho_n))$. For any $\sigma\in\mathcal{S}_n$, we have $\sigma(\mathcal{C}_k^n)=\mathcal{C}_k^n$ and $c_{\sigma(\delta)}^{C_n}=c_\delta^{C_n}$. Thus (15) can also be rewritten

$$U_{\lambda,\mu}(q)=q^{|\delta|/2}\sum_{\sigma\in\mathcal{S}_n}(-1)^{l(\sigma)}\sum_{\delta\in\mathcal{C}_k^n}c_\delta^{C_n}\mathcal{P}_q^{A_{n-1}}(\sigma(\lambda+\rho_n+\delta)-(\mu+\rho_n))=\sum_{\delta\in\mathcal{C}_k^n}c_\delta^{C_n}q^{\frac{|\mu|-|\lambda|}{2}}K_{\lambda+\delta,\mu}^{A_{n-1}}(q).$$

The proof is similar for $u_{\lambda,\mu}(q)$. ■
By setting $q=1$ in the above relations we obtain the following expressions of the multiplicities $U_{\lambda,\mu}$ and $u_{\lambda,\mu}$ in terms of Kostka numbers.

Corollary 6.2.2

$$\left\{ \begin{array}{l} U_{\lambda,\mu} = \sum_{\delta \in \mathcal{C}_k^n} c_\delta^{C_n} K_{\lambda,\mu-\delta}^{A_{n-1}} = \sum_{\delta \in \mathcal{C}_k^n} c_\delta^{C_n} K_{\lambda+\delta,\mu}^{A_{n-1}} \\ v_{\lambda,\mu} = \sum_{\delta \in \mathcal{D}_k^n} c_\delta^{D_n} K_{\lambda,\mu-\delta}^{A_{n-1}} = \sum_{\delta \in \mathcal{D}_k^n} c_\delta^{D_n} K_{\lambda+\delta,\mu}^{A_{n-1}} \end{array} \right. .$$

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