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A duality between $q$-multiplicities in tensor products and
$q$-multiplicities of weights for the root systems $B, C$ or $D$

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Abstract

Starting from Jacobi-Trudi’s type determinental expressions for the Schur functions of types $B, C$ and $D$, we define a natural $q$-analogue of the multiplicity $[V(\lambda) : M(\mu)]$ when $M(\mu)$ is a tensor product of row or column shaped modules defined by $\mu$. We prove that these $q$-multiplicities are equal to certain Kostka-Foulkes polynomials related to the root systems $C$ or $D$. Finally we express the corresponding multiplicities in terms of Kostka numbers.

1 Introduction

Given two partitions $\lambda$ and $\mu$ of length $n$, the Kostka number $K_{\lambda,\mu}^{A_n}$ is equal to the dimension of the weight space $\mu$ in the finite dimensional irreducible $sl_{n+1}$-module $V(\lambda)$ of highest weight $\lambda$. The Schur duality is a classical result in representation theory establishing that $K_{\lambda,\mu}^{A_n}$ is also equal to the multiplicity of $V(\lambda)$ in the tensor products

$$V(\mu_1 \Lambda_1) \otimes \cdots \otimes V(\mu_n \Lambda_1) \text{ and } V(\Lambda_{\mu'_1}) \otimes \cdots \otimes V(\Lambda_{\mu'_m})$$

where $\mu' = (\mu'_1, \ldots, \mu'_m)$ is the conjugate partition of $\mu$ and the $\Lambda_i$’s $i = 1, \ldots, n-1$ are the fundamental weights of $sl_{n+1}$. Another way to define $K_{\lambda,\mu}^{A_n}$ is to use the Jacobi-Trudi identity which gives a determinantal expression of the Schur function $s_\mu = \text{char}(V(\mu))$ in terms of the characters $h_k = \text{char}(V(k\Lambda_1))$ of the $k$-th symmetric power representation. This formula can be rewritten

$$s_\mu = \prod_{1 \leq i < j \leq n} (1 - R_{i,j}) h_\mu$$

where $h_\mu = h_{\mu_1} \cdots h_{\mu_n}$ and the $R_{i,j}$ are the raising operators (see 3.2). Then one can prove that it makes sense to write

$$h_\mu = \prod_{1 \leq i < j \leq n} (1 - R_{i,j})^{-1} s_\mu$$

which gives the decomposition of $h_\mu$ on the basis of Schur functions. From this decomposition we derive the following expression for $K_{\lambda,\mu}^{A_n}$:

$$K_{\lambda,\mu}^{A_n} = \sum_{\sigma \in S_n} (-1)^{l(\sigma)} \mathcal{P}^{A_n}(\sigma(\lambda + \rho) - (\mu + \rho))$$

where $S_n$ is the symmetric group of order $n$ and $\mathcal{P}^{A_n}$ the ordinary Kostant’s partition function defined from the equality:

$$\prod_{\alpha \text{ positive root}} \frac{1}{(1 - x^\alpha)} = \sum_{\beta} \mathcal{P}^{A_n}(\beta)x^\beta$$
with \( \beta \) running on the set of nonnegative integral combinations of positive roots of \( sl_n \).

There exists a \( q \)-analogue \( K^A_{\lambda,\mu}(q) \) of \( K^A_{\lambda,\mu} \) obtained by replacing the ordinary Kostant’s partition function \( \mathcal{P}^A \) by its \( q \)-analogue \( \mathcal{P}_q^A \) satisfying

\[
\prod_{\alpha \text{ positive root}} \frac{1}{(1 - q^{x_\alpha})} = \sum_\beta \mathcal{P}_q^A(\beta) x^\beta.
\]

So we have

\[
K^A_{\lambda,\mu}(q) = \sum_{\sigma \in S_n} (-1)^{l(\sigma)} \mathcal{P}_q^A(\sigma(\lambda + \rho) - (\mu + \rho)) \tag{4}
\]

which is a polynomial in \( q \) with nonnegative integer coefficients \([8],[9]\). In \([11]\), Nakayashiki and Yamada have shown that \( K^A_{\lambda,\mu}(q) \) can also be computed from the combinatorial \( R \) matrix corresponding to Kashiwara’s crystals associated to some \( U_q(sl_n) \)-modules.

For \( g = so_{2n+1}, sp_{2n} \) or \( so_{2n} \) there also exist expressions similar to \([3]\) for the multiplicities \( K^g_{\lambda,\mu} \) of the weight \( \mu \) in the finite dimensional irreducible module \( V(\lambda) \) but a so simple duality as for \( sl_n \) does not exist although it is possible to obtain certain duality results between multiplicities of weights and tensor product multiplicities of representations by using dual pairs of algebraic groups \([5]\).

This implies that the weight quantifications of weight multiplicities and tensor product multiplicities cannot coincide for the root systems \( B_n, C_n \) and \( D_n \). The Kostka-Fouleks polynomials \( K^g_{\lambda,\mu}(q) \) are the \( q \)-analogues of \( K^g_{\lambda,\mu} \) defined in \([4]\) by quantifying the partition function corresponding to the root system associated to \( g \) (see \([2,2]\)). In \([13]\), Hatayama, Kuniba, Okado and Takagi have introduced for type \( C_n \) a quantification \( X^C_{\lambda,\mu}(q) \) of the multiplicity of \( V(\lambda) \) in the tensor product

\[
W(\mu_1 A_1) \otimes \cdots \otimes W(\mu_n A_1)
\]

where for any \( i = 1, \ldots, n \),

\[
W(\mu_i A_1) = V(\mu_i A_1) \oplus V((\mu_i - 2) A_1) \oplus \cdots \oplus V((\mu_i \mod 2) A_1).
\]

This quantification is based on the determination of the combinatorial \( R \) matrix of some \( U'_q(g) \)-crystals in the spirit of \([11]\). Note that there also exist \( q \)-multiplicities for the \( sp_2 \)-module \( V(\lambda) \) in a tensor product

\[
V(\Lambda_1) \otimes^k V(\Lambda_2) \otimes^l
\]

where \( k,l \) are positive integers obtained by Yamada \([17]\).

In this paper we first use Jacobi-Trudi’s type determinantal expressions for the Schur functions associated to \( g \) to introduce \( q \)-analogues of the multiplicity of \( V(\lambda) \) in the tensor products

(i) : \( h(\mu) = V(\mu_1 A_1) \otimes \cdots \otimes V(\mu_n A_1) \), \( \delta(\mu) = W(\mu_1 A_1) \otimes \cdots \otimes W(\mu_n A_1) \)

(ii) : \( e(\mu) = V(\Lambda_{\mu_1}) \otimes \cdots \otimes V(\Lambda_{\mu_n}) \), \( \epsilon(\mu) = W(\Lambda_{\mu_1}) \otimes \cdots \otimes W(\Lambda_{\mu_n}) \) with \( n \geq |\mu| \)

where

\[
\left\{ \begin{array}{l}
W(\mu_i A_1) = V(\mu_i A_1) \oplus V((\mu_i - 2) A_1) \oplus \cdots \oplus V((\mu_i \mod 2) A_1) \\
W(\Lambda_k) = V(\Lambda_k) \oplus V(\Lambda_{k-2}) \oplus \cdots \oplus V(\Lambda_{k \mod 2})
\end{array} \right.
\]

With the condition \( n \geq |\mu| \) for (ii), these multiplicities are independent of the Lie algebra \( g \) of type \( B_n, C_n \) or \( D_n \) considered. When \( q = 1 \), we recover a remarkable property already used by Koike and Terada in \([3]\). Next we prove that these \( q \)-multiplicities are in fact equal to Kostka-Fouleks
polynomials associated to the root systems of types C and D. It is possible to extend the definition (4) of the Kostka-Foulkes polynomials associated to the root system $A_n$ by replacing $\mu$ by $\gamma \in \mathbb{N}^n$ where $\gamma$ is not a partition. In this case $K_{\gamma}^{A_n}(q)$ may have nonnegative coefficients but $K_{\lambda,\gamma}^{A_n}(1)$ is equal to the dimension of the weight space $\gamma$ in $V(\lambda)$. Now if we extend (4) by replacing $\lambda$ by $\xi \in \mathbb{N}^n$, the polynomial $K_{\xi,\gamma}^{A_n}(q)$ is equal up to a sign to a Kostka-Foulkes polynomial $K_{\nu,\mu}^{A_n}(q)$ where $\nu$ is a partition. We obtained two expressions of the $q$-multiplicities defined above respectively in terms of the polynomials $K_{\lambda,\gamma}^{A_n}(q)$ and $K_{\xi,\mu}^{A_n}(q)$. By specializing at $q = 1$, this yields expressions of the corresponding multiplicities in terms of Kostka numbers.

In section 1 we recall the background on the root systems $B_n, C_n$ and $D_n$ and the corresponding Kostka-Foulkes polynomials. We review in section 2 the determinantal identities for Schur functions that we need in the sequel and we introduce the formalism suggested in (3) to prove the expressions of Schur functions in terms of raising and lowering operators implicitly contain in (3). Thank to this formalism we are able to obtain expressions for multiplicities similar to (3). We quantify these multiplicities to obtain the desired $q$-analogues in section 3. We prove in Section 4 two duality theorems between our $q$-analogues and certain Kostka-Foulkes polynomials of types $C$ and $D$. Finally we establish formulas expressing the associated multiplicities in terms of Kostka numbers.

Notation: In the sequel we frequently define similar objects for the root systems $B_n, C_n$ and $D_n$. When they are related to type $B_n$ (resp. $C_n, D_n$), we implicitly attach to them the label $B$ (resp. the labels $C, D$). To avoid cumbersome repetitions, we sometimes omit the labels $B, C$ and $D$ when our definitions or statements are identical for the three root systems.

Note: While writing this work, I have been informed that Shimozono and Zabrocki [16] have introduced independently and by using creating operators essentially the same tensor power multiplicities. Thanks to this formalism they recover in particular Jacobi-Trudi’s type determinantal expressions of the Schur functions associated to the root systems $B, C$ and $D$ which constitute the starting point of this article.

2 Background on the root systems $B_n, C_n$ and $D_n$

2.1 Convention for the positive roots

Consider an integer $n \geq 1$. The weight lattice for the root system $C_n$ (resp. $B_n$ and $D_n$) can be identified with $P_{C_n} = \mathbb{Z}^n$ (resp. $P_{B_n} = P_{D_n} \left( \frac{\mathbb{Z}}{2} \right)^n$) equipped with the orthonormal basis $\varepsilon_i, i = 1, ..., n$. We take for the simple roots

$$\begin{cases} 
\alpha_{n}^{B_n} = \varepsilon_n \\
\alpha_{n}^{C_n} = 2\varepsilon_n \\
\alpha_{n}^{D_n} = \varepsilon_n + \varepsilon_{n-1} 
\end{cases} \text{ and } \begin{cases} 
\alpha_{i}^{B_n} = \varepsilon_i - \varepsilon_{i+1}, i = 1, ..., n - 1 \text{ for the root system } B_n \\
\alpha_{i}^{C_n} = \varepsilon_i - \varepsilon_{i+1}, i = 1, ..., n - 1 \text{ for the root system } C_n \\
\alpha_{i}^{D_n} = \varepsilon_i - \varepsilon_{i+1}, i = 1, ..., n - 1 \text{ for the root system } D_n 
\end{cases} \tag{5}$$

Then the set of positive roots are

$$\begin{cases} 
R_{B_n}^+ = \{ \varepsilon_i - \varepsilon_j, \varepsilon_i + \varepsilon_j \text{ with } 1 \leq i < j \leq n \} \cup \{ \varepsilon_i \text{ with } 1 \leq i \leq n \} \text{ for the root system } B_n \\
R_{C_n}^+ = \{ \varepsilon_i - \varepsilon_j, \varepsilon_i + \varepsilon_j \text{ with } 1 \leq i < j \leq n \} \cup \{ 2\varepsilon_i \text{ with } 1 \leq i \leq n \} \text{ for the root system } C_n \\
R_{D_n}^+ = \{ \varepsilon_i - \varepsilon_j, \varepsilon_i + \varepsilon_j \text{ with } 1 \leq i < j \leq n \} \text{ for the root system } D_n 
\end{cases}$$

Denote respectively by $P_{B_n}^+, P_{C_n}^+$ and $P_{D_n}^+$ the sets of dominant weights of $so_{2n+1}, sp_{2n}$ and $so_{2n}$. 

3
Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a partition with $n$ parts. We will classically identify $\lambda$ with the dominant weight $\sum_{i=1}^{n} \lambda_i \varepsilon_i$. Note that there exists dominant weights associated to the orthogonal root systems whose coordinates on the basis $\varepsilon_i$, $i = 1, \ldots, n$ are not positive integers (hence which can not be regarded as a partition). For each root system of type $B_n, C_n$ or $D_n$, the set of weights having nonnegative integer coordinates on the basis $\varepsilon_1, \ldots, \varepsilon_n$ can be identify with the set $\pi^+_n$ of partitions of length $n$. For any partition $\lambda$, the weights of the finite dimensional $so_{2n+1}, sp_{2n}$ or $so_{2n}$-module of highest weight $\lambda$ are all in $\pi_n \cong \mathbb{Z}^n$. For any $\alpha \in \pi_n$ we write $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

The conjugate partition of the partition $\lambda$ is denoted $\lambda'$ as usual. Consider $\lambda, \mu$ two partitions of length $n$ and set $m = \text{max}(\lambda_1, \mu_1)$. Then by adding to $\lambda'$ and $\mu'$ the required numbers of parts 0 we will consider them as partitions of length $m$.

The Weyl group $W_{B_n} = W_{C_n}$ of $so_{2n+1}$ and $sp_{2n}$ is identified to the sub-group of the permutation group of the set $\{\overline{1}, \overline{2}, 1, 2, \ldots, n\}$ generated by $s_i = (i, i+1)(\overline{i}, \overline{i+1}), i = 1, \ldots, n-1$ and $s_n = (n, \overline{n})$ where for $a \neq b$ $(a, b)$ is the simple transposition which switches $a$ and $b$. We denote by $l_B$ the length function corresponding to the set of generators $s_i$, $i = 1, \ldots, n$.

The Weyl group $W_{D_n}$ of $so_{2n}$ is identified to the sub group of $W_{B_n}$ generated by $s_i = (i, i+1)(\overline{i}, \overline{i+1}), i = 1, \ldots, n-1$ and $s'_n = (n, n-1)(n-1, \overline{n})$. We denote by $l_D$ the length function corresponding to the set of generators $s_n$ and $s_i$, $i = 1, \ldots, n-1$.

Note that $W_{D_n} \subset W_{B_n}$ and any $w \in W_{B_n}$ verifies $w(\overline{i}) = w(i)$ for $i \in \{1, \ldots, n\}$. The action of $w$ on $\beta = (\beta_1, \ldots, \beta_n) \in P_n$ is given by
\[ w \cdot (\beta_1, \ldots, \beta_n) = (\beta^w_1, \ldots, \beta^w_n) \]
where $\beta^w_i = \beta_{w(i)}$ if $\sigma(i) \in \{1, \ldots, n\}$ and $\beta^w_i = -\beta_{w(\overline{i})}$ otherwise.

The half sums $\rho_{B_n}, \rho_{C_n}$ and $\rho_{D_n}$ of the positive roots associated to each root system $B_n, C_n$ and $D_n$ verify:
\[ \rho_{B_n} = (n - \frac{1}{2}, n - \frac{3}{2}, \ldots, \frac{1}{2}), \rho_{C_n} = (n, n-1, \ldots, 1) \text{ and } \rho_{B_n} = (n-1, n-2, \ldots, 0). \]

In the sequel we identify the symmetric group $S_n$ with the sub group of $W_{B_n}$ or $W_{D_n}$ generated by the $s_i$'s, $i = 1, \ldots, n-1$.

### 2.2 Schur functions and Kostka-Foukmues polynomials

We now briefly review the notions of Schur functions and Kostka-Foulkes polynomials associated to the roots systems $B_n, C_n$ and $D_n$ and refer the reader to [12] for more details. For any weight $\beta = (\beta_1, \ldots, \beta_n) \in \pi_n$ we set $x^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n}$ where $x_1, \ldots, x_n$ are fixed indeterminates. We set
\[ a_{B_n}^\beta = \sum_{w \in W_{B_n}} (-1)^{l(w)} (w \cdot x^\beta) \]
where $w \cdot x^\mu = x^{w(\mu)}$. The Schur function $s_{B_n}^\beta$ is defined as in [12] by
\[ s_{B_n}^\beta = \frac{a_{B_n}^{\beta + \rho_{B_n}}}{a_{\rho_{B_n}}^{\rho_{B_n}}}. \]
When \( \nu \in \pi_n^+ \), \( s_{\nu}^{B_n} \) is the Weyl character of \( V(\nu) \) the finite dimensional irreducible module with highest weight \( \nu \). For any \( w \in W_{B_n} \), the dot action of \( w \) on \( \beta \in \pi_n \) is defined by
\[
w \cdot \beta = w \cdot (\beta + \rho_{B_n}) - \rho_{B_n}.
\]

We have the following straightening law for the Schur functions. For any \( \beta \in \pi_n \), \( s_{\beta}^{B_n} \) is a nonnegative integer, write \( R_{\pi_n} \) for any \( \pi \in \pi_n \) and write \( K[\pi_n] \) for the \( \mathbb{K} \)-module generated by the \( x^\beta \), \( \beta \in \pi_n \). Set \( C_{B_n} = K[\pi_n]^{W_{B_n}} = \{ f \in K[\pi_n], w \cdot f = f \text{ for any } w \in W_{B_n} \} \). Then \( \{ s_{\beta}^{B_n}, \nu \in \pi_n^+ \} \) is a basis of \( K[\pi_n] \). We define \( s_{\beta}^{C_n} \) and \( s_{\beta}^{D_n} \) belonging to \( C_{C_n} = C_{B_n} \) and \( C_{D_n} \) in the same way and we obtain similarly that \( \{ s_{\beta}^{C_n}, \nu \in \pi_n^+ \} \) and \( \{ s_{\beta}^{D_n}, \nu \in \pi_n^+ \} \) are respectively bases of \( C_{C_n} \) and \( C_{D_n} \).

The \( q \)-analogue \( P_{q}^{B_n} \) of Kostant’s partition function corresponding to the root system \( B_n \) is defined by the equality
\[
\prod_{\alpha \in R_{\pi_n}^+} \frac{1}{1 - q^{x\alpha}} = \sum_{\beta \in \pi_n} P_{q}^{B_n}(\beta) x^\beta.
\]

Note that \( P_{q}^{B_n}(\beta) = 0 \) if \( \beta \) is not a linear combination of positive roots of \( R_{\pi_n}^+ \) with nonnegative coefficients. We write similarly \( P_{q}^{C_n} \) and \( P_{q}^{D_n} \) for the \( q \)-partition functions associated respectively to the root systems \( C_n \) and \( D_n \). Given \( \lambda \) and \( \mu \) two partitions of length \( n \), the Kostka-Foulkes polynomials of types \( B_n, C_n \) and \( D_n \) are then respectively defined by
\[
K_{\lambda,\mu}^{B_n}(q) = \sum_{\sigma \in W_{B_n}} (-1)^{(\sigma)} P_{q}^{B_n}(\sigma(\lambda + \rho_{B_n}) - (\mu + \rho_{B_n})),
\]
\[
K_{\lambda,\mu}^{C_n}(q) = \sum_{\sigma \in W_{C_n}} (-1)^{(\sigma)} P_{q}^{C_n}(\sigma(\lambda + \rho_{C_n}) - (\mu + \rho_{C_n})),
\]
\[
K_{\lambda,\mu}^{D_n}(q) = \sum_{\sigma \in W_{D_n}} (-1)^{(\sigma)} P_{q}^{D_n}(\sigma(\lambda + \rho_{D_n}) - (\mu + \rho_{D_n})).
\]

Remarks:

(i) : We have \( K_{\lambda,\mu}(q) = 0 \) when \(|\lambda| \neq |\mu| \).

(ii) : When \(|\lambda| = |\mu| \), \( K_{\lambda,\mu}^{B_n}(q) = K_{\lambda,\mu}^{C_n}(q) = K_{\lambda,\mu}^{D_n}(q) = K_{\lambda,\mu}^{A_{n-1}}(q) \) that is, the Kostka-Foulkes polynomials associated to the root systems \( B_n, C_n \) and \( D_n \) are Kostka-Foulkes polynomials associated to the root system \( A_{n-1} \).

3 Determinantal identities and multiplicities of representations

3.1 Determinantal identities for Schur functions

Consider \( k \in \mathbb{Z} \). When \( k \) is a nonnegative integer, write \((k)_n = (k, 0, \ldots, 0)\) for the partition of length \( n \) with a unique non-zero part equal to \( k \). Then set
\[
h_k^{B_n} = s_{(k)_n}^{B_n}, h_k^{C_n} = s_{(k)_n}^{C_n}, h_k^{D_n} = s_{(k)_n}^{D_n}
\]
and
\[
H_k^{B_n} = h_k^{B_n} + h_{k-2}^{B_n} + \cdots + h_{k \mod 2}^{B_n}, H_k^{C_n} = h_k^{C_n} + h_{k-2}^{C_n} + \cdots + h_{k \mod 2}^{C_n},
\]
\[
H_k^{D_n} = h_k^{D_n} + h_{k-2}^{D_n} + \cdots + h_{k \mod 2}^{D_n}.
\]
When $k$ is a negative integer we set $h_{k}^{B_{n}} = h_{k}^{C_{n}} = h_{k}^{D_{n}} = 0$ and $H_{k}^{B_{n}} = H_{k}^{C_{n}} = H_{k}^{D_{n}} = 0$. For any $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{Z}^n$

$$u_{\alpha}^{B_{n}} = \det \begin{pmatrix} h_{\alpha_1}^{B_{n}} & h_{\alpha_1+1}^{B_{n}} + h_{\alpha_1}^{B_{n-1}} & \cdots & h_{\alpha_1+n-1}^{B_{n}} + h_{\alpha_1-n}^{B_{n}} \\ h_{\alpha_2}^{B_{n}} & h_{\alpha_2+1}^{B_{n}} + h_{\alpha_2}^{B_{n-2}} & \cdots & h_{\alpha_2+n-1}^{B_{n}} + h_{\alpha_2-n}^{B_{n}} \\ \vdots & \vdots & \ddots & \vdots \\ h_{\alpha_{n-1}}^{B_{n}} & h_{\alpha_{n-1}+1}^{B_{n}} + h_{\alpha_{n-1}}^{B_{n-2}} & \cdots & h_{\alpha_{n-1}+n-1}^{B_{n}} + h_{\alpha_{n-1}-n}^{B_{n}} \\ h_{\alpha_n}^{B_{n}} + h_{\alpha_n-n}^{B_{n}} & \cdots & \cdots & \cdots \end{pmatrix}. \quad (6)$$

By using the equalities $h_{k}^{B_{n}} = H_{k}^{B_{n}} - h_{k}^{B_{n-2}}$ and simple computations on determinants we have also

$$u_{\alpha}^{B_{n}} = \det \begin{pmatrix} H_{\alpha_1}^{B_{n}} - H_{\alpha_1-2}^{B_{n}} & H_{\alpha_1}^{B_{n}} - H_{\alpha_1-1}^{B_{n}} & \cdots & H_{\alpha_1+n-1}^{B_{n}} - H_{\alpha_1-n}^{B_{n}} \\ H_{\alpha_2}^{B_{n}} - H_{\alpha_2-3}^{B_{n}} & H_{\alpha_2}^{B_{n}} - H_{\alpha_2-1}^{B_{n}} & \cdots & H_{\alpha_2+n-2}^{B_{n}} - H_{\alpha_2-n}^{B_{n}} \\ \vdots & \vdots & \ddots & \vdots \\ H_{\alpha_{n-1}}^{B_{n}} - H_{\alpha_{n-1}-2}^{B_{n}} & H_{\alpha_{n-1}}^{B_{n}} - H_{\alpha_{n-1}-1}^{B_{n}} & \cdots & H_{\alpha_{n-1}+n-2}^{B_{n}} - H_{\alpha_{n-1}-n}^{B_{n}} \\ H_{\alpha_n}^{B_{n}} - H_{\alpha_n-2n-2}^{B_{n}} & \cdots & \cdots & \cdots \end{pmatrix}. \quad (7)$$

We define $u_{\alpha}^{C_{n}}$ and $u_{\alpha}^{D_{n}}$ similarly by replacing $h_{k}^{B_{n}}$ respectively by $h_{k}^{C_{n}}$ and $h_{k}^{D_{n}}$.

Consider $p$ and $n$ two integers such that $n \geq 1$. When $p$ is nonnegative and $n \geq p$, write $(1^{p})_n = (1, ..., 1, 0, ..., 0)$ for the partition of length $n$ having $p$ non zero parts equal to 1. We set

$$\begin{cases} e_p^n = s_{(1^{p})_n}^B = s_{(1^{p})_n}^C = s_{(1^{p})_n}^D = s_{(1^{p})_n} \text{ if } 0 \leq p \leq n \\ e_p^n = e_{2p-n}^B = e_{2p-n}^C = e_{2p-n}^D = e_{2p-n} \text{ if } n + 1 \leq p \leq 2n \\ e_p^n = e_{p-n}^B = e_{p-n}^C = e_{p-n}^D = e_{p-n} \text{ otherwise} \end{cases}$$

and

$$E_{k}^{B_{n}} = e_{k}^{B_{n}} + e_{k-2}^{B_{n}} + \cdots + e_{k \mod 2}^{B_{n}}; \quad E_{k}^{C_{n}} = e_{k}^{C_{n}} + e_{k-2}^{C_{n}} + \cdots + e_{k \mod 2}^{C_{n}}; \quad E_{k}^{D_{n}} = e_{k}^{D_{n}} + e_{k-2}^{D_{n}} + \cdots + e_{k \mod 2}^{D_{n}}.$$ 

For any $\beta = (\beta_1, ..., \beta_n) \in \mathbb{Z}^n$

$$v_{\beta}^{B_{n}} = \det \begin{pmatrix} e_{\beta_1}^{B_{n}} & e_{\beta_1+1}^{B_{n}} & \cdots & e_{\beta_1+n-1}^{B_{n}} + e_{\beta_1-n}^{B_{n}} \\ e_{\beta_2}^{B_{n}} & e_{\beta_2+1}^{B_{n}} & \cdots & e_{\beta_2+n-2}^{B_{n}} + e_{\beta_2-n}^{B_{n}} \\ \vdots & \vdots & \ddots & \vdots \\ e_{\beta_{n-1}}^{B_{n}} & e_{\beta_{n-1}+1}^{B_{n}} & \cdots & e_{\beta_{n-1}+n-2}^{B_{n}} + e_{\beta_{n-1}-n}^{B_{n}} \\ e_{\beta_n}^{B_{n}} + e_{\beta_n-n}^{B_{n}} & \cdots & \cdots & \cdots \end{pmatrix}.$$ 

By using the equalities $e_{k}^{B_{n}} = E_{k}^{B_{n}} - E_{k}^{B_{n-2}}$ and simple computations on determinants we have also

$$v_{\beta}^{B_{n}} = \det \begin{pmatrix} E_{\beta_1}^{B_{n}} - E_{\beta_1}^{B_{n-2}} & E_{\beta_1}^{B_{n}} - E_{\beta_1}^{B_{n-1}} & \cdots & E_{\beta_1-n}^{B_{n}} - E_{\beta_1-n}^{B_{n}} \\ E_{\beta_2}^{B_{n}} - E_{\beta_2}^{B_{n-3}} & E_{\beta_2}^{B_{n}} - E_{\beta_2}^{B_{n-2}} & \cdots & E_{\beta_2-n}^{B_{n}} - E_{\beta_2-n}^{B_{n}} \\ \vdots & \vdots & \ddots & \vdots \\ E_{\beta_{n-1}}^{B_{n}} - E_{\beta_{n-1}}^{B_{n-2}} & E_{\beta_{n-1}}^{B_{n}} - E_{\beta_{n-1}}^{B_{n-2}} & \cdots & E_{\beta_{n-1}-n}^{B_{n}} - E_{\beta_{n-1}-n}^{B_{n}} \\ E_{\beta_n}^{B_{n}} - E_{\beta_n-n}^{B_{n}} & \cdots & \cdots & \cdots \end{pmatrix}.$$ 

The determinants $v_{\beta}^{C_{n}}, v_{\beta}^{D_{n}}$ are defined similarly.
Proposition 3.1.1 (see [3]) Consider λ a partition of length n and suppose that \( \lambda' = (\lambda'_1, \ldots, \lambda'_m) \) is a partition of length m. Then \( u_\lambda = s_\lambda \) and \( v_{\lambda'} = s_{\lambda'} \).

Lemma 3.1.2 (straightening law for \( u_\alpha \) and \( v_\beta \))

Consider \( \alpha \in \pi_n \) then

\[
u_\alpha = \begin{cases} (-1)^{l(\sigma)}u_\lambda & \text{if there exists } \sigma \in S_n \text{ and } \lambda \in \pi_n^+ \text{ such that } \sigma \circ \alpha = \lambda \\ 0 & \text{otherwise} \end{cases}
\]

Consider \( \beta \in \pi_m \) then

\[
u_\beta = \begin{cases} (-1)^{l(\sigma)}v_\nu & \text{if there exists } \sigma \in S_m \text{ and } \nu \in \pi_m^+ \text{ such that } \sigma \circ \alpha = \nu \\ 0 & \text{otherwise} \end{cases}
\]

Proof. By commuting the rows \( i \) and \( i + 1 \) in the determinant (\( \mathbf{\bar{5}} \)) we see that \( u_{\sigma, 0} = -u_\alpha \). This implies that \( u_{\sigma, 0} = (-1)^{l(\sigma)}u_\alpha \) for any \( \sigma \in S_n \). Then it follows from the definition of the dot action that \( u_\alpha = 0 \) or there exists \( \gamma \in \pi_n \) and \( \sigma \in S_n \) such that \( \gamma_1 \geq \cdots \geq \gamma_n \) and \( \gamma = \sigma \circ \alpha \). In this last case we have \( u_\alpha = (-1)^{l(\sigma)}u_\gamma \). Now if there exists a negative \( \gamma_i \), \( u_\gamma = 0 \) since all the \( H_k \) which appear in the lowest row of (\( \mathbf{\bar{5}} \)) are equal to 0. Thus \( \gamma \) is a partition. The proof is similar for \( v_\beta \). □

3.2 Determinantal identities in terms of raising and lowering operators

Let \( \mathcal{L}_n = \mathbb{K}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}] \) be the ring of formal series in the indeterminates \( x_1, x_1^{-1}, \ldots, x_n, x_n^{-1} \). We consider the two following determinants

\[
\delta_n(\alpha) = \det \begin{pmatrix}
\frac{x_1^\alpha}{1}, & \frac{x_1^{\alpha+1}}{1} + \frac{x_1^{\alpha-1}}{1}, & \cdots, & \frac{x_1^{\alpha+n-1}}{1} + \frac{x_1^{\alpha-n+1}}{1} \\
\frac{x_2^\alpha}{1}, & \frac{x_2^{\alpha+2}}{1} + \frac{x_2^{\alpha-2}}{1}, & \cdots, & \frac{x_2^{\alpha+n-2}}{1} + \frac{x_2^{\alpha-n+2}}{1} \\
\cdots & \cdots & \cdots & \cdots \\
\frac{x_n^{\alpha-n+1}}{1}, & \frac{x_n^{\alpha-n+2}}{1} + \frac{x_n^{\alpha-n}}{1}, & \cdots, & \frac{x_n^{\alpha+n-2n+2}}{1}
\end{pmatrix}
\]

and

\[
\Delta_n(\alpha) = \det \begin{pmatrix}
\frac{x_1^{\alpha_1}}{1} - \frac{x_1^{\alpha_2}}{1}, & \frac{x_1^{\alpha+1}}{1} - \frac{x_1^{\alpha_1-1}}{1}, & \cdots, & \frac{x_1^{\alpha+n-1}}{1} - \frac{x_1^{\alpha-n+1}}{1} \\
\frac{x_2^{\alpha_1}}{1} - \frac{x_2^{\alpha_2}}{1}, & \frac{x_2^{\alpha+2}}{1} - \frac{x_2^{\alpha_2-2}}{1}, & \cdots, & \frac{x_2^{\alpha+n-2}}{1} - \frac{x_2^{\alpha-n+2}}{1} \\
\cdots & \cdots & \cdots & \cdots \\
\frac{x_n^{\alpha-n+1}}{1}, & \frac{x_n^{\alpha-n+2}}{1} - \frac{x_n^{\alpha-n}}{1}, & \cdots, & \frac{x_n^{\alpha+n-2n+2}}{1}
\end{pmatrix}
\]

From a simple computation we derive the equalities:

\[
\delta_n(\alpha) = \prod_{1 \leq i < j \leq n} \left( 1 - \frac{x_i}{x_j} \right) \prod_{1 \leq r < s \leq n} \left( 1 - \frac{1}{x_i x_j} \right) x^\alpha \quad \text{and} \quad \Delta_n(\alpha) = \prod_{1 \leq i < j \leq n} \left( 1 - \frac{x_i}{x_j} \right) \prod_{1 \leq r < s \leq n} \left( 1 - \frac{1}{x_i x_j} \right) x^\alpha. \tag{8}
\]

We set \( h_\alpha = h_{\alpha_1} \cdots h_{\alpha_n}, H_\alpha = H_{\alpha_1} \cdots H_{\alpha_n}, e_\alpha = e_{\alpha_1} \cdots e_{\alpha_n} \) and \( E_\alpha = E_{\alpha_1} \cdots E_{\alpha_n} \).

Remarks

(i) For any partition \( \mu \) of length \( n \), \( h_\mu \) is the character of \( h(\mu) = V(\mu_1 \Lambda_1) \otimes \cdots \otimes V(\mu_n \Lambda_1) \) and \( H_\mu \) is the character of \( S(\mu) = W(\mu_1 \Lambda_1) \otimes \cdots \otimes W(\mu_n \Lambda_1) \) where for any \( k \in \mathbb{N}, W(k \Lambda) = V(k \Lambda_1) \oplus V((k - 2) \Lambda_1) \oplus \cdots \oplus V((k \text{ mod } 2) \Lambda_1) \).

(ii) For any partition \( \mu \) of length \( n \) such that \( \mu' \) is of length \( m \), \( e_{\mu'} \) is the character of \( e(\mu) = V(\Lambda_{\mu'_1} \Lambda) \otimes \cdots \otimes V(\Lambda_{\mu'_m} \Lambda) \) and \( E_{\mu'} \) is the character of \( E(\mu) = W(\Lambda_{\mu'_1} \Lambda) \otimes \cdots \otimes W(\Lambda_{\mu'_m} \Lambda) \) where for any \( k \in \mathbb{N} \) with \( k \leq n \), \( W(\Lambda_k) = V(\Lambda_k) \oplus V(\Lambda_{k-2}) \oplus \cdots \oplus V(\Lambda_{k \text{ mod } 2}) \).
For the root system $B_n$ we introduce six linear maps $h_{B_n}, H_{B_n}, u_{B_n}$ and $e_{B_n}, E_{B_n}, v_{B_n}$ as follows:

\[
\begin{align*}
&\{ h_{B_n} : \mathcal{L}_n \to \mathcal{C}_{B_n}, \quad x^\alpha \mapsto h_\alpha \} , \\
&\{ H_{B_n} : \mathcal{L}_n \to \mathcal{C}_{B_n}, \quad x^\alpha \mapsto H_\alpha \} , \\
&\{ u_{B_n} : \mathcal{L}_n \to \mathcal{C}_{B_n}, \quad x^\alpha \mapsto u_\alpha \}, \\
&\{ e_{B_n} : \mathcal{L}_n \to \mathcal{C}_{B_n}, \quad x^\alpha \mapsto e_\alpha \} , \\
&\{ E_{B_n} : \mathcal{L}_n \to \mathcal{C}_{B_n}, \quad x^\alpha \mapsto E_\alpha \} , \\
&\{ v_{B_n} : \mathcal{L}_n \to \mathcal{C}_{B_n}, \quad x^\alpha \mapsto v_\alpha \} .
\end{align*}
\]

Note that these maps are not ring homomorphisms. For the roots systems $C_n$ and $D_n$ we define respectively the maps $h_{C_n}, H_{C_n}, u_{C_n}, e_{C_n}, E_{C_n}, v_{C_n}$ and $h_{D_n}, H_{D_n}, u_{D_n}, e_{D_n}, E_{D_n}, v_{D_n}$ similarly.

Let $\omega_n$ and $\Omega_n$ be the endomorphisms of $\mathcal{L}_n$ corresponding respectively to the multiplication by

\[
\phi_n = \prod_{1 \leq i < j \leq n} (1 - \frac{x_i}{x_j}) \prod_{1 \leq i < j \leq n} (1 - \frac{1}{x_i x_j}) \quad \text{and} \quad \Phi_n = \prod_{1 \leq i < j \leq n} (1 - \frac{x_i}{x_j}) \prod_{1 \leq i < j \leq n} (1 - \frac{1}{x_i x_j}).
\]

Since $\phi_n^{-1}$ and $\Phi_n^{-1}$ belong to $\mathcal{L}_n$, $\omega_n$ and $\Omega_n$ are the automorphisms of $\mathcal{L}_n$ corresponding to the multiplication by $\phi_n^{-1}$ and $\Phi_n^{-1}$.

**Proposition 3.2.1** We have

1. $u_n = h_n \cdot \omega_n$ and $u_n = H_n \cdot \Omega_n$,

2. $v_n = e_n \cdot \omega_n$ and $v_n = E_n \cdot \Omega_n$.

**Proof.** 1 : We have seen that $h_n$ is not a ring-homomorphism. Nevertheless we have by definition of the $h_\alpha$

\[
h_n(x^\alpha) = h_n(x_1^\alpha) \cdots h_n(x_n^\alpha) = h_{\alpha_1} \cdots h_{\alpha_n}.
\]

More generally if $P_1, \ldots, P_n$ are polynomials respectively in the indeterminates $x_1, \ldots, x_n$, we have

\[
h_n(P_1(x_1) \cdots P_n(x_n)) = h_n(P_1(x_1)) \cdots h_n(P_n(x_n))
\]

by linearity of $h_n$. We can write

\[
\delta_n(\alpha) = \sum_{\sigma \in S_n} (-1)^{l(\sigma)} x_{\sigma(1)}^{\alpha_1 - \sigma(1) + 1} x_{\sigma(2)}^{\alpha_2 - \sigma(2) + 2} + x_{\sigma(2)}^{\alpha_2 - \sigma(2) + 2} \cdots (x_{\sigma(n)}^{\alpha_n - \sigma(n) + n} + x_{\sigma(n)}^{\alpha_n - \sigma(n) - n + 2})
\]

and by the previous argument

\[
h_n(\delta_n(\alpha)) = \sum_{\sigma \in S_n} (-1)^{l(\sigma)} h_{\alpha_1 - \sigma(1) + 1}(h_{\alpha_2 - \sigma(2) + 2} + h_{\alpha_2 - \sigma(2) + 2}) \cdots (h_{\alpha_n - \sigma(n) + n} + h_{\alpha_n - \sigma(n) - n + 2}) = u_\alpha
\]

where the last equality follows from \([\mathfrak{f}]\). By \([\mathfrak{g}]\) we have $\delta_n(\alpha) = \omega_n(x^\alpha)$. Thus by applying $h_n$ to this equality we obtain $h_n(\omega_n(x^\alpha)) = u_\alpha = u_n(x^\alpha)$. Hence $u_n = h_n \cdot \omega_n$. We derive the equality $u_n = H_n \cdot \Omega_n$ in a similar way starting from

\[
\Delta_n(\alpha) = \sum_{\sigma \in S_n} (-1)^{l(\sigma)} (x_{\sigma(1)}^{\alpha_1 - \sigma(1) + 1} + x_{\sigma(1)}^{\alpha_1 - \sigma(1) - 1}) \cdots (x_{\sigma(n)}^{\alpha_n - \sigma(n) + n} + x_{\sigma(n)}^{\alpha_n - \sigma(n) - n}).
\]

2 : The arguments are the same than in 1 once replacing the characters $h$ and $H$ respectively by the characters $e$ and $E$. ■

Consider $\alpha = (\alpha_1, \ldots, \alpha_n) \in \pi_n$ and two integers $i, j$ such that $1 \leq i \leq j \leq n$. The raising operator $R_{i,j}$ and the lowering operator $L_{i,j}$ are respectively defined on $\pi_n$ by $R_{i,j}(\alpha) = \alpha + \varepsilon_i - \varepsilon_j$ and $L_{i,j}(\alpha) = \alpha - \varepsilon_i - \varepsilon_j$. From the previous lemma we obtain:

\[8\]
Corollary 3.2.2 For any partition $\mu = (\mu_1, ..., \mu_n)$ we have

$$s_\mu = \left( \prod_{1 \leq i < j \leq n} (1 - R_{i,j}) \prod_{1 \leq r < s \leq n} (1 - L_{r,s}) \right) h_\mu, \quad s_\mu = \left( \prod_{1 \leq i < j \leq m} (1 - R_{i,j}) \prod_{1 \leq r < s \leq m} (1 - L_{r,s}) \right) H_\mu,$$

where $\mu' = (\mu'_1, ..., \mu'_m)$ is the conjugate partition of $\mu$.

**Proof.** Let us write

$$\phi_n = \prod_{1 \leq i < j \leq n} \left( 1 - \frac{x_i}{x_j} \right) \prod_{1 \leq r < s \leq n} (1 - \frac{1}{x_r x_s}) = \sum_{\alpha \in \pi_n} a(\alpha) x^\alpha.$$

Then by 1 of Proposition 3.2.1, we have for any $\mu \in \pi_n^+$,

$$u_n(x^\mu) = h_n \left( \sum_{\alpha \in \pi_n} a(\alpha) x^{\alpha + \mu} \right) = \sum_{\alpha \in \pi_n} a(\alpha) h_{\alpha + \mu} = u_\mu = s_\mu,$$

where the last equality follows from Proposition 3.1.1. This is exactly equivalent to

$$s_\mu = \left( \prod_{1 \leq i < j \leq n} (1 - R_{i,j}) \prod_{1 \leq r < s \leq n} (1 - L_{r,s}) \right) h_\mu.$$

The arguments are essentially the same for the other equalities. \[\blacksquare\]

### 3.3 Expressions for the multiplicities of representations

Write

$$\phi_n^{-1} = \sum_{\alpha \in \pi_n} f(\alpha) x^\alpha \quad \text{and} \quad \Phi_n^{-1} = \sum_{\alpha \in \pi_n} F(\alpha) x^\alpha.$$

From Lemma 3.2.1 we deduce that $h_n = u_n \circ \omega_n^{-1}$ and $H_n = u_n \circ \Omega_n^{-1}$. By applying these identities to $x^\mu$ where $\mu$ is a partition of length $n$ with $\mu'$ of length $m$ we obtain as in Corollary 3.2.2

$$h_\mu = \left( \prod_{1 \leq i < j \leq n} \frac{1}{1 - R_{i,j}} \prod_{1 \leq r < s \leq n} \frac{1}{1 - L_{r,s}} \right) s_\mu, \quad H_\mu = \left( \prod_{1 \leq i < j \leq m} \frac{1}{1 - R_{i,j}} \prod_{1 \leq r < s \leq m} \frac{1}{1 - L_{r,s}} \right) s_\mu,$$

$$e_{\mu'} = \left( \prod_{1 \leq i < j \leq m} \frac{1}{1 - R_{i,j}} \prod_{1 \leq r < s \leq m} \frac{1}{1 - L_{r,s}} \right) s_\mu \quad \text{and} \quad E_{\mu'} = \left( \prod_{1 \leq i < j \leq m} \frac{1}{1 - R_{i,j}} \prod_{1 \leq r < s \leq m} \frac{1}{1 - L_{r,s}} \right) s_\mu.$$

These relations must be understood as a short way to write

$$h_\mu = \sum_{\alpha \in \pi_n} f(\alpha) u_{\mu + \alpha}, \quad H_\mu = \sum_{\alpha \in \pi_n} F(\alpha) u_{\mu + \alpha},$$

$$e_{\mu'} = \sum_{\beta \in \pi_m} f(\alpha) v_{\mu' + \beta}, \quad E_{\mu'} = \sum_{\beta \in \pi_m} F(\alpha) v_{\mu' + \beta}.$$

For any positive integer $n$ write $\rho_t = (n, n - 1, ..., 1)$. 

9
Proposition 3.3.1 Consider a partition $\mu$ of length $n$ such that $\mu'$ has length $m$. Then for the three root systems $B_n, C_n$ and $D_n$ we have:

\[
\begin{align*}
& (i) : \quad h_\mu = \sum_{\lambda \in \pi_n} \sum_{\sigma \in S_n} (-1)^{|\sigma|} f(\sigma(\lambda + \rho_n) - \mu - \rho_n) u_\lambda \\
& H_\mu = \sum_{\lambda \in \pi_n} \sum_{\sigma \in S_n} (-1)^{|\sigma|} F(\sigma(\lambda + \rho_n) - \mu - \rho_n) u_\lambda ,
\end{align*}
\]

\[
\begin{align*}
& (ii) : \quad e^\mu = \sum_{\nu \in \pi_n^+} \sum_{\sigma \in S_n} (-1)^{|\sigma|} f(\sigma(\mu + \rho_n) - \mu' - \rho_n) v_\nu \\
& E^\mu = \sum_{\nu \in \pi_n^+} \sum_{\sigma \in S_n} (-1)^{|\sigma|} F(\sigma(\mu + \rho_n) - \mu' - \rho_n) v_\nu .
\end{align*}
\]

Proof. (i) : Note first that the above relations do not depend on the root system considered. Indeed for any nonnegative integer $n$, we have $\rho_{B_m} = \rho_n - (\frac{1}{2}, \ldots, \frac{1}{2})$, $\rho_{C_n} = \rho_n$ and $\rho_{D_m} = \rho_n - (1, \ldots, 1)$. Thus $\sigma(\lambda + \rho_{B_n}) - \mu - \rho_{B_n} = \sigma(\lambda + \rho_{C_n}) - \mu - \rho_{C_n} = \sigma(\lambda + \rho_{D_n}) - \mu - \rho_{D_n} = \sigma(\lambda + \rho_n) - \mu - \rho_n$. We have

\[ h_\mu = \sum_{\alpha \in \pi_n} f(\alpha) u_{\mu + \alpha} . \]

From Lemma 3.1.2 we deduce that for any $\alpha \in \pi_n$ we have $u_{\mu + \alpha} = 0$ or there exits a partition $\lambda$ such that $\mu + \alpha = \sigma(\lambda + \rho_n) - \rho_n$ and $u_{\mu + \alpha} = (-1)^{|\alpha|} u_\lambda$. By setting $\alpha = \sigma(\lambda + \rho_n) - \mu - \rho_n$ in the above sum we obtain $h_\mu = \sum_{\lambda \in \pi_n} \sum_{\sigma \in S_n} (-1)^{|\sigma|} f(\sigma(\lambda + \rho_n) - \mu - \rho_n) u_\lambda$. The arguments are similar for the other assertions. \ \[ \Box \]

From relations (i) and by using the fact that $u_\lambda = s_\lambda$ for any partition $\lambda$ of length $n$, we derive the equalities

\[ h_\mu = \sum_{\lambda \in \pi_n} u_{\lambda,\mu} s_\lambda \quad \text{and} \quad H_\mu = \sum_{\lambda \in \pi_n} U_{\lambda,\mu} s_\lambda \]

where

\[ u_{\lambda,\mu} = \sum_{\sigma \in S_n} (-1)^{|\sigma|} f(\sigma(\lambda + \rho_n) - \mu - \rho_n) \quad \text{and} \quad U_{\lambda,\mu} = \sum_{\sigma \in S_n} (-1)^{|\sigma|} F(\sigma(\lambda + \rho_n) - \mu - \rho_n) \] (9)

are respectively the multiplicities of $V(\lambda)$ in $h(\mu)$ and $H(\mu)$. Note that $u_{\lambda,\mu} = 0$ and $U_{\lambda,\mu} = 0$ unless $|\mu| \geq |\lambda|$.

For the relations (ii) the situation is more complicated since the partitions $\nu$ obtained by applying straightening laws to the $v_{\mu'+\beta}$ yields polynomials $v_\nu$ where $\nu \in \pi_m$ is a partition of length $m$ so can not be necessarily regarded as the conjugate partition of a partition $\lambda \in \pi_n$. The straightening law of Lemma 3.1.2 implies that $|\nu| = |\mu'|$. Since $|\mu| = |\mu'|$, this problem disappear if we suppose $n \geq |\mu|$ for we will have $\nu_1 \leq |\nu| \leq n$ and thus $\nu' \in \pi_n$. We can then set $\nu = \lambda'$ with $\lambda \in \pi_n$ and obtain

\[ e^\mu = \sum_{\lambda \in \pi_n} v_{\lambda,\mu} s_\lambda \quad \text{and} \quad E^\mu = \sum_{\lambda \in \pi_n} V_{\lambda,\mu} s_\lambda . \]

We deduce that

\[
\begin{align*}
& v_{\lambda,\mu} = u_{\lambda',\mu'} = \sum_{\sigma \in S_m} (-1)^{|\sigma|} f(\sigma(\lambda' + \rho_m) - \mu' - \rho_m) \quad (10) \\
& V_{\lambda,\mu} = U_{\lambda',\mu'} = \sum_{\sigma \in S_m} (-1)^{|\sigma|} F(\sigma(\lambda' + \rho_m) - \mu' - \rho_m) \quad (11)
\end{align*}
\]

are respectively the multiplicities of $V(\lambda)$ in the tensor products $e(\mu)$ and $E(\mu)$ when $n \geq |\mu|$.
4 Quantification of the multiplicities

4.1 The functions $f_q$ and $F_q$

Set

$$
\phi_n(q) = \prod_{1 \leq i < j \leq n} (1 - q \frac{x_i}{x_j}) \prod_{1 \leq r < s \leq n} (1 - q \frac{x_r}{x_s}) \quad \text{and} \quad \Phi_n(q) = \prod_{1 \leq i < j \leq n} (1 - q \frac{x_i}{x_j}) \prod_{1 \leq r < s \leq n} (1 - q \frac{x_r}{x_s}).
$$

The functions $f_q$ and $F_q$ are obtained by considering the formal series expansions of $\phi_n^{-1}(q)$ and $\Phi_n^{-1}(q)$. Namely we have

$$
\phi_n^{-1}(q) = \sum_{\alpha \in \pi_n} f_q(\alpha) x^\alpha \quad \text{and} \quad \Phi_n^{-1}(q) = \sum_{\alpha \in \pi_n} F_q(\alpha) x^\alpha.
$$

(12)

4.2 Some $q$-analogues of multiplicities of $V(\lambda)$ in $\mathfrak{h}(\mu)$, $\mathfrak{f}(\mu)$, $\mathfrak{e}(\mu)$ or $\mathfrak{E}(\mu)$

Given $\lambda$ and $\mu$ two partitions of length $n$, let $c_{\lambda,\mu}(q)$ and $C_{\lambda,\mu}(q)$ be the two polynomials defined by

$$
\lambda, \mu \quad \text{and} \quad u_{\lambda,\mu}(q) = \sum_{\sigma \in \mathcal{S}_n} (-1)^{l(\sigma)} f_q(\sigma + \rho_n - \mu) \quad \text{and} \quad U_{\lambda,\mu}(q) = \sum_{\sigma \in \mathcal{S}_n} (-1)^{l(\sigma)} F_q(\sigma + \rho_n - \mu).
$$

Then from the equalities (9), (10) and (11) we obtain:

**Proposition 4.2.1** Let $\lambda$ and $\mu$ be two partitions of length $n$. Then

1. $u_{\lambda,\mu}(q)$ and $U_{\lambda,\mu}(q)$ are $q$-analogues of the multiplicity of the representation $V(\lambda)$ in $\mathfrak{h}(\mu)$ and $\mathfrak{f}(\mu)$.

2. $v_{\lambda,\mu}(q) = u_{\lambda,\mu'}(q)$ and $\psi_{\lambda,\mu}(q) = U_{\lambda,\mu'}(q)$ are $q$-analogues of the multiplicity of the representation $V(\lambda)$ in $\mathfrak{e}(\mu)$ and $\mathfrak{E}(\mu)$ when the condition $n \geq |\mu|$ is satisfied.

The following example is obtained from the explicit computation of the function $f_q$ when $n = 2$.

**Example 4.2.2** Consider $\mu$ a partition of length 2 and set $\mathcal{E}_\mu = \{ \lambda \in \pi_2^+, \lambda = (\mu_1 + r - s, \mu_2 - r - s), \quad s \in \{0, ..., \mu_2\}, \quad r \in \{0, ..., \mu_2 - s\} \}$. Then for any partition $\lambda$ of length 2 we have:

$$
u_{\lambda,\mu}(q) = \begin{cases} 
q^{\mu_1 - \lambda_1} & \text{if } \lambda \in \mathcal{E}_\mu \\
0 & \text{otherwise} 
\end{cases}.
$$

**Remarks**

(i) : It follows from the definition of the $q$-functions $f_q$ and $F_q$ that $u_{\lambda,\mu}(q) = U_{\lambda,\mu}(q) = 0$ if $|\lambda| > |\mu|$.

(ii) : It is not trivial from the very definitions that $u_{\lambda,\mu}(q)$ and $U_{\lambda,\mu}(q)$ are polynomials in $q$ with nonnegative integer coefficients. This property will be proved in Section 5.1.3.
5 The duality theorems

5.1 A duality theorem for the $q$-multiplicities in $\mathfrak{h}(\mu)$ and $\mathfrak{h}(\mu)$

For any nonnegative integer $n$, set $\kappa_n = (1, ..., 1) \in \pi_n$.

**Lemma 5.1.1** Consider $\lambda, \mu$ two partitions of length $n$ such that $|\lambda| \geq |\mu|$. Let $k$ be any integer such that $k \geq \frac{|\lambda|-|\mu|}{2}$. Then we have

$$K_{\lambda+k\kappa_n, \mu+k\kappa_n}(q) = \sum_{\sigma \in S_n} (-1)^{l(\sigma)} p_q(\sigma(\lambda + \rho_n) - (\mu + \rho_n))$$

(13)

where the sum is indexed by the elements of the symmetric group $S_n$.

**Proof.** Since $P_q(\alpha) = 0$ if $\alpha$ is not a linear combination of positive roots with nonnegative coefficients, we have $P_q(\alpha) = 0$ for any $\alpha \in \pi_n$ such that $|\alpha| < 0$. Consider $\delta = (\delta_1, ..., \delta_n) \in \pi_n$ and $w \in W_n$. Write $w(\delta) = (\delta_1^w, ..., \delta_n^w)$ and denote by $E_{w,\delta} = \{i_1, ..., i_p\}$ the set of the indices $i_k$ such that $\delta_i^w$ and $\delta_i^w$ have opposite signs. Define the sum $S_{w,\delta} = \sum_{i_k \in E_{w,\delta}} \delta_i$. Then $|w(\delta)| = |\delta| - 2S_{w,\delta}$. Now consider $k$ a nonnegative integer and set $\delta = (\lambda + \rho_n + k\kappa_n)$. We have $|w(\lambda + \rho_n + k\kappa_n)| = |(\lambda + \rho_n + k\kappa_n)| - 2S_{w,\lambda+\rho_n} + |(\mu + \rho_n + k\kappa_n)| - 2kp$. Thus we obtain

$$|w(\lambda + \rho_n + k\kappa_n) - (\mu + \rho_n + k\kappa_n)| = |(\lambda + \rho_n + k\kappa_n)| - 2S_{w,\lambda+\rho_n} - |(\mu + \rho_n + k\kappa_n)| - 2kp = |\lambda| - |\mu| - 2S_{w,\lambda+\rho_n} - 2kp.$$  

When $w \notin S_n$, we have $p \geq 1$ and $S_{w,\lambda+\rho_n} \geq 1$ since the coordinates of $\lambda + \rho_n$ are all positive. Hence $|w(\lambda + \rho_n + k\kappa_n) - (\mu + \rho_n + k\kappa_n)| < |\lambda| - |\mu| - 2k$ and is negative as soon as $k \geq \frac{|\lambda|-|\mu|}{2}$. For such an integer $k$ the sum defining $K_{\lambda+k\kappa_n, \mu+k\kappa_n}(q)$ normally running on $W_n$ can be restricted to (13) and we obtain

$$K_{\lambda+k\kappa_n, \mu+k\kappa_n}(q) = \sum_{\sigma \in S_n} (-1)^{l(\sigma)} p_q(\sigma(\lambda + \rho_n + k\kappa_n) - (\mu + \rho_n + k\kappa_n)).$$

Since $\sigma \in S_n$, we have $\sigma(k\kappa_n) = k\kappa_n$. Thus

$$K_{\lambda+k\kappa_n, \mu+k\kappa_n}(q) = \sum_{\sigma \in S_n} (-1)^{l(\sigma)} p_q(\sigma(\lambda + \rho_n) - (\mu + \rho_n)).$$

We define the involution $I$ on $\pi_n$ by $I(\alpha_1, ..., \alpha_n) = (-\alpha_n, ..., -\alpha_1)$ for any $\alpha = (\alpha_1, ..., \alpha_n) \in \pi_n$.

**Lemma 5.1.2** For any $\alpha = (\alpha_1, ..., \alpha_n) \in \pi_n$ we have

$$f_q(\alpha) = P_q^{D_n}(I(\alpha)) \text{ and } F_q(\alpha) = P_q^{C_n}(I(\alpha))$$

where $P_q^{B_n}$ and $P_q^{D_n}$ are the $q$-partition functions associated respectively to the root systems $B_n$ and $D_n$.  

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By applying \( I \) to the equality
\[
\prod_{\alpha \in R_{\mathcal{C}^n}} \frac{1}{1-q x^\alpha} = \sum_{\beta \in \pi_n} \mathcal{P}_{q^\beta}^C(I(\beta)) x^\beta
\]
we obtain
\[
\prod_{1 \leq i < j \leq n} \frac{1}{1-q \varepsilon_j x_i} \prod_{1 \leq r \leq s \leq n} \frac{1}{1-q x_r x_s} = \sum_{\beta \in \pi_n} \mathcal{P}_{q^\beta}^C(I(\beta)) x^\beta.
\]
Set \( \alpha = I(\beta) \). The equality becomes
\[
\Phi_n^{-1}(q) = \sum_{\alpha \in \pi_n} \mathcal{P}_{q^\alpha}^C(I(\alpha)) x^\alpha
\]
and from the definition (see [2]) of the function \( F_q \), we obtain \( \mathcal{P}_{q^\beta}^C(I(\alpha)) = F_q(\alpha) \). The assertion with \( f_q \) is proved in the same way by considering the root system \( D_n \).

Given \( \sigma \in \mathcal{S}_n \), denote by \( \sigma^* \) the permutation defined by
\[
\sigma^*(k) = \sigma(n-k+1).
\]
For any \( i \in \{1, ..., n-1\} \), we have \( s_i^* = s_{n-i} \). The following Lemma is straightforward:

**Lemma 5.1.3** The map \( \sigma \to \sigma^* \) is an involution of the group \( \mathcal{S}_n \). Moreover we have \( \sigma(I(\beta)) = I(\sigma^*(\beta)) \) and \( l(\sigma) = l(\sigma^*) \) for any \( \beta \in \pi_n, \sigma \in \mathcal{S}_n \).

**Lemma 5.1.4** Let \( \lambda, \mu \) two partitions of length \( n \) and \( \sigma \in \mathcal{S}_n \). Then
\[
(-1)^l(\sigma) f_q(\sigma(\lambda + \rho_n) - (\mu + \rho_n)) = (-1)^{l(\sigma^*)} \mathcal{P}_{q^\lambda}^D(I(\lambda) + \rho_n) - (I(\mu) + \rho_n))
\]
and
\[
(-1)^l(\sigma) F_q(\sigma(\lambda + \rho_n) - (\mu + \rho_n)) = (-1)^{l(\sigma^*)} \mathcal{P}_{q^{\lambda^*}}^C(I(\lambda) + \rho_n) - (I(\mu) + \rho_n)).
\]

**Proof.** Since \( l(\sigma) = l(\sigma^*) \), it suffices to prove the equalities
\[
f_q(\sigma(\lambda + \rho_n) - (\mu + \rho_n)) = \mathcal{P}_{q^\lambda}^D(I(\lambda) + \rho_n) - (I(\mu) + \rho_n))
\]
and
\[
F_q(\sigma(\lambda + \rho_n) - (\mu + \rho_n)) = \mathcal{P}_{q^{\lambda^*}}^C(I(\lambda) + \rho_n) - (I(\mu) + \rho_n)).
\]
Set \( P = \mathcal{P}_{q^{\lambda^*}}^C(I(\sigma(\lambda) + \rho_n) - (I(\mu) - \rho_n)) \). From the above Lemma we deduce
\[
P = \mathcal{P}_{q^\lambda}^D(I(\sigma(\lambda) + \sigma^*(\rho_n) - I(\mu) - \rho_n)).
\]
Now an immediate computation shows that \( \sigma^*(\rho_n) - \rho_n = I(\sigma(\rho_n) - \rho_n) \). Thus we derive
\[
P = \mathcal{P}_{q^{\lambda^*}}^C(I(\sigma(\lambda + \rho_n) - (\mu + \rho_n)) = F_q(\sigma(\lambda + \rho_n) - (I(\mu) - \rho_n))
\]
where the last equality follows from Lemma 5.1.3.

We obtain the equality \( f_q(\sigma(\lambda + \rho_n) - (\mu + \rho_n)) = \mathcal{P}_{q^\lambda}^D(I(\lambda) + \rho_n) - (I(\mu) + \rho_n) \) in a similar way.
We obtain similarly the equality by Lemma 5.1.1. Since
\[ \hat{\sigma} U \]
the definition of \( \sigma \).
Then the Kostka-Foulkes polynomials associated to the root system Corollary 5.1.8
We also recover a property of the Kostka-Foulkes polynomials associated to the root system Corollary 5.1.7
By using the fact that the Kostka-Foulkes polynomials have nonnegative integer coefficients \([9]\) we
Theorem 5.1.5 Consider \( \lambda, \mu \) two partitions of length \( n \) and set \( m = \max(\lambda_1, \mu_1) \). Let \( k \) be any integer such that \( k \geq \frac{|\mu| - |\lambda|}{2} \). Then \( \hat{\lambda} = (m - \lambda_n, \ldots, m - \lambda_1) \) and \( \hat{\mu} = (m - \mu_n, \ldots, m - \mu_1) \) are partitions of length \( n \) and
\[
\begin{align*}
  u_{\lambda, \mu}(q) &= K_{\lambda + k\kappa_n, \hat{\mu} + k\kappa_n}^D_n(q) \\
  U_{\lambda, \mu}(q) &= K_{\lambda + k\kappa_n, \hat{\mu} + k\kappa_n}^C_n(q)
\end{align*}
\]

Proof. First \( \hat{\lambda} \) and \( \hat{\mu} \) are clearly partitions of length \( n \) since \( m = \max(\lambda_1, \mu_1) \). It follows from the definition of \( U_{\lambda, \mu}(q) \) and the above lemma that
\[
U_{\lambda, \mu}(q) = \sum_{\sigma \in S_n} (-1)^{l(\sigma)} F_q(\sigma(\lambda + \rho_n) - \mu - \rho_n) = \sum_{\sigma^* \in S_n} (-1)^{l(\sigma^*)} P^C_q(\sigma^*(I(\lambda) + \rho_n)) - (I(\mu) + \rho_n)).
\]
Then by Lemma 5.1.3 we obtain
\[
U_{\lambda, \mu}(q) = \sum_{\sigma \in S_n} (-1)^{l(\sigma)} P^C_q(\sigma(I(\lambda) + \rho_n)) - (I(\mu) + \rho_n)).
\]
We have \( \sigma(I(\lambda) + \rho_n + m\kappa_n) = \sigma(I(\lambda) + \rho_n) + m\kappa_n \) since \( \sigma \in S_n \). So we can write
\[
U_{\lambda, \mu}(q) = \sum_{\sigma \in S_n} (-1)^{l(\sigma)} P^C_q(\sigma(I(\lambda) + m\kappa_n + \rho_n)) - (I(\mu) + m\kappa_n + \rho_n)).
\]
Since \( \hat{\lambda} = I(\lambda) + m\kappa_n \) and \( \hat{\mu} = I(\mu) + m\kappa_n \) we derive
\[
U_{\lambda, \mu}(q) = \sum_{\sigma \in S_n} (-1)^{l(\sigma)} P^C_q(\sigma(\hat{\lambda} + \rho_n) - (\hat{\mu} + \rho_n)) = K_{\lambda + k\kappa_n, \hat{\mu} + k\kappa_n}^C_n(q)
\]
by Lemma 5.1.4.
We obtain similarly the equality \( u_{\lambda, \mu}(q) = K_{\lambda + k\kappa_n, \hat{\mu} + k\kappa_n}^D_n(q) \) by replacing \( P^C_q \) by \( P^D_q \). [8]

Example 5.1.6 Consider \( \mu = (4, 2, 1) \) and \( \lambda = (2, 1, 0) \). We have \( m = 4 \), \( \hat{\mu} = (3, 2, 0) \) and \( \hat{\lambda} = (4, 3, 2) \). We choose \( k = 2 \). Then we obtain the equalities
\[
\begin{align*}
  u_{\lambda, \mu}(q) &= K_{(6, 5, 4), (5, 4, 2)}^{D_n}(q) = q^3 + q^2 \\
  U_{\lambda, \mu}(q) &= K_{(6, 5, 4), (5, 4, 2)}^{C_n}(q) = q^5 + 2q^4 + 3q^3 + 2q^2
\end{align*}
\]
By using the fact that the Kostka-Foulkes polynomials have nonnegative integer coefficients [8] we obtain the following corollary.

Corollary 5.1.7 The polynomials \( u_{\lambda, \mu}(q) \) and \( U_{\lambda, \mu}(q) \) have nonnegative integers coefficients.

We also recover a property of the Kostka-Foulkes polynomials associated to the root system \( A_{n-1} \) proved in [8].

Corollary 5.1.8 Consider \( \lambda, \mu \) two partitions of length \( n \) such that \( |\lambda| = |\mu| \) and set \( m = \max(\lambda_1, \mu_1) \). Then the Kostka-Foulkes polynomials associated to the root system \( A_{n-1} \) verifies
\[
K^{A_{n-1}}_{\lambda, \mu}(q) = K^{A_{n-1}}_{\hat{\lambda}, \hat{\mu}}(q)
\]
where \( \hat{\lambda} = (m - \lambda_n, \ldots, m - \lambda_1) \) and \( \hat{\mu} = (m - \mu_n, \ldots, m - \mu_1) \).
Proof. Suppose that $\beta$ is a linear combination of $I(R_{C_n}^+)$ with nonnegative coefficients such that $|\beta| = 0$. Then $\beta$ is necessarily a linear combination of the roots $\varepsilon_i - \varepsilon_j$, $1 \leq i < j \leq n$ with nonnegative coefficients (see [4]) that is, a linear combination with nonnegative coefficients of the positive roots associated to the root system $A_{n-1}$. This implies that

$$f_q(\beta) = F_q(\beta) = P_{q}^{A_{n-1}}(\beta)$$

where $P_{q}^{A_{n-1}}$ is the $q$-partition function associated to the root system $A_{n-1}$. For any $\sigma \in S_n$, we have $|\sigma(\lambda + \rho_n) - (\mu + \rho_n)| = 0$ since $|\lambda| = |\mu|$. Thus

$$f_q(\sigma(\lambda + \rho_n) - (\mu + \rho_n)) = F_q(\sigma(\lambda + \rho_n) - (\mu + \rho_n)) = P_{q}^{A_{n-1}}(\sigma(\lambda + \rho_n) - (\mu + \rho_n))$$

and the multiplicities $u_{\lambda,\mu}(q)$ and $U_{\lambda,\mu}(q)$ coincide with the Kostka-Foulkes polynomial $K_{\lambda,\mu}^{A_{n-1}}(q)$ when $|\lambda| = |\mu|$. Moreover by applying Theorem 5.1.5 with $|\lambda| = |\mu|$ and $k = 0$, we obtain $U_{\lambda,\mu}(q) = K_{\lambda,\mu}^{C_n}(q) = K_{\lambda,\mu}^{A_{n-1}}(q)$ where the last equality is due to the fact that the Kostka-Foulkes polynomials of types $B_n, C_n$ or $D_n$ are Kostka-Foulkes polynomials associated to the root system $A_{n-1}$ when $|\lambda| = |\mu|$. So we derive the equality $K_{\lambda,\mu}^{A_{n-1}}(q) = K_{\lambda,\mu}^{A_{n-1}}(q)$.

We have seen that $U_{\lambda,\mu}(q)$ can be regarded as a $q$-analogue of the multiplicity of the representation $V(\lambda)$ in $\mathcal{F}_q^{C_n}(\mu)$. In [3], Hatayama, Kuniba, Okado and Takagi have introduced another quantification $X_{\lambda,\mu}(q)$ of this multiplicity based on the determination of the combinatorial $R$ matrix of the $U'_q(C_n^{(1)})$-crystals $B_k$. Considered as the crystal graph of the $U_q(C_n)$-module $M_k$, $B_k$ can be identified with

$$B((k\Lambda_1) \oplus B((k-2)\Lambda_1) \oplus \cdots \oplus B(k \mod 2\Lambda_1)$$

where for any $i \in \{k, k-2, \ldots, k \mod 2\}$, $B((k\Lambda_1)$ is the crystal graph of the irreducible finite dimensional $U_q(C_n)$-module of highest weight $k\Lambda_1$. Note that the character of $M_k$ is equal to $H_k^{C_n}$.

Recall that the combinatorial $R$ matrix associated to crystals $B_k$ is equivalent to the description of the crystal graph isomorphisms

$$\left\{ \begin{array}{l}
B_l \otimes B_k \cong B_k \otimes B_l \\
B_l \otimes B_2 \rightarrow b' \otimes b''
\end{array} \right.$$

together with the energy function $H$ on $B_l \otimes B_k$. The multiplicity of $V(\lambda)$ in $\mathcal{F}_q^{C_n}(\mu)$ is then equal to the number of highest weight vertices of weight $\lambda$ in the crystal $B_\mu = B_{\mu_1} \otimes \cdots \otimes B_{\mu_n}$. Then $X_{\lambda,\mu}(q)$ is defined by

$$X_{\lambda,\mu}(q) = \sum_{b \in E_\lambda} q^{\sum_{0 \leq i < j \leq n} H(b_{i} \otimes b^{(i+1)}_{j})}$$

where $E_\lambda$ is the set of highest weight vertices $b = b_1 \otimes \cdots \otimes b_n$ in $B_\mu$ of highest weight $\lambda$, $b^{(i)}_{j}$ is determined by the crystal isomorphism

$$B_{\mu_1} \otimes B_{\mu_{i+1}} \otimes B_{\mu_{i+2}} \otimes \cdots \otimes B_{\mu_{j}} \rightarrow B_{\mu_1} \otimes B_{\mu_2} \otimes B_{\mu_{i+1}} \cdots \otimes B_{\mu_{j-1}},$$

$$b_1 \otimes b_{i+1} \otimes \cdots \otimes b_j \rightarrow b^{(i)}_{j} \otimes b'_{i} \otimes \cdots \otimes b'_{j-1}$$

and for any $j = 1, \ldots, n$, $H(b_0 \otimes b^{(1)}_{j})$ depends only on $b^{(1)}_{j}$.

Many computations suggest the following conjecture

**Conjecture 5.1.9** For any partition $\lambda$ and $\mu$ of length $n$ with $|\mu| \geq |\lambda|$,

$$U_{\lambda,\mu}(q) = q^{|\mu|-|\lambda|}X_{\lambda,\mu}(q).$$

Note that the conjecture is in particular true for all the examples given in the tables of [3].
5.2 A duality theorem for the $q$-multicities in $\varepsilon(\mu)$ and $\mathcal{C}(\mu)$

Consider $\lambda, \mu$ two partitions of length $l$ such that $l \geq |\mu| \geq |\lambda|$. Write $n = \max(\lambda_1, \mu_1)$. Then by adding to $\lambda'$ and $\mu'$ the required numbers of parts $0$ we can consider them as partitions of length $n$. Set $m = \max(\lambda'_1, \mu'_1)$. We define the partitions $\widetilde{\lambda}$ and $\mu''$ belonging to $\pi_n$ by $\widetilde{\lambda} = (m - \lambda'_n, \ldots, m - \lambda'_1)$ and $\mu'' = (m - \mu'_n, \ldots, m - \mu'_1)$.

**Theorem 5.2.1** With the above notations, we have for any integer $k \geq \frac{|\mu| - |\lambda|}{2}$

\[
\left\{
\begin{array}{l}
(i) : v_{\lambda,\mu}(q) = K_{\widetilde{\lambda} + k\kappa_n, \mu'' + k\kappa_n}^{D_n}(q) \\
(ii) : V_{\lambda,\mu}(q) = K_{C_{\lambda}''}^{C_{\mu}''} = (\lambda_1, \ldots, m - \lambda'_1, \mu_1, \ldots, m - \mu'_1)\end{array}\right.
\]

**Proof.** Since $l \geq |\mu|$, we have by Proposition 4.2.1 the equality $v_{\lambda,\mu}(q) = u_{\lambda,\mu'}(q)$. Moreover we have $m \geq \max(\lambda'_1, \mu'_1)$ and $k \geq \frac{|\mu| - |\lambda|}{2}$ for $|\lambda'| = |\lambda|$ and $|\mu'| = |\mu|$. Hence by applying Theorem 5.1.3 we obtain $v_{\lambda,\mu}(q) = K_{\widetilde{\lambda} + k\kappa_n, \mu'' + k\kappa_n}^{D_n}(q)$ where $\widetilde{\lambda} = (m - \lambda'_n, \ldots, m - \lambda'_1)$ and $\mu'' = (m - \mu'_n, \ldots, m - \mu'_1) = \hat{\mu}$. So (i) is proved. We obtain (ii) similarly. \[\blacksquare\]

**Example 5.2.2** For $\lambda = (2, 1, 0, 0, 0)$ and $\mu = (2, 2, 1, 0, 0)$ we have $l = 5, n = 2$. Moreover $\lambda' = (2, 1), \mu' = (3, 2)$ and $m = 3$. So $\widetilde{\lambda} = (2, 1)$ and $\hat{\mu} = (1, 0)$. Hence for $k = 1$

\[
\left\{
\begin{array}{l}
(i) : v_{\lambda,\mu}(q) = K_{(3,2),(2,1)}^{(3,2),(2,1)}(q) = q \\
(ii) : V_{\lambda,\mu}(q) = K_{(3,2),(2,1)}^{(3,2),(2,1)}(q) = q^2 + q
\end{array}\right.
\]

**Remark** When $\lambda, \mu$ are considered as weights associated to the root system $C_l$, the above theorem is essentially the quantification of a duality result explicitely by Foulle from results of [3] for the dual pair $(Sp(2l), Sp(2n))$.

6 Identities for the $q$-multiplicities $U_{\lambda,\mu}(q)$ and $u_{\lambda,\mu}(q)$

6.1 A relations between $q$-partition functions

Consider a nonnegative integer $k$ and define the finite sets

\[
\left\{
\begin{array}{l}
C_k^n = \{ \beta \in \pi_n, \beta = \sum_{1 \leq r \leq s \leq n} c_{r,s}^n (\varepsilon_r + \varepsilon_s) \text{ with } c_{r,s}^n \geq 0 \text{ and } |\beta| = 2k \} \\
D_k^n = \{ \beta \in \pi_n, \beta = \sum_{1 \leq r < s \leq n} c_{r,s}^n (\varepsilon_r + \varepsilon_s) \text{ with } c_{r,s}^n \geq 0 \text{ and } |\beta| = 2k \}
\end{array}\right.
\]

Note that each $\beta \in C_k^n$ (resp. $\beta \in D_k^n$) verifies $|\beta| = 2 \sum_{1 \leq r \leq s \leq n} c_{r,s}^n$ (resp. $|\beta| = 2 \sum_{1 \leq r < s \leq n} c_{r,s}^n$). This implies that

\[
\prod_{1 \leq r \leq s \leq n} \frac{1}{1 - q} = \sum_{k \geq 0, \beta \in C_k^n} c^n_{\beta} q^k x^\beta \quad \text{and} \quad \prod_{1 \leq r < s \leq n} \frac{1}{1 - q} = \sum_{k \geq 0, \beta \in D_k^n} c^n_{\beta} q^k x^\beta
\]

where $c^n_{\beta}$ (resp. $c^n_{\beta}$) is the number of ways to decompose $\beta$ as $\beta = \sum_{1 \leq r \leq s \leq n} c_{r,s}^n (\varepsilon_r + \varepsilon_s)$ (resp. $\beta = \sum_{1 \leq r < s \leq n} c_{r,s}^n (\varepsilon_r + \varepsilon_s)$) with $c_{r,s}^n \geq 0$.

**Lemma 6.1.1** For any $\beta \in \pi_n$ with $|\beta| = 2k \geq 0$, we have

\[
F_q(\beta) = \sum_{\delta \in C_k^n} c^n_q q^k \mathcal{P}^A_q (\beta + \delta) \quad \text{and} \quad f_q(\beta) = \sum_{\delta \in D_k^n} c^n_q q^k \mathcal{P}^A_q (\beta + \delta).
\]
Suppose \( u \) is equal to the dimension of the weight space of weight \( \lambda, \mu \).

Proof. We have:

\[
\prod_{1 \leq i < j \leq n} \frac{1}{1 - q^{x_{ij}}} \prod_{1 \leq i \leq n} \frac{1}{1 - q^{x_{ii}}} = \sum_{\eta \in \pi_n} \sum_{\delta \in \pi_n} c^\eta_{\delta} q^{\|\delta\|/2} \mathcal{P}_q^{A_n}(\eta),x^{\eta-\delta}
\]

which implies the equality \( F_q(\beta) = \sum_{\eta - \delta = \beta} c^\eta_{\delta} q^{\|\delta\|/2} \mathcal{P}_q^{A_n}(\eta) \). Since \( \mathcal{P}_q^{A_n}(\eta) = 0 \) when \( |\eta| \neq 0 \), we can suppose \( |\eta| = 0 \) and \( |\delta| = |\beta| \) in the previous sum. Then \( \delta \in C_n^\eta \) and the result follows immediately. The proof for \( f_q(\beta) \) is similar.

\[ \diamondsuit \]

6.2 Expressions of the multiplicities \( u_{\lambda, \mu} \) and \( U_{\lambda, \mu} \) in terms of Kostka numbers

Suppose that \( \xi \) and \( \gamma \) belong to \( \pi_n \). Then we can define the polynomial

\[
K^{A_n-1}_{\xi, \gamma}(q) = \sum_{\sigma \in S_n} (-1)^{l(\sigma)} \mathcal{P}_q^{A_n-1}(\sigma(\xi + \rho_n) - (\gamma + \rho_n)).
\]

Note that the coefficients of \( K^{A_n-1}_{\xi, \gamma}(q) \) may be negative. When \( \xi = \lambda \) is a partition, \( K^{A_n-1}_{\lambda, \gamma}(1) \) is equal to the dimension of the weight space of weight \( \gamma \) in \( V(\lambda) \). When \( \gamma = \mu \) is a partition, we have

\[
\begin{cases}
K^{A_n-1}_{\xi, \mu}(q) = (-1)^{l(\tau)} K^{A_n-1}_{\tau, \mu}(q) & \text{if } \xi = \tau \circ (\nu) \text{ with } \tau \in S_n \text{ and } \nu \text{ a partition} \\
0 & \text{otherwise}
\end{cases}
\]

Proposition 6.2.1 Consider \( \lambda, \mu \) two partitions of length \( n \) such that \( k = |\mu| - |\lambda| \geq 0 \). Then

\[
u_{\lambda, \mu}(q) = \sum_{\delta \in D_n^\lambda} c^\eta_{\delta} q^{\|\delta\|} K^{A_n-1}_{\lambda, \mu}(q) = \sum_{\delta \in D_n^\lambda} c^\eta_{\delta} q^{\|\delta\|} K^{A_n-1}_{\lambda, \mu}(q) \quad \text{and}
\]

\[
u_{\lambda, \mu}(q) = \sum_{\delta \in C_n^{\lambda, \mu}} c^\eta_{\delta} q^{\|\delta\|} K^{A_n-1}_{\lambda, \mu}(q) = \sum_{\delta \in C_n^{\lambda, \mu}} c^\eta_{\delta} q^{\|\delta\|} K^{A_n-1}_{\lambda, \mu}(q).
\]

Proof. By definition we have

\[
u_{\lambda, \mu}(q) = \sum_{\sigma \in S_n} (-1)^{l(\sigma)} \mathcal{P}_q^{A_n-1}(\sigma(\lambda + \rho_n) - (\mu + \rho_n)).
\]

Hence from the above lemma we derive

\[
u_{\lambda, \mu}(q) = \sum_{\delta \in C_k^\lambda} c^\eta_{\delta} q^{\|\delta\|/2} \sum_{\sigma \in S_n} (-1)^{l(\sigma)} \mathcal{P}_q^{A_n-1}(\sigma(\lambda + \rho_n) - (\mu - \delta + \rho_n)) \quad (15)
\]

which yields the first desired equality since \( K^{A_n-1}_{\lambda, \mu-\delta}(q) = \sum_{\sigma \in S_n} (-1)^{l(\sigma)} \mathcal{P}_q^{A_n-1}(\sigma(\lambda + \rho_n) - (\mu - \delta + \rho_n)) \).

For any \( \sigma \in S_n \), we have \( \sigma(C_n^\lambda) = C_n^\lambda \) and \( c^\sigma_{\tau(\delta)} = c^\sigma_{\delta} \). Thus (15) can also be rewritten

\[
u_{\lambda, \mu}(q) = q^{\|\delta\|/2} \sum_{\sigma \in S_n} (-1)^{l(\sigma)} \sum_{\delta \in C_k^\mu} c^\eta_{\delta} \mathcal{P}_q^{A_n-1}(\sigma(\lambda + \rho_n + \delta) - (\mu + \rho_n)) = \sum_{\delta \in C_k^\mu} c^\eta_{\delta} q^{\|\delta\|/2} K^{A_n-1}_{\lambda, \mu+\delta}(q).
\]

The proof is similar for \( u_{\lambda, \mu}(q) \). \( \diamondsuit \)

By setting \( q = 1 \) in the above relations we obtain the following expressions of the multiplicities \( U_{\lambda, \mu} \) and \( u_{\lambda, \mu} \) in terms of Kostka numbers.
Corollary 6.2.2
\[
\begin{align*}
U_{\lambda,\mu} &= \sum_{\delta \in C} c^n_C K^{A_{n-1}}_{\lambda+\delta,\mu} \\
v_{\lambda,\mu} &= \sum_{\delta \in D} c^n_D K^{A_{n-1}}_{\lambda+\delta,\mu}
\end{align*}
\]

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