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Modified Bernstein Polynomials and Jacobi Polynomials in \( q \)-Calculus

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Abstract

We introduce here a generalization of the modified Bernstein polynomials for Jacobi weights using the \( q \)-Bernstein basis proposed by G.M. Phillips to generalize classical Bernstein Polynomials. The function is evaluated at points which are in geometric progression in \( ]0, 1[ \). Numerous properties of the modified Bernstein Polynomials are extended to their \( q \)-analogues: simultaneous approximation, pointwise convergence even for unbounded functions, shape-preserving property, Voronovskaya theorem, self-adjointness. Some properties of the eigenvectors, which are \( q \)-extensions of Jacobi polynomials, are given.

Keywords: \( q \)-Bernstein, \( q \)-Jacobi, Bernstein-Durrmeyer, totally positive, simultaneous approximation.

AMS subject classification: 41A10, 41A25, 41A36
1 Introduction

G.M. Phillips has proposed a generalization of Bernstein polynomials based on the \(q\)-integers (cf. \[9\]). We introduce here a \(q\)-analogue of the operators which are often called Bernstein Durrmeyer polynomials and denoted \(M_{\alpha,\beta}^{n,1}\) (cf. \[3\],\[2\]).

In all the paper, we shall assume that \(q \in ]0,1[\) and, \(\alpha,\beta > -1\) (part 5 excepted).

For any integer \(n\), and a function \(f\) defined on \(]0,1[\) we set

\[
M_{\alpha,\beta}^{n,q} f(x) = \sum_{k=0}^{n} f_{\alpha,\beta}^{n,k,q} b_{n,k,q}(x) \tag{1}
\]

where each \(f_{\alpha,\beta}^{n,k,q}\) is a mean of \(f\) defined by Jackson integrals. The polynomials \(b_{n,k,q}(x)\) are \(q\)-analogues of the Bernstein basis polynomials and are defined by

\[
b_{n,k,q}(x) = \binom{n}{k}_q x^k (1-x)^{n-k}, \text{ with } \binom{n}{k}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!}, \text{ (}q\text{-binomial coefficient)}\]

\(k = 0, \ldots, n\). They verify \(\sum_{k=0}^{n} b_{n,k,q}(x) = 1\) (cf. \[3\]).

We follow the definitions and notations of (\[7\]).

For any real \(a\), \([a]_q = (1-q^a)/(1-q)\), \(\Gamma_q(a+1) = (1-q)_q^a/(1-q)\)^a,

(if \(a \in \mathbb{N}\) the \(q\)-integer \([a]_q\) is \([a]_q = 1+q+\cdots+q^{a-1}\) and \(\Gamma_q(a+1) = ([a]_q)!\));

\(1-x)_q^a = \prod_{j=0}^{\infty} (1-q^jx)/\prod_{j=0}^{\infty} (1-q^{j+a}x)\) and consequently \((1-x)^{a+b}_q = (1-x)_{q}^a (1-q^a x)_{q}^b\) holds for any \(b\), \((1-x)_{q}^m = \prod_{j=0}^{m} (1-q^{j+a})\) if \(m\) is integer.

The notations will be simplified as much as possible, the superscript \(\alpha,\beta\) and the index \(q\) when \(q\) is fixed, will be suppressed in some proofs.

We introduce the positive bilinear form:

\[
\langle f, g \rangle_{\alpha,\beta}^{q} = q^{(\alpha+1)(\beta+1)} (1-q) \sum_{i=0}^{\infty} q^i q^{i\alpha} (1-q^{i+1})_{q}^\beta f(q^{i+\beta+1}) g(q^{i+\beta+1}), \tag{2}
\]
whenever it is defined. It can be written under the form of two definite $q$-integrals

$$\langle f, g \rangle_{q}^{\alpha, \beta} = \int_{0}^{q_{t+1}} t^{\alpha}(1-q^{-\beta}t)_{q}^{\beta} f(t)g(t)d_{q}t$$

and

$$\langle f, g \rangle_{q}^{\alpha, \beta} = q^{(\alpha+1)(\beta+1)} \int_{0}^{1} t^{\alpha}(1-qt)_{q}^{\beta} f(q^{\beta+1}t)g(q^{\beta+1}t)d_{q}t.$$ 

(the definite $q$-integral of a function $f$ is

$$\int_{a}^{0} f(x)d_{q}x = a(1-q)\sum_{i=0}^{\infty} q^{i}f(q^{i}a)$$ (cf. [7]))

**Definition 1** We set in formula (3):

$$f_{n,k,q}^{\alpha, \beta} = \frac{\langle b_{n,k,q}, f \rangle_{q}^{\alpha, \beta}}{\langle b_{n,k,q}, 1 \rangle_{q}^{\alpha, \beta}} = \frac{\int_{0}^{1} t^{k+\alpha} (1-qt)^{n-k+\beta} f(q^{\beta+1}t) d_{q}t}{\int_{0}^{1} t^{k+\alpha} (1-qt)^{n-k+\beta} d_{q}t}, \quad k = 0, \ldots, n,$$ (3)

**to define**

$$M_{n,q}^{\alpha, \beta}f(x) = \sum_{k=0}^{n} \frac{\langle b_{n,k,q}, f \rangle_{q}^{\alpha, \beta}}{\langle b_{n,k,q}, 1 \rangle_{q}^{\alpha, \beta}} b_{n,k,q}(x).$$ (4)

We see that the polynomial $M_{n,q}^{\alpha, \beta}f$ is well defined if there exists $\gamma \geq 0$ such that $x^{\gamma}f(x)$ is bounded on $]0, A]$ for some $A \in ]0, 1]$ and $\alpha > \gamma - 1$. Indeed, $x^{\alpha}f(x)$ is then $q$-integrable for the weight $w_{q}^{\alpha, \beta}(x) = x^{\alpha}(1-qx)^{\beta}$. We will say, in this case, that $f$ satisfies the condition $C(\alpha)$. Also $\langle f, g \rangle_{q}^{\alpha, \beta}$ is well defined if the product $fg$ satisfies $C(\alpha)$, particularly if $f^{2}$ and $g^{2}$ do it.

In many cases, the limit of $M_{n,q}^{\alpha, \beta}f(x)$ when $q$ tends to 1 is:

$$M_{n,q}^{\alpha, \beta}f(x) = \sum_{k=0}^{n} \left( \int_{0}^{1} t^{k+\alpha} (1-t)^{n-k+\beta} f(t) dt \right) \int_{0}^{1} t^{k+\alpha} (1-t)^{n-k+\beta} dt \right) b_{n,k}(x)$$

with $b_{n,k}(x) = \binom{n}{k} x^{k}(1-x)^{n-k}$.

Numerous properties of the operator $M_{n,1}^{\alpha, \beta}$ will be extended to $M_{n,q}^{\alpha, \beta}$ in this paper.
2 First properties

For any \( n \in \mathbb{N} \), the operator \( M_{n,q}^{\alpha,\beta} \) has the following properties.

- It is linear, positive and it preserves the constants so it is a contraction

\[
( \sup_{x \in [0,1]} |M_{n,q}^{\alpha,\beta} f(x)| \leq \sup_{x \in [0,1]} |f(x)| ).
\]

- It is self-adjoint: \( \langle M_{n,q}^{\alpha,\beta} f, g \rangle_q^{\alpha,\beta} = \langle f, M_{n,q}^{\alpha,\beta} g \rangle_q^{\alpha,\beta} \).

- It preserves the degrees of the polynomials of degree \( \leq n \).

The first properties are consequences of the definition. The last one follows after the following proposition since \( D_q x^p = \lceil p \rceil x^{p-1} \).

**Proposition 1** If \( f \) verifies the condition \( C(\alpha) \), we have:

\[
D_q M_{n,q}^{\alpha,\beta} f(x) = \frac{[n]_q}{(n + \alpha + \beta + 2)_q} q^{\alpha + \beta + 2} M_{n-1,q}^{\alpha+1,\beta+1} \left( D_q f \left( \frac{\cdot}{q} \right) \right) (qx), x \in [0,1],
\]

where the \( q \)-derivative of a function \( f \) is \( D_q f(x) = \frac{f(qx) - f(x)}{(q-1)x} \) if \( x \neq 0 \).

(When \( f' \) is continuous on \([0,1]\), the limit of formula (5) is, when \( q \) tends to 1,

\[
\left( M_{n,1}^{\alpha,\beta} f \right)'(x) = n(n + \alpha + \beta + 2)^{-1} M_{n-1,1}^{\alpha+1,\beta+1} (f') (x) \) (cf. \([3]\)).
\]

**Proof.** We compute \( D_{b_n,k}(x) = [n] (b_{n-1,k-1}(qx)/q^{k-1} - b_{n-1,k}(qx)/q^k) \) if 

\[
1 \leq k \leq n-1 \text{ and } D_{b_{n,0}}(x) = -[n] b_{n-1,0}(qx), D_{b_{n,n}}(x) = [n] b_{n-1,n-1}(qx)/q^{n-1}
\]

to get

\[
D M_{n,q}^{\alpha,\beta} f(x) = [n] \sum_{k=0}^{n-1} b_{n-1,k}(qx)(f_{n,k+1}^{\alpha,\beta} - f_{n,k}^{\alpha,\beta})/q^k.
\]

We denote \( \psi_{n,k}^{\alpha,\beta}(t) = t^{k+\alpha}(1 - qt)^{n-k+\beta}, k = 0, \ldots, n \).

Recall that the \( q \)-derivative of \( g_1 g_2 \) is \( D_q (g_1 g_2)(x) = D_q g_1(x) g_2(qx) + g_1(x) D_q g_2(x) \).

The \( q \)-Beta functions are \( B_q(u,v) = \int_0^1 t^{u-1}(1 - qt)^{v-1}d_q t = \Gamma_q(u) \Gamma_q(v)/\Gamma_q(u+v) \).

The function \( \psi_{n-1,k}^{\alpha+1,\beta+1}(t) f(q^{\alpha+1} t), t \in [0,1], \) extended by 0 in 0 is continuous at 0.
Hence we may use a \( q \)-integration by parts to write, for \( k = 0, \ldots, n - 1 \):

\[
B_q(k + \alpha + 2, n - k + \beta + 1) [n + \alpha + \beta + 2] (f_{n,k+1}^{\alpha,\beta} - f_{n,k}^{\alpha,\beta}) = 
- q^{k+\alpha} \int_0^1 (D \psi_{n,k+1}^{\alpha,\beta+1}) (f(q^{\beta+1} t) d_q t = \int_0^1 q^{k+\alpha+\beta+2} \psi_{n-1,k}^{\alpha,\beta+1} (t) (D f)(q^{\beta+1} t) d_q t 
- \left[ \psi_{n-1,k}^{\alpha+1,\beta+1} (\frac{q}{q}) f(q^{\beta+1} t) \right]_0^1.
\]

Hence \( (f_{n,k+1}^{\alpha,\beta} - f_{n,k}^{\alpha,\beta}) = \frac{q^{\alpha+\beta+2+k}}{[\alpha+\alpha+\beta+2]} \int_0^1 (1-q)^{n-k+\beta} (D f)(q^{\beta+1} t) d_q t \) and

\[
DM_n^{\alpha,\beta} f(x) = \left[ \frac{n}{n+\alpha+\beta+2} \right] \sum_{k=0}^{n-1} \left( b_{n-1,k}^{\alpha,\beta} D f(\frac{n}{q}) \right)_{\alpha+1,\beta+1} b_{n-1,k}(x).
\]

**Theorem 1** The following equality holds for any \( x \in [0,1] \):

\[
M_n^{\alpha,\beta} f(x) = \sum_{j=0}^\infty \Phi_{j,n,q}^{\alpha,\beta}(x) f(q^{j+\beta+1})
\]

where, \( \Phi_{j,n,q}^{\alpha,\beta}(x) = \sum_{k=0}^n v_k b_{n,k,q}(q^{j+\beta+1}) b_{n,k,q}(x) \),

\[
u_k = (1-q) j^{(\alpha+1)} (1-q^{j+1})^\beta, \quad j \in \mathbb{N},
\]

\[
v_k^{-1} = q^{k(\beta+1)} \binom{n}{k} B_q(k + \alpha + 1, n - k + \beta + 1), \quad k = 0, \ldots, n.
\]

Moreover, for any \( r \in \mathbb{N} \), the sequence \( \Phi_{r,n,q}^{\alpha,\beta}, \Phi_{r-1,n,q}^{\alpha,\beta}, \ldots, \Phi_{0,n,q}^{\alpha,\beta} \) is totally positive, that is to say, the collocation matrix \( (\Phi_{r-j,n,q}(x_i))_{i=1,\ldots,m,j=0,\ldots,r} \) is totally positive for any family \( (x_i), 0 \leq x_1 < \ldots < x_m \leq 1 \).

**Proof.** We set \( \Phi_j = \Phi_{j,n,q}^{\alpha,\beta}, b_{n,k,q} = b_k, c = q^{\beta+1} \). The formulae (3) come by writing the definite \( q \)-integrals \( \langle b_k, f \rangle_q^{\alpha,\beta} \) as discrete sums in (3) and the Beta integrals \( \langle b_k, 1 \rangle_q^{\alpha,\beta} = \binom{n}{k} B_q(k + \alpha + 1, n - k + \beta + 1), k = 0, \ldots, n. \)

For the total positivity of the \( \Phi_j \), we have to prove that, for any \( m \in \mathbb{N} \) and any two families \( (x_i)_{i=1,\ldots,m}, (j_k)_{k=1,\ldots,m} \), with \( 0 \leq x_1 \leq \ldots \leq x_m \leq 1, \ 0 \leq j_m \leq \ldots \leq j_1 \), the determinant \( \det(\Phi_{j_i}(x_i))_{i=1,\ldots,m,j=1,\ldots,m} \) is non negative. From the multilinearity of
the determinants, there is a basic composition formula for the discrete sums (cf. [8]).

We have $\det(\Phi_j(x_i))_{i=1,\ldots,m,l=1,\ldots,m} = \prod_{l=1}^m u_j E$ where

$$E = \det(\sum_{k=0}^n v_k b_k (cq^j b_k(x_i)))_{i=1,\ldots,m}\det(b_k (cq^j b_k(x_i)))_{i=1,\ldots,m,l=1,\ldots,m} = \prod_{l=1}^m u_j E$$

$$= \sum_{k_1=0}^n \cdots \sum_{k_m=0}^n v_{k_1} \cdots v_{k_m} \det(b_k (cq^j b_k(x_i)))_{i=1,\ldots,m,l=1,\ldots,m}$$

$$= \sum_{k_1=0}^n \cdots \sum_{k_m=0}^n v_{k_1} \cdots v_{k_m} b_{k_1}(x_1) \cdots b_{k_m}(x_m) \det(b_k (cq^j b_k(x_i)))_{i=1,\ldots,m,l=1,\ldots,m}$$

$$= \sum_{0 \leq k_1 \leq \ldots \leq k_m \leq n} v_{k_1} \cdots v_{k_m} \det(b_{k_1}(x_1))_{i=1,\ldots,m,l=1,\ldots,m} \det(b_k (cq^j b_k(x_i)))_{i=1,\ldots,m,l=1,\ldots,m}.$$

We know that the $q$-Bernstein basis is totally positive (cf. [8]). Hence we have

$$\det(b_k (x_i))_{i=1,\ldots,m,l=1,\ldots,m} \geq 0 \text{ and also } \det(b_k (cq^j b_k(x_i)))_{i=1,\ldots,m,l=1,\ldots,m} \geq 0,$$

so $E$ is non-negative and the result follows.

Corollary 1 The number of sign changes of the polynomial $M_{n,q}^{\alpha,\beta} f$ on $]0,1[$ is not greater than the number of sign changes of the function $f$.

Proof. For any $r \in \mathbb{N}$, the sequence $\Phi_r, \Phi_{r-1}, \ldots, \Phi_0$ is totally positive. We deduce that the number of sign changes of the polynomial $\sum_{j=0}^r \Phi_j(x) f(q^{j+\beta+1})$ is not greater than the number of sign changes of the sequence $f(q^{j+\beta+1})$, $j = 0, \ldots, r$, hence not greater than the number of sign changes of the function $f$ in $]0,1[$ (cf. [8]).

When $r$ tends to infinity this property is preserved hence is true for $M_{n,q} f$.

Corollary 2 Let $f$ be a function satisfying the condition $C(\alpha)$.

1. If $f$ is increasing (respectively decreasing), then the function $M_{n,q}^{\alpha,\beta} f$ is increasing (respectively decreasing).

2. If $f$ is convex, then the function $M_{n,q}^{\alpha,\beta} f$ is convex.
Proof. 1) If \( f \) is monotone, for any \( s \in \mathbb{R} \) the function \( f - s \) has at most one sign change. Hence \( M_n^{\alpha,\beta}(f - s) = M_n^{\alpha,\beta}f - s \) has at most one sign change and \( M_n^{\alpha,\beta}f \) is monotone. If \( f \) is increasing, \( Df(q) \) is positive on \([0, q]\). Since the operators \( M_n^{\alpha,\beta} \) are positive, we obtain for \( x \in [0, 1] \), \( M_{n-1}^{\alpha+1,\beta+1}(Df(q))(qx) \geq 0 \), and, using (3), \( DM_n^{\alpha,\beta}f(x) \geq 0 \). So the function \( M_n^{\alpha,\beta}f \) is increasing.

2) Let suppose the function \( f \) is convex. Since \( M_n^{\alpha,\beta} \) preserves the degree of the polynomials, for any real numbers \( \gamma_1, \gamma_2 \) there exist \( \delta_1, \delta_2 \) and a function \( g \) such that \( g(x) = f(x) - \delta_1x - \delta_2 \) and \( M_n^{\alpha,\beta}f(x) - \gamma_1x - \gamma_2 = M_n^{\alpha,\beta}g(x) \). The number of sign changes of \( g \) being at most two, it is the same for \( M_n^{\alpha,\beta}g \). Hence \( M_n^{\alpha,\beta}f \) is convex or concave. Moreover, if a function \( \varphi \) is convex (respectively concave), \( D^2\varphi(x) = \frac{q^3}{(q-1)^2x^2}(\varphi(q^2x) - [2] \varphi(qx) + q\varphi(x)) \) is \( \geq 0 \) (respectively \( \leq 0 \)). Hence \( M_{n-2}^{\alpha+2,\beta+2}(D^2\varphi(q))(q^2x) \geq 0 \). Using (3) two times we obtain \( D^2M_n^{\alpha,\beta}\varphi(x) \geq 0 \) and \( M_n^{\alpha,\beta}\varphi \) is not concave. 

3 Convergence properties

Theorem 2 If \( f \) is continuous on \([0, 1]\),

\[
\| M_{n,q}^{\alpha,\beta}f - f \|_{\infty} \leq C_{\alpha,\beta} \omega \left( f, \frac{1}{\sqrt{|n|q}} \right),
\]

where \( \| f \|_{\infty} \) is the uniform norm of \( f \) on \([0, 1]\) and \( \omega(f,.) \) is the usual modulus of continuity of \( f \), the constant \( C_{\alpha,\beta} \) being independent of \( n, q, f \).

Proof. As \( M_n^{\alpha,\beta} \) is positive, O. Shisha and B. Mond theorem can be applied. It is sufficient to prove that the order of approximation of \( f \) by \( M_n^{\alpha,\beta}f \) is \( O\left(\frac{1}{|n|} \right) \) for the
functions $f_i(x) = x^i$, $i = 0, 1, 2$. We compute the polynomials $M_{n}^{\alpha, \beta} f_i$, $i = 1$ and 2, with the help of (3) by $q$-integrations.

$$[n + \alpha + \beta + 2] (M_{n}^{\alpha, \beta} f_1(x) - x) = q^{\beta + 1} [\alpha + 1] - x [\alpha + \beta + 2]$$

and

$$[n + \alpha + \beta + 2] [n + \alpha + \beta + 3] (M_{n}^{\alpha, \beta} f_2(x) - x^2) =$$

$$[n] [2] x (q^{\alpha + 2 \beta + 3} [\alpha + 2] (1 - x) - [\beta + 1] x) + [\alpha + \beta + 3] [\alpha + \beta + 2] x^2 + q^{2 \beta + 2} [\alpha + 2] [\alpha + 1].$$

The result follows since $0 < q < 1$ and $0 \leq [a] \leq \max(a, 1)$ if $a \geq 0$.  

**Remark 1**

In order to have uniform convergence for all continuous functions on $[0, 1]$, it is sufficient to have $\lim_{n \to \infty} M_{n,q}^{\alpha, \beta} f_i = f_i$ for $i = 1, 2$, hence $\lim_{n \to \infty} 1/ [n]_q = 0$. This is realized if and only if $q = q_n$ and $\lim_{n \to \infty} q_n = 1$. Indeed, for every $n \in \mathbb{N}$, in both cases $q^n < 1/2$ and $q^n \geq 1/2$, we have $1 - q < 1/ [n]_q \leq 2 \max(1 - q, \ln 2/n)$. To maximize the order of approximation by the operator $M_{n,q}^{\alpha, \beta}$, we are interested to have $[n]_q$ of the same order as $n$, that is to say to have $\rho n < [n]_q \leq n$, for some $\rho > 0$, property which holds with the following property $S$ for $(q_n)$.

**Definition 2** The sequence $(q_n)_{n \in \mathbb{N}}$, has the property $S$ if and only if there exists $N \in \mathbb{N}$ and $c > 0$ such that, for any $n > N$, $1 - q_n < c/n$.

**Lemma 1**

The property $S$ holds if and only if the property $P_1$ (respectively $P_2$) holds where:

$P_1$ is "There exists $N_1 \in \mathbb{N}$ and $c_1 > 0$ such that, for any $n > N_1$, $[n]_q \geq c_1 n$?",

$P_2$ is "There exists $N_2 \in \mathbb{N}$ and $c_2 > 0$ such that, for any $n > N_2$, $q^n \geq c_2$?".

**Proof.** For any $n \in \mathbb{N}$, the function $\xi(x) = (1 - x^n)/(1 - x)$ is increasing on $[0, 1]$. If $S$ holds, we have, for any $n > N_1 = N$, $[n]_q = \xi(q_n) \geq \xi(1 - c/n) \geq n(1 - e^{-c})/c$. 

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and \( P_1 \) follows. If \( P_1 \) holds, we have, for any \( n > N = N_1, 1/(1-q_n) \geq [n]_{q_n} \geq c_1 n \) and \( S \) follows. If \( P_2 \) holds, we have, for any \( n > N = N_2, n(1-q_n) \leq -n \ln q_n < -\ln c_2 \) and \( S \) follows. If \( S \) holds, there exists \( N_2 > N \) such that, if \( n > N_2 \), \( 1/(1-q_n) < e^{-2n(1-q_n)} > e^{-2c} \) and \( P_2 \) follows. \( \blacksquare \)

**Theorem 3** If the function \( f \) is continuous at the point \( x \in [0,1] \), then,

\[
\lim_{n \to \infty} M_{n,q_n}^{\alpha,\beta} f(x) = f(x)
\]  

(7)

in the two following cases:

1. If \( f \) is bounded on \([0,1]\) and the sequence \((q_n)\) is such that \( \lim_{n \to \infty} q_n = 1 \),

2. If there exist real numbers \( \alpha', \beta' \geq 0 \) and a real \( \kappa' > 0 \) such that, for any \( x \in [0,1], |x^{\alpha'}(1-x)^{\beta'} f(x)| \leq \kappa' \), \( \alpha' < \alpha + 1 \), \( \beta' < \beta + 1 \) and the sequence \((q_n)\) owns the property \( S \).

**Theorem 4** If the function \( f \) admits a second derivative at the point \( x \in [0,1] \) then, in the cases 1 and 2 of theorem 3, we have the Voronovskaya-type limit:

\[
\lim_{n \to \infty} [n]_{q_n} (M_{n,q_n}^{\alpha,\beta} f(x) - f(x)) = \frac{d}{dx} \left( x^{\alpha + 1}(1-x)^{\beta + 1} f'(x) \right) / x^{\alpha}(1-x)^{\beta} .
\]  

(8)

(The limit operator is the Jacobi differential operator for the weight \( x^{\alpha}(1-x)^{\beta} \))

For the proofs of theorems 3 and 4 we need some preparation.

**Proposition 2** We set, for any \( n, m \in \mathbb{N} \setminus \{0\} \) and \( x \in [0,1], q \in [1/2,1], \)

\[
T_{n,m,q}(x) = \sum_{k=0}^{n} b_{n,k,q}(x) \int_{0}^{1} t^{k+\alpha} (1-qt)^{n-k+\beta} (x-t)^m d_q t
\]  

(9)
For any \( m \), there exists a constant \( K_m > 0 \), independent of \( n \) and \( q \), such that:

\[
\sup_{x \in [0,1]} |T_{n,m,q}(x)| \leq \begin{cases} 
    K_m / [n]_q^{m/2} & \text{if } m \text{ is even}, \\
    K_m / [n]_q^{(m+1)/2} & \text{if } m \text{ is odd}.
\end{cases}
\]

To prove this proposition we consider the lemmas 2 and 3.

**Lemma 2** We set, for any \( n, m \in \mathbb{N} \) and \( x \in [0,1] \),

\[
T_{n,m,q}^1(x) = \sum_{k=0}^{n} b_{n,k,q}(x) \int_0^1 t^{k+\alpha} (1 - qt)^{n-k+\beta} (x - t)^m d_q t.
\]

The following recursion formula holds for any \( q \in [1/2, 1] \) and \( m \geq 2 \):

\[
[n + m + \alpha + \beta + 2]_q q^{-2m-1} T_{n,m+1,q}^1(x) = \]

\[
(-x(1-x)D_q T_{n,m,q}^1(x) + T_{n,m,q}^1(x)(p_{1,m}(x) + x(1-q)[n + \alpha + \beta]_q [m + 1]_q q^{1-\alpha-m})
\]

\[
+T_{n,m-1,q}^1(x)p_{2,m}(x) + T_{n,m-2,q}^1(x)p_{3,m}(x)(1-q),
\]

(10)

where the polynomials \( p_{i,m}(x) \), \( i = 1, 2 \) and 3 are uniformly bounded with regard to \( n \) and \( q \).

**Proof.** 1) We set \( \psi_k(t) = t^{k+\alpha} (1 - qt)^{n-k+\beta} \) and \( l_k(x) = b_{n,k}(x) \int_0^1 \psi_k(t)(x - t)^m d_q t \), \( k = 0, \ldots, n \) and \( T_{n,m,q}^1 = T_{m}^1 \).

We compute

\[
x(1-x)D_q T_{n}^1(x) = x(1-x) [m] \sum_{k=0}^{n} l_k(x) \int_0^1 \psi_k(t)(x - t)^m d_q t
\]

\[
+ \sum_{k=0}^{n} l_k(x) \int_0^1 \psi_k(t)(qx - t)^m ([k] - [n]) x d_q t = A + B.
\]

We have \( A = x(1-x) [m] T_{m-1}^1(x) \) and

\[
B = q^{-\alpha} \sum_{k=0}^{n} l_k(x) \int_0^1 (D_q \psi_k)(t)(1 - qt)(qx - t)^m d_q t
\]

\[
- q^{1-\alpha-m} [n + \alpha + \beta] \sum_{k=0}^{n} l_k(x) \int_0^1 \psi_k(t)(qx - t)^{m+1} d_q t
\]

\[
+ (x([n + \alpha + \beta] q^{2-\alpha-m} - [n]) + [-\alpha]) \sum_{k=0}^{n} l_k(x) \int_0^1 \psi_k(t)(qx - t)^m d_q t
\]

\[
= B_1 + B_2 + B_3.
\]
We \( q \)-integrate by parts, setting \( \sigma(t) = \left(\frac{t}{q}(1-t)(qx - \frac{t}{q})^m\right) \). The \( q \)-integral in \( B_1 \) becomes \( \int_0^1 D_q \psi_k(t) t(1-qt)(qx-t)^m d_q t = [\psi_k(t) \sigma(t)]_0^1 - \int_0^1 \psi_k(t) (D_q \sigma)(t) d_q t \) for each \( k = 0, \ldots, n, \)

We expand \( \sigma(t) = q^{-2m}(x - \frac{1}{q})^{m+2} + q^{-2m}([3] - q^{m+2}) x - q^m(x - \frac{1}{q})^{m+1} \)

\(+ q^{-2m+1}(x(q^{m-1} + [m] (1 - q)q^m) - x^2(1 + [2] q(1 - q) [m]) (x - \frac{1}{q})^m \)

\(- q^{2m+3}x^2 [m] (1 - q)(q^{m-2} - x)(x - \frac{1}{q})^{m-1}. \)

We obtain \( B_1 = -q^{-\alpha-2m-1}[m + 1] T_n(x) - [m + 1] ([3] x - q^{m+2} x - q^m) T_{m+1}^n(x) \)

\(- q^{-\alpha-2m} [m] x(q^{m-1} + [1] [m] q^m - x(1 + q(1 - q) [2] [m]) T_{m+1} \)

\(+ q^{-\alpha+2-2m} [m - 1] x^2 [m] (q^{m-2} - x)(1 - q) T_{m-2}^1(x). \)

Moreover we have \( B_2 = -q^{1-\alpha-m} [n + \alpha + \beta] (T_{n,m+1}^1(x) - (1 - q) [m + 1] x T_{n,m}^1(x)), \)

\( B_3 = (x(q^{\alpha} [\beta - m + 2] - [2 - \alpha - m]) + [-\alpha])(T_{n,m}^1(x) - (1 - q) [m] x T_{n,m-1}^1(x)). \)

**Lemma 3** For any \( m \in \mathbb{N}, q \in [1/2, 1], x \in [0, 1] \), the expansion of \( (x-t)^m \) on the Newton basis at the points \( x/q^i, i = 0, \ldots, m - 1 \) is:

\[
(x-t)^m = \sum_{k=1}^{m} d_{m,k}(1-q)^{m-k}(x-t)^k_q, \quad (11)
\]

where the coefficients \( d_{m,k} \), verify \( |d_{m,k}| \leq d_m, k = 1, \ldots, m, \) and \( d_m \) does not depend on \( x, t, q. \)

**Proof.** For \( m = 1 \), it is obvious. If for some \( m \geq 1 \), the relation (11) holds, we write \( x-t = q^{-k}((x - q^k t) - (1 - q) [k] x) \) for \( k = 1, \ldots, m \) and we obtain \( (x-t)^{m+1} \)

\[ = \sum_{k=1}^{m+1} d_{m+1,k}(1-q)^{m+1-k}(x-t)^k_q \]

with \( d_{m+1,k} = q^{-k}(q d_{m,k-1} - [k] d_{m,k}) \) if \( k = 1, \ldots, m \)

and \( d_{m+1,m+1} = q^{-m}d_{m,m} \). Since \( |d_{m,k}| \leq d_m \) we have \( |d_{m+1,k}| \leq d_{m+1} = 2^{-m}(m+1)d_m, \)

\( k = 1, \ldots, m+1. \)
Proof of the proposition 2

At first we prove that, for any \( x \), \(|T_n^1(x)| \leq H_m/\lceil n \rceil^{m/2}\) if \( m \) is even (respectively \( \leq H_m/\lceil n \rceil^{(m+1)/2}\) if \( m \) is odd), where \( H_m \) does not depend on \( n, q, x \). We have \( T_{n,0}^1(x) = 1 \) and the formulae for \( M_n^\alpha f_i, i = 1, 2 \) of the proof of theorem 2 give the result for \( m = 1 \) and 2. The product \( \lceil n + \alpha + \beta \rceil (1 - q) = 1 - q^{n+\alpha+\beta} \) is positive and bounded by \( \max(1, |1 - 2^{-(1+\alpha+\beta)}|) \). If the result is true for some \( p \geq 2 \), \( p \) odd (respectively even) and any \( m \leq p \), the result for \( p + 1 \) follows from the recursion formula (10) of lemma 2.

Then, we write, for any \( n, m \in \mathbb{N} \) and \( x \in [0,1] \), using lemma 3 and \( 1 - q < 1/\lceil n \rceil \),
\[
|T_{n,m,q}(x)| \leq d_n \sum_{k=0}^{m} (1 - q)^{m-k} |T_k^1(x)| \leq d_n (T_m^1(x) + \sum_{k=1}^{m-1} \lceil n \rceil^{m-k} H_k \lceil n \rceil^{-k/2}) \\
\leq d_n T_m^1(x) + \sum_{k=1}^{m-1} H_k \lceil n \rceil^{-(m+1)/2} \text{ and the result follows.}
\]

Now the following lemma is the key.

**Lemma 4** Let \( (q_n) \) be a sequence owning the property \( S \), \( x \in ]0,1[ \) and \( \delta \in ]0,1[ \), \( \delta < \min(x, 1-x) \). Let \( \alpha, \beta, \alpha', \beta' \) be real numbers such that \( \alpha', \beta' \geq 0, \alpha > \alpha' - 1, \beta > \beta' - 1 \). We set \( \varphi(t) = t^{-\alpha'}(1-t)^{-\beta'}, t \in ]0,1[ \) and \( I_{x,\delta}(t) = 1 \) if \( |t-x| > \delta, I_{x,\delta}(t) = 0 \) elsewhere.

The sequence \( E_n(x, \delta) = \sum_{k=0}^{n} b_{n,k,q_n}(x) \int_0^1 t^{k+\alpha}(1-q_n t)^{n-k+\beta} \varphi(q_n^{\beta+1} t) I_{x,\delta}(t) d q_n t \) verifies \( \lim_{n \to \infty} n E_n(x, \delta) = 0 \) for any \( x \) and \( \delta \) such that \( 0 < \delta < x < 1 - \delta \).

**Proof.** Let \( \overline{\alpha} \) (respectively \( \overline{\beta} \)) be the smallest integer such that \( \overline{\alpha} \geq \alpha \) (respectively \( \overline{\beta} \geq \beta \)) and \( \tau \) (respectively \( \tau' \)) be a real number such that
\[
\tau > \frac{\overline{\alpha} + \overline{\beta} + 2}{\alpha - \alpha' + 1} \quad \text{(respectively } \tau' > \frac{\overline{\alpha} + \overline{\beta} + 2}{\beta - \beta' + 1}).
\]
For any $k = 0, \ldots, n$, we have $\int_0^1 t^{k+\alpha}(1-qt)^{n-k-\beta}dqt \geq \int_0^1 t^{k+\alpha}(1-qt)^{n-k-\beta}dqt$

$$= \frac{[k+\alpha][n-k-\beta]}{[n+\alpha+\beta+1]} \geq \left[\eta\right]^{-1}(n + \alpha + \beta + 1 - \alpha + \beta + 1).$$

We set $I_{x,\delta}(t) = 1$ if $0 < t < x - \delta$ and $I_{x,\delta}(t) = I_{x,\delta}(t) = 0$ elsewhere.

We split the interval $(0, 1)$ introducing $e_n = 1/n^\tau, e'_n = 1 - 1/n^\tau, n \in \mathbb{N}$, and we define, using again $\psi_{n,k,q}(t) = t^{k+\alpha}(1-qt)^{n-k-\beta}$ and $l_{n,k,q}(x) = b_{n,k,q}(x) / \int_0^1 \psi_{n,k,q}(t)dqt$ of lemma 3. $A_1^n = \sum_{k=0}^n l_{n,k,q}(x) \int_0^{e_n} t^{k+\alpha-\alpha'}(1-qt)^{n-k-\beta}dqt, t,$

$$A_2^n = \sum_{k=0}^n l_{n,k,q}(x) \int_0^{e_n} t^{k+\alpha-\alpha'}(1-qt)^{n-k-\beta}I_{x,\delta}(t)dqt, t,$$

$$A_3^n = \sum_{k=0}^n l_{n,k,q}(x) \int_0^{e_n} t^{k+\alpha}(1-qt)^{n-k-\beta}I_{x,\delta}(t)dqt, t,$$

$$A_4^n = \sum_{k=0}^n l_{n,k,q}(x) \int_0^{e_n} t^{k+\alpha}(1-qt)^{n-k-\beta}I_{x,\delta}(t)dqt, t.$$  

If $t > x$, (respectively if $t < x$) then $t^{\alpha'} < x^{\alpha'}$ (respectively $(1 - q_n^{\beta+1})^{\beta'} > (1 - q_n^{\beta+1})^{\beta'}$).

Hence, we have $E_n(x, \delta) \leq (1/2)^{-\beta+1}(1-x)^{-\beta}(A_1^n + A_2^n) + x^{\alpha'}(A_3^n + A_4^n)$ if $q_n \geq 1/2$, and it is sufficient to prove $\lim_{n \to \infty} nA_i^n = 0$ for $i = 1, 2, 3, 4$.

If $q_n^{\beta+1} \geq c$ and $e_n < 1/2$, we have for $k = 0, \ldots, n$, $\int_0^{e_n} t^{k+\alpha-\alpha'}(1-qt)^{n-k-\beta}dqt$

$$= q_n^{-k(\beta+1)} \left[ \left[ \frac{n}{k} \right] \right]^{-1} \int_0^{e_n} b_{n,k,q}(q_n^{\beta+1}t)^{\alpha-\alpha'}(1-qt)^{\beta}dqt \leq \left[ \left[ \frac{n}{k} \right] \right]^{-1} \gamma_1 c_n^{\alpha-\alpha'+1},$$

where $\gamma_1$ does not depend on $k, n, x$, since $q_n^{(\beta+1)k} \geq c^{\beta+1}, 0 \leq b_{n,k,q}(q_n^{\beta+1}t) \leq 1$, and $(1-qt)^{\beta} \leq 1$ if $\beta \geq 0$ and $t \in [0, 1]$, (respectively $(1-qt)^{\beta} \leq (1-e_n)^{-1} \leq 2$ if $\beta < 0$ and $t \in [0, e_n]$). Hence, we have $A_1^n \leq \gamma_1 (n + \alpha + \beta + 1)^{\alpha + \beta + 1}n^{-\gamma(\alpha + \alpha'+1)}$. The choice of $\tau$ and the property $S$ on $(q_n)$ give $\lim_{n \to \infty} nA_1^n = 0$.

We choose $m \in \mathbb{N}$ such that $m > \tau + 1$ and we write

$$A_2^n \leq n^{\tau} e^{-2m} \sum_{k=0}^n l_{n,k,q}(x) \int_0^{e_n} t^{k+\alpha}(1-qt)^{n-k-\beta}(x-t)^{2m}dqt,$$

$$\leq n^{\tau} e^{-2m} T_{n,2m,q}(x) \leq K_{2m} e^{-2m} n^{\tau} - m, \text{ hence } \lim_{n \to \infty} nA_2^n = 0 \text{ by the choice of } m.$$

Now we have, if $t < e'_n, (1-qt^{\beta+1})^{\beta'} > (1-e'_n)^{\beta'} \geq n^{-\gamma' \beta'}, \text{ hence}$
\[ A^3_n \leq n^{r'\beta} \sum_{k=0}^{n} l_{n,k,q_n}(x) \int_0^{e'_n} t^{k+\alpha}(1-q_n t)^{n-k+\beta} I^+_{x,\delta}(t)dq_n t. \]

We choose \( m' \in \mathbb{N} \) such that \( m' > r'\beta + 1 \) to have \( A^3_n \leq n^{r'\beta}\delta^{-2m'} \sum_{k=0}^{n} l_{n,k,q_n}(x) \int_0^{1} t^{k+\alpha}(1-q_n t)^{n-k+\beta}(x-t)^{2m'}dq_n t \)

\[ \leq n^{r'\beta}\delta^{-2m'}T_{n,2m',q_n} \leq K_{2m}\delta^{-2m'}n^{r'\beta-m'}, \] hence \( \lim_{n \to \infty} nA^3_n = 0 \) by the choice of \( m' \).

To finish, we prove that \( (1-q_n^{\beta-\beta'+1} t)^{\delta'} \leq (1-q_n^{3+1} t)^{\delta'} \) for any \( t \in [0,1] \).

If \( 0 \leq \beta' < 1 \), we use the \( q \)-binomial formula (cf. [1]) and the inequalities

\[-\beta' \leq -\beta' \text{ and } -\beta' + k / [k] \geq (-\beta' + k) / k \]

for any integer \( k \geq 1 \). In the other cases, if \( l \) is the integer such that \( l \leq \beta' < l + 1 \), we use the rules of product of \( q \)-binomials to write

\[(1-q_n^{\beta-\beta'+1} t)^{\delta'} = (1-q_n^{\beta-\beta'+1} t)^l (1-q_n^{\beta-\beta'+1} t)^{l-1} \text{ and the result follows.} \]

Then, with the same rules, we write, for any \( k = 0, \ldots, n \) and \( t \in [0,1] \),

\[(1-q_n t)^{n-k+\beta} = (1-q_n t)^{\beta'-l} (1-q_n^{\beta-\beta'+1} t)^{\delta'} (1-q_n^{\beta+1} t)^{n-k} \]

and

\[(1-q_n t)^{n-k+\beta} / (1-q_n^{3+1} t)^{\delta'} \leq (1-q_n t)^{\beta'-l} (1-q_n^{3+1} t)^{n-k}. \]

We deduce if \( q_n^\alpha \geq c \) and

\[ e'_n > 1/2, \quad A^4_n \leq \sum_{k=0}^{n} q_{n-1}^{(\beta+1)k} [k]_{q_n}^{-1} l_{n,k,q_n}(x) \int_0^{1} t^{\alpha} (1-q_n t)^{\beta'-l} b_{n,k,q_n}(q_n^{\beta+1} t)dq_n t \]

\[ \leq \gamma_2(n+\alpha+\beta'+1)\beta'\delta + (1-e'_n)^{\beta-\beta'+1} \text{ where } \gamma_2 \text{ does not depend on } k, n, x. \]

The choice of \( e'_n \) and \( \tau' \) gives \( \lim_{n \to \infty} nA^4_n = 0 \). \( \blacksquare \)

**Proof of theorem 3**

Suppose \( f \) is continuous at \( x \in ]0,1[ \). Let \( \varepsilon > 0 \) be an arbitrary real number. There exists \( \delta' > 0 \) such that \( |f(x) - f(t)| < \varepsilon \) for any \( t \in [0,1] \) such that \( |x - t| < \delta' \). Let \( \delta = \delta'/2 \) and \( N' \in \mathbb{N} \) such that \( (1-q_n^{3+1} x) < \delta \) for \( n > N' \). Then we have, if \( |x - t| < \delta \) and \( n > N' \), the inequalities

\[-\delta < -q_n^{3+1} \delta < x - q_n^{3+1} t = q_n^{3+1} (x-t) + (1-q_n^{3+1}) x < 2\delta \]

and \( |f(x) - f(q_n^{3+1} t)| < \varepsilon. \)

Hence, we have, if \( |f| \) is bounded by \( \kappa \), \( |f(x) - f(q_n^{3+1} t)| < \varepsilon + 2\kappa I_{x,\delta}(t) \) and, in
the case 2, \(|f(x) - f(q_n^{\alpha+1}t)| < \varepsilon + (|f(x)| + \kappa'(q_n^{\alpha+1}t)^{-\alpha'}(1 - q_n^{\alpha+1}t)^{-\beta'} I_{x,\delta}(t).\)

We apply the operator \(M_{\alpha,\beta}^{n,q_n}\) at the function \(h_x(t) = f(t) - f(x)\).

We have \(|M_{\alpha,\beta}^{n,q_n} f(x) - f(x)| = |M_{\alpha,\beta}^{n,q_n} h_x(x)| \leq (M_{\alpha,\beta}^{n,q_n} |h_x|) (x)\)
\[
\leq \begin{cases} 
\varepsilon + 2\kappa T_{n,2,q_n}(x)/\delta^2 & \text{in the case 1,} \\
\varepsilon + |f(x)| T_{n,2,q_n}(x)/\delta^2 + \kappa' E_n(x, \delta, q_n) & \text{in the case 2.}
\end{cases}
\]

The second term (respectively and the third term in the case 2) of the right hand side vanishes when \([n]_{q_n}\) tends to infinity. Since \(\lim_{n \to \infty} 1/[n]_{q_n} = 0\) in both cases (remark 1), the result follows. \(\blacksquare\)

**Proof of theorem 4**

We write Taylor formula at the point \(x,\)

\(f(t) = f(x) + (t-x)f'(x) + (t-x)^2 f''(x)/2 + (t-x)^2 \varepsilon(t-x)\) where \(\lim_{u \to 0} \varepsilon(u) = 0.\)

We apply the operator \(M_{\alpha,\beta}^{n,q_n}\) at the function \(f\) of the variable \(t\) to obtain

\(M_{\alpha,\beta}^{n,q_n} f(x) - f(x) = -f'(x)T_{n,1,q_n}(x) + \frac{f''(x)}{2} T_{n,2,q_n}(x) + R_n(x)\) where

\(R_n(x) = M_{\alpha,\beta}^{n,q_n} \zeta_x(x)\) with \(\zeta_x(t) = (t-x)^2 \varepsilon(t-x).\) We use \(\lim_{q \to 1} [a]_q = a\) for any \(a \in \mathbb{R}\) and we verify, with the help of the formulæ of the proof of theorem 2, that

\(\lim_{[n]_{q_n} \to \infty} [n]_{q_n} T_{n,1,q_n}(x) = (\alpha + \beta + 2)x - \alpha - 1\) and \(\lim_{[n]_{q_n} \to \infty} [n]_{q_n} T_{n,2,q_n}(x) = 2x(1-x).\)

So, to obtain the result, we have to prove that \(\lim_{[n]_{q_n} \to \infty} [n]_{q_n} R_n(x) = 0.\) We proceed in the same manner as in the proof of theorem 3. For any arbitrary \(\eta > 0,\) we can find \(\delta > 0\) such that, for \(n\) great enough, \(\varepsilon(x-t) < \eta\) if \(|x - q_n^{\alpha+1}t| < \delta.\)

We obtain the inequality \(|\zeta_x(t)| \leq \eta(x-t)^2 + (\rho_x + |f(t)|) I_{x,\delta}(q^{-\beta+1}t)\) for any \(t \in ]0, 1[,\) where \(\rho_x\) is independent of \(t\) and \(\delta.\) We deduce

\([n]_{q_n} |R_n(x)| \leq \begin{cases} 
[n]_{q_n} (\eta T_{n,2,q_n}(x) + (\rho_x + \kappa) T_{n,4,q_n}(x)/\delta^4) & \text{in the case 1,} \\
[n]_{q_n} (\eta T_{n,2,q_n}(x) + \rho_x T_{n,4,q_n}(x)/\delta^4) + \kappa' n E_n(x, \delta) & \text{in the case 2.}
\end{cases}\)
The right hand side tends to $2\eta x(1-x)$ when $n$ (hence $[n]_{q_n}$) tends to infinity and is as small as wanted. □

Remark 2

1) We see that the best order of approximation in (8) is in $1/\|n\|_{q_n}$. If $1 - q_n = 1/n^\gamma$ with $0 < \gamma < 1$, then $\lim_{n \to \infty} [n]_{q_n}/n^\gamma = 1$, hence $[n]_{q_n}$ can be replaced by $n^\gamma$ in (8). If $1 - q_n = 1/n \log n$ or $1/n^\gamma$ with $\gamma > 1$, then $\lim_{n \to \infty} [n]_{q_n}/n = 1$, $[n]_{q_n}$ can be replaced by $n$ and we refound exactly the Voronovskaya-limit property of $M_{n,1}^{\alpha,\beta}(x)$ (case 1).

2) In the case 2, the theorems 3 and 4 are valid for $M_{n,1}^{\alpha,\beta}$, if $wf$ is Lebesgue integrable on $[0,1]$, and this result is new. (In the proof the Jackson integrals have to be replaced by Lebesgue integrals).

Theorem 5  If $f'$ is continuous on $[0,1]$ and $q > 1/2$, then

$$
\|D_q(M_{n,q}^{\alpha,\beta}f) - f'\|_\infty \leq C'_{\alpha,\beta} \left( \omega \left(f', \frac{1}{\sqrt{\|n\|_q}} \right) + \omega \left(f', 1 - q \right) + \frac{[\alpha + \beta + 2]}{[n]_q} \|f'\|_\infty \right),
$$

where $C'_{\alpha,\beta}$ is a constant independent of $n, q, f$.

Hence, if $\lim_{n \to \infty} q_n = 1$, then $\lim_{n \to \infty} D_{q_n}(M_{n,q_n}^{\alpha,\beta}f) = f'(x)$ uniformly on $[0,1]$.

Proof. We write, using (3), for any $x \in [0,1]$,

$$
DM_n^{\alpha,\beta}f(x) - f'(x) = \frac{[n]}{[n + \alpha + \beta + 2]} \left( M_{n-1}^{\alpha+1,\beta+1} \left(Df \left(\frac{\cdot}{q} \right) \right) (qx) - Df(x) + Df(x) - f'(x) \right) + \left( \frac{[n]}{[n + \alpha + \beta + 2]} - 1 \right) f'(x).
$$

Since $0 < \frac{[n]}{[n + \alpha + \beta + 2]} < 1$, we have $|D(M_n^{\alpha,\beta}f(x)) - f'(x)|$.

The theorem (2) for the function $Df \left(\frac{\cdot}{q} \right)$ gives

$$
\left| M_{n-1}^{\alpha+1,\beta+1} \left(Df \left(\frac{\cdot}{q} \right) \right) (qx) - Df \left(\frac{\cdot}{q} \right)(qx) \right| \leq C_{\alpha+1,\beta+1} \omega \left(Df \left(\frac{\cdot}{q} \right), \frac{1}{\sqrt{\|n-1\|}} \right). Moreover
$|Df(x) - f'(x)| = |f'(y) - f'(x)|$ for some $y$ with $qx < y < x$ hence $|y - x| < 1 - q$ and $|Df(x) - f'(x)| \leq \omega(f', 1 - q)$. The modulus of continuity of $Df\left(\frac{z}{q}\right)$ is linked with the modulus of continuity of $f'$. Indeed, for any $y_i \in [0, 1]$ and $i = 1, 2$, there exists $z_i$, such that, $y_i < z_i < y_i/q$ and $Df\left(\frac{z}{q}\right)(y_i) = f'(z_i)$. As $|z_1 - z_2| \leq |y_1 - y_2|/q + (1 - q)/q$ we get $\omega\left(Df\left(\frac{z}{q}\right), t\right) = \sup_{|y_1 - y_2| < t} \left|Df\left(\frac{z}{q}\right)(y_1) - Df\left(\frac{z}{q}\right)(y_2)\right| \leq \sup_{|y_1 - y_2| < t} \left(|f'(z_1) - f'(z_2)|\right) \leq 2(\omega(f', t) + \omega(f', 1 - q))$ and the result follows. ■

**Corollary 3** If $f'$ is continuous on $[0, 1]$ and $1 - q_n = o(1/n^4)$, then

$$\lim_{n \to \infty} (M_{n,q_n}^{\alpha,\beta} f)'(x) = f'(x) \text{ uniformly on } [0, 1].$$

**Proof.** For any $x \in [0, 1]$, there exists $u \in (q_n x, x)$ such that

$$D_{q_n}(M_{n,q_n}^{\alpha,\beta} f)(x) = (M_{n,q_n}^{\alpha,\beta} f)'(u) \text{ and } (x - u) < 1 - q_n.$$

Hence $\left|D_{q_n}(M_{n,q_n}^{\alpha,\beta} f)(x) - (M_{n,q_n}^{\alpha,\beta} f)'(x)\right| \leq (1 - q_n) \left|(M_{n,q_n}^{\alpha,\beta} f)''(v)\right|$ for some $v$ and $\|D_{q_n}(M_{n,q_n}^{\alpha,\beta} f) - (M_{n,q_n}^{\alpha,\beta} f)'\|_\infty \leq n^4(1 - q_n) \|M_{n,q_n}^{\alpha,\beta} f\|_\infty \leq n^4(1 - q_n) \|f\|_\infty$, via Markov inequality. ■

### 4 self-adjointness properties

In this part $q \in [0, 1]$ is independent of $n$.

On the space of polynomials $\langle \cdot, \cdot \rangle_q^{\alpha,\beta}$ is an inner product. Let $\left(P_{r,q}^{\alpha,\beta}\right)_{r \in \mathbb{N}}$ be the sequence of the orthogonal polynomials for $\langle \cdot, \cdot \rangle_q^{\alpha,\beta}$ such that degree of $P_{r,q}^{\alpha,\beta} = r$ and $P_{r,q}^{\alpha,\beta}(0) = \left[r + \alpha\right]_q / \left[r\right]_q^{\alpha + 1}$. We set $\nu_r = \left(P_{r,q}^{\alpha,\beta}, P_{r,q}^{\alpha,\beta}\right)_q^{1/2}$.

We define $U_q^{\alpha,\beta}$ which is a $q$-analogue of the Jacobi differential operator by:

$$U_q^{\alpha,\beta} f(x) = D_q\left(x^{\alpha+1}(1 - q^{-\beta-1}) P^{\alpha+1}_{q,\nu} D_q f\left(\frac{z}{q}\right)\right) / x^{\alpha}(1 - q^{\beta}) x^{\beta}. \quad (12)$$
Proposition 3 The operator $U_{\alpha,\beta}^q$ is self-adjoint for $\langle ., . \rangle_{\alpha,\beta}^q$. It preserves the space of polynomials of degree $r \in \mathbb{N}$. Consequently, for any $r \in \mathbb{N}$, $P_{r,q}^{\alpha,\beta}$ is eigenvector of $U_{\alpha,\beta}^q$ for the eigenvalue $\mu_{r,q}^{\alpha,\beta} = -q^{-\beta-r} [r]_q [r + \alpha + \beta + 1]_q$.

Proof.

$U_{\alpha,\beta}^q$ is a $q$-differential operator of order 2. We compute with $q$-binomial relations,

$$U_{\alpha,\beta}^q f(x) = (-q^{\alpha+\beta} [\beta+1] x + [\alpha+1] (1 - q^{-\beta-1} x)) Df(x) + (1 - q^{-\beta-1} x) q D^2 f(x),$$

hence the operator $U_{\alpha,\beta}^q$ preserves the degree of polynomials. If $f$ and $g$ are polynomials $\langle U f, g \rangle$ is well defined. We write, since the $q$-integration by parts is valid,

$$\langle U_{\alpha,\beta}^q f, g \rangle = \left[ x^\alpha+1 (1 - q^{-\beta-1} x) [\beta+1] Df(x) g(x) \right]_0^{q^{\beta+1}} - \int_0^{q^{\beta+1}} (qx)^{\alpha+1} (1 - q^{-\beta} x)^{\beta+1} Df(x) Dg(x) d_q x,$$

and the first term vanishes. We compute $U_{\alpha,\beta}^q f(x)$ for $f(x) = x^r$ to obtain

$$U_{\alpha,\beta}^q f(x) = q^{1-r} [r] ([\alpha+r] x^{r-1} (1-x) - q^{-\beta-1} [\beta+1] x^r)$$

where the coefficient of $x^r$ is the eigenvalue $\mu_{r,q}^{\alpha,\beta}$. ■

Proposition 4 The eigenvectors of the operators $M_{n,q}^{\alpha,\beta}$, $n \in \mathbb{N}$, are the polynomials $P_{r,q}^{\alpha,\beta}$, $r \in \mathbb{N}$ and, if $f$ satisfies $c(\alpha)$, we have

$$M_{n,q}^{\alpha,\beta} f = \sum_{r=0}^{n} \chi_{n,r}^{\alpha,\beta} (f, P_{r,q}^{\alpha,\beta}) P_{r,q}^{\alpha,\beta} / n^2$$

with the eigenvalues

$$\chi_{n,r}^{\alpha,\beta} = q^{r(r+\alpha+\beta+1)} [n]! \Gamma_q (n + \alpha + \beta + 2) / [n-r]! \Gamma_q (n + r + \alpha + \beta + 2)$$

if $r \leq n$, $\chi_{n,r}^{\alpha,\beta} = 0$ otherwise.

Proof. Since $M_n$ is self adjoint and preserve the degree of polynomials, the orthogonal polynomials $P_r$ are eigenvectors. The eigenvalue $\chi_{n,r}^{\alpha,\beta}$ is obtained by computing $M_n f(x)$ for $f(x) = x^r$.

We use (3) $r$ times to get

$$D^r M_n f(x) = \frac{q^{r(\alpha+\beta+r+1)} [r]! [n] \ldots [n-r+1]}{[n+\alpha+\beta+2] \ldots [n+r+\alpha+\beta+1]}.$$

Corollary 4 1. For any $n, m \in \mathbb{N}$, the operators $M_{n,q}^{\alpha,\beta}$ and $M_{m,q}^{\alpha,\beta}$ commute on the space of functions satisfying $C(\alpha)$. 

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2. For any $n \in \mathbb{N}$, the operators $M_{n,q}^{\alpha,\beta}$ and $U_{q}^{\alpha,\beta}$ commute on the space of functions $f$ such that $f'$ is defined in a neighborhood of 0 and is continuous at the point 0.

Proof. 2) For any $r \in \mathbb{N}$ the $q$-integrals $\langle f, U P_r \rangle$ and $\langle U f, P_r \rangle$ are well-defined if $f'$ is continuous at the point 0. We go from one to the other by two $q$-integrations by parts which are valid because $\lim_{x \to 0} D f(\frac{x}{q}) = f'(0)$. Then we write

$$UM_n f = \sum_{r=0}^{n} \lambda_{n,r} \langle f, P_r \rangle \mu_r P_r / \nu_r^2 = \sum_{r=0}^{n} \lambda_{n,r} \langle U f, P_r \rangle P_r / \nu_r^2 = M_n U f.$$  \[ \text{Remark 3} \]

This proposition and its corollary open a field to study $\lim_{n \to \infty} M_{n,q}^{\alpha,\beta} f$ for $q$ fixed.

Formally we have $\lim_{n \to \infty} M_{n,q}^{\alpha,\beta} f = S_q^{\alpha,\beta} f = \sum_{r=0}^{\infty} q^{(r+\alpha+\beta+1)} \langle f, P_{r,q}^{\alpha,\beta} \rangle / \nu_r^2$ and $\lim_{n \to \infty} M_{n,q}^{\alpha,\beta} Q = \sum_{r=0}^{\deg Q} q^{(r+\alpha+\beta+1)} \langle Q, P_{r,q}^{\alpha,\beta} \rangle / \nu_r^2$ if $Q$ is a polynomial.

So $\lim_{n \to \infty} M_{n,q}^{\alpha,\beta} f = f$, if and only if $f$ is a constant. Moreover, $\lambda_{n-1,r}^{\alpha,\beta} - \lambda_{n,r}^{\alpha,\beta} = q^{n+\beta} \lambda_{n,r}^{\alpha,\beta} P_r^{\alpha,\beta} / [n][n+\alpha+\beta+1]$, $r \in \mathbb{N}$, hence $M_{n-1}^{\alpha,\beta} f - M_n^{\alpha,\beta} f = q^{n+\beta} \sum_{k=n+1}^{\infty} q^{k+\beta} / [k][k+\alpha+\beta+1] U_q^{\alpha,\beta} M_n^{\alpha,\beta} f$ and it is easy to prove (cf. [2]) that, when $f'$ is defined in a neighborhood of 0, continuous at 0, $\|M_{n,q}^{\alpha,\beta} f - S_q^{\alpha,\beta} f\|_{\infty} \leq \gamma_n \sup_{x \in [0,1]} \left| U_q^{\alpha,\beta} f(x) \right|$, where $\gamma_n = \sum_{k=n+1}^{\infty} q^{k+\beta} / [k][k+\alpha+\beta+1] \sim q^{n+\beta+1} / [n][n+\alpha+\beta+1]$. Of course $U_q^{\alpha,\beta} f$ has to be bounded on $[0,q^{\beta+2}]$, which is true, for example, if $f$ is bounded on $[0,1]$ and continuous on $[0,A]$ for some $A < 1$.

Proposition 5 The polynomials $P_{r,q}^{\alpha,\beta}$ are $q$-extensions of Jacobi polynomials for the weight $x^\alpha (1-x)^\beta$ denoted $P_r^{\alpha,\beta}, r \in \mathbb{N}$. They own the following properties which are the $q$-analogues of the well-known properties of Jacobi polynomials.

1. For any $r \in \mathbb{N}$, $\lim_{q \to 1} P_{r,q}^{\alpha,\beta} = P_r^{\alpha,\beta}$,
2. For any \( r \in \mathbb{N} \), the polynomial \( P_{r,q}^{\alpha,\beta} \) is a \( q \)-hypergeometric function (cf. [4]):

\[
P_{r,q}^{\alpha,\beta}(x) = \binom{\alpha+r}{r} q^r \sum_{k=0}^{\infty} \frac{\Gamma_q(\alpha+r+1)\Gamma_q(\beta+r+1)}{\Gamma_q(\alpha+1)\Gamma_q(\beta+1)} x^{\alpha+k}(1-q^{-\beta}x)^{\beta+r-k} q^c_k.
\]

So we have \( P_{r,q}^{\alpha,\beta}(x) = \binom{\alpha+r}{r} p_r(q^{-\beta-1}x; q^{\alpha+1}, q^{\beta+1}; q) \), where \( p_r(x; u, v : q) \) is the shifted little \( q \)-Jacobi polynomial of degree \( r \) (cf. [3], p.592).

3. They verify a \( q \)-analogue of Rodrigues formula:

\[
P_{r,q}^{\alpha,\beta}(x) = \frac{1}{[r]!} D_q^r \left( x^{\alpha+r}(1-q^{-\beta-r}x)^{\beta+r} \right). \]

4. We have the relation for the \( q \)-derivative:

\[
D_q P_{r,q}^{\alpha,\beta} \left( \frac{1}{q} \right) = -q^{-\beta-r} [r + \alpha + \beta + 1] P_{r-1,q}^{\alpha+1,\beta+1}. \]

**Proof.** 2) We look for the analytic solutions of the equation \( U_q^{\alpha,\beta} f - \mu_{r,q}^{\alpha,\beta} f = 0 \).

We write \( f(x) = \sum_{k=0}^{\infty} a_k x^k \) and \( U_q^{\alpha,\beta} f(x) - \mu_{r,q}^{\alpha,\beta} f(x) = \left[ \alpha + 1 \right] a_1 - \mu_{r,q}^{\alpha,\beta} a_0 + \left[ \beta \right] a_0 \) \( q^{-\beta} \sum_{k=1}^{\infty} \left( \left[ k+1 \right] [k + \alpha + 1] q^k a_{k+1} - \left[ \left[ k \right] [k + \alpha + 1] - \mu_{r,q}^{\alpha,\beta} q^{k+1} a_k \right) q^{-k} x^k \right. \). We obtain \( \frac{a_{k+1}}{a_k} = -q^{-r-\beta} \frac{\left[ r \right] - \left[ k \right]}{\left[ k+1 \right] [k + \alpha + 1]} = q^{-\beta} R(q^k) \), for any \( k \in \mathbb{N} \), with

\[
R(t) = \frac{(q^r - 1)(q^{r+\alpha+\beta+1} - t^{-1})}{(q^r - t^{-1})(q^{\alpha+1} - t^{-1})}\]

and the result follows.

3) For any polynomial \( Q \) of degree < \( r \), we verify, with the help of \( q \)-integrations by parts that \( \left< \frac{D_q^r \left( x^{\alpha+1}(1-q^{-\beta-r}x)^{\beta+r} \right)}{x^\alpha(1-q^{-\beta}x)^\beta} \right>_r \), \( Q_q^{\alpha,\beta} = 0 \).

We compute \( P_{r,q}^{\alpha,\beta}(0) \) by using a \( q \)-extension of Leibniz formula. We write

\[
D_q^r \left( x^{\alpha+1}(1-q^{-\beta-r}x)^{\beta+r} \right) = \sum_{k=0}^{r} \binom{r}{k} \frac{\Gamma_q(\alpha+r+1)\Gamma_q(\beta+r+1)}{\Gamma_q(\alpha+1)\Gamma_q(\beta+1)} x^{\alpha+k}(1-q^{-\beta}x)^{\beta+r-k} q^c_k,
\]

where \( c_k = k(k+\alpha-\beta-r+(k-1)/2) \) and \( D_q^r \left( x^{\alpha+1}(1-q^{-\beta-r}x)^{\beta+r} \right) = \sum_{k=0}^{r} \binom{r}{k} \frac{\Gamma_q(\alpha+r+1)\Gamma_q(\beta+r+1)}{\Gamma_q(\alpha+1)\Gamma_q(\beta+1)} x^{\alpha+k}(1-q^{-\beta-r+k}x)^{\beta+r-k} q^c_k = A(x) \).

We obtain \( A(0) = \frac{\Gamma_q(\alpha+r+1)\Gamma_q(\beta+r+1)}{\Gamma_q(\alpha+1)\Gamma_q(\beta+1)} \) \( [r]! \) \( \left[ \frac{r+\alpha}{r} \right] \) hence \( P_{r,q}^{\alpha,\beta}(x) = A(x)/[r]! \)
1) We take \( \lim_{q \to 1} A(x) = \sum_{k=0}^{r} \binom{r}{k} \frac{\Gamma(\alpha+r+1)\Gamma(\beta+r+1)}{\Gamma(\alpha+k+1)\Gamma(\beta+r-k+1)} x^k (1-x)^{r-k} \). It is \( r!P_r^\alpha(x) \) (Rodrigues formula).

4) We use (5) to prove that \( D_q P_{r,q}^{\alpha,\beta}(x) \) is eigenvector of \( M_{r-1,q}^{\alpha+1,\beta+1} \). Hence, it is equal to \( P_{r-1,q}^{\alpha+1,\beta+1} \) up to a constant. We compute \( D_q P_{r,q}^{\alpha,\beta}(0) = a_1 = \mu_{r,q}^{\alpha,\beta} a_0 \), hence
\[
 a_1 = -\left[ \left( \frac{r+\alpha}{r} \right) q^{-\beta-r} \right] [r + \alpha + \beta + 1] / [\alpha + 1] = -q^{-\beta-r} [r + \alpha + \beta + 1] \left[ \frac{r+\alpha}{r-1} \right] \]
and the result follows.  

5 The case \( \alpha = \beta = -1 \)

In this part, we study the operators \( M_{n,q}^{-1,-1} \). They are built with \( M_{n+1,q}^{0,0} \) as kantorovich operators are built with Bernstein operators (formula (13)).

Definition 3

The operator \( M_{n,q}^{-1,-1} \) is defined by replacing \( \alpha = \beta \) by \(-1\) in formula (4). It is:

\[
 M_{n,q}^{-1,-1} f(x) = \sum_{k=0}^{n} f_{n,k,q}^{-1,-1} p_{n,k,q}(x)
\]

with \( f_{n,0,q}^{-1,-1} = f(0) \), \( f_{n,n,q}^{-1,-1} = f(1) \) and the coefficients \( f_{n,k,q}^{-1,-1} \) for \( k = 1, \ldots, n-1 \), are given by (3) taking \( \alpha = \beta = -1 \).

The bilinear form is
\[
 \langle f, g \rangle_{q}^{-1,-1} = \int_{0}^{1} f(t) g(t) d_q t.
\]

The polynomial \( M_{n,q}^{-1,-1} f \) is well defined for any function \( f \) defined on \([0,1]\), bounded in a neighborhood of 0 (condition \( C(-1) \)). It verifies \( M_{n,q}^{-1,-1} f(0) = f(0) \) and \( M_{n,q}^{-1,-1} f(0) = f(1) \), hence it preserves the affine functions.

Proposition 6 If the function \( f \) is continuous on \([0,1]\), then
\[
 \lim_{\alpha \to -1} M_{n,q}^{\alpha,\alpha} f(x) = M_{n,q}^{-1,-1} f(x) \text{ for any } x \in [0,1].
\]
\textbf{Proof.} The $q$-binomial coefficients $b_{n,k,q}(x)$ are positive and form a partition of the unity. Hence it is sufficient to prove that $\lim_{\alpha \to -1} f_{n,k,q}^{\alpha,\alpha} = f_{n,k,q}^{1-1}$ for any $k$. For $k = 1, \ldots, n - 1$, we compute

$$
\begin{align*}
\frac{f_{n,k,q}^{\alpha,\alpha} - f_{n,k,q}^{1-1}}{\int_0^1 (1 - qt)^{-k+\alpha} f(t) dt} & = \frac{f_{n,k,q}^{1} (1 - qt)^{-k+\alpha} (f(q^{\alpha+1} t) - f(t)) dt}{B_q(k+\alpha+1,n-k+\alpha+1)} \\
+ \frac{\int_0^1 (1 - qt)^{-k+\alpha} f(t) dt}{B_q(k+\alpha+1,n-k+\alpha+1)} & = F_1 + F_2 + F_3.
\end{align*}
$$

We consider $I_k = \frac{\int_0^1 (1 - qt)^{-k+\alpha} (f(q^{\alpha+1} t) - f(t)) dt}{B_q(k+\alpha+1,n-k+\alpha+1)}$, with $\tilde{f}_0(t) = f(0), \tilde{f}_n(t) = f(1)$ and $\tilde{f}_k(t) = f(t), k = 1, \ldots, n - 1$ and prove that $\lim_{\alpha \to -1} I_k = 0$, for $k = 0, \ldots, n$.

We use the additivity of the modulus of continuity of $f$, Beta integrals and we set $\delta = [\alpha + 1] / [n + 2\alpha + 2]$. We have $|f(q^{\alpha+1} t) - f(0)| \leq \omega(f, t) \leq \omega(f, \delta)(1 + t/\delta)$, hence $|I_0| \leq \omega(f, \delta)(1 + \frac{1}{\delta}) \int_0^1 t^{\alpha+1} (1 - qt)^{\alpha+1} dt \int_0^1 t^{\alpha} (1 - qt)^{\alpha+1} dt \leq 2\omega(f, \frac{[\alpha+1]}{[n+2\alpha+2]})$.

For $k = n$, we have $|f(q^{\alpha+1} t) - f(1)| \leq \omega(f, 1 - q^{\alpha+1})$, hence

$$
|I_n| \leq \omega(f, \delta)(1 + \frac{1}{\delta}) \int_0^1 t^{\alpha+n} (1 - qt)^{\alpha+1} dt \int_0^1 t^{\alpha+n} (1 - qt)^{\alpha+1} dt \leq 2\omega(f, \frac{[\alpha+1]}{[n+2\alpha+2]}).
$$

For $k = 1, \ldots, n - 1$, we have $|f(q^{\alpha+1} t) - f(t)| \leq \omega(f, 1 - q^{\alpha+1})$ and

$$
|I_k| \leq \omega(f, 1 - q^{\alpha+1}).
$$

As $f_{n,0,q}^{\alpha,\alpha} - f(0) = I_0$ and $f_{n,n,q}^{\alpha,\beta} - f(1) = I_n$, the result follows for $k = 0$ and $n$.

For the other cases, $F_1 = I_k$ vanishes when $\alpha$ tends to $-1$. The upper term of $F_2$ is the $q$-integral $(1 - q) \sum_{j=0}^n q^{j(k+\alpha+1)} (1 - q^{j+1})^{n-k+\alpha} f(q^j)$. This serie is uniformly convergent, hence its limit when $\alpha$ tends to $-1$ is the upper term of $F_3$. At last the lower term of $F_2$ tends to the lower term of $A_3$ because $\Gamma_q$ is continuous. ■

Numerous properties shown in the case $\alpha, \beta > -1$ are still true if $\alpha = \beta = -1$.

Some of them are given in the following.

\textbf{Proposition 7} \textit{If the function $f$ is continuous at the points 0 and 1, and verifies the}
condition C(-1), we have:

\[ D_q M_{n,q}^{-1,1} f(x) = M_{n-1,q}^{0,0} \left( D_q f \left( \frac{x}{q} \right) \right) (qx), x \in [0, 1]. \]  \hspace{1cm} (13)

**Proof.** The expressions for \((f_{n,k+1}^{\alpha,\beta} - f_{n,k}^{\alpha,\beta})\) in proposition (1) hold if \(\alpha = \beta = -1\) and \(k = 1, \ldots, n - 2\). For the two other terms we have

\[ [n - 1] (f_{n,1}^{1,-1} - f(0)) = -[n^{-1}(f(t) - f(1))]_0^1 + \int_0^1 (1 - qt)^{n-1} D_q f(t) dt \]  \hspace{1cm} \text{and}

\[ [n - 1] (f_{n,n-1}^{1,-1} - f(1)) = -[(1 - t)^{n-1}(f(t) - f(0))]_0^1 + \int_0^1 (qt)^{n-1} D_q f(t) dt. \]

The first terms vanish since \(f\) is continuous at 0 and 1. \(\square\)

**Theorem 6**

1. \(M_{n,q}^{1,1} f(x) = \sum_{j=0}^{\infty} \Phi_{j,n,q}^{-1,1}(x) f(q^j),\) where \(\Phi_{j,n,q}^{1,1}\) is defined in formula (2) with \(\alpha = \beta = -1\). The sequence \(\Phi_{1,n,q}^{1,1}, \Phi_{2,n,q}^{1,1}, \ldots, \Phi_{n,n,q}^{1,1}\) is totally positive. Consequently the operator \(M_{n,q}^{1,1}\) diminishes the number of sign changes and preserves the monotony.

2. If \(f\) is continuous on \([0, 1]\), then \(\|M_{n,q}^{1,1} f - f\|_\infty \leq \text{Cte } \omega \left( f, \frac{1}{\sqrt[n]{q}} \right),\) \(\text{ (theorem }3)\).

3. If \(\lim_{n \to \infty} q_n = 1\) and if the function \(f\) is bounded on \([0, 1]\), then

(a) \(\lim_{n \to \infty} M_{n,q_n}^{1,1} f(x) = f(x)\) if \(f\) is continuous at the point \(x \in ]0, 1[\), \(\text{ (theorem }3)\).

(b) \(\lim_{n \to \infty} \left[ n \right] q_n \left( M_{n,q_n}^{1,1} f(x) - f(x) \right) = f''(x)\) if the function \(f\) admits a second derivative at the point \(x \in ]0, 1[\), \(\text{ (theorem }4)\).
References


