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# CLUSTER ALGEBRAS FROM CLUSTER CATEGORIES

PHILIPPE CALDERO AND FRÉDÉRIC CHAPOTON

ABSTRACT. Cluster categories are triangulated categories which were recently introduced in connection with the theory of cluster algebras. In this article, we prove that some cluster algebras of type ADE can be recovered from the data of the corresponding cluster category. This also provides some explicit formulas for cluster variables.

## 1. INTRODUCTION

Cluster algebras were introduced in [FZ02] by S. Fomin and A. Zelevinsky in connection with the theory of dual canonical bases and total positivity. Coordinate rings of many varieties from Lie group theory – semisimple Lie groups, homogeneous spaces, generalized Grassmannian, double Bruhat cells, Schubert varieties – have a structure of cluster algebra, at least conjecturally, see [BFZ04, Sco03]. One of the goals of the theory is to provide a general framework for the study of canonical bases of these coordinate rings and their  $q$ -deformations.

A cluster algebra  $\mathcal{A}$  of rank  $n$  is a subalgebra of the field  $\mathbb{Q}(u_1, \dots, u_n)$ . It is defined from a distinguished set of generators, called cluster variables, constructed by an induction process from a antisymmetrizable matrix  $B$ , see Section 2.1. The Laurent phenomenon asserts that  $\mathcal{A}$  is a subalgebra of  $\mathbb{Q}[u_1^{\pm 1}, \dots, u_n^{\pm 1}]$ . There exists a notion of compatibility between two cluster variables; maximal subsets of pairwise compatible cluster variables are called clusters. All clusters have the same cardinality, which is the rank of the cluster algebra.

A cluster algebra is of *finite type* if the number of cluster variables is finite. The classification of cluster algebras of finite type [FZ03] is a fundamental step in the theory. The main result is that these cluster algebras come from an antisymmetrized Cartan matrix of finite type, see Section 2.2. Moreover, in this case the cluster variables are in correspondence with the set of almost positive roots  $\Phi_{\geq -1}$ , *i.e.* positive roots or opposed simple roots, of the root system.

The Gabriel theorem asserts that the set of indecomposable representations of a quiver  $Q$  of Dynkin type is in bijection with the set  $\Phi_+$  of positive roots. The cluster category  $\mathcal{C}$  was constructed in [BMR<sup>+</sup>, CCS04] as an extension of the category  $\text{Mod}_k(Q)$  such that the set of indecomposable objects of  $\mathcal{C}$  is in bijection with  $\Phi_{\geq -1}$ . The category  $\mathcal{C}$  is not abelian in general, but it is a triangulated category, [Kel03]. In [BMR<sup>+</sup>], this category is studied in depth. The authors give a correspondence between cluster variables and indecomposable objects of  $\mathcal{C}$ . They prove that the compatibility of two cluster variables correspond to the vanishing of the Ext groups; hence, clusters correspond to so-called ext-configurations. They prove that there exist many analogies between finite cluster algebras and cluster categories, but the

properties of the correspondence are mostly conjectural, see [BMR<sup>+</sup>, Conjecture 9.3].

In [CCS04], the authors prove that the denominators of cluster variables can be calculated from  $\mathcal{C}$  in type A. They give a combinatorial/geometric approach of  $\mathcal{C}$  in the spirit of Teichmüller spaces, [FG03].

The main question behind both articles is: can one realize the cluster algebra as a “Hall algebra” of the category  $\mathcal{C}$  in some sense? In this article, the cluster variable associated to an indecomposable  $kQ$ -module, in fact to any module, is explicitly given, see Theorem 3.4. This result is interesting from different angles.

1. It strengthens the relations between the category  $\mathcal{C}$  and the algebra  $\mathcal{A}$ .
2. We obtain here explicit expressions for cluster variables, instead of inductive ones. These expressions are in terms of Euler-Poincaré characteristic of Grassmannians of submodules. Note that these characteristics can be easily calculated in the  $A_n$  case, see Example 3.2. They can also be calculated in a combinatorial way in the  $D_n$  case.
3. We believe that these explicit expressions can be used to show that cluster variables have a positive Laurent expansion in any cluster associated to a Dynkin type quiver. This would prove a positivity conjecture in this case. Recall that this conjecture is known to hold only for a distinguished seed so far.
4. The expression gives the possibility to quantify cluster algebras in the Ringel-Hall algebras spirit: the Euler-Poincaré characteristic should be replaced by a polynomial which counts  $\mathbb{F}_q$ -rational points on the variety.

In the sequel, we give a conjectural expression for cluster variables associated to a multiplicity-free indecomposable module over any quiver of simply-laced finite cluster type. As a special case, this conjecture enables, in  $A_n$  type, to calculate in a combinatorial way the cluster variables in terms of any cluster.

To conclude, we give a connection between our theorem, the geometric realization of [CCS04], and the Coxeter-Conway friezes [CC73].

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## 2. RECOLLECTION FROM CLUSTER ALGEBRAS AND CLUSTER CATEGORIES

2.1. In this section, we give basic definitions and Theorems concerning cluster algebras, see [FZ02, FZ03, BFZ04]. The cluster algebras in this article are defined on a trivial semigroup of coefficients, and will be called *reduced* cluster algebras. The recollection below is transposed into the framework of reduced cluster algebras.

Let  $n$  be a positive integer and let  $B = (b_{ij})$  be a square matrix in  $M_n(\mathbb{Z})$ . We say that  $B$  is antisymmetrizable if there exists a diagonal matrix in  $M_n(\mathbb{N})$  such that  $DB$  is antisymmetric. We introduce the field  $\mathcal{F} := \mathbb{Q}(u_1, \dots, u_n)$ , with transcendence base  $\underline{u} := (u_1, \dots, u_n)$ . A couple  $(\underline{x}, B)$ , where  $\underline{x} = (x_1, \dots, x_n)$  is a transcendence base of  $\mathcal{F}$  and where  $B$  is an antisymmetrizable matrix with integer coefficients, will be called a seed. In the sequel, we will identify the lines and the columns of the matrix  $B$  with the elements of  $\underline{x}$ .

Fix a seed  $(\underline{x}, B)$ ,  $B = (b_{xy})$ , and  $z$  in the base  $\underline{x}$ . Let  $z'$  in  $\mathcal{F}$  be such that

$$(1) \quad zz' = \prod_{b_{xz} > 0} x^{b_{xz}} + \prod_{b_{xz} < 0} x^{-b_{xz}}.$$

This is the so-called exchange relation. Now, set  $\underline{x}' := \underline{x} - \{z\} \cup \{z'\}$  and  $B' = (b'_{xy})$  such that

$$(2) \quad b'_{xy} = \begin{cases} -b_{xy} & \text{if } x = z \text{ or } y = z, \\ b_{xy} + 1/2(|b_{xz}| |b_{zy}| + b_{xz} |b_{zy}|) & \text{otherwise.} \end{cases}$$

Then, it is known that  $(\underline{x}', B')$  is also a seed. We say that this seed is the mutation of the seed  $(\underline{x}, B)$  in the direction  $z$ . We also say that  $z$  and  $z'$  form an exchange pair. It is easily seen that the mutation of the seed  $(\underline{x}', B')$  in the direction  $z'$  is  $(\underline{x}, B)$ . We can define the equivalence relation generated by  $(\underline{x}, B) \sim (\underline{x}', B')$  if  $(\underline{x}', B')$  is a mutation of  $(\underline{x}, B)$ .

We assign to a antisymmetrizable matrix  $B$  a  $\mathbb{Q}$ -algebra in the following way.

**Definition 2.1.** The reduced cluster algebra  $\mathcal{A}(B)$  associated to the antisymmetrizable matrix  $B$  is the subalgebra of  $\mathcal{F}$  generated all  $\underline{x}$  such that  $(\underline{x}, B) \sim (\underline{x}, B')$ . Such  $\underline{x}$  are called clusters and the elements of  $\underline{x}$  are called cluster variables.

*Remark 2.2.* More generally, see [BFZ04], cluster algebras are associated to rectangular matrices in  $M_{n,m}(\mathbb{Z})$ . We will not be concerned with such algebras in this article.

Note the so-called *Laurent phenomenon*, see [FZ02]:

**Theorem 2.3.** *Let  $B$  be a antisymmetrizable matrix in  $M_n(\mathbb{Z})$ , then  $\mathcal{A}(B)$  is a subalgebra of  $\mathbb{Q}[u_i^{\pm 1}, 1 \leq i \leq n]$ .*

2.2. This section is concerned with *finite* reduced cluster algebras, *i.e.* cluster algebras with a finite number of cluster variables.

Let  $\Delta$  be a Dynkin diagram of rank  $n$  and let  $A_\Delta$  be its Cartan matrix. We denote by  $\Phi$ , resp.  $\Phi_+$ , the root system, resp. the set of positive roots, associated to  $\Delta$ . Let  $\alpha_i, 1 \leq i \leq n$ , be the simple roots and let  $\mathcal{Q}$  be the  $\mathbb{Z}$ -lattice generated by them. We also denote by  $\Phi_{\geq -1}$  the set of *almost positive* roots  $\Phi_+ \cup \{-\alpha_1, \dots, -\alpha_n\}$ .

We have the following fundamental Theorem, see [FZ03]:

**Theorem 2.4.** *A reduced cluster algebra  $\mathcal{A}$  is finite if and only if there exists a seed  $(\underline{x}, B)$  of  $\mathcal{A}$  such that the Cartan counterpart of the matrix  $B$  is a Cartan matrix of finite type.*

In the Theorem, the Cartan counterpart of a matrix  $B = (b_{ij})$  in  $M_n(\mathbb{Z})$  is the matrix  $A = (a_{ij})$  with

$$(3) \quad a_{ij} = \begin{cases} 2 & \text{if } i = j, \\ -|b_{ij}| & \text{if } i \neq j. \end{cases}$$

These algebras will be called finite reduced cluster algebras. Actually, the Theorem of Fomin and Zelevinsky is more precise. The correspondence  $\mathcal{A} \mapsto \Delta$  of the Theorem provides a bijection from the set of finite reduced cluster algebras into the set of Dynkin diagrams of finite type. Hence, to a Dynkin diagram  $\Delta$  of type A to G, we can associate a unique algebra  $\mathcal{A}(\Delta)$ ; this is the reduced cluster algebra of the corresponding type.

Let us go further in the study of the algebra  $\mathcal{A}(\Delta)$ . Let  $\text{Cl}_\Delta$  be the set of cluster variables of  $\mathcal{A}(\Delta)$ .

**Theorem 2.5.** *Fix a Dynkin diagram of finite type  $\Delta$ . Then, there exists a bijection  $\beta$  from the set  $\text{Cl}_\Delta$  to  $\Phi_{\geq -1}$  that sends  $u_i$  to  $-\alpha_i$ .*

2.3. We present in this section a recollection on quiver representations. We fix a field  $k$  which can be either the finite field  $\mathbb{F}_q$  or the field  $\mathbb{C}$  of complex numbers. From now on, let  $\Delta$  be a Dynkin diagram of simply laced finite type and  $Q$  be an oriented quiver with underlying graph  $\Delta$ . We index by  $I = \{1, \dots, n\}$  the set of its vertices. We consider the category  $kQ = \text{Mod}_k(Q)$  of finite dimensional  $k$ -representations of the quiver  $Q$ . We denote by  $\overline{\text{Mod}}_k(Q)$ , resp.  $\text{Ind}_k(Q)$ , the set of isoclasses, resp. indecomposable modules, of  $\text{Mod}_k(Q)$ . We also denote by  $P_i$ , resp.  $S_i$ ,  $1 \leq i \leq n$ , the projective, resp. simple, modules of the category. Recall that, by the Theorem of Gabriel, the set  $\text{Ind}_k(Q)$  does not depend on the field  $k$  and is in natural bijection with the set of positive roots  $\Phi_+$ . This bijection sends the simple modules  $S_i$  to the simple roots  $\alpha_i$ . Hence, the Grothendieck group of the category  $\text{Mod}_k(Q)$  is naturally identified with the lattice  $\mathcal{Q}$ . We define the map  $\underline{\dim} : \text{Mod}_k(Q) \rightarrow \mathcal{Q}$  which associate to a module  $M$  its class  $[M]$  in the Grothendieck group.

We define for  $M, N$  in  $\text{Mod}_k(Q)$ :

$$(4) \quad [M, N] = \dim \text{Hom}_{kQ}(M, N), \quad [M, N]^1 = \dim \text{Ext}_{kQ}^1(M, N),$$

$$(5) \quad \langle M, N \rangle = [M, N] - [M, N]^1.$$

It is known that  $\langle, \rangle$  can be defined on the Grothendieck group  $\mathcal{Q}$ , this is the Euler form.

Fix  $\underline{e}$  in  $\mathcal{Q}$  and let  $M$  be in  $\text{Mod}_k(Q)$ . We define the  $\underline{e}$ -Grassmannian  $\text{Gr}_{\underline{e}}(M)_k$  of the module  $M$  on  $k$ :

$$(6) \quad \text{Gr}_{\underline{e}}(M)_k := \{N, N \in \text{Mod}_k(Q), N \subset M, \underline{\dim}(N) = \underline{e}\}.$$

When the fixed field is clear, we will omit the index  $k$  in the notation. Note that  $\text{Gr}_{\underline{e}}(M)$  can be realized as a closed subvariety of the classical Grassmannian  $\text{Gr}_e(M)$ , where  $e = \sum_i e_i$ . Hence, the variety  $\text{Gr}_{\underline{e}}(M)$  is projective. We define in an obvious way the variety  $\text{Gr}_{\underline{e}}(M)$ , for  $M$  in  $\overline{\text{Mod}}_k(Q)$ , and we set  $\text{Gr}(M) := \bigcup_{\underline{e}} \text{Gr}_{\underline{e}}(M)$ . So, in the sequel the Grassmannian will always mean ‘‘Grassmannian of submodules’’.

2.4. The cluster category  $\mathcal{C} = \mathcal{C}_{\Delta}$  has been introduced in [BMR<sup>+</sup>] and [CCS04]. As proved in these articles, the cluster category  $\mathcal{C}_{\Delta}$  is strongly linked with the finite cluster algebra  $\mathcal{A}(\Delta)$ . As a first example of this relation, the set of indecomposable objects of  $\mathcal{C}_{\Delta}$  is in bijection with  $\Phi_{\geq -1}$ , so it is in bijection with the cluster variables of  $\mathcal{A}(\Delta)$ .

Let  $Q$  be as above. In the sequel, we denote by  $B_Q$  be the antisymmetric matrix in  $M_n(\mathbb{Z})$  such that  $b_{ij} = 1$  if  $i \rightarrow j$  in  $Q$  and 0 if  $i$  and  $j$  are not connected. Let  $\mathcal{D}\text{Mod}_k(Q)$  be the derived category of  $\text{Mod}_k(Q)$ . Note that as  $\text{Mod}_k(Q)$  is hereditary, the indecomposable objects of  $\mathcal{D}\text{Mod}_k(Q)$  are the shifts of  $\text{Ind}_k(Q)$ . We define the functor  $F$  of  $\mathcal{D}\text{Mod}_k(Q)$  by  $F : M \mapsto \tau^{-1}SM$ , where  $S$  is the shift and  $\tau$  is the Auslander-Reiten translation. The category  $\mathcal{C}_Q$  is the category of orbits of  $\mathcal{D}\text{Mod}_k(Q)$  by  $F$ , see [BMR<sup>+</sup>]. It is a triangulated category [Kel03], but it is not abelian in general. A nice property of this category is that the functor  $\text{Ext}^1$  is symmetric. This category does not depend on the orientation of  $Q$ , but only on  $\Delta$ . It will be denoted by  $\mathcal{C}_{\Delta}$  or just  $\mathcal{C}$ . The set  $\text{Ind}_k(Q)$  embeds naturally in the set  $\text{Ind } \mathcal{C}$

of indecomposable objects of  $\mathcal{C}$ . Moreover,  $\text{Ind } \mathcal{C} = \text{Ind}_k(Q) \cup \{SP_i, 1 \leq i \leq n\}$ . We have a choice of bijections such that the following diagram is commutative:

$$(7) \quad \begin{array}{ccc} \Phi_+ & \longrightarrow & \Phi_{\geq -1} \\ \downarrow \simeq & & \downarrow \simeq \\ \text{Ind}_k(Q) & \longrightarrow & \text{Ind } \mathcal{C}, \end{array}$$

where the horizontal arrows are the natural embeddings. An important Theorem of [BMR<sup>+</sup>] asserts that the ext-configurations of  $\text{Ind } \mathcal{C}$ , *i.e.* maximal subsets of  $\text{Ind } \mathcal{C}$  with trivial pairwise extension, correspond to clusters via the bijections  $\text{Ind}(\mathcal{C}_\Delta) \simeq \Phi_{\geq -1} \simeq \text{Cl}_\Delta$ . Note also that the ext-configurations of  $\text{Ind}_k(Q)$  are ext configurations of  $\text{Ind } \mathcal{C}$  which belong to  $\text{Ind}_k(Q)$ .

2.5. In the previous section, we have seen a correspondence between indecomposable objects of  $\mathcal{C}$  and cluster variables. We will see in this section properties of the Auslander-Reiten translation  $\tau$  in this correspondence. First of all, let us recall some basic facts on the Auslander-Reiten theory, see [ARS95].

Let  $\Gamma_Q$  be the Auslander-Reiten quiver of  $\text{Mod}_k(Q)$ . Recall that its set of vertices is  $\text{Ind}_k(Q)$  and the arrows are given by irreducible morphisms of the category. The AR-quiver  $\Gamma_{\mathcal{C}}$  of  $\mathcal{C}$  is defined in the same way.

Let  $M$  be a non projective module in  $\text{Ind}_k(Q)$  and let  $M' = \tau M \in \text{Ind}_k(Q)$  be its AR-translated. We consider the direct sum  $B$  of indecomposable modules  $B_j$  such that  $M' \rightarrow B_j$  in  $\Gamma_Q$ . Then,  $B$  is also the direct sum of indecomposable modules  $B_j$  such that  $B_j \rightarrow M$  in  $\Gamma_Q$  and we have the following exact sequence of modules:

$$(8) \quad 0 \longrightarrow \tau M \longrightarrow B \xrightarrow{\sigma} M \longrightarrow 0.$$

Moreover, this exact sequence is almost split in the following sense: each morphism  $N \rightarrow M$  which is not a split epimorphism factors through  $\sigma$ .

The AR-quivers of  $\text{Mod}_k(Q)$  are well known and can be explicitly described, see [Gab80].

The AR-quiver  $\Gamma_{\mathcal{C}}$  is a slight extension of  $\Gamma_Q$ . Indeed, see [BMR<sup>+</sup>], each exact sequence as in (8) gives rise in the triangulated category  $\mathcal{C}$  to a triangle

$$(9) \quad \tau M \rightarrow B \rightarrow M \rightarrow S\tau M,$$

where the first two morphisms are composed with irreducible morphisms. In other terms, the embedding  $\text{Ind}_k(Q) \subset \text{Ind } \mathcal{C}$  provides an embedding  $\Gamma_Q \subset \Gamma_{\mathcal{C}}$  as a full subquiver. In order to describe  $\Gamma_{\mathcal{C}}$ , it is sufficient to note that:

There exists an arrow  $SP_i \rightarrow M$  in  $\mathcal{C}$  if and only if

$$(10) \quad M = \begin{cases} SP_k & k \rightarrow i \text{ in } Q, \\ P_j & i \rightarrow j \text{ in } Q, \end{cases}$$

and there exists an arrow  $M \rightarrow SP_i$  in  $\mathcal{C}$  if and only if

$$(11) \quad M = \begin{cases} P_k & k \rightarrow i \text{ in } Q, \\ SP_j & i \rightarrow j \text{ in } Q. \end{cases}$$

In the following proposition, we denote by  $x_M$  the cluster variable corresponding to the indecomposable object  $M$  of  $\text{Ind } \mathcal{C}$ .

**Proposition 2.6.** *Let  $\mathcal{C}$  be the cluster category of type A, D or E.*

- (i) Let  $M$  be an indecomposable object of  $\mathcal{C}$ . Then  $x_M$  and  $x_{\tau M}$  form an exchange pair.  
(ii) With the notation above, we have

$$(12) \quad x_{\tau M} x_M = \prod_j x_{B_j} + 1.$$

*Proof.* (i) is a direct application of [BMR<sup>+</sup>, Proposition 7.6].

Fix  $M$  in  $\text{Ind } \mathcal{C}$ . Then, there exists a oriented quiver  $Q$  and a sink  $i$  of  $Q$  such that in the equivalence  $\mathcal{C} \simeq \mathcal{D} \text{Mod}_k(Q)/F$ , the object  $M$  is identified with the simple projective  $P_i = S_i$ . We know that this equivalence provides a natural algebra isomorphism  $\mathcal{A}(\Delta) \simeq \mathcal{A}(B_Q)$ . Hence, and by the discussion above, see (10) and (11), it is enough to prove that

$$(13) \quad x_{SP_i} x_{S_i} = \prod_{i \rightarrow j} x_{SP_j} + 1.$$

Let us prove the equality. Set  $B_Q = (b_{ij})$ . The exchange relation gives

$$(14) \quad x_{SP_i} x_{S_i} = \prod_{b_{ji}=1} x_{SP_j} + \prod_{b_{ji}=-1} x_{SP_j}.$$

As  $i$  is a sink of  $Q$ , the second term is one. Moreover,  $b_{ji} = 1$  if and only if  $j \rightarrow i$  in  $Q$ . So, we have the claimed equality.  $\square$

Remark that (ii) is a particular case of [BMR<sup>+</sup>, Conjecture 9.3].

### 3. THE MAIN THEOREM

3.1. In this section, we give realizations of any finite reduced cluster algebra from the category  $\mathcal{C}$ ; for each oriented quiver  $Q$  with underlying Dynkin diagram  $\Delta$  of type A-D-E, we realize  $\mathcal{A}(\Delta)$  from  $\mathcal{C}_Q$ . Actually, we recover the algebra  $\mathcal{A}(B_Q)$  from the category  $\mathcal{C}_Q$ . In the sequel, we fix a quiver  $Q$  of type A-D-E.

Recall that  $\mathcal{F} = \mathbb{Q}(u_i, 1 \leq i \leq n)$ . For each  $M$  in  $\overline{\text{Mod}}_k(Q)$  with dimension vector  $\underline{\dim}(M) = \underline{m} = \sum_i m_i \alpha_i$ , set

$$(15) \quad X_M = \sum_{\underline{e}} \chi(\text{Gr}_{\underline{e}}(M)) \prod_i u_i^{-\langle \underline{e}, \alpha_i \rangle - \langle \alpha_i, \underline{m} - \underline{e} \rangle},$$

where  $\chi$  is the Euler-Poincaré characteristic of the complex Grassmannian. Remark that the sum is finite since the dimension vectors  $\underline{e} = \sum_i e_i \alpha_i$  which occur in the sum verify  $0 \leq e_i \leq m_i$ . We now illustrate with examples.

**Example 3.1.** Suppose that  $Q$  is the following alternated orientation for  $A_3$ :

$$(16) \quad 1 \longrightarrow 2 \longleftarrow 3.$$

Then, the indecomposable modules of  $\text{Mod}_k(Q)$  are  $S_1, S_2, S_3, P_1, P_3, I_2$ , where  $[I_2] = [S_1] + [S_2] + [S_3]$ . The AR quiver  $\Gamma_{\mathcal{C}}$  has the following shape:

$$(17) \quad \begin{array}{ccccccc} & & SP_3 & & P_3 & & S_1 \\ & \nearrow & & \searrow & & \searrow & \nearrow \\ SP_2 & & & & S_2 & & I_2 & & S_1 \\ & \searrow & & \nearrow & & \nearrow & \searrow & & \\ & & SP_1 & & P_1 & & S_3 & & \\ & & & & & & & & \end{array} .$$

We compute explicitly the  $X_M$  using formula (15). In the following sums, the terms are ordered by  $\sum e_i$ .

$$(18) \quad X_{S_2} = \frac{1}{u_2} + \frac{u_1 u_3}{u_2} = \frac{1 + u_1 u_3}{u_2},$$

$$(19) \quad X_{P_3} = \frac{u_1}{u_2} + \frac{1}{u_2 u_3} + \frac{1}{u_3} = \frac{1 + u_2 + u_1 u_3}{u_2 u_3}, \quad X_{P_1} = \frac{u_3}{u_2} + \frac{1}{u_2 u_1} + \frac{1}{u_1} = \frac{1 + u_2 + u_1 u_3}{u_2 u_1},$$

$$(20) \quad X_{I_2} = \frac{1}{u_2} + \frac{1}{u_1 u_3} + \frac{1}{u_1 u_3} + \frac{1}{u_1 u_2 u_3} + \frac{u_2}{u_1 u_3} = \frac{1 + 2u_2 + u_2 + u_1 u_3}{u_1 u_2 u_3},$$

$$(21) \quad X_{S_1} = \frac{1}{u_1} + \frac{u_2}{u_1} = \frac{1 + u_2}{u_1}, \quad X_{S_3} = \frac{1}{u_3} + \frac{u_2}{u_3} = \frac{1 + u_2}{u_3}.$$

**Example 3.2.** If  $Q$  is an oriented quiver of type  $A_n$ , and if  $M$  is an indecomposable module of  $\text{Mod}_k(Q)$ , then  $\chi(\text{Gr}_{\underline{e}}(M)) = 0$  or  $1$ . More precisely, the indecomposable  $kQ$ -modules correspond to connected full subquivers of  $Q$ . Let  $Q_M$  be the quiver corresponding to  $M$  and let  $V_M = \{p, p+1, \dots, m\}$  be the set of vertices. Then, the submodules  $N$  of  $M$  correspond to subsets  $V_N$  of  $V_M$  such that the following property holds:  $i \in V_N$  and  $i \rightarrow j \Rightarrow j \in V_N$ .

Now, if  $N$  is a submodule of  $M$  with dimension vector  $\underline{n}$ , then  $\text{Gr}_{\underline{n}}(M) = 1$ .

**Example 3.3.** We consider the following quiver  $Q$  of type  $D_4$ :

$$(22) \quad \begin{array}{ccc} & 3 & \\ & \downarrow & \\ 1 & \longrightarrow 2 & \longleftarrow 4. \end{array}$$

Let  $M$  be the indecomposable module with maximal dimension, *i.e.*  $[M] = [S_1] + [S_3] + [S_4] + 2[S_2]$ . Then, we have  $\text{Gr}_{\alpha_2} M = \mathbb{P}^1$  and so  $\chi(\text{Gr}_{\alpha_2} M) = 2$ .

The module  $M$  has 13 submodules:  $0, S_2, 2S_2, P_1, P_3, P_4, P_1 + S_2, P_3 + S_2, P_4 + S_2, P_1 + P_3, P_1 + S_2, P_1 + P_4, P_3 + P_4, M$ . But,  $S_2$  has "multiplicity" 2. That gives

$$(23) \quad X_M = \frac{(1 + u_2)^3 + 2u_1 u_3 u_4 + 3u_1 u_2 u_3 u_4 + u_1^2 u_3^2 u_4^2}{u_1 u_2^2 u_3 u_4}.$$

Let  $E_Q$  be the  $\mathbb{Q}[u_i, 1 \leq i \leq n]$ -submodule of  $\mathcal{F}$  generated by  $X_M, M \in \overline{\text{Mod}}_k(Q)$ , then:

**Theorem 3.4.** *For each quiver  $Q$  of type A-D-E,  $E_Q$  is a subalgebra of  $\mathcal{F}$ . It identifies with the subalgebra  $\mathcal{A}(B_Q) = \mathcal{A}(\Delta)$  of  $\mathcal{F}$ . Up to this identification, the set of cluster variables of  $\mathcal{A}(B_Q)$  is given by  $\{u_i, 1 \leq i \leq n\} \cup \{X_M, M \in \text{Ind}_k(Q)\}$ .*

Note that in particular, the algebra  $E_Q$  does not depend on  $Q$  but only on  $\Delta$ . The subsections below are devoted to the proof of this Theorem.



3.2. In order to calculate the Euler-Poincaré characteristic of Grassmannians, we will use the following classical Lemma which is an application of Lefschetz's trace formula.

**Lemma 3.5.** *Let  $X_{\mathbb{Z}}$  be a variety defined on  $\mathbb{Z}$ . We denote by  $X_{\mathbb{C}}$ , resp.  $X_{\mathbb{F}_q}$  the corresponding variety defined on  $\mathbb{C}$ , resp.  $\mathbb{F}_q$ . Suppose that there exists a polynomial  $P$  of with integral coefficients such that  $|X_{\mathbb{F}_q}| = P(q)$  for each  $q = p^m$ ,  $m \in \mathbb{N}$ . Then, the Euler-Poincaré characteristic of  $X_{\mathbb{C}}$  is given by  $\chi(X_{\mathbb{C}}) = P(1)$ .*

Note that the Grassmannians discussed above verify the hypothesis of the Lemma. Indeed, they are defined on  $\mathbb{Z}$  and their cardinality is given by sums of Hall polynomials.

3.3. We prove here that  $E_Q$  is a subalgebra of  $\mathcal{F}$ . Actually, we will prove the following:

**Proposition 3.6.** *Fix  $\underline{g}$  in  $\mathcal{Q}$ . For all  $M, N$  in  $\overline{\text{Mod}}_k(Q)$ , we have*

$$(24) \quad \chi(\text{Gr}_{\underline{g}}(M \oplus N)) = \sum_{\underline{e} + \underline{f} = \underline{g}} \chi(\text{Gr}_{\underline{e}}(M)) \chi(\text{Gr}_{\underline{f}}(N)).$$

By the bilinearity of the Euler form, this Proposition implies

**Corollary 3.7.** *For all  $M, N$  in  $\overline{\text{Mod}}_k(Q)$ , we have  $X_M X_N = X_{M \oplus N}$ . Hence,  $E_Q$  is a subalgebra of  $\mathcal{F}$ . It is the  $\mathbb{Q}$ -subalgebra generated by  $\{u_i, 1 \leq i \leq n\} \cup \{X_M, M \in \text{Ind}_k(Q)\}$ .*

By Lemma 3.5, Proposition 3.6 can be obtained by counting points on  $\mathbb{F}_q$  varieties. Set  $k = \mathbb{F}_q$ . Fix two  $kQ$ -modules  $M$  and  $N$ . Let  $\pi : M \oplus N \rightarrow N$  be the projection on the second factor.

Fix a submodule  $A$  of  $M$  and a submodule  $B$  of  $N$ . Let us introduce

$$(25) \quad \text{Gr}_{A,B}(M \oplus N) := \{L \in \text{Gr}(M \oplus N), L \cap M = A \text{ and } \pi(L) = B\}.$$

**Lemma 3.8.** *Fix a submodule  $A$  of  $M$  and a submodule  $B$  of  $N$ . Then, there exists a bijection:*

$$(26) \quad \text{Hom}_{kQ}(B, M/A) \rightarrow \text{Gr}_{A,B}(M \oplus N),$$

which maps the morphism  $f$  to  $L_f = (Id + f)(B)$  viewed as a subspace of  $M \oplus N$ .

*Proof.* We first claim that the map is well defined.

Fix a morphism  $f$  in  $\text{Hom}_{kQ}(B, M/A)$ . Then, the space  $L_f$  is the union of  $b + f(b)$ ,  $b \in B$  seen as classes modulo  $A$ . The fact that  $f$  is a morphism of  $kQ$ -modules implies that  $L_f$  is a submodule of  $M \oplus N$ . Moreover,  $\pi(L_f) = B$  by construction. So, in order to prove the claim, it remains to prove that  $L_f \cap M = A$ . Taking  $b = 0$  gives that  $A \subset L_f$ , so  $A \subset L_f \cap M$ . Conversely, if  $x \in L_f \cap M$ , then  $x \in b + f(b)$ . So,  $x = b + m$ , with  $b \in B$  and  $m \in M$ . Applying  $\pi$  gives  $b = 0$ . Hence,  $x \in f(0) = A$ .

Let us prove that the map is injective. Suppose  $f$  and  $g$  in  $\text{Hom}_{kQ}(B, M/A)$  such that  $L := (Id + f)(B) = (Id + g)(B)$ . We know that  $L \cap M = A$ , so, for each  $b$  in  $B$ , there exists an element  $x_b$  in  $L$  such that  $\pi(x_b) = b$ , and this element is unique up to an element of  $A$ . By hypothesis, for such  $b$ , there exists  $b'$  in  $B$  such that  $(Id + f)(b) = (Id + g)(b')$ . By the uniqueness, we have  $x_b + A = x_{b'} + A$ , and applying  $\pi$  gives  $b = b'$ . In the identity above, we get  $f(b) = g(b)$  as desired.

Let us prove that the map is surjective. Let  $L$  be in  $\text{Gr}_{A,B}(M \oplus N)$ . For each  $b$  in  $B$ , construct the element  $x_b$  as above. Then, we can define  $f_L : b \mapsto x_b - b \in M/A$ . It is easily verified that  $L \mapsto f_L$  provides the reverse map.  $\square$

*Proof of the Proposition.* For  $g$  in  $\mathcal{Q}$ , consider the map

$$(27) \quad \zeta_g : \text{Gr}_g(M \oplus N) \rightarrow \coprod_{\underline{e} + \underline{f} = g} \text{Gr}_{\underline{e}}(M) \times \text{Gr}_{\underline{f}}(N), L \mapsto (L \cap M, \pi(X)).$$

This map is clearly surjective:  $\zeta_g(A \oplus B) = (A, B)$ . Moreover, the Lemma above proves that  $\zeta_g^{-1}(A, B)$  has  $q^{[B, M/A]}$  elements. Now, the Proposition is a direct consequence of Lemma 3.5.

3.4. Now, we need to understand the natural set of generators of the algebra  $E_Q$ ,  $\{u_i, 1 \leq i \leq n\} \cup \{X_M, M \in \text{Ind}_k(Q)\}$  by Corollary 3.7. We want to prove that it is precisely the set of cluster variables of  $\mathcal{A}(B_Q)$ . By construction the variables  $u_i, 1 \leq i \leq n$ , are cluster variables. Now, for each indecomposable module  $M$  in  $\text{Mod}_k(Q)$ , let  $\nu(M)$  be the smallest integer such that  $\tau^{\nu(M)}M = 0$  in  $\text{Mod}_k(Q)$ . We want to prove by induction on  $\nu(M)$  that  $X_M$  is a cluster variable of  $\mathcal{A}(B_Q)$ . The case  $\nu(M) = 0$  correspond to the projective case. By Section 2.5, in this case, we have to prove:

**Lemma 3.9.** *For all  $i, 1 \leq i \leq n$ , we have*

$$(28) \quad u_i X_{P_i} = \prod_{i \rightarrow j} X_{P_j} \prod_{k \rightarrow i} u_k + 1.$$

*Proof.* Set  $\underline{d}_i := \underline{\dim}(P_i)$ . It is known that the radical  $\text{Rad } P_i$  verifies the following:

- (i)  $P_i / \text{Rad } P_i = S_i$ ,
- (ii)  $M \subset P_i \Leftrightarrow M \subset \text{Rad } P_i$  or  $M = P_i$ ,
- (iii)  $\text{Rad } P_i = \bigoplus_{i \rightarrow j} P_j$ .

By (i), we have

$$(29) \quad X_{\text{Rad } P_i} = \sum_{\underline{e}} \chi(\text{Gr}_{\underline{e}} \text{Rad } P_i) \prod_l u_l^{-\langle \underline{e}, \alpha_l \rangle - \langle \alpha_l, \underline{d}_i - \alpha_i - \underline{e} \rangle}.$$

Using the fact that the Euler form satisfies

$$(30) \quad \langle \alpha_k, \alpha_i \rangle = \begin{cases} 1 & \text{if } k = i, \\ -1 & \text{if } k \rightarrow i, \\ 0 & \text{else,} \end{cases}$$

one gets

$$(31) \quad X_{\text{Rad } P_i} = \sum_{\underline{e}} \chi(\text{Gr}_{\underline{e}} \text{Rad } P_i) \left( \prod_l u_l^{-\langle \underline{e}, \alpha_l \rangle - \langle \alpha_l, \underline{d}_i - \underline{e} \rangle} \right) \left( \prod_{k \rightarrow i} u_k^{-1} \right) u_i.$$

By (ii), we have

$$(32) \quad X_{P_i} = \sum_{\underline{e}} \chi(\text{Gr}_{\underline{e}} \text{Rad } P_i) \prod_l u_l^{-\langle \underline{e}, \alpha_l \rangle - \langle \alpha_l, \underline{d}_i - \underline{e} \rangle} + u_i^{-1}.$$

Comparing with (31) gives

$$(33) \quad X_{P_i} = X_{\text{Rad } P_i} \left( \prod_{k \rightarrow i} u_k \right) u_i^{-1} + u_i^{-1}.$$

The Lemma is now a consequence of (iii) and Corollary 3.7.  $\square$

3.5. We prove here the induction discussed in Section 3.4. What we need to prove is that for all non projective indecomposable  $kQ$ -module  $N$ , if  $X_{\tau N}$  is a cluster variable of  $\mathcal{A}(\Delta)$ , then  $X_N$  is also a cluster variable. By Section 2.5, what we have to prove is

**Proposition 3.10.** *Suppose that  $M, N$  are indecomposable modules and*

$$(34) \quad 0 \longrightarrow M \xrightarrow{\iota} B \xrightarrow{\pi} N \longrightarrow 0$$

*is an almost split exact sequence, then  $X_{M \oplus N} = X_M X_N = X_B + 1$ .*

Remark that  $M \oplus N$  and  $B$  are the middle terms  $Y$  of respectively a split sequence and an almost split sequence  $0 \rightarrow M \rightarrow Y \rightarrow N \rightarrow 0$ . The reader may view the Formula above as a “difference” between split and almost split. The proof of the Proposition is an adaptation of the proof in Section 3.3 in the almost split case.

*Proof.* Set  $\underline{m} = \underline{\dim}M$ ,  $\underline{n} = \underline{\dim}N$ . Recall that  $M = \tau N$ . We have

$$(35) \quad X_{M \oplus N} = \sum_{\underline{e}} \chi(\text{Gr}_{\underline{e}}(M \oplus N)) \prod_i u_i^{-\langle \underline{e}, \alpha_i \rangle - \langle \alpha_i, \underline{m} + \underline{n} - \underline{e} \rangle}.$$

By Lemma 3.5,  $X_{M \oplus N}$  can be seen as a polynomial of  $\mathbb{Z}[u_i^{\pm 1}][q]$  evaluated at  $q = 1$ :

$$(36) \quad X_{M \oplus N} = \left( \sum_L \prod_i u_i^{-\langle \underline{\dim}L, \alpha_i \rangle - \langle \alpha_i, \underline{m} + \underline{n} - \underline{\dim}L \rangle} \right) |_{q=1},$$

where  $L$  runs over the set of submodules of  $M \oplus N$  and  $k = \mathbb{F}_q$ . In this Formula the term corresponding to the submodule  $L = 0 \oplus N$  in  $M \oplus N$  is

$$(37) \quad \prod_i u_i^{-\langle \underline{n}, \alpha_i \rangle - \langle \alpha_i, \underline{m} \rangle} = 1,$$

by the Serre duality formula.

As in the proof of Proposition 3.6, our Proposition follows from Lemma below.  $\square$

**Lemma 3.11.** *Consider the map*

$$(38) \quad \zeta_{\underline{g}} : \text{Gr}_{\underline{g}}(B) \rightarrow \prod_{\underline{e} + \underline{f} = \underline{g}} \text{Gr}_{\underline{e}}(M) \times \text{Gr}_{\underline{f}}(N), L \mapsto (\iota^{-1}(L), \pi(L)).$$

*The fiber of a point  $(A, C)$  is empty if  $(A, C) = (0, N)$ , and is an affine space of dimension  $[C, M/A]$  if not.*

*Proof.* Let us prove the case  $(A, C) = (0, N)$ . Suppose that  $L \subset B$ ,  $\pi(L) = N$  and  $\iota^{-1}(L) = 0$ . Then,  $L \simeq N$ . The embedding provides a non trivial morphism  $N \rightarrow B$ . This is impossible because the category  $\text{Mod}_k(Q)$  is unidirected.

Suppose now  $(A, C) \neq (0, N)$ . If  $C$  is not equal to  $N$ , then, the “almost split” property implies that  $\pi$  has a section  $C \rightarrow B$ . We are in the split case and the proof is as in Section 3.3.

It remains to prove the case where  $A \neq 0$  and  $C = N$ . Since  $0 \neq A \subset M$ , we have  $[N, A]^1 = [A, M] \neq 0$ . There exists a non split exact sequence  $0 \rightarrow A \rightarrow E \rightarrow$

$N \rightarrow 0$ . We claim that we have following commutative diagram :

$$(39) \quad \begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & M & \longrightarrow & E & \xrightarrow{\mu} & C \longrightarrow 0 \\ & & \downarrow & & \downarrow \varphi & & \parallel \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C \longrightarrow 0. \end{array}$$

Indeed,  $\varphi$  exists by the almost split property and it is a monomorphism by an easy diagram chasing.

Hence,  $E \in \zeta_g^{-1}(A, N)$  and the fiber is non empty.

Let  $\mu : E \rightarrow C$  as in the diagram above. We sketch a proof of the sequel which is very similar to the proof in Section 3.3. We consider the map  $\text{Hom}(N, M/A) \rightarrow \zeta_g^{-1}(A, N)$ , by  $f \mapsto (\mu^{-1} + f)(N)$ . This map is well defined. It is a bijection and its converse map is  $L \mapsto f_L : n \mapsto \varpi^{-1}(n) - \mu^{-1}(n)$ , where  $\varpi$  is the canonical surjection  $L \rightarrow L/\iota(A)$ .  $\square$

#### 4. A CONJECTURE IN THE MULTIPLICITY-FREE CASE

4.1. Let us consider a quiver  $Q$  of finite cluster type, as introduced in [CCS04]. This can be one of the quivers of Dynkin type considered before, but many other quivers arise in the mutation process starting from a Dynkin quiver. These quivers can be defined from the matrix  $B$  of a seed  $(\underline{x}, B)$  in a simply-laced finite reduced cluster algebra by reversing the procedure used for the Dynkin quivers in Section 2.4.

Then it is expected in general and known in type  $A$  [CCS04] that there is a correspondence between cluster variables (other than the initial ones) for the seed associated to  $Q$  and indecomposables of the category of modules over  $Q$  with some relations.

Let  $M$  be an indecomposable object in the category of modules over the quiver  $Q$  with relations. Assume that  $M$  is multiplicity-free, that is  $\dim(M_i) \leq 1$  for all  $i$ . This implies that a submodule  $N$  of  $M$  is determined by its dimension vector. Abusing notation, we will denote a module  $N$  and its support by the same letter, and by  $\text{Gr}_N(M)$  the Grassmannian which in these cases is either empty or a point.

Let  $\mathbf{E}_M$  be the set of arrows of  $Q$  between vertices of  $M$  such that the associated morphism in  $M$  is zero. For  $e$  in  $\mathbf{E}_M$ , let  $s(e)$  and  $t(e)$  be the source and target of  $e$ . In type  $A$ , the set  $\mathbf{E}_M$  is always empty. We will display later an example of cluster quiver and a module over it in type  $D_4$  where  $\mathbf{E}_M$  is not empty.

*Conjecture 4.1.* The cluster variable  $X_M$  has the following expression:

$$(40) \quad \frac{1}{\prod_{i \in M} u_i} \sum_{N \subset M} \frac{\prod_{i \in N} \left( \prod_{i \rightarrow j} u_j \right) \prod_{i \in M/N} \left( \prod_{j \rightarrow i} u_j \right)}{\prod_{i \notin M, i \rightarrow M, M \rightarrow i} u_i \prod_{e \in \mathbf{E}_M} u_{s(e)} u_{t(e)}},$$

where the sum runs over submodules  $N$  of  $M$ .

One can recognize in the left factor of this Formula the expression for what should be the denominator of the cluster variable. Therefore the remaining part should be a formula for the numerator.

4.2. In this section, we check that the previous conjecture is fully compatible with the main Theorem of this article.

So let  $Q$  be an oriented Dynkin quiver and  $M$  be a multiplicity-free indecomposable module over  $Q$ . In particular the category of modules over  $Q$  is hereditary. As  $M$  is assumed indecomposable, there can not be any vanishing edge in the support of  $M$ , hence  $\mathbf{E}_M$  is empty. As  $Q$  has no loop, there is no vertex outside  $M$  with arrows in  $Q$  to  $M$  and from  $M$ . Hence Formula (40) simplifies to

$$(41) \quad \frac{1}{\prod_{i \in M} u_i} \sum_{N \subset M} \prod_{i \in N} \left( \prod_{i \rightarrow j} u_j \right) \prod_{i \in M/N} \left( \prod_{j \rightarrow i} u_j \right).$$

On the other hand, Formula (15) can be reformulated using injective and projective resolutions and the known Euler-Poincaré characteristic as the following expression:

$$(42) \quad X_M = \sum_{N \subset M} \frac{[P_1^N] [I_1^{M/N}]}{[P_0^N] [I_0^{M/N}]},$$

where

$$(43) \quad 0 \longrightarrow P_1^N \longrightarrow P_0^N \longrightarrow N \longrightarrow 0$$

and

$$(44) \quad 0 \longrightarrow M/N \longrightarrow I_0^{M/N} \longrightarrow I_1^{M/N} \longrightarrow 0$$

are projective and injective resolutions and the brackets mean replacing the direct sum of projective modules  $P_i$  or injective modules  $I_i$  by the corresponding product of variables  $u_i$ .

Let us fix a submodule  $N$  of  $M$  and denote by  $\mathbf{S}(N)$  the set of sources of the quiver underlying  $N$ .

Using the simple shapes of the module  $N$  and the quiver  $Q$ , one can describe the projective resolution, to get that

$$(45) \quad \frac{[P_1^N]}{[P_0^N]} = \frac{\prod_{i \in N} \left( \prod_{j \notin N, i \rightarrow j} u_j \right) \prod_{j \in N \setminus \mathbf{S}(N)} u_j^{\mathbf{N}(j)-1}}{\prod_{j \in \mathbf{S}(N)} u_j},$$

where  $\mathbf{N}(j)$  is the number of arrows inside  $N$  with target  $j$ .

This becomes

$$(46) \quad \prod_{i \in N} \left( \prod_{j \notin N, i \rightarrow j} u_j \right) \prod_{j \in N} u_j^{\mathbf{N}(j)-1}.$$

Then it follows that

$$(47) \quad \frac{[P_1^N]}{[P_0^N]} = \frac{\prod_{i \in N} \left( \prod_{i \rightarrow j} u_j \right)}{\prod_{j \in N} u_j}.$$

A similar argument for the injective resolution of  $M/N$  completes the check that Formula (40) is compatible with Formula (15).

4.3. Let us give two simple examples of Formula (40) outside the scope of the previous comparison. Let us consider first the following cluster quiver  $Q$  of type  $A_3$ :

$$(48) \quad \begin{array}{ccc} & 1 & \\ & \uparrow & \searrow \\ & 2 & \longleftarrow 3, \end{array}$$

with the relations  $f_{1,2} f_{2,3} = f_{2,3} f_{3,1} = f_{3,1} f_{1,2} = 0$  according to [CCS04]. Let  $M$  be the indecomposable multiplicity-free module with support  $\{1, 2\}$ . Then Formula (40) gives

$$(49) \quad X_M = \left( \frac{1}{u_1 u_2} \right) \frac{(1)(u_3 u_1) + (u_3)(u_3) + (u_3 u_2)(1)}{u_3} = \frac{u_1 + u_2 + u_3}{u_1 u_2},$$

which is the correct expression.

Let us consider now the following cluster quiver  $Q$  of type  $D_4$ :

$$(50) \quad \begin{array}{ccc} 1 & \longrightarrow & 2 \\ \downarrow & \swarrow & \downarrow \\ 3 & \longrightarrow & 4, \end{array}$$

with relations  $f_{1,2} f_{2,4} = f_{1,3} f_{3,4}$  and  $f_{2,4} f_{4,1} = f_{4,1} f_{1,2} = f_{4,1} f_{1,3} = f_{1,3} f_{3,4} = 0$ , as conjectured in [CCS04]. One can easily check that these relations provides the right category of modules for the conjectures made in [CCS04] to hold true. Let  $M$  be the following indecomposable module:

$$(51) \quad \begin{array}{ccc} k & \xrightarrow{\text{Id}} & k \\ \downarrow \text{Id} & \swarrow 0 & \downarrow \text{Id} \\ k & \xrightarrow{\text{Id}} & k. \end{array}$$

Note that the set  $E_M$  contains the diagonal arrow.

Then using Formula (40), one gets that  $X_M$  is equal to

$$\frac{(1)(u_1^2 u_2 u_3 u_4) + (u_1)(u_1^2 u_4) + (1+1)(u_1 u_4)(u_1 u_4) + (u_1 u_4^2)(u_4) + (u_1 u_2 u_3 u_4^2)(1)}{(u_1 u_2 u_3 u_4)(u_1 u_4)},$$

which simplifies to the correct expression:

$$(52) \quad X_M = \frac{(u_1 + u_4)^2 + u_2 u_3 (u_1 + u_4)}{u_1 u_2 u_3 u_4}.$$

## 5. COXETER-CONWAY FRIEZES

We give here an interpretation of Coxeter-Conway friezes [CC73], which follows directly from the main theorem.

5.1. Following Conway and Coxeter, we construct a frieze from a triangulation of the  $(n+3)$ -gon. The construction is the following.

We consider a triangulation  $T$  of the  $(n+3)$ -gon, *i.e.* a maximal set of non crossing diagonals of the polygon. Note that each maximal set has exactly  $n$  diagonal. To each vertex  $k$ ,  $k \in \mathbb{Z}/(n+3)\mathbb{Z}$ , of the polygon, let  $d_k$  be the number of diagonals

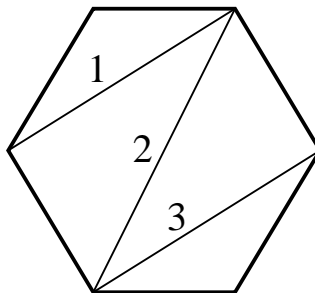


FIGURE 1.

containing the vertex  $k$ . Set  $m_k = d_k + 1$ . We construct a frieze filled with numbers in the following way.

We place a row  $R_0$  filled with 1. Then, we place above it a staggered row filled with  $m_k$ ,  $k \in \mathbb{Z}/(n+3)\mathbb{Z}$ . Then, we fill further staggered rows above such that each diamond

$$(53) \quad \begin{array}{ccc} & b & \\ a & & d \\ & c & \end{array}$$

verifies  $ad = 1 + bc$ , until we reach again a row filled with 1 only.

**Example 5.1.** Consider the triangulation  $T$  of Figure 1. Then, the corresponding frieze is

$$(54) \quad \begin{array}{cccccccc} & & 1 & & 1 & & 1 & & 1 & & \\ & & & 1 & & 3 & & 2 & & 1 & \\ & 1 & & 2 & & 5 & & 1 & & & \\ & & 1 & & 3 & & 2 & & 1 & & \\ & 1 & & 1 & & 1 & & 1 & & & \end{array}$$

Now, we make a connection with another construction. In [CCS04], the authors define for each triangulation  $T$  of the  $(n+3)$ -gon a quiver  $Q_T$  in the following way. Let  $D_i$ ,  $1 \leq i \leq n$ , be the diagonals of  $T$ . The set of vertices of the quiver  $Q_T$  is  $\{1, \dots, n\}$ , and  $i \rightarrow j$  if and only if  $D_i$  and  $D_j$  are edges of a triangle in  $T$  and if the angle from  $D_i$  to  $D_j$  is counterclockwise. We can choose the triangulation such that  $Q_T$  is any orientation of the Dynkin diagram of type  $A_n$ . In this case, we can define the categories  $\text{Mod}_k(Q)$  and  $\mathcal{C}$  as before.

For any indecomposable object  $M$  of  $\mathcal{C}$ , let  $x_M$  be the following number

$$(55) \quad x_M = X_M |_{u_1=\dots=u_n=1}.$$

We have the proposition:

**Proposition 5.2.** *Let  $T$  be a triangulation of the  $(n+3)$ -gon such the associated quiver  $Q_T$  is an orientation of the Dynkin diagram of type  $A_n$ . Let  $\gamma_T$  be obtained from the AR-quiver of  $\Gamma_{Q_T}$  by replacing each  $M$  by the number  $x_M$ . Then,  $\gamma_T$  is the Coxeter-Conway frieze associated to  $T$ .*

*Remark 5.3.* The reader can check the proposition on an example by comparing Example 3.1 and Example 5.1.

*Proof.* We sketch the proof of the proposition. In [CCS04, par. 5], the authors define a bijection from the set of diagonals of the  $(n+3)$ -gon to the set of indecomposable objects of  $\mathcal{C}$ . And moreover, the objects of the first line of  $\Gamma_{\mathcal{C}}$  correspond to diagonals of type  $[k-1, k+1]$ ,  $k \in \mathbb{Z}/(n+3)\mathbb{Z}$ . Let  $M_k$  be the object corresponding to  $[k-1, k+1]$ . By [CCS04, par. 5]  $M_k$  is an object of  $\text{Mod}_k(Q)$  if and only if the diagonal  $[k-1, k+1]$  intersects  $T$ . Moreover, in this case the quiver associated to  $M_k$  as in Example 3.2 is unidirected. So,  $M_k$  has exactly  $\dim M_k + 1 = \times_{M_k}$  submodules. Moreover, by [CCS04], the dimension of  $M_k$  is exactly the number of diagonals of  $T$  intersected by  $[k-1, k+1]$ . So,  $\times_{M_k} = m_k$ . Now, the diamond relations correspond to exchange relations as in Proposition 3.10.  $\square$

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