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# THE KORTEWEG - DE VRIES - KAWAHARA EQUATION IN A BOUNDARY DOMAIN APPROXIMATING THE QUARTER-PLANE PROBLEM

JUAN CARLOS CEBALLOS<sup>A</sup>, MAURICIO SEPÚLVEDA<sup>B</sup>, AND OCTAVIO VERA<sup>C</sup>

**ABSTRACT.** We are concerned with the initial-boundary-value problem associated to the Korteweg - de Vries - Kawahara (KdVK) equation which is a transport equation perturbed by dispersive terms of 3rd and 5th order. The (KdVK) equation appears in several fluids dynamics problems. We obtain local smoothing effects that are uniform with respect to the size of the interval. We also propose a simple finite different scheme for the problem and prove its stability. Finally we give some numerical examples.

Keywords and phrases: Evolution equations, gain in regularity, weighted Sobolev space, numerical methods.

Mathematics Subject Classification: Primary35Q53; Secondary 47J353

## 1. INTRODUCTION

We study the following equation of Korteweg-de Vries-Kawahara type in a boundary domain of  $\mathbb{R}$

$$(\text{KdVK}) = \begin{cases} u_t + \eta u_{xxxxx} + u_{xxx} + uu_x + u_x = 0, & x \in [0, L], \quad t \in [0, T], \\ u(0, t) = g(t), & t \in [0, T[, \\ u_x(L, t) = 0, \quad u_{xx}(L, t) = 0, \quad u_{xxx}(L, t) = 0, \quad u_{xxxx}(L, t) = 0, & t \in [0, T[, \\ u(x, 0) = u_0(x) \end{cases}$$

where  $L > 0$ ,  $T \in ]0, \infty[$  and  $\eta \in \mathbb{R}$  is a constant. Equation (KdVK) is the boundary version of the Korteweg-de Vries-Kawahara equation posed in  $\mathbb{R}$

$$u_t + \eta u_{xxxxx} + u_{xxx} + uu_x = 0$$

with  $-\infty < x < +\infty$ ,  $t \in [0, T]$ ,  $T$  is an arbitrary positive time, and  $\eta \in \mathbb{R}$ . The above equation is a particular case of the Benney-Lin equation derived by Benney [2] and later by Lin [8] (see also [10]). It describes one dimensional evolutions of small but finite amplitude long waves in various problems in fluid dynamics. This also can be seen as an hybrid of the well known fifth order Korteweg-de Vries(KdV) equation or Kawahara equation. For comprehensive descriptions of results pertaining to the (KdVK) equation, the reader may consult the review articles of [3, 9, 11] and references therein.

This paper is organized as follows: In section 2 outlines briefly the notation and terminology to be used subsequently and presents a statement of the principal result. In section 3 we consider the linear problem and we will find *a priori* estimates. In section 4 we obtain estimates that are independent of  $L$  for the

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non-homogeneous linear system. In section 5 we prove the existence of a time  $T_{min}$  depending only on  $\|g\|_{H^1(0,T)}$  and  $\|u_0\|_{L^2((1+x^2)dx)}$  but not on  $L$  such that  $u^L$  exists on  $[0, T_{min}]$  thanks to uniform (with respect to  $L$ ) smoothing effects. In section 6 we present a finite different scheme for the initial-boundary-value problem (KdVK); we prove its stability and present some numerical experiments. The paper concludes with some commentary concerning aspects not covered in the present study. Our main result reads as follows (Theorem 5.6)

**Theorem 1.1.** *(Existence and Uniqueness) Let  $\eta \leq 0$ ,  $u_0 \in L^2((1+x^2)dx)$ ,  $g \in H_{loc}^1(\mathbb{R}^+)$  and  $0 < L < +\infty$ . Then there exists a unique weak maximal solution of (KdVK) defined over  $[0, T_L]$ . Moreover, there exists  $T_{min} > 0$  independent of  $L$ , depending only on  $\|u_0\|_{L^2(0,L)}$  and  $\|g\|_{H^1(0,T)}$  such that  $T_L \geq T_{min}$ . The solution  $u$  depends continuously on  $u_0$  and  $g$  in the following sense: Let a sequence  $u_0^n \rightarrow u_0$  in  $L^2((1+x^2)dx)$ , let a sequence  $g^n \rightarrow g$  in  $H_{loc}^1(\mathbb{R}^+)$  and denote by  $u^n$  the solution with data  $(u_0^n, g^n)$  and  $T_L^n$  its existence time. Then*

$$\liminf_{n \rightarrow +\infty} T_L^n \geq T_L$$

and for all  $t < T_L$ ,  $u^n$  exists on the interval  $[0, T]$  if  $n$  is large enough and  $u^n \rightarrow u$  in  $\mathbb{H}_T$ .

## 2. PRELIMINARIES

Let  $\Omega$  a bounded domain in  $\mathbb{R}$ . For any real  $p$  in the range  $[1, \infty]$ ,  $L^p(\Omega)$  denotes the collection of real-valued Lebesgue measurable  $p$ th-power absolutely integrable functions defined on  $\Omega$ . As usual,  $L^\infty(\Omega)$  denotes the essentially bounded real-valued functions defined on  $\Omega$ . These spaces get their usual norms,

$$\|u\|_{L^p(\Omega)} = \left( \int_{\Omega} |u(x)|^p dx \right)^{1/p} \quad \text{for } 1 \leq p < +\infty$$

and

$$\|u\|_{L^\infty(\Omega)} = \operatorname{ess\,sup}_{x \in \Omega} |u(x)|$$

For an arbitrary Banach space  $X$ , the associated norm will be denoted  $\|\cdot\|_X$ . The following spaces will intervene in the subsequent analysis. For any real  $p$  in the range  $[1, \infty]$  and  $-\infty \leq a < b \leq +\infty$  the notation  $L^p(a, b; X)$  denotes the Banach space of measurable functions  $u : (a, b) \rightarrow X$  whose norms are  $p$ th-power integrable(essentially bounded if  $p = +\infty$ ). These spaces get their norms,

$$\|u\|_{L^p(a, b; X)}^p = \int_a^b \|u(\cdot, t)\|_X^p dt \quad \text{for } 1 \leq p < +\infty.$$

and

$$\|u\|_{L^\infty(a, b; X)} = \operatorname{ess\,sup}_{t \in (a, b)} \|u(\cdot, t)\|_X \quad \text{for } p = +\infty.$$

We assume that for  $0 < L < +\infty$  the initial data  $u_0$  verifies  $u_0 \in L^2(0, L)$  and  $x u_0 \in L^2(0, L)$ . Then we define

$$\|u_0\|_{L^2((1+x^2)dx)} = \sqrt{\int_0^L u_0^2 (1+x^2) dx}.$$

We introduce the spaces  $\mathbb{E}$  and  $\mathbb{H}$  defined by

$$\mathbb{E} := \{f \in L^1(0, T; L^2((1+x^2)dx)), \sqrt{t}f \in L^2(0, T; L^2((1+x^2)dx))\}.$$

This space is endowed with the norm

$$\begin{aligned} \|f\|_{\mathbf{E}} &= \int_0^T \sqrt{\int_0^L f^2(x, t)(1+x^2)dx} dt + \sqrt{\int_0^T \int_0^L t f^2(x, t)(1+x^2)dx} dt \\ &= \int_0^T \|f\|_{L^2((1+x^2)dx)} dt + \|\sqrt{t}f\|_{L^2(0, T; L^2((1+x^2)dx))} \\ &= \|f\|_{L^1(0, T; L^2((1+x^2)dx))} + \|\sqrt{t}f\|_{L^2(0, T; L^2((1+x^2)dx))} \end{aligned}$$

Let  $T > 0$ . We define

$$\begin{aligned} \mathbf{H}_T &:= \{u \in C([0, T]; L^2((1+x^2)dx)), u_x \in C([0, T]; L^2((1+x)dx)), \\ &\quad \sqrt{t}u_x \in L^2([0, T]; L^2((1+x)dx)), \sqrt{t}u_{xx} \in L^2([0, T]; L^2(0, L)), \\ &\quad \sqrt{t}u_{xxx} \in L^2([0, T]; L^2(0, L))\}. \end{aligned}$$

This space is endowed with the norm

$$\begin{aligned} \|u\|_{\mathbf{H}} &= \|u\|_{L^\infty(0, T; L^2((1+x^2)dx))} + \|u_x\|_{L^2(0, T; L^2((1+x)dx))} \\ &\quad + \|\sqrt{t}u_x\|_{L^\infty(0, T; L^2((1+x)dx))} + \|\sqrt{t}u_{xx}\|_{L^2(0, T; L^2(0, L))} + \|\sqrt{t}u_{xxx}\|_{L^2(0, T; L^2(0, L))}. \end{aligned}$$

In the chapter 5, we need to use the infinitesimal generator of a  $C_0$  semi-group  $S(t)$  (see the classical reference [1]). In this sense, we consider the generic inhomogeneous initial value problem

$$\begin{cases} \frac{\partial u}{\partial t}(t) = Au(t) + f(t), & t > 0, \\ u(0) = x, \end{cases}$$

where  $f : [0, T] \rightarrow X$  and  $A$  is the infinitesimal generator of a  $C_0$  semi-group  $S(t)$  so that the corresponding homogeneous equation, *i.e.*, the equation with  $f \equiv 0$ , has a unique solution for every initial value  $x \in D(A)$ .

We need also some well-known inequalities given by :

**Lemma 2.1.** *Let  $u \in L^2((1+x^2)dx)$ ,  $v \in H^1(0, T)$ ,  $w \in L^2(0, T; L^2((1+x^2)dx))$ , and  $z \in H^1(0, L)$  with  $z(0) = 0$ . Then, we have the following inequalities*

$$\|u\|_{L^2((1+x^2)dx)}^2 \leq 3 \|u\|_{L^2((1+x^2)dx)}^2 \quad (2.1)$$

$$\|v\|_{L^\infty(0, T)} \leq (c + \sqrt{T}) \|v\|_{H^1(0, T)} \quad (2.2)$$

$$\|w\|_{L^1(0, T; L^2((1+x^2)dx))} \leq \sqrt{T} \|w\|_{L^2(0, T; L^2((1+x^2)dx))} \quad (2.3)$$

$$\|z\|_{L^\infty(dx)} \leq 2 \|z\|_{L^2(dx)} \|z_x\|_{L^2(dx)} \quad (2.4)$$

Finally, we note by  $c$ , a generic constants, not necessarily the same at each occasion, which depends in an increasing way on the indicated quantities.

### 3. UNIFORM ESTIMATES ON THE SOLUTIONS TO THE LINEAR HOMOGENEOUS PROBLEM

In this section we consider the linear problem for the equation of Korteweg-de Vries-Kawahara type

$$(\text{KdVK})_L \begin{cases} u_t + \eta u_{xxxxx} + u_{xxx} + u_x = 0, \\ u(0, t) = 0, \quad t \in [0, T[, \\ u_x(L, t) = 0, \quad u_{xx}(L, t) = 0, \quad u_{xxx}(L, t) = 0, \quad u_{xxxx}(L, t) = 0, \quad t \in [0, T[, \\ u(x, 0) = u_0(x) \end{cases}$$

with  $x \in [0, L]$ ,  $t \in [0, T]$ ,  $T$  an arbitrary positive time, and  $\eta \in \mathbb{R}$  is a constant.

**Proposition 3.1.** *Let  $u_0, u_{0x} \in L^2(0, L)$  and  $\eta \leq 0$ . Then there exists a continuous  $t \rightarrow c(t)$  such that*

$$\|u\|_{L^\infty(0, T; L^2((1+x^2)dx))} \leq c(T) \|u_0\|_{L^2(0, L)} \quad (3.1)$$

$$\|u_x\|_{L^\infty(0, T; L^2((1+x)dx))} \leq c(T) \|u_{0x}\|_{L^2(0, L)} \quad (3.2)$$

$$\|u_x\|_{L^2(0, T; L^2(0, L))} \leq c(T) \|u_0\|_{L^2(0, L)} \quad (3.3)$$

$$\|u(L, t)\|_{L^2(0, T)} \leq c(T) \|u_0\|_{L^2(0, L)} \quad (3.4)$$

$$\|u\|_{L^2(0, L)} \leq c(T) \|u_0\|_{L^2(0, L)} \quad (3.5)$$

$$\|u_x(0, t)\|_{L^2(0, T)} \leq c(T) \|u_0\|_{L^2(0, L)} \quad (3.6)$$

$$\|u_x(L, t)\|_{L^2(0, T)} \leq c(T) \|u_0\|_{L^2(0, L)} \quad (3.7)$$

$$\|u_x\|_{L^2(0, L)} \leq c(T) \|u_{0x}\|_{L^2(0, L)} \quad (3.8)$$

$$\|u_x(0, t)\|_{L^2(0, T)} \leq c(T) \|u_{0x}\|_{L^2(0, L)} \quad (3.9)$$

$$\|u_{xx}(0, t)\|_{L^2(0, T)} \leq c(T) \|u_{0x}\|_{L^2(0, L)} \quad (3.10)$$

$$\|\sqrt{x} u_x\|_{L^2(0, L)} \leq c(T) \|u_{0x}\|_{L^2(0, L)} \quad (3.11)$$

$$\|u_{xx}\|_{L^2(0, T; L^2((0, L)))} \leq c(T) \|u_{0x}\|_{L^2(0, L)} \quad (3.12)$$

$$\|\sqrt{t} u_x\|_{L^\infty(0, T; L^2((0, L)))} \leq c(T) \|u_0\|_{L^2(0, L)} \quad (3.13)$$

$$\|\sqrt{t} u_x(0, t)\|_{L^2(0, T)} \leq c(T) \|u_0\|_{L^2(0, L)} \quad (3.14)$$

$$\|\sqrt{t} u_{xx}(0, t)\|_{L^2(0, T)} \leq c(T) \|u_0\|_{L^2(0, L)} \quad (3.15)$$

$$\|\sqrt{t} u_x\|_{L^\infty(0, T; L^2((1+x)dx))} \leq c(T) \|u_0\|_{L^2(0, L)} \quad (3.16)$$

$$\|\sqrt{t} u_{xx}\|_{L^2(0, T; L^2((0, L)))} \leq c(T) \|u_0\|_{L^2(0, L)} \quad (3.17)$$

Additionally, if  $\eta < 0$ , we have

$$\|u_{xx}\|_{L^2(0, T; L^2(0, L))} \leq c(T) \|u_0\|_{L^2(0, L)} \quad (3.18)$$

$$\|u_{xxx}\|_{L^2(0, T; L^2(0, L))} \leq c(T) \|u_0\|_{L^2(0, L)} \quad (3.19)$$

$$\|u_{xx}(0, t)\|_{L^2(0, T)} \leq c(T) \|u_0\|_{L^2(0, L)} \quad (3.20)$$

$$\|u_{xxx}(0, t)\|_{L^2(0, T)} \leq c(T) \|u_{0x}\|_{L^2(0, L)} \quad (3.21)$$

$$\|u_{xxx}\|_{L^2(0, T; L^2((0, L)))} \leq c(T) \|u_{0x}\|_{L^2(0, L)} \quad (3.22)$$

$$\|\sqrt{t} u_{xxx}(0, t)\|_{L^2(0, T)} \leq c(T) \|u_0\|_{L^2(0, L)} \quad (3.23)$$

$$\|\sqrt{t} u_{xxx}\|_{L^2(0, T; L^2((0, L)))} \leq c(T) \|u_0\|_{L^2(0, L)} \quad (3.24)$$

*Proof.* Multiplying  $(\text{KdVK})_L$  by  $u$  and integrating over  $x \in (0, L)$  we have

$$\int_0^L uu_t dx + \eta \int_0^L uu_{xxxx} dx + \int_0^L uu_{xxx} dx + \int_0^L uu_x dx = 0$$

each term is treated separately, integrating by parts we obtain

$$\frac{d}{dt} \int_0^L u^2 dx + 2\eta u_x(0, t) u_{xxx}(0, t) - \eta u_{xx}^2(0, t) + u_x^2(0, t) + u^2(L, t) = 0$$

where

$$\frac{d}{dt} \int_0^L u^2 dx + u^2(L, t) + u_x^2(0, t) - \eta [u_x(0, t) - u_{xxx}(0, t)]^2 - \eta u_{xx}^2(0, t) = -\eta [u_x^2(0, t) + u_{xxx}^2(0, t)]$$

hence

$$\frac{d}{dt} \int_0^L u^2 dx + u^2(L, t) + u_x^2(0, t) - \eta u_{xx}^2(0, t) \leq -\eta [u_x^2(0, t) + u_{xxx}^2(0, t)]. \quad (3.25)$$

Integrating (3.25) over  $(0, t)$  we have

$$\begin{aligned} & \int_0^L u^2 dx + \int_0^t u^2(L, s) ds + \int_0^t u_x^2(0, s) ds - \eta \int_0^t u_{xx}^2(0, s) ds \leq \int_0^L u_0^2 dx \\ & - \eta \left[ \int_0^t u_x^2(0, s) ds + \int_0^t u_{xxx}^2(0, s) ds \right] \end{aligned}$$

then

$$\begin{aligned} & \|u\|_{L^2(0, L)}^2 + \|u_x(0, t)\|_{L^2(0, T)}^2 + \|u(L, t)\|_{L^2(0, T)}^2 - \eta \|u_{xx}(0, t)\|_{L^2(0, T)}^2 \leq \|u_0\|_{L^2(0, L)}^2 \\ & - \eta [L \|u_x(0, t)\|_{L^\infty(0, T)}^2 + L \|u_{xxx}(0, t)\|_{L^\infty(0, T)}^2] \leq c(t) \|u_0\|_{L^2(0, L)}^2 \end{aligned}$$

hence

$$\|u\|_{L^2(0, L)}^2 + \|u_x(0, t)\|_{L^2(0, T)}^2 + \|u(L, t)\|_{L^2(0, T)}^2 - \eta \|u_{xx}(0, t)\|_{L^2(0, T)}^2 \leq c(t) \|u_0\|_{L^2(0, L)}^2. \quad (3.26)$$

Therefore we obtain (3.5), (3.6), (3.7), and (3.20).

By other hand, multiplying  $(\text{KdVK})_L$  by  $xu$  and integrating over  $x \in (0, L)$  we obtain

$$\int_0^L xuu_t dx + \eta \int_0^L xuu_{xxxx} dx + \int_0^L xuu_{xxx} dx + \int_0^L xuu_x dx = 0$$

each term is treated separately, integrating by parts we obtain

$$\frac{d}{dt} \int_0^L xu^2 dx - \int_0^L u^2 dx + 3 \int_0^L u_x^2 dx + Lu^2(L, t) - 5\eta \int_0^L u_{xx}^2 dx - 4\eta u_x(0, t)u_{xx}(0, t) = 0$$

there

$$\begin{aligned} & \frac{d}{dt} \int_0^L xu^2 dx - \int_0^L u^2 dx + 3 \int_0^L u_x^2 dx + Lu^2(L, t) - 5\eta \int_0^L u_{xx}^2 dx \\ & - 2\eta [u_x(0, t) + u_{xx}(0, t)]^2 = -\eta [u_x^2(0, t) + u_{xx}^2(0, t)] \end{aligned}$$

where

$$\frac{d}{dt} \int_0^L xu^2 dx + 3 \int_0^L u_x^2 dx + Lu^2(L, t) - 5\eta \int_0^L u_{xx}^2 dx \leq \int_0^L u^2 dx - \eta [u_x^2(0, t) + u_{xx}^2(0, t)]$$

and using (3.5) we have

$$\frac{d}{dt} \int_0^L xu^2 dx + 3 \int_0^L u_x^2 dx + Lu^2(L, t) - 5\eta \int_0^L u_{xx}^2 dx \leq c(t) \|u_0\|_{L^2(0, L)}^2 - \eta [u_x^2(0, t) + u_{xx}^2(0, t)] \quad (3.27)$$

Integrating (3.27) over  $(0, t)$  and using similar calculus to above we have

$$\begin{aligned} & \int_0^L xu^2 dx + 3 \int_0^t \int_0^L u_x^2 dx ds + L \int_0^t u^2(L, s) ds - 5\eta \int_0^t \int_0^L u_{xx}^2 dx ds \leq \int_0^L xu_0^2 dx \\ & + \|u_0\|_{L^2(0, L)}^2 \int_0^t c(s) ds - \eta \left[ \int_0^t u_x^2(0, s) ds + \int_0^t u_{xx}^2(0, s) ds \right] \\ & \leq L \|u_0\|_{L^2(0, L)}^2 + \|u_0\|_{L^2(0, L)}^2 \int_0^t c(s) ds - \eta \left[ \int_0^t u_x^2(0, s) ds + \int_0^t u_{xx}^2(0, s) ds \right] \leq c(t) \|u_0\|_{L^2(0, L)}^2 \end{aligned}$$

then

$$\int_0^L xu^2 dx + 3\|u_x\|_{L^2(0,T;L^2(0,L))}^2 + L\|u(L,t)\|_{L^2(0,T)}^2 - 5\eta\|u_{xx}\|_{L^2(0,T;L^2(0,L))}^2 \leq c(t)\|u_0\|_{L^2(0,L)}^2. \quad (3.28)$$

This way (3.3), (3.4), and (3.18) follows.

Multiplying  $(\text{KdVK})_L$  by  $u_{xx}$  and integrating over  $x \in (0, L)$  we have

$$\int_0^L u_{xx}u_t dx + \eta \int_0^L u_{xx}u_{xxxxx} dx + \int_0^L u_{xx}u_{xxx} dx + \int_0^L u_{xx}u_x dx = 0$$

each term is treated separately, integrating by parts we obtain

$$\frac{d}{dt} \int_0^L u_x^2 dx + 2\eta u_{xx}(0,t)u_{xxx}(0,t) - \eta u_{xxx}^2(0,t) + u_x^2(0,t) + u_x^2(L,t) = 0$$

hence

$$\begin{aligned} & \frac{d}{dt} \int_0^L u_x^2 dx + u_x^2(0,t) + u_x^2(L,t) - \eta u_{xxx}^2(0,t) - \eta [u_{xx}(0,t) - u_{xxxx}(0,t)]^2 \\ & = -\eta [u_{xx}^2(0,t) + u_{xxxx}^2(0,t)] \end{aligned}$$

then

$$\frac{d}{dt} \int_0^L u_x^2 dx + u_x^2(0,t) + u_x^2(L,t) - \eta u_{xxx}^2(0,t) \leq -\eta [u_{xx}^2(0,t) + u_{xxxx}^2(0,t)]. \quad (3.29)$$

Integrating (3.29) over  $(0, t)$  we have

$$\begin{aligned} & \int_0^L u_x^2 dx + \int_0^t u_x^2(0,s) ds + \int_0^t u_x^2(L,s) ds - \eta \int_0^t u_{xxx}^2(0,s) ds \leq \int_0^t u_x^2(0,s) ds \\ & - \eta \left[ \int_0^t u_{xx}^2(0,s) ds + \int_0^t u_{xxxx}^2(0,s) ds \right]. \end{aligned}$$

then

$$\begin{aligned} & \int_0^L u_x^2 dx + \|u_x(0,t)\|_{L^2(0,T)}^2 + \|u_x(L,t)\|_{L^2(0,T)}^2 - \eta \|u_{xxx}(0,t)\|_{L^2(0,T)}^2 \leq \|u_x(0)\|_{L^2(0,L)}^2 \\ & - \eta [T \|u_{xx}(0,t)\|_{L^\infty(0,T)}^2 + T \|u_{xxxx}(0,t)\|_{L^\infty(0,T)}^2] \leq c(t) \|u_x(0)\|_{L^2(0,L)}^2 \end{aligned}$$

therefore

$$\begin{aligned} & \int_0^L u_x^2 dx + \|u_x(0,t)\|_{L^2(0,T)}^2 + \|u_x(L,t)\|_{L^2(0,T)}^2 \\ & - \eta \|u_{xxx}(0,t)\|_{L^2(0,T)}^2 \leq c(t) \|u_x(0)\|_{L^2(0,L)}^2. \end{aligned} \quad (3.30)$$

This way (3.8), (3.9), (3.10), (3.21) follows.

Differentiating  $(\text{KdVK})_L$  in the variable  $x$  we obtain

$$u_{xt} + \eta u_{xxxxx} + u_{xxx} + u_{xx} = 0$$

Multiplying by  $xu_x$  and integrating over  $x \in (0, L)$  we obtain

$$\int_0^L xu_x u_{xt} dx + \eta \int_0^L xu_x u_{xxxxx} dx + \int_0^L xu_x u_{xxx} dx + \int_0^L xu_x u_{xx} dx = 0$$

each term is treated separately, integrating by parts we obtain

$$\begin{aligned} & \frac{d}{dt} \int_0^L x u_x^2 dx + 2u_x(0, t)u_{xx}(0, t) - \int_0^L u_x^2 dx + 3 \int_0^L u_{xx}^2 dx \\ & + 2\eta u_x(0, t)u_{xxxx}(0, t) - 4\eta u_{xx}(0, t)u_{xxx}(0, t) - 5\eta \int_0^L u_{xxx}^2 dx = 0 \end{aligned}$$

then

$$\begin{aligned} & \frac{d}{dt} \int_0^L x u_x^2 dx + 2u_x(0, t)u_{xx}(0, t) - \int_0^L u_x^2 dx + 3 \int_0^L u_{xx}^2 dx \\ & - \eta[u_x(0, t) - u_{xxxx}(0, t)]^2 - 2\eta[u_{xx}(0, t) - u_{xxx}(0, t)]^2 - 5\eta \int_0^L u_{xxx}^2 dx \\ & = -\eta[u_x^2(0, t) + u_{xxxx}^2(0, t)] - 2\eta[u_{xx}^2(0, t) + u_{xxx}^2(0, t)] \end{aligned}$$

hence

$$\begin{aligned} & \frac{d}{dt} \int_0^L x u_x^2 dx + [u_x(0, t) + u_{xx}(0, t)]^2 - \int_0^L u_x^2 dx + 3 \int_0^L u_{xx}^2 dx \\ & - \eta[u_x(0, t) - u_{xxxx}(0, t)]^2 - 2\eta[u_{xx}(0, t) - u_{xxx}(0, t)]^2 - 5\eta \int_0^L u_{xxx}^2 dx \\ & = -\eta[u_x^2(0, t) + u_{xxxx}^2(0, t)] - 2\eta[u_{xx}^2(0, t) + u_{xxx}^2(0, t)] + u_x^2(0, t) + u_{xx}^2(0, t). \end{aligned}$$

Performing straightforward calculus

$$\begin{aligned} & \frac{d}{dt} \int_0^L x u_x^2 dx - \int_0^L u_x^2 dx + 3 \int_0^L u_{xx}^2 dx - 5\eta \int_0^L u_{xxx}^2 dx \leq -\eta[u_x^2(0, t) + u_{xxxx}^2(0, t)] \\ & - 2\eta[u_{xx}^2(0, t) + u_{xxx}^2(0, t)] + u_x^2(0, t) + u_{xx}^2(0, t) \end{aligned}$$

hence

$$\begin{aligned} & \frac{d}{dt} \int_0^L x u_x^2 dx + 3 \int_0^L u_{xx}^2 dx - 5\eta \int_0^L u_{xxx}^2 dx \leq \int_0^L u_x^2 dx - \eta[u_x^2(0, t) + u_{xxxx}^2(0, t)] \\ & - 2\eta[u_{xx}^2(0, t) + u_{xxx}^2(0, t)] + u_x^2(0, t) + u_{xx}^2(0, t) \end{aligned}$$

then

$$\begin{aligned} & \frac{d}{dt} \int_0^L x u_x^2 dx + 3 \int_0^L u_{xx}^2 dx - 5\eta \int_0^L u_{xxx}^2 dx \leq \|u_x\|_{L^2(0, L)}^2 - \eta[u_x^2(0, t) + u_{xxxx}^2(0, t)] \\ & - 2\eta[u_{xx}^2(0, t) + u_{xxx}^2(0, t)] + u_x^2(0, t) + u_{xx}^2(0, t). \end{aligned} \quad (3.31)$$

Integrating (3.31) over  $(0, t)$  and using 3.8) we obtain

$$\int_0^L x u_x^2 dx + 3 \int_0^t \int_0^L u_{xx}^2 dx ds - 5\eta \int_0^t \int_0^L u_{xxx}^2 dx ds \leq c(t) \|u_{0x}\|_{L^2(0, L)}^2$$

where

$$\int_0^L x u_x^2 dx + 3 \|u_{xx}\|_{L^2(0, T; L^2(0, L))}^2 - 5\eta \|u_{xxx}\|_{L^2(0, T; L^2(0, L))}^2 \leq c(t) \|u_{0x}\|_{L^2(0, L)}^2. \quad (3.32)$$

and (3.11), (3.12), (3.22) follows.



Multiplying  $(\text{KdVK})_L$  by  $x^2u$  and integrating over  $x \in (0, L)$  we have

$$\int_0^L x^2 u u_t dx + \eta \int_0^L x^2 u u_{xxxx} dx + \int_0^L x^2 u u_{xxx} dx + \int_0^L x^2 u u_x dx = 0$$

each term is treated separately, integrating by parts we obtain

$$\frac{d}{dt} \int_0^L x^2 u^2 dx + 2u^2(L, t) + L^2 u^2(L, t) + 6 \int_0^L x u_x^2 dx - 2 \int_0^L x u^2 dx - 10\eta \int_0^L x u_{xx}^2 dx + 6\eta u_x^2(0, t) = 0$$

where

$$\begin{aligned} & \frac{d}{dt} \int_0^L x^2 u^2 dx + 2u^2(L, t) + L^2 u^2(L, t) + 6 \int_0^L x u_x^2 dx - 10\eta \int_0^L x u_{xx}^2 dx = -2 \int_0^L x u^2 dx - 6\eta u_x^2(0, t) \\ & \leq 2L \int_0^L u^2 dx - 6\eta u_x^2(0, t). \end{aligned} \quad (3.33)$$

Integrating (3.33) over  $(0, t)$  and using (3.5) we obtain

$$\begin{aligned} & \int_0^L x^2 u^2 dx + (2 + L^2) \|u(L, t)\|_{L^2(0, T)}^2 + 6 \int_0^t \int_0^L x u_x^2 dx ds \\ & - 10\eta \int_0^t \int_0^L x u_{xx}^2 dx ds \leq c(t) \|u_0\|_{L^2(0, L)}^2. \end{aligned} \quad (3.34)$$

This way

$$\|xu\|_{L^2(0, L)} \leq c(t) \|u_0\|_{L^2(0, L)} \quad (3.35)$$

$$\|u(L, t)\|_{L^2(0, T)} \leq c(t) \|u_0\|_{L^2(0, L)} \quad (3.36)$$

$$\|\sqrt{x} u_x\|_{L^2(0, T; L^2(0, L))} \leq c(t) \|u_0\|_{L^2(0, L)} \quad (3.37)$$

$$\|\sqrt{x} u_{xx}\|_{L^2(0, T; L^2(0, L))} \leq c(t) \|u_0\|_{L^2(0, L)} \quad (\text{if } \eta < 0) \quad (3.38)$$

Adding (3.25) with (3.33), using (3.5) and performing straightforward calculus we obtain

$$\begin{aligned} & \|u\|_{L^2((1+x^2)dx)}^2 + (3 + L^2) \|u(L, t)\|_{L^2(0, T)}^2 + \|u_x(0, t)\|_{L^2(0, T)}^2 + 6 \int_0^t \int_0^L x u_x^2 dx ds \\ & - \eta \|u_{xx}(0, t)\|_{L^2(0, T)}^2 - 10\eta \int_0^t \int_0^L x u_{xx}^2 dx ds \leq c(t) \|u_0\|_{L^2(0, L)}^2 \end{aligned} \quad (3.39)$$

and (3.1) follows. Adding (3.30) with (3.32), using (3.8) and performing straightforward calculus we have

$$\begin{aligned} & \|u_x\|_{L^2((1+x)dx)}^2 + \|u_x(0, t)\|_{L^2(0, T)}^2 + \|u_{xx}(0, t)\|_{L^2(0, T)}^2 - \eta \|u_{xxx}(0, t)\|_{L^2(0, T)}^2 \\ & + 3 \|u_{xx}\|_{L^2(0, T; L^2(0, L))}^2 - 5\eta \|u_{xxx}\|_{L^2(0, T; L^2(0, L))}^2 \leq c(t) \|u_0\|_{L^2(0, L)}^2 \end{aligned} \quad (3.40)$$

and (3.2), (3.19) follows.

Multiplying (3.29) by  $t$  and integrating the resulting expression in the temporal variable over  $(0, t)$  and using (3.3) we have

$$\begin{aligned} & t \frac{d}{dt} \int_0^L u_x^2 dx + t u_x^2(0, t) + t u_{xx}^2(0, t) - \eta t u_{xxx}^2(0, t) \leq -\eta [t u_{xx}^2(0, t) + t u_{xxx}^2(0, t)] \quad / \int_0^t ds \\ & \int_0^t s \left[ \frac{d}{ds} \int_0^L u_x^2 dx \right] ds + \int_0^t s u_x^2(0, s) ds + \int_0^t s u_{xx}^2(0, s) ds - \eta \int_0^t s u_{xxx}^2(0, s) ds \\ & \leq -\eta \int_0^t s u_{xx}^2(0, s) ds - \eta \int_0^t s u_{xxx}^2(0, s) ds \end{aligned}$$

then

$$\begin{aligned} & t \int_0^L u_x^2 dx - \int_0^t \left[ \int_0^L u_x^2 dx \right] ds + \int_0^t s u_x^2(0, s) ds + \int_0^t s u_{xx}^2(0, s) ds - \eta \int_0^t s u_{xxx}^2(0, s) ds \\ & \leq -\eta \int_0^t s [u_{xx}^2(0, s) + u_{xxxx}^2(0, s)] ds \end{aligned}$$

hence

$$\begin{aligned} & t \int_0^L u_x^2 dx + \int_0^t s u_x^2(0, s) ds + \int_0^t s u_{xx}^2(0, s) ds - \eta \int_0^t s u_{xxx}^2(0, s) ds \\ & \leq \int_0^t \int_0^L u_x^2 dx ds - \eta \int_0^t s [u_{xx}^2(0, s) + u_{xxxx}^2(0, s)] ds \end{aligned} \quad (3.41)$$

where

$$\begin{aligned} & t \int_0^L u_x^2 dx + \|\sqrt{t} u_x(0, t)\|_{L^2(0, T)}^2 + \|\sqrt{t} u_{xx}(0, t)\|_{L^2(0, T)}^2 - \eta \|\sqrt{t} u_{xxx}(0, t)\|_{L^2(0, T)}^2 \\ & \leq c(t) \|u_0\|_{L^2(0, L)}^2 \end{aligned} \quad (3.42)$$

and (3.13) follows.

Taking the product of (3.31) by  $t$ , integrating the resulting expression in the temporal variable over  $(0, t)$ , using (3.8) and integrating by parts we have

$$t \int_0^L x u_x^2 dx + 3 \|\sqrt{t} u_{xx}\|_{L^2(0, T; L^2(0, L))}^2 - 5 \eta \|\sqrt{t} u_{xxx}\|_{L^2(0, T; L^2(0, L))}^2 \leq c(t) \|u_0\|_{L^2(0, L)}^2 \quad (3.43)$$

Adding (3.42) with (3.43)

$$\begin{aligned} & t \int_0^L u_x^2(1+x) dx + \|\sqrt{t} u_x(0, t)\|_{L^2(0, T)}^2 + \|\sqrt{t} u_{xx}(0, t)\|_{L^2(0, T)}^2 - \eta \|\sqrt{t} u_{xxx}(0, t)\|_{L^2(0, T)}^2 \\ & + 3 \|\sqrt{t} u_{xx}\|_{L^2(0, T; L^2(0, L))}^2 - 5 \eta \|\sqrt{t} u_{xxx}\|_{L^2(0, T; L^2(0, L))}^2 \leq c(t) \|u_0\|_{L^2(0, L)}^2 \end{aligned}$$

where

$$\begin{aligned} & \|\sqrt{t} u_x\|_{L^2((1+x)dx)}^2 + \|\sqrt{t} u_x(0, t)\|_{L^2(0, T)}^2 + \|\sqrt{t} u_{xx}(0, t)\|_{L^2(0, T)}^2 - \eta \|\sqrt{t} u_{xxx}(0, t)\|_{L^2(0, T)}^2 \\ & + 3 \|\sqrt{t} u_{xx}\|_{L^2(0, T; L^2(0, L))}^2 - 5 \eta \|\sqrt{t} u_{xxx}\|_{L^2(0, T; L^2(0, L))}^2 \leq c(t) \|u_0\|_{L^2(0, L)}^2. \end{aligned} \quad (3.44)$$

and (3.14), (3.15), (3.23), (3.16), (3.17), (3.24) follows.  $\square$

**Remark 3.1.** *Let us recall how one obtains (3.2) which is the more surprising estimate since it gives a smoothing effect.*

#### 4. NON-HOMOGENEOUS LINEAR ESTIMATES

Thanks to standard duality techniques, we obtain estimates that are independent of  $L$  for the non homogeneous linear system

$$(KdVK)_{NH} \begin{cases} v_t + \eta v_{xxxxx} + v_{xxx} + v_x = f(x, t), & x \in [0, L], \quad t \in [0, T[, \\ v(0, t) = 0, & t \in [0, T[, \\ v_x(L, t) = 0, \quad v_{xx}(L, t) = 0, \quad v_{xxx}(L, t) = 0, \quad v_{xxxx}(L, t) = 0, & t \in [0, T[, \\ v(x, 0) = v_0(x) \end{cases}$$

**Proposition 4.1.** *Let  $v_0 \in L^2(0, L)$ ,  $f \in \mathbf{E}$  and  $\eta \leq 0$ . Then there exists a continuous  $t \rightarrow c(t)$  such that*

$$\|v\|_{L^\infty(0, T; L^2((1+x^2)dx))} \leq c(T) \|f\|_{L^1(0, T; L^2((1+x^2)dx))} \quad (4.1)$$

$$\|v_x\|_{L^\infty(0, T; L^2((1+x)dx))} \leq c(T) \|f\|_{L^1(0, T; L^2((1+x^2)dx))} \quad (4.2)$$

$$\|v_x\|_{L^2(0, T)} \leq c(T) \|f\|_{L^1(0, T; L^2((1+x^2)dx))} \quad (4.3)$$

$$\|\sqrt{t} v_x\|_{L^\infty(0, T; L^2(1+x)dx)} \leq c(T) \|f\|_{\mathbf{E}} \quad (4.4)$$

$$\|\sqrt{t} v_x(0, t)\|_{L^2(0, T)} \leq c(T) \|f\|_{\mathbf{E}} \quad (4.5)$$

$$\|\sqrt{t} v_{xx}(0, t)\|_{L^2(0, T)} \leq c(T) \|f\|_{\mathbf{E}} \quad (4.6)$$

$$\|\sqrt{t} v_x\|_{L^2(0, T; L^2(0, T))} \leq c(T) \|f\|_{\mathbf{E}} \quad (4.7)$$

$$\|v_x\|_{L^\infty(0, T; L^2((1+x)dx))} \leq c(T) \|f\|_{L^2(0, T; L^2((1+x^2)dx))} \quad (4.8)$$

$$\begin{aligned} \|v_x(0, t)\|_{L^2(0, T)}^2 + \|v_{xx}(0, t)\|_{L^2(0, T)}^2 \\ - \eta \|v_{xxx}(0, t)\|_{L^2(0, T)}^2 \leq c(T) \|f\|_{L^2(0, T; L^2((1+x^2)dx))}^2 \end{aligned} \quad (4.9)$$

$$\|v_{xx}\|_{L^2(0, T; L^2(0, L))} \leq c(T) \|f\|_{L^2(0, T; L^2((1+x^2)dx))} \quad (4.10)$$

*Proof.* Adding (3.25) with (3.33) we have

$$\begin{aligned} & \frac{d}{dt} \int_0^L v^2(1+x^2)dx + (3+L)v^2(L, t) + v_x^2(0, t) + 6 \int_0^L x v_x^2 dx - \eta v_{xx}(0, t) - 10\eta \int_0^L x v_{xx}^2 dx \\ & \leq \int_0^L v f(1+x^2)dx - \eta [v_x^2(0, t) + v_{xxx}^2(0, t)] - 2L \int_0^L v^2 dx - 6\eta v_x^2(0, t) \\ & \leq 2 \|v\|_{L^2((1+x^2)dx)} \|f\|_{L^2((1+x^2)dx)} - \eta [v_x^2(0, t) + v_{xxx}^2(0, t)] - 2L \int_0^L v^2 dx - 6\eta v_x^2(0, t). \end{aligned}$$

Integrating the above equation over  $(0, t)$  we obtain

$$\begin{aligned} & \int_0^L v^2(1+x^2)dx + (3+L) \int_0^t v^2(L, s)ds + \int_0^t v_x^2(0, s)ds + 6 \int_0^t \int_0^L x v_x^2 dx ds - \eta \int_0^t v_{xx}(0, s)ds \\ & - 10\eta \int_0^t \int_0^L x v_{xx}^2 dx ds \leq \int_0^L v_0^2(1+x^2)dx + 2 \int_0^t \|v\|_{L^2((1+x^2)dx)} \|f\|_{L^2((1+x^2)dx)} ds \\ & - \eta \left[ \int_0^t v_x^2(0, s)ds + \int_0^t v_{xxx}^2(0, s)ds \right] - 2L \int_0^t \int_0^L v^2 dx ds - 6\eta \int_0^t v_x^2(0, s)ds \end{aligned}$$

where

$$\begin{aligned} & \|v\|_{L^\infty(0, T; L^2(1+x^2)dx)} + (3+L) \|v(L, t)\|_{L^2(0, T)}^2 + \|v_x(0, t)\|_{L^2(0, T)}^2 + 6 \int_0^t \int_0^L x v_x^2 dx ds \\ & - \eta \|v_{xx}(0, t)\|_{L^2(0, T)}^2 - 10\eta \int_0^t \int_0^L x v_{xx}^2 dx ds \leq (1+L^2) \|v_0\|_{L^2(0, L)} \\ & + 2 \|v\|_{L^\infty(0, T; L^2(1+x^2)dx)} \|f\|_{L^1(0, T; L^2((1+x^2)dx))} - \eta [\|v_x(0, t)\|_{L^2(0, T)}^2 + \|v_{xxx}(0, t)\|_{L^2(0, T)}^2] \\ & - 2L \|v\|_{L^2(0, T; L^2(0, L))} - 6\eta \|v_x(0, t)\|_{L^2(0, T)}^2 \end{aligned} \quad (4.11)$$

and using straightforward calculus (4.1) follows.

For  $(\text{KdVK})_{NH}$  we have(see [1] for the construction of this semi-group).

$$\begin{aligned}
v(x, t) &= \int_0^t S(t-s)f(x, s)ds \quad / \partial_x \\
v_x(x, t) &= \int_0^t \partial_x S(t-s)f(x, s)ds \quad / \|\cdot\|_{L^2((1+x)dx)} \\
\|v_x(x, t)\|_{L^2((1+x)dx)} &\leq \int_0^t \|\partial_x S(t-s)f(x, s)\|_{L^2((1+x)dx)}ds \quad / \psi(t) \in L^2(0, T) \\
\|v_x(x, t)\|_{L^2((1+x)dx)}\psi(t) &\leq \left[ \int_0^t \|\partial_x S(t-s)f(x, s)\|_{L^2((1+x)dx)}ds \right] \psi(t) \quad / \int_0^T dt \\
\int_0^T \|v_x(x, t)\|_{L^2((1+x)dx)}\psi(t)dt &\leq \int_0^T \left[ \int_0^t \|\partial_x S(t-s)f(x, s)\|_{L^2((1+x)dx)}ds \right] \psi(t)dt \quad / |\cdot| \\
\left| \int_0^T \|v_x(x, t)\|_{L^2((1+x)dx)}\psi(t)dt \right| &\leq \int_0^T \left[ \int_0^t \|\partial_x S(t-s)f(x, s)\|_{L^2((1+x)dx)}ds \right] |\psi(t)|dt.
\end{aligned}$$

Using Fubini's theorem and Cauchy-Schwartz inequality we have

$$\begin{aligned}
\left| \int_0^T \|v_x(x, t)\|_{L^2((1+x)dx)}\psi(t)dt \right| &\leq \int_0^T \int_0^T |\psi(t)| \|\partial_x S(t-s)f(x, s)\|_{L^2((1+x)dx)} dt ds \\
&\leq \int_0^T \left[ \int_0^T |\psi(t)|^2 dt \right]^{1/2} \left[ \int_0^T \|\partial_x S(t-s)f(x, s)\|_{L^2((1+x)dx)}^2 dt \right]^{1/2} ds \\
&\leq \left[ \int_0^T |\psi(t)|^2 dt \right]^{1/2} \int_0^T \left[ \int_0^T \|\partial_x S(t-s)f(x, s)\|_{L^2((1+x)dx)}^2 dt \right]^{1/2} ds \\
&\leq M \left[ \int_0^T |\psi(t)|^2 dt \right]^{1/2} \int_0^T \left[ \int_0^T \|f(x, s)\|_{L^2((1+x)dx)}^2 dt \right]^{1/2} ds \\
&\leq M \left[ \int_0^T |\psi(t)|^2 dt \right]^{1/2} \int_0^T \|f(x, s)\|_{L^2((1+x)dx)} \left[ \int_0^T dt \right]^{1/2} ds \\
&\leq M T^{1/2} \left[ \int_0^T |\psi(t)|^2 dt \right]^{1/2} \int_0^T \|f(x, s)\|_{L^2((1+x)dx)} ds \\
&\stackrel{(2.1)}{\leq} \sqrt{3} M T^{1/2} \left[ \int_0^T |\psi(t)|^2 dt \right]^{1/2} \int_0^T \|f(x, s)\|_{L^2((1+x^2)dx)} ds \\
&\leq c \left[ \int_0^T |\psi(t)|^2 dt \right]^{1/2} \|f\|_{L^1(0, T; L^2((1+x^2)dx))}
\end{aligned}$$

therefore

$$\left| \int_0^T \|v_x(x, t)\|_{L^2((1+x)dx)}\psi(t)dt \right| \leq c \left[ \int_0^T |\psi(t)|^2 dt \right]^{1/2} \|f\|_{L^1(0, T; L^2((1+x^2)dx))}. \quad (4.12)$$

If  $\psi(t) = \|v_x(x, t)\|_{L^2((1+x)dx)}$  then

$$\left| \int_0^T \|v_x(x, t)\|_{L^2((1+x)dx)}^2 dt \right| \leq c \left[ \int_0^T \|v_x(x, t)\|_{L^2((1+x)dx)}^2 dt \right]^{1/2} \|f\|_{L^1(0, T; L^2((1+x^2)dx))}$$

then

$$\|v_x(x, t)\|_{L^2(0, T; L^2((1+x)dx))}^2 \leq c \|v_x(x, t)\|_{L^2(0, T; L^2((1+x)dx))} \|f\|_{L^1(0, T; L^2((1+x^2)dx))}$$

hence

$$\|v_x(x, t)\|_{L^2(0, T; L^2((1+x)dx))} \leq c \|f\|_{L^1(0, T; L^2((1+x^2)dx))} \quad (4.13)$$

and (4.2) follows.

Multiplying  $(\text{KdVK})_{NH}$  by  $v_{xx}$  and integrate over  $x \in (0, L)$  we have

$$\int_0^L v_{xx} v_t dx + \eta \int_0^L v_{xx} v_{xxxxx} dx + \int_0^L v_{xx} v_{xxx} dx + \int_0^L v_x v_{xx} dx = \int_0^L f(x, t) dx.$$

Each term is treated separately, integrating by parts and using straightforward calculus we obtain

$$\begin{aligned} \frac{d}{dt} \int_0^L v_x^2 dx - \eta [v_{xx}(0, t) - v_{xxxx}(0, t)]^2 + v_x^2(0, t) + v_{xx}^2(0, t) - \eta v_{xxx}^2(0, t) &= -2 \int_0^L f(x, t) v_{xx} dx \\ - \eta [v_{xx}^2(0, t) + v_{xxxx}^2(0, t)] \end{aligned}$$

hence

$$\frac{d}{dt} \int_0^L v_x^2 dx + v_x^2(0, t) + v_{xx}^2(0, t) - \eta v_{xxx}^2(0, t) \leq -2 \int_0^L v_{xx} f(x, t) dx - \eta [v_{xx}^2(0, t) + v_{xxxx}^2(0, t)] \quad (4.14)$$

then

$$\begin{aligned} \frac{d}{dt} \int_0^L v_x^2 dx + v_x^2(0, t) + v_{xx}^2(0, t) - \eta v_{xxx}^2(0, t) &\leq \int_0^L v_{xx}^2 dx + \int_0^L f^2(x, t) dx \\ - \eta [v_{xx}^2(0, t) + v_{xxxx}^2(0, t)]. \end{aligned} \quad (4.15)$$

Differentiating  $(\text{KdVK})_{NH}$  in  $x$ -variable we have

$$v_{xt} + \eta v_{xxxxxx} + v_{xxxx} + v_{xx} = \partial_x f(x, t).$$

Multiplying the above equation by  $xv_x$  and integrate over  $x \in (0, L)$  we have

$$\int_0^L xv_x v_{xt} dx + \eta \int_0^L xv_x v_{xxxxxx} dx + \int_0^L xv_x v_{xxxx} dx + \int_0^L xv_x v_{xx} dx = \int_0^L xv_x \partial_x f(x, t) dx.$$

Each term is treated separately, integrating by parts we obtain

$$\begin{aligned} \frac{d}{dt} \int_0^L xv_x^2 dx + [v_x(0, t) + v_{xx}(0, t)]^2 - 2\eta [v_{xx}(0, t) + v_{xxx}(0, t)]^2 - \eta [v_x(0, t) - v_{xxxx}(0, t)]^2 \\ - 5\eta \int_0^L v_{xxx}^2 dx + 3 \int_0^L v_{xx}^2 dx - \int_0^L v_x^2 dx = -2 \int_0^L v_x f(x, t) dx - 2 \int_0^L xv_{xx} f(x, t) dx \\ + v_x^2(0, t) + v_{xx}^2(0, t) - 2\eta [v_{xx}^2(0, t) + v_{xxx}^2(0, t)] - \eta [v_x^2(0, t) + v_{xxxx}^2(0, t)] \end{aligned}$$

where

$$\begin{aligned} \frac{d}{dt} \int_0^L xv_x^2 dx - 5\eta \int_0^L v_{xxx}^2 dx + 3 \int_0^L v_{xx}^2 dx &\leq \int_0^L v_x^2 dx - 2 \int_0^L v_x f(x, t) dx - 2 \int_0^L xv_{xx} f(x, t) dx \\ + v_x^2(0, t) + v_{xx}^2(0, t) - 2\eta [v_{xx}^2(0, t) + v_{xxx}^2(0, t)] &- \eta [v_x^2(0, t) + v_{xxxx}^2(0, t)] \end{aligned}$$

then

$$\begin{aligned} & \frac{d}{dt} \int_0^L x v_x^2 dx - 5\eta \int_0^L v_{xxx}^2 dx + 3 \int_0^L v_{xx}^2 dx \leq 2 \int_0^L v_x^2 dx + \int_0^L f^2(x, t) dx \\ & \int_0^L v_{xx}^2 dx + \int_0^L x^2 f^2(x, t) dx - 2\eta [v_{xx}^2(0, t) + v_{xxx}^2(0, t)] - \eta [v_x^2(0, t) + v_{xxxx}^2(0, t)] \\ & + v_x^2(0, t) + v_{xx}^2(0, t) \end{aligned}$$

hence

$$\begin{aligned} & \frac{d}{dt} \int_0^L x v_x^2 dx - 5\eta \int_0^L v_{xxx}^2 dx + 3 \int_0^L v_{xx}^2 dx \leq 2 \int_0^L v_x^2 dx + \int_0^L f^2(x, t)(1+x^2) dx \\ & \int_0^L v_{xx}^2 dx - 2\eta [v_{xx}^2(0, t) + v_{xxx}^2(0, t)] - \eta [v_x^2(0, t) + v_{xxxx}^2(0, t)] + v_x^2(0, t) + v_{xx}^2(0, t) \end{aligned} \quad (4.16)$$

Multiplying (4.15) by two and adding (4.16) we have

$$\begin{aligned} & \frac{d}{dt} \int_0^L (1+x) v_x^2 dx + 3 \int_0^L v_{xx}^2 dx + 2v_x^2(0, t) + 2v_{xx}^2(0, t) - 2\eta v_{xxx}^2(0, t) - 5\eta \int_0^L v_{xxx}^2 dx \\ & \leq 2 \int_0^L v_x^2 dx + 2 \int_0^L v_{xx}^2 dx + 2 \int_0^L f^2(x, t) dx + \int_0^L v_x^2 dx + \int_0^L x^2 f^2(x, t) dx \\ & - \eta [v_{xx}^2(0, t) + v_{xxx}^2(0, t)] - 2\eta [v_{xx}^2(0, t) + v_{xxx}^2(0, t)] - \eta [v_x^2(0, t) + v_{xxxx}^2(0, t)] \\ & \leq 2 \int_0^L v_x^2 dx + 2 \int_0^L v_{xx}^2 dx + c \int_0^L f^2(x, t)(1+x^2) dx - \eta [v_{xx}^2(0, t) + v_{xxx}^2(0, t)] \\ & - 2\eta [v_{xx}^2(0, t) + v_{xxx}^2(0, t)] - \eta [v_x^2(0, t) + v_{xxxx}^2(0, t)] \end{aligned}$$

where

$$\begin{aligned} & \frac{d}{dt} \int_0^L v_x^2(1+x) dx + \int_0^L v_{xx}^2 dx - 2\eta v_{xxx}^2(0, t) - 5\eta \int_0^L v_{xxx}^2 dx \leq 2 \int_0^L v_x^2 dx + c \int_0^L f^2(x, t)(1+x^2) dx \\ & - \eta [v_{xx}^2(0, t) + v_{xxx}^2(0, t)] - 2\eta [v_{xx}^2(0, t) + v_{xxx}^2(0, t)] - \eta [v_x^2(0, t) + v_{xxxx}^2(0, t)]. \end{aligned} \quad (4.17)$$

Multiplying (4.17) by  $t$  and integrate in the temporal variable over  $(0, t)$  we have

$$\begin{aligned} & \int_0^t s \left[ \frac{d}{ds} \int_0^L v_x^2(1+x) dx \right] ds + \int_0^t s \int_0^L v_{xx}^2 dx ds - 2\eta \int_0^t s v_{xxx}^2(0, s) ds - 5\eta \int_0^t s \int_0^L v_{xxx}^2 dx ds \\ & \leq 2 \int_0^t s \int_0^L v_x^2 dx ds + c \int_0^t s \int_0^L f^2(x, t)(1+x^2) dx ds - \eta \left[ \int_0^t s v_{xx}^2(0, s) ds + \int_0^t s v_{xxx}^2(0, s) ds \right] \\ & - 2\eta \left[ \int_0^t s v_{xx}^2(0, s) ds + \int_0^t s v_{xxx}^2(0, s) ds \right] - \eta \left[ \int_0^t s v_x^2(0, s) ds + \int_0^t s v_{xxxx}^2(0, s) ds \right]. \end{aligned}$$

then

$$\begin{aligned}
& t \int_0^L v_x^2(1+x)dx + \int_0^t s \int_0^L v_{xx}^2 dx ds - 2\eta \int_0^t s v_{xxx}^2(0, s) ds - 5\eta \int_0^t s \int_0^L v_{xxx}^2 dx ds \\
& \leq \int_0^t \int_0^L v_x^2(1+x) dx ds + 2 \int_0^t s \int_0^L v_x^2 dx ds + c \int_0^t s \int_0^L f^2(x, t)(1+x^2) dx ds \\
& - \eta \left[ \int_0^t s v_{xx}^2(0, s) ds + \int_0^t s v_{xxx}^2(0, s) ds \right] - 2\eta \left[ \int_0^t s v_{xx}^2(0, s) ds + \int_0^t s v_{xxx}^2(0, s) ds \right] \\
& - \eta \left[ \int_0^t s v_x^2(0, s) ds + \int_0^t s v_{xxx}^2(0, s) ds \right].
\end{aligned}$$

hence

$$\begin{aligned}
& \|\sqrt{t} v_x\|_{L^2((1+x)dx)}^2 + \int_0^t s \|v_{xx}\|_{L^2(0, L)}^2 ds - 2\eta \int_0^t s v_{xxx}^2(0, s) ds - 5\eta \int_0^t \|v_{xxx}\|_{L^2(0, L)}^2 ds \\
& \leq \|v_x\|_{L^2(0, T; L^2((1+x)dx))}^2 + 2 \int_0^t s \|v_x\|_{L^2(0, L)}^2 ds + c \|\sqrt{t} f\|_{L^2(0, T; L^2((1+x^2)dx))}^2 \\
& - \eta \left[ \int_0^t s v_{xx}^2(0, s) ds + \int_0^t s v_{xxx}^2(0, s) ds \right] - 2\eta \left[ \int_0^t s v_{xx}^2(0, s) ds + \int_0^t s v_{xxx}^2(0, s) ds \right] \\
& - \eta \left[ \int_0^t s v_x^2(0, s) ds + \int_0^t s v_{xxx}^2(0, s) ds \right]
\end{aligned}$$

finally

$$\begin{aligned}
& \|\sqrt{t} v_x\|_{L^2((1+x)dx)}^2 + \|\sqrt{t} v_{xx}\|_{L^2(0, T; L^2(0, L))}^2 - 2\eta \|\sqrt{t} v_{xxx}(0, t)\|_{L^2(0, T)}^2 - 5\eta \|v_{xxx}\|_{L^2(0, T; L^2(0, L))}^2 \\
& \leq \|v_x\|_{L^2(0, T; L^2((1+x)dx))}^2 + 2 \|\sqrt{t} v_x\|_{L^2(0, T; L^2(0, L))}^2 + c \|\sqrt{t} f\|_{\mathbb{E}}^2 \\
& - \eta \left[ \int_0^t s v_{xx}^2(0, s) ds + \int_0^t s v_{xxx}^2(0, s) ds \right] - 2\eta \left[ \int_0^t s v_{xx}^2(0, s) ds + \int_0^t s v_{xxx}^2(0, s) ds \right] \\
& - \eta \left[ \int_0^t s v_x^2(0, s) ds + \int_0^t s v_{xxx}^2(0, s) ds \right]. \tag{4.18}
\end{aligned}$$

Using straightforward calculus the other terms are estimates the similar form that section one and the proposition follows.  $\square$

## 5. NONLINEAR CASE

We now prove in this section a local existence result for the nonlinear system

$$(\text{KdVK}) \begin{cases} u_t + \eta u_{xxxxx} + u_{xxx} + uu_x + u_x = 0, & x \in [0, L], \quad t \in [0, T], \\ u(0, t) = g(t), & t \in [0, T[, \\ u_x(L, t) = 0, \quad u_{xx}(L, t) = 0, \quad u_{xxx}(L, t) = 0, \quad u_{xxxx}(L, t) = 0, & t \in [0, T[, \\ u(x, 0) = u_0(x) \end{cases}$$

where  $L > 0$ ,  $T \in ]0, \infty[$  and  $\eta \in \mathbb{R}$  is a constant. Let  $\xi$  be a smooth function defined over  $\mathbb{R}^+$  such that  $\xi(0) = 1$  and  $\xi(x) = 0$ ,  $\forall x \geq 1$ .

**Definition 5.1.** A weak solution of (KdVK) on  $[0, T]$  is a function  $u(x, t) \in \mathbb{H}_T$  such that

$$w(x, t) = u(x, t) - \xi(x)g(t)$$

satisfy

$$w(x, t) = S(t)w_0(x) - \int_0^t \{\xi g_s + \eta \xi_{xxxxx}g(s) + g(s)(\xi_{xxx} + \xi_x) + (w + \xi g(s))(w_x + \xi_x g(s))\} ds$$

where  $w_0(x) = u_0(x) - \xi(x)g(0)$ .

Let  $w$  be defined by

$$w(x, t) = u(x, t) - \xi(x)g(t)$$

then replacing in (KdVK) we have

$$w_t + \xi g_t + \eta [w_{xxxxx} + \xi_{xxxxx}g] + w_{xxx} + \xi_{xxx}g + (w + \xi g)(w_x + \xi_x g) + w_x + \xi_x g = 0$$

then

$$\begin{aligned} w_t + \eta w_{xxxxx} + w_{xxx} + w_x &= -[\xi g_t + \eta \xi_{xxxxx}g + g(\xi_{xxx} + \xi_x) + (w + \xi g)(w_x + \xi_x g)] \\ &= -F(w, w_x, g). \end{aligned}$$

This way, system (KdVK) becomes

$$(\text{KdVK})_\xi \begin{cases} w_t + \eta w_{xxxxx} + w_{xxx} + w_x = -F(w, w_x, g), & x \in [0, L], \quad t \in [0, T], \\ w(0, t) = 0, & t \in [0, T[, \\ w_x(L, t) = 0, \quad w_{xx}(L, t) = 0, \quad w_{xxx}(L, t) = 0, \quad w_{xxxx}(L, t) = 0, & t \in [0, T[, \\ w(x, 0) = u_0(x) - \xi(x)g(0) \end{cases}$$

where  $F(w, w_x, g) = \xi g_t + \eta \xi_{xxxxx}g + g(\xi_{xxx} + \xi_x) + (w + \xi g)(w_x + \xi_x g)$ . Indeed, we have transformed the original problem (KdVK) into a problem with a Dirichlet boundary condition  $g = 0$ .

We write  $w(x, t) = S(t)w_0(x) - \int_0^t S(t-s)F(w, w_x, g)(s)ds$  where  $S(t)$  is the linear semi group. We introduce the following functional  $\mathcal{L}$  defined by

$$\mathcal{L}(w_0, g, w) = S(t)w_0(x) - \int_0^t S(t-s)F(w, w_x, g)(s)ds$$

**Lemma 5.1.** *There exists a constant  $c(T)$  depending on  $T$ , independent of  $L$  such that for all  $u_0 \in L^2(0, L)$  and  $\eta \leq 0$*

$$\|S(t)u_0\|_{\mathbb{H}} \leq c(T) \|u_0\|_{L^2(0, L)}$$

and the map  $t \rightarrow c(t)$  is continuous.

*Proof.* It is an consequence of inequalities (3.1), (3.2), (3.16) and (3.17). □

**Lemma 5.2.** *There exists a constant  $c(T)$  depending on  $T$ , independent of  $L$  such that for all  $f \in \mathbb{E}$  and  $\eta \leq 0$*

$$\left\| \int_0^t S(t-s)f(s)ds \right\|_{\mathbb{H}} \leq c(T) \|f\|_{\mathbb{E}}$$

and the map  $t \rightarrow c(t)$  is continuous.

*Proof.* It is an consequence of inequalities (4.1), (4.2), (4.4) and (4.5). □

**Lemma 5.3.** *For all  $w \in H^1([0, L])$  such that  $w(0) = 0$ , one has*

$$\|\sqrt{x}w\|_{L^\infty(0, L)} \leq 5 \left( \sqrt{\|w\|_{L^2((1+x)dx)}} \sqrt{\|w_x\|_{L^2((1+x)dx)}} + \|w\|_{L^2((1+x)dx)} \right) \quad (5.1)$$



**Theorem 5.1.** *Let  $\eta \leq 0$ . Suppose that  $w_0, z_0 \in L^2(0, L)$ ,  $g$  and  $h$  are in  $H_{loc}^1(\mathbb{R}^+)$ , then there exists a continuous function  $t \rightarrow c(t)$  such that for all  $T \in [0, T_0]$ ,*

$$\begin{aligned} \|\mathcal{L}(w_0, g, w) - \mathcal{L}(z_0, h, z)\|_{\mathbb{H}} &\leq c(T) \|w_0 - z_0\|_{L^2(0, L)} \\ &+ c(T) \sqrt{T} (\|g\|_{H^1(0, T)} + \|h\|_{H^1(0, T)} + 2 + \|w\|_{\mathbb{H}}) \|g - h\|_{H^1(0, T)} \\ &+ c(T) T^{1/4} \|w - z\|_{\mathbb{H}} (\|w\|_{\mathbb{H}} + \|z\|_{\mathbb{H}}) \\ &+ c(T) \sqrt{T} \|w - z\|_{\mathbb{H}} (\|h\|_{H^1(0, T)} + \|w\|_{\mathbb{H}} + \|z\|_{\mathbb{H}}). \end{aligned} \quad (5.2)$$

*Proof.* We have

$$\mathcal{L}(w_0, g, w) - \mathcal{L}(z_0, h, z) = S(t)(w_0 - z_0)(x) - \int_0^t S(t-s)[F(w, w_x, g) - F(z, z_x, h)](s) ds. \quad (5.3)$$

Let

$$\begin{aligned} H(s) &= F(w, w_x, g)(s) - F(z, z_x, h)(s) \\ &= \xi g_t + \eta \xi_{xxxx} g + g(\xi_{xxx} + \xi_x) + (w + \xi g)(w_x + \xi_x g) \\ &\quad - \xi h_t - \eta \xi_{xxxx} h - h(\xi_{xxx} + \xi_x) - (z + \xi h)(z_x + \xi_x h) \\ &= \xi \xi_x (g^2 - h^2) + (g - h)(\xi_x + \xi_{xxx}) + \xi(g_t - h_t) + \eta \xi_{xxxx} (g - h) \\ &\quad + \xi_x w g - \xi_x z h + \xi w_x g - \xi z_x h + w w_x - z z_x \\ &= F_1 + F_2 + F_3 \end{aligned} \quad (5.4)$$

where

$$\begin{aligned} F_1 &= \xi \xi_x (g^2 - h^2) + (g - h)(\xi_x + \xi_{xxx}) + \xi(g_t - h_t) + \eta \xi_{xxxx} (g - h) \\ F_2 &= \xi_x w g - \xi_x z h + \xi w_x g - \xi z_x h \\ F_3 &= w w_x - z z_x. \end{aligned}$$

Hence in (5.3) we have

$$\mathcal{L}(w_0, g, w) - \mathcal{L}(z_0, h, z) = S(t)(w_0 - z_0)(x) - \int_0^t S(t-s)H(s) ds.$$

Using Lemma 5.1 and 5.2 we have

$$\begin{aligned} \|\mathcal{L}(w_0, g, w) - \mathcal{L}(z_0, h, z)\|_{\mathbb{H}} &= \left\| S(t)(w_0 - z_0)(x) - \int_0^t S(t-s)H(s) ds \right\|_{\mathbb{H}} \\ &\leq \|S(t)(w_0 - z_0)(x)\|_{\mathbb{H}} + \left\| \int_0^t S(t-s)H(s) ds \right\|_{\mathbb{H}} \\ &\leq c(T) \|w_0 - z_0\|_{L^2(0, L)} + c(T) \|H\|_{\mathbb{E}} \\ &= c(T) \|w_0 - z_0\|_{L^2(0, L)} \\ &\quad + c(T) \left( \|H\|_{L^1(0, T; L^2(1+x^2)dx)} + \|\sqrt{t} H\|_{L^2(0, T; L^2(1+x^2)dx)} \right) \end{aligned}$$

We estimate separately the three parts of  $H$ .

$$\begin{aligned} \|F_1\|_{L^2((1+x^2)dx)} &= \|\xi \xi_x (g^2 - h^2) + (g - h)(\xi_x + \xi_{xxx}) + \eta \xi_{xxxx} (g - h) + \xi(g_t - h_t)\|_{L^2((1+x^2)dx)} \\ &\leq \|[\xi \xi_x (g + h) + (\xi_x + \xi_{xxx}) + \eta \xi_{xxxx}](g - h)\|_{L^2((1+x^2)dx)} + \|\xi(g_t - h_t)\|_{L^2((1+x^2)dx)} \\ &\leq c(2 + |g(t)| + |h(t)|) |g(t) - h(t)| + c |g_t(t) - h_t(t)| \end{aligned}$$

because  $\xi$  is compactly supported. Moreover and using  $H^1(0, T) \hookrightarrow L^\infty(0, T)$ , one gets

$$\begin{aligned} \|F_1\|_{\mathbb{E}} &\leq \sqrt{T} \|F_1\|_{L^2(0, T; L^2((1+x^2)dx))} \\ &\leq c\sqrt{T} (2 + \|g\|_{H^1(0, T)} + \|h\|_{H^1(0, T)}) \|g - h\|_{H^1(0, T)}. \end{aligned} \quad (5.5)$$

Now we estimate the term  $F_2$ .

$$\begin{aligned} F_2 &= \xi_x w g - \xi_x z h + \xi w_x g - \xi z_x h \\ &= \xi_x w (g - h) + \xi_x (w - z) h + \xi w_x (g - h) + \xi (w_x - z_x) h \end{aligned}$$

then

$$\begin{aligned} \|F_2\|_{L^2(0, T; L^2((1+x^2)dx))} &= \|\xi_x w (g - h)\|_{L^2(0, T; L^2((1+x^2)dx))} + \|\xi_x (w - z) h\|_{L^2(0, T; L^2((1+x^2)dx))} \\ &+ \|\xi w_x (g - h)\|_{L^2(0, T; L^2((1+x^2)dx))} + \|\xi (w_x - z_x) h\|_{L^2(0, T; L^2((1+x^2)dx))}. \end{aligned}$$

Using that  $\xi$  is compactly supported and  $H^1(0, T) \hookrightarrow L^\infty(0, T)$  we have

$$\begin{aligned} \|F_2\|_{L^2(0, T; L^2((1+x^2)dx))} &\leq c \|g - h\|_{L^\infty(0, T)} \|w\|_{L^2(0, T; L^2((1+x)dx))} + c \|h\|_{L^\infty(0, T)} \|w - z\|_{L^2(0, T; L^2((1+x)dx))} \\ &+ c \|g - h\|_{L^\infty(0, T)} \|w_x\|_{L^2(0, T; L^2((1+x)dx))} + c \|h\|_{L^\infty(0, T)} \|w_x - z_x\|_{L^2(0, T; L^2((1+x)dx))} \\ &\leq c \|g - h\|_{H^1(0, T)} \|w\|_{L^2(0, T; L^2((1+x)dx))} + c \|h\|_{H^1(0, T)} \|w - z\|_{L^2(0, T; L^2((1+x)dx))} \\ &+ c \|g - h\|_{H^1(0, T)} \|w_x\|_{L^2(0, T; L^2((1+x)dx))} + c \|h\|_{H^1(0, T)} \|w_x - z_x\|_{L^2(0, T; L^2((1+x)dx))}. \end{aligned}$$

Therefore

$$\|F_2\|_{\mathbb{E}} \leq c\sqrt{T} (\|w\|_{\mathbb{H}} \|g - h\|_{H^1(0, T)} + \|w - z\|_{\mathbb{H}} \|h\|_{H^1(0, T)}). \quad (5.6)$$

We estimate the term  $F_3$ .

$$\begin{aligned} F_3 &= w w_x - z z_x = (w - z) w_x + z (w_x - z_x) \\ \implies \|w w_x - z z_x\|_{\mathbb{E}} &= \|(w - z) w_x + z (w_x - z_x)\|_{\mathbb{E}} \leq \|(w - z) w_x\|_{\mathbb{E}} + \|z (w_x - z_x)\|_{\mathbb{E}} \\ \implies \|F_3\|_{\mathbb{E}} &\leq \|(w - z) w_x\|_{\mathbb{E}} + \|z (w_x - z_x)\|_{\mathbb{E}} \\ &= \|(w - z) w_x\|_{L^1(0, T; L^2((1+x^2)dx))} + \|\sqrt{t} (w - z) w_x\|_{L^2(0, T; L^2((1+x^2)dx))} \\ &+ \|z (w_x - z_x)\|_{L^1(0, T; L^2((1+x^2)dx))} + \|\sqrt{t} z (w_x - z_x)\|_{L^2(0, T; L^2((1+x^2)dx))} \\ &= \|(w - z) w_x\|_{L^1(0, T; L^2((1+x^2)dx))} + \|z (w_x - z_x)\|_{L^1(0, T; L^2((1+x^2)dx))} \\ &+ \|\sqrt{t} (w - z) w_x\|_{L^2(0, T; L^2((1+x^2)dx))} + \|\sqrt{t} z (w_x - z_x)\|_{L^2(0, T; L^2((1+x^2)dx))} \\ &= F_{1,3} + F_{2,3} \end{aligned}$$

where

$$F_{1,3} = \|(w - z) w_x\|_{L^1(0, T; L^2((1+x^2)dx))} + \|z (w_x - z_x)\|_{L^1(0, T; L^2((1+x^2)dx))}$$

and

$$F_{2,3} = \|\sqrt{t} (w - z) w_x\|_{L^2(0, T; L^2((1+x^2)dx))} + \|\sqrt{t} z (w_x - z_x)\|_{L^2(0, T; L^2((1+x^2)dx))}$$

then using that if  $a \geq 0$  and  $b \geq 0$  then  $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$  we have

$$\begin{aligned}
F_{1,3} &= \|(w-z)w_x\|_{L^1(0,T;L^2((1+x^2)dx))} + \|z(w_x - z_x)\|_{L^1(0,T;L^2((1+x^2)dx))} \\
&= \int_0^T \|(w-z)w_x\|_{L^2((1+x^2)dx)} dt + \int_0^T \|(w_x - z_x)z\|_{L^2((1+x^2)dx)} dt \\
&= \int_0^T \sqrt{\int_0^L (w-z)^2 w_x^2 (1+x^2) dx} dt + \int_0^T \sqrt{\int_0^L (w_x - z_x)^2 z^2 (1+x^2) dx} dt \\
&= \int_0^T \sqrt{\int_0^L (w-z)^2 w_x^2 dx} + \int_0^L (w-z)^2 w_x^2 x^2 dx} dt \\
&\quad + \int_0^T \sqrt{\int_0^L (w_x - z_x)^2 z^2 dx} + \int_0^L (w_x - z_x)^2 z^2 x^2 dx} dt \\
&\leq \int_0^T \left( \sqrt{\int_0^L (w-z)^2 w_x^2 dx} + \sqrt{\int_0^L (w-z)^2 w_x^2 x^2 dx} \right) dt \\
&\quad + \int_0^T \left( \sqrt{\int_0^L (w_x - z_x)^2 z^2 dx} + \sqrt{\int_0^L (w_x - z_x)^2 z^2 x^2 dx} \right) dt \tag{5.7}
\end{aligned}$$

but

$$\begin{aligned}
\int_0^L (w-z)^2 w_x dx &\leq \|w-z\|_{L^\infty(0,L)}^2 \int_0^L w_x^2 dx \\
&\leq \|w-z\|_{L^\infty(0,L)}^2 \int_0^L w_x^2 (1+x) dx \\
&\stackrel{(2.4)}{\leq} 2 \|w-z\|_{L^2((1+x)dx)} \|w_x - z_x\|_{L^2((1+x)dx)} \|w_x\|_{L^2((1+x)dx)} \tag{5.8}
\end{aligned}$$

and

$$\begin{aligned}
\int_0^L (w-z)^2 w_x^2 x^2 dx &\leq \|x(w-z)^2\|_{L^\infty(0,L)} \int_0^L x w_x^2 dx \\
&\leq \|\sqrt{x}(w-z)\|_{L^\infty(0,L)}^2 \int_0^L w_x^2 (1+x) dx
\end{aligned}$$

then

$$\|x(w-z)w_x\|_{L^2(0,L)}^2 \leq \|\sqrt{x}(w-z)\|_{L^\infty(0,L)}^2 \int_0^L w_x^2 (1+x) dx$$

hence using Lemma 5.3

$$\begin{aligned}
\|x(w-z)w_x\|_{L^2(0,L)} &\leq \|\sqrt{x}(w-z)\|_{L^\infty(0,L)} \sqrt{\int_0^L w_x^2 (1+x) dx} \\
&\leq 5 \left( \sqrt{\|w-z\|_{L^2((1+x)dx)}} \sqrt{\|w_x - z_x\|_{L^2((1+x)dx)} + \|w-z\|_{L^2((1+x)dx)}} \right) \\
&\quad \times \|w_x\|_{L^2((1+x)dx)}. \tag{5.9}
\end{aligned}$$

This way, using (5.8) and (5.9) for the first term in (5.7) we obtain

$$\begin{aligned}
& \int_0^T \left( \sqrt{\int_0^L (w-z)^2 w_x^2 dx} + \sqrt{\int_0^L (w-z)^2 w_x^2 x^2 dx} \right) dt \leq \int_0^T \sqrt{\int_0^L (w-z)^2 w_x^2 dx} dt \\
& + \int_0^T \sqrt{\int_0^L (w-z)^2 w_x^2 x^2 dx} dt \leq \sqrt{2} \int_0^T \|w-z\|_{L^2((1+x)dx)}^{1/2} \|w_x - z_x\|_{L^2((1+x)dx)}^{1/2} \|w_x\|_{L^2((1+x)dx)} dt \\
& + 5 \int_0^T \left( \|w-z\|_{L^2((1+x)dx)}^{1/2} \|w_x - z_x\|_{L^2((1+x)dx)}^{1/2} + \|w-z\|_{L^2((1+x)dx)} \right) \|w_x\|_{L^2((1+x)dx)} dt \\
& \leq c \int_0^T \|w-z\|_{L^2((1+x)dx)}^{1/2} \|w_x - z_x\|_{L^2((1+x)dx)}^{1/2} \|w_x\|_{L^2((1+x)dx)} dt \\
& + 5 \int_0^T \|w-z\|_{L^2((1+x)dx)} \|w_x\|_{L^2((1+x)dx)} dt \leq c \|w-z\|_{\mathbb{H}} \int_0^T \|w_x - z_x\|_{L^2((1+x)dx)}^{1/2} \|w_x\|_{L^2((1+x)dx)} dt \\
& + 5 \|w-z\|_{\mathbb{H}} \int_0^T \|w_x\|_{L^2((1+x)dx)} dt \leq c(\sqrt{T} + T^{1/4}) \|w-z\|_{\mathbb{H}} \|w\|_{\mathbb{H}}.
\end{aligned}$$

Therefore

$$\int_0^T \left( \sqrt{\int_0^L (w-z)^2 w_x^2 dx} + \sqrt{\int_0^L (w-z)^2 w_x^2 x^2 dx} \right) dt \leq c(\sqrt{T} + T^{1/4}) \|w-z\|_{\mathbb{H}} \|w\|_{\mathbb{H}}. \quad (5.10)$$

Using the similar technique for the second term we obtain

$$\int_0^T \left( \sqrt{\int_0^L (w_x - z_x)^2 z^2 dx} + \sqrt{\int_0^L (w_x - z_x)^2 z^2 x^2 dx} \right) dt \leq c(\sqrt{T} + T^{1/4}) \|w-z\|_{\mathbb{H}} \|w\|_{\mathbb{H}}. \quad (5.11)$$

This way from (5.10) and (5.11) we have

$$F_{1,3} \leq c(\sqrt{T} + T^{1/4}) \|w-z\|_{\mathbb{H}} \|w\|_{\mathbb{H}}. \quad (5.12)$$

Now we estimate the term  $F_{2,3}$ . Using the Lemma 5.3, we estimate the following term

$$\begin{aligned}
\|x(w-z)w_x\|_{L^2(0,L)} &= \|\sqrt{x}(w-z)\sqrt{x}w_x\|_{L^2(0,L)} \\
&\leq \|\sqrt{x}(w-z)\|_{L^\infty(0,L)} \|\sqrt{x}w_x\|_{L^2(0,L)} \\
&\leq 5 \left( \sqrt{\|w-z\|_{L^2((1+x)dx)}} \sqrt{\|w_x - z_x\|_{L^2((1+x)dx)}} + \|w-z\|_{L^2((1+x)dx)} \right) \\
&\quad \times \|\sqrt{x}w_x\|_{L^2(0,L)} \\
&\leq 5 \sqrt{\|w-z\|_{L^2((1+x)dx)}} \sqrt{\|w_x - z_x\|_{L^2((1+x)dx)}} \|\sqrt{x}w_x\|_{L^2(0,L)} \\
&+ 5 \|w-z\|_{L^2((1+x)dx)} \|\sqrt{x}w_x\|_{L^2(0,L)} \\
&\leq 5 \sqrt{\|w-z\|_{L^2((1+x)dx)}} \sqrt{\|w_x - z_x\|_{L^2((1+x)dx)}} \|w_x\|_{L^2((1+x)dx)} \\
&+ 5 \|w-z\|_{L^2((1+x)dx)} \|w_x\|_{L^2((1+x)dx)}, \quad (5.13)
\end{aligned}$$

hence from (5.13)

$$\begin{aligned}
\|\sqrt{t} x (w - z) w_x\|_{L^2(0, T; L^2(0, L))} &= \|\sqrt{t} \sqrt{x} (w - z) \sqrt{x} w_x\|_{L^2(0, T; L^2(0, L))} \\
&\leq 5 \sqrt{\|w - z\|_{L^\infty(0, T; L^2((1+x)dx))}} \|\sqrt{t} w_x\|_{L^\infty(0, T; L^2((1+x)dx))} \\
&\quad \times \sqrt{\|w_x - z_x\|_{L^2(0, T; L^2((1+x)dx))}} \sqrt{T} \\
&\quad + 5 \|w - z\|_{L^\infty(0, T; L^2((1+x)dx))} \|\sqrt{t} w_x\|_{L^2(0, T; L^2((1+x)dx))} \\
&\leq c \|w - z\|_{\mathbb{H}}^{1/2} \|w\|_{\mathbb{H}} \sqrt{\|w_x - z_x\|_{L^2(0, T; L^2((1+x)dx))}} \sqrt{T} \\
&\quad + c \|w - z\|_{\mathbb{H}} \sqrt{T} \|w\|_{\mathbb{H}} \\
&\leq c \sqrt{T} \|w - z\|_{\mathbb{H}} \|w\|_{\mathbb{H}}, \quad \forall 0 \leq t \leq T.
\end{aligned} \tag{5.14}$$

This way from (5.14), the first term in  $F_{2,3}$  is estimated by

$$\|\sqrt{t} (w - z) w_x\|_{L^2(0, T; L^2((1+x^2)dx))} \leq c \sqrt{T} \|w - z\|_{\mathbb{H}} \|w\|_{\mathbb{H}} \tag{5.15}$$

and the similar form we estimate the second term in  $F_{2,3}$ .

$$\|\sqrt{t} (w_x - z_x) z\|_{L^2(0, T; L^2((1+x^2)dx))} \leq c \sqrt{T} \|w - z\|_{\mathbb{H}} \|w\|_{\mathbb{H}}. \tag{5.16}$$

Therefore the Theorem 5.1 follows.  $\square$

Let

$$T_0 > 0, \quad r = 2c(T_0) \left( \|w_0\|_{L^2(0, L)} + \sqrt{T_0} (2 + \|g\|_{H^1(0, T)}) \right) \|g\|_{H^1(0, T)}.$$

We denote by  $\mathbb{B}_r(0)$  the ball of center zero and of radius  $r$  in  $\mathbb{H}_T$ .

**Theorem 5.2.** *Let  $\eta \leq 0$  and  $g \in H_{loc}^1(\mathbb{R}^+)$ . Then, there exists a time  $T_1 \in (0, T_0]$  such that the application  $(w_0, g, w) \rightarrow \mathcal{L}(w_0, g, w)$  maps the ball  $\mathbb{B}_r$  into itself.*

*Proof.* Using (5.2) with  $z_0 = 0, h = 0, z = 0$ , one gets

$$\begin{aligned}
\|\mathcal{L}(w_0, g, w)\|_{\mathbb{H}} &\leq c(T_0) \|w_0\|_{L^2(0, L)} + c(T_0) \sqrt{T_0} (\|w\|_{\mathbb{H}} + \|g\|_{H^1(0, T)} + 2) \|g\|_{H^1(0, T)} \\
&\quad + c(T_0) T_0^{1/4} \|w\|_{\mathbb{H}}^2 + c(T_0) \sqrt{T_0} \|w\|_{\mathbb{H}}^2 \\
&\leq c(T_0) \|w_0\|_{L^2(0, L)} + c(T_0) \sqrt{T_0} (\|g\|_{H^1(0, T)} + 2) \|g\|_{H^1(0, T)} \\
&\quad + c(T_0) \sqrt{T_0} \|w\|_{\mathbb{H}} \|g\|_{H^1(0, T)} + c(T_0) T_0^{1/4} \|w\|_{\mathbb{H}}^2 + c(T_0) \sqrt{T_0} \|w\|_{\mathbb{H}}^2 \\
&\leq c(T_0) \left( \|w_0\|_{L^2(0, L)} + \sqrt{T_0} (\|g\|_{H^1(0, T)} + 2) \right) \|g\|_{H^1(0, T)} \\
&\quad + c(T_0) \sqrt{T_0} \|w\|_{\mathbb{H}} \|g\|_{H^1(0, T)} + c(T_0) T_0^{1/4} \|w\|_{\mathbb{H}}^2 + c(T_0) \sqrt{T_0} \|w\|_{\mathbb{H}}^2 \\
&= \frac{r}{2} + c(T_0) \sqrt{T_0} \|w\|_{\mathbb{H}} \|g\|_{H^1(0, T)} + c(T_0) T_0^{1/4} \|w\|_{\mathbb{H}}^2 + c(T_0) \sqrt{T_0} \|w\|_{\mathbb{H}}^2.
\end{aligned}$$

If  $w \in \mathbb{B}_r(0)$ , then

$$\|\mathcal{L}(w_0, g, w)\|_{\mathbb{H}} \leq \frac{r}{2} + c\sqrt{T_0} r \|g\|_{H^1(0, T)} + cT_0^{1/4} r^2 + c\sqrt{T_0} r^2.$$

Choosing  $T_0$  such that

$$c \left( \sqrt{T_0} r \|g\|_{H^1(0, T)} + T_0^{1/4} r^2 + \sqrt{T_0} r^2 \right) \leq \frac{r}{2}$$

we have that  $(w_0, g, w) \rightarrow \mathcal{L}(w_0, g, w)$  maps the ball  $\mathbb{B}_r(0)$  into itself.  $\square$

**Theorem 5.3.** *Let  $\eta \leq 0$  and  $g \in H_{loc}^1(\mathbb{R}^+)$ . Then, there exists a time  $T_2 \in (0, T_1]$  such that the application  $(w_0, g, w) \rightarrow \mathcal{L}(w_0, g, w)$  is a contraction over  $(\mathbb{B}_r(0), \|\cdot\|_{\mathbb{H}})$ .*

*Proof.* Using (5.2) with  $w_0 = z_0$  and  $g = h$  we have

$$\begin{aligned} \|\mathcal{L}(w_0, g, w) - \mathcal{L}(w_0, g, z)\|_{\mathbb{H}} &\leq c(T) T^{1/4} \|w - z\|_{\mathbb{H}} (\|w\|_{\mathbb{H}} + \|z\|_{\mathbb{H}}) \\ &\quad + c(T) \sqrt{T} \|w - z\|_{\mathbb{H}} (\|g\|_{H^1(0, T)} + \|w\|_{\mathbb{H}} + \|z\|_{\mathbb{H}}). \end{aligned}$$

so that, if  $w, z \in \mathbb{B}_r(0)$  then

$$\begin{aligned} \|\mathcal{L}(w_0, g, w) - \mathcal{L}(w_0, g, z)\|_{\mathbb{H}} &\leq c(T) T^{1/4} \|w - z\|_{\mathbb{H}} 2r + c(T) \sqrt{T} \|w - z\|_{\mathbb{H}} (\|g\|_{H^1(0, T)} + 2r) \\ &\leq c \left( 2r T^{1/4} + 2r \sqrt{T} + \sqrt{T} \|g\|_{H^1(0, T)} \right) \end{aligned}$$

then the application  $(w_0, g, w) \rightarrow \mathcal{L}(w_0, g, w)$  is a contraction over  $(\mathbb{B}_r(0), \|\cdot\|_{\mathbb{H}})$ .  $\square$

**Theorem 5.4.** *Let  $\eta \leq 0$ . Then there exists a unique  $w$  defined in  $\mathbb{H}_T$ , weak solution of (KdVK).*

*Proof.* To apply the Banach's fixed point theorem for  $(w_0, g, w) \rightarrow \mathcal{L}(w_0, g, w)$  on  $\mathbb{B}_r(0)$ , which is a complete metric space and yields local existence and uniqueness.  $\square$

**Theorem 5.5.** *If  $\eta \leq 0$ . Then the solution  $w$  depends continuously on  $w_0 \in L^2((1+x^2)dx)$  and  $g \in H_{loc}^1(\mathbb{R}^+)$ .*

*Proof.* From (5.2) for small times, one gets

$$\begin{aligned} \|w - z\|_{\mathbb{H}} &\leq c(T) \|w_0 - z_0\|_{L^2(0, L)} + c(T) \sqrt{T} (\|g\|_{H^1(0, T)} + \|h\|_{H^1(0, T)} + 2 + \|w\|_{\mathbb{H}}) \|g - h\|_{H^1(0, T)} \\ &\quad + c(T) T^{1/4} \|w - z\|_{\mathbb{H}} (\|w\|_{\mathbb{H}} + \|z\|_{\mathbb{H}}) + c(T) \sqrt{T} \|w - z\|_{\mathbb{H}} (\|h\|_{H^1(0, T)} + \|w\|_{\mathbb{H}} + \|z\|_{\mathbb{H}}). \end{aligned}$$

Then if  $w_0 \rightarrow w_0$  in  $L^2((1+x^2)dx)$  and if  $g \rightarrow h$  in  $H^2(0, T)$  one gets that  $w \rightarrow z$  in  $\mathbb{H}_T$ . These results were obtained locally in time. But, since the time interval where this result holds depends only on  $\|w_0\|_{L^2(0, L)}$  and  $\|g\|_{H^1(0, T)}$ , it can be extended as long the solution exists. The proof follows.  $\square$

Finally as consequence of theorems 5.4 and 5.5 we have our main result:

**Theorem 5.6.** *(Existence and Uniqueness) Let  $\eta \leq 0$ ,  $u_0 \in L^2((1+x^2)dx)$ ,  $g \in H_{loc}^1(\mathbb{R}^+)$  and  $0 < L < +\infty$ . Then there exists a unique weak maximal solution of (KdVK) defined over  $[0, T_L]$ . Moreover, there exists  $T_{min} > 0$  independent of  $L$ , depending only on  $\|u_0\|_{L^2(0, L)}$  and  $\|g\|_{H^1(0, T)}$  such that  $T_L \geq T_{min}$ . The solution  $u$  depends continuously on  $u_0$  and  $g$  in the following sense: Let a sequence  $u_0^n \rightarrow u_0$  in  $L^2((1+x^2)dx)$ , let a sequence  $g^n \rightarrow g$  in  $H_{loc}^1(\mathbb{R}^+)$  and denote by  $u^n$  the solution with data  $(u_0^n, g^n)$  and  $T_L^n$  its existence time. Then*

$$\liminf_{n \rightarrow +\infty} T_L^n \geq T_L$$

and for all  $t < T_L$ ,  $u^n$  exists on the interval  $[0, T]$  if  $n$  is large enough and  $u^n \rightarrow u$  in  $\mathbb{H}_T$ .

**Remark 5.1.** *All the estimates obtained in the previous section can be applied to the quarter plane problem ( $L = +\infty$ ) since they are uniform with respect to  $L$ . This gives*

**Corollary 5.1.** *Let  $\eta \leq 0$ ,  $u_0 \in L^2((1+x^2)dx)$ ,  $g \in H^1_{loc}(\mathbb{R}^+)$  and  $L = +\infty$ . Then there exists a unique solution of (KdVK) in the space*

$$\begin{aligned} \overline{\mathbb{H}}_T &= \{u \in C([0, T]; L^2(\mathbb{R}^+; (1+x^2)dx)) \text{ such that} \\ &u_x \in C(0, T; L^2(\mathbb{R}^+; (1+x^2)dx)), \sqrt{t} u_x \in L^\infty(0, T; L^2(\mathbb{R}^+; (1+x)dx)) \\ &\sqrt{t} u_{xx} \in L^2(0, T; L^2(\mathbb{R}^+; dx)), \sqrt{t} u_{xxx} \in L^2(0, T; L^2(\mathbb{R}^+; dx))\}. \end{aligned}$$

for some  $T > 0$ .

**Remark 5.2.** *We consider the Korteweg-de Vries-Kawahara equation in a quarter plane*

$$(KdVK)_{QP} = \begin{cases} u_t + \eta u_{xxxx} + u_{xxx} + uu_x + u_x = 0, & x \geq 0, \quad t \geq 0, \\ u(0, t) = g(t), & t \geq 0 \\ u(x, 0) = u_0(x), & x \geq 0. \end{cases}$$

*In this context, we are interested in the following result: Consider a family of initial values  $u_0^L \in L^2([0, L])$  such that*

$$\sup_{L \rightarrow \infty} \int_0^L |u_0^L(x)|^2 (1+x^2) dx$$

*and such that  $u_0^L \rightarrow u_0$  in  $L^2_{loc}(\mathbb{R}^+)$  strongly. Then, for all  $T > 0$ , if  $L$  is large enough,  $u^L$  the solution of (KdVK) with initial data  $u_0^L$  is defined on  $[0, T]$  and  $u^L \rightarrow u$  in  $L^p(0, T; L^2_{loc}(\mathbb{R}^+))$  strongly for all  $1 \leq p < +\infty$ , where  $u$  is a solution of (KdVK) $_{QP}$  with initial value  $u_0$ .*

*In order to prove this question, we need similar results for Korteweg-de Vries-Kawahara equation as those obtained by J. Bona and R. Whinter [5, 6, 4] for the Korteweg-de Vries equation.*

In the next section we present numerical results for regular solutions of the KdV-Kawahara equation (KdVK).

## 6. NUMERICAL SIMULATIONS

We consider finite differences based on the unconditionally stable schemes for simulation of solitons described in [7].

**Description of the scheme.** We note by  $u_i^n$  and  $v_i^n$  the approximate value of  $u(i\Delta x, n\Delta t)$  and  $v(i\Delta x, n\Delta t)$ , solutions of the nonlinear problem (KdVK) and the linear problem (KdVK) $_L$  respectively, where  $\Delta x$  is the space-step, and  $\Delta t$  is the time-step, for  $i = 0, \dots, N$ , and  $n = 0, \dots, M$ . Let the discrete space :

$$X_N = \{u = (u_0, u_1, \dots, u_N) \in \mathbb{R}^{N+1} \mid \text{with } u_0 = 0 \text{ and } u_N = u_{N-1} = u_{N-2} = u_{N-3} = u_{N-4}\}$$

and  $(D^+u)_i = \frac{u_{i+1} - u_i}{\Delta x}$  and  $(D^-u)_i = \frac{u_i - u_{i-1}}{\Delta x}$  the classical difference operators (here, we suppose  $u_{-1} = 0$ ). In order to obtain a positive matrix we have to chose a particular discretization. The numerical scheme for the approximation of the nonlinear problem (KdVK) reads as following :

$$\frac{u^{n+1} - u^n}{\Delta t} + Au^{n+1} + \frac{1}{2}D^- [u^n]^2 = 0 \quad (6.1)$$

where

$$A = \eta D^+ D^+ D^+ D^- D^- + D^+ D^+ D^- + \frac{1}{2}(D^+ + D^-).$$

In the same way, the numerical scheme for Linear problem  $(\text{KdVK})_L$  reads as following :

$$\frac{v^{n+1} - v^n}{\Delta t} + Av^{n+1} = 0 \quad (6.2)$$

In order to obtain estimates for the solution of the numerical scheme for the linear case, we have the following Lemma describing the quadratic forms associated to the different matrices.

**Lemma 6.1.** *For all  $u \in X_N$ , we have*

$$\frac{1}{2}((D^+ + D^-)u, u) = 0 \quad (6.3)$$

$$(D^+D^+D^-u, u) = \frac{1}{2} \left( \frac{1}{\Delta x} [D^-u]_1^2 + \Delta x |D^+D^-u|^2 \right) \quad (6.4)$$

$$(D^+D^+D^+D^-D^-u, u) = -\frac{1}{2} \left( \frac{1}{\Delta x} [D^-D^-u]_1^2 + \Delta x |D^+D^-D^-u|^2 \right) \quad (6.5)$$

where  $(u, v) = \sum_{i=0}^N u_i v_i$  and  $|u| = \sqrt{(u, u)}$ .

*Proof.* The matrix  $\frac{1}{2}(D^+ + D^-)$  is clearly an antisymmetric matrix, and we have the first identity of the Lemma.

The second and third equality is a consequence of the following identity

$$(D^+, z, z) = \frac{1}{2} \left( [z_{N+1}]^2 - [z_1]^2 - \Delta x |D^+z|^2 \right) \quad (6.6)$$

for all  $z \in \mathbb{R}^{N+1}$ , which is easy to prove directly from the formula

$$(a - b)a = \frac{1}{2}(a^2 - b^2) + \frac{1}{2}(a - b)^2. \quad (6.7)$$

We remark that  $(D^+D^+D^-u, u) = -(D^+D^-u, D^-u)$  for all  $u \in X_N$ , and we apply (6.6) with  $z = D^-u$  in order to obtain (6.4).

In the same way, we remark that  $(D^+D^+D^+D^-D^-u, u) = (D^+D^-D^-u, D^-D^-u)$  for all  $u \in X_N$ , and we apply (6.6) with  $z = D^-u$  in order to obtain (6.5).  $\square$

**Corollary 6.1.** *If  $\eta \leq 0$ , the matrix  $I + kA$  is definite positive, and for any  $u^n \in X_N$  there exists a unique  $u^{n+1}$  solution of (6.1). We have the existence and uniqueness solution for the linear case (6) too.*

The following estimate shows that the numerical schemes (6.1) (6) are  $l^2$ -stables.

**Proposition 6.1.** *For any  $v^n \in X_N$  verifying the linear scheme (6), we have*

$$\begin{aligned} |v^n|^2 &+ \Delta t \sum_{k=0}^{n-1} |v^{k+1} - v^k|^2 \\ &+ \frac{\Delta t}{\Delta x} \sum_{k=1}^n [D^-v^k]_1^2 + \Delta t \Delta x \sum_{k=1}^n |D^+D^-v^k|^2 \\ &- \eta \left( \frac{\Delta t}{\Delta x} \sum_{k=1}^n [D^-D^-v^k]_1^2 + \Delta t \Delta x \sum_{k=1}^n |D^+D^-D^-v^k|^2 \right) = |v^0|^2. \end{aligned}$$

*Proof.* We multiply the numerical scheme (6.1) by  $v^{n+1}$ . Then, applying the Lemma 6.1, and the equality (6.7), with  $a = v^{k+1}$  and  $b = v^k$ , we deduce the identity of the proposition.  $\square$



In order to obtain the unconditional stability for the nonlinear version of the scheme, we will find a discrete estimates that is equivalent to that of (3.28) (see Proposition 3.1). Let us denote by  $x$  the sequence  $x_i = i\Delta x$ . We have the following Lemma :

**Lemma 6.2.** *For all  $u \in X_N$ , we have*

$$\frac{1}{2}((D^+ + D^-)u, xu) = \frac{1}{4}\Delta x^2|D^+u|^2 - \frac{1}{4}u_N^2 - \frac{1}{2}|u|^2 \quad (6.8)$$

$$\begin{aligned} (D^+D^+D^-u, xu) &= \frac{3}{2}\Delta t|D^-u|^2 \\ &\quad + \frac{1}{2}\Delta t\Delta x^2 \left( \sum_{i=2}^{N-3} (i-1)|D^+D^-u|^2 + [D^+D^-u^k]_1^2 \right) \end{aligned} \quad (6.9)$$

$$\begin{aligned} (D^+D^+D^+D^-D^-u, xu) &= -\frac{5}{2}\Delta t|D^-D^-u|^2 - \frac{1}{2}\Delta t\Delta x^2 \left( \sum_{i=2}^{N-3} (i-1)|D^+D^-D^-u^k|^2 \right. \\ &\quad \left. - 2(N-2)[D^+D^-D^-u]_{N-2}^2 - 2[D^+D^-D^-u^k]_1^2 \right) \end{aligned} \quad (6.10)$$

where  $(xu)_i = i\Delta x u_i$ .

*Proof.* We note that  $(D^-xu)_i = (xD^-u)_i + \frac{1}{\Delta x}u_{i-1}$ . Then

$$((D^+ + D^-)u, xu) = -\sum_{i=1}^N u_i x_i ((D^-u)_i + u_{i-1}) + (xD^-u, u) = -(D^+u, u) - |u|^2.$$

Using the identity (6.6) in this last relation, we conclude (6.8). Following the same idea and applying the same identity (6.6) is easy to prove (6.9) and (6.10)  $\square$

**Proposition 6.2.** *For any  $v^n \in X_N$  verifying the linear scheme (6), we have*

$$\begin{aligned} x|v^n|^2 &+ \Delta t \sum_{k=0}^{n-1} |\sqrt{x}(v^{k+1} - v^k)|^2 \\ &+ \left( 3\Delta t + \frac{1}{2}\Delta x^2 \right) \sum_{k=1}^n |D^-v^k|^2 + \Delta t\Delta x^2 \sum_{k=1}^n \sum_{i=2}^{N-3} (i-1)|D^+D^-v^k|^2 \\ &- 5\eta\Delta t \sum_{k=1}^n |D^-D^-v^k|^2 - \eta\Delta t\Delta x^2 \sum_{k=1}^n \sum_{i=2}^{N-3} (i-1)|D^+D^-D^-v^k|^2 \\ &+ \Delta t\Delta x^2 \sum_{k=0}^{n-1} \left( [D^+D^-v^k]_1^2 - 2\eta(N-2)[D^+D^-D^-v^k]_{N-2}^2 - 2\eta[D^+D^-D^-v^k]_1^2 \right) \\ &= x|v^0|^2 + \sum_{k=1}^n \left( |v^k|^2 + \frac{1}{2}v_N^2 \right). \end{aligned}$$

*Proof.* We multiply the numerical scheme (6.1) by  $xv^{n+1}$ . Then, applying the Lemma 6.2, and the equality (6.7), with  $a = \sqrt{x}v^{k+1}$  and  $b = \sqrt{x}v^k$ , we deduce the identity of the proposition.  $\square$

From the identities of Proposition 6.1 and 6.2 and following similar analysis of the proof of Proposition 3.1, we deduce the following result :

**Proposition 6.3.** *If  $\eta \leq 0$ , there exists a constant  $C > 0$  (independent of  $\Delta t$  and  $\Delta x$ ) such that for any  $v^p \in X_N$  verifying the linear scheme (6), it satisfies :*

$$\begin{aligned} \sup_{k=0, \dots, p} |v^k|^2 &\leq C|v_0|^2 \\ \Delta t \sum_{k=0}^p |D^- u^k|^2 &\leq C|v_0|^2. \end{aligned}$$

Moreover, if  $\eta < 0$  we have

$$\Delta t \sum_{k=0}^p |D^- D^- u^k|^2 \leq C|u_0|^2.$$

Now, let us introduce the non-homogeneous linear scheme approximating the solution of the (KdVK)<sub>NH</sub> problem :

$$\frac{v^{n+1} - v^n}{\Delta t} + Av^{n+1} = f^n.$$

The existence proof of the continuous case studied in the previous sections applies in the discrete non-homogeneous linear case and the discrete nonlinear case for any discretization of the non-linear part, in particular for  $f_n = \frac{1}{2}D^- [u^n]^2$ . Thus, we obtain the following result of convergence :

**Theorem 6.1.** *For any  $u^n \in X_N$  verifying the non-linear scheme (6.1), with  $\alpha = 1$ , and  $\eta \leq 0$ , there exists  $\varepsilon_0 > 0$  such that, if  $\Delta t \leq \varepsilon_0$ , then there exists  $T > 0$  and a constant  $C = C(T) > 0$  (independent of  $\Delta t$  and  $\Delta x$ ) such that :*

$$\begin{aligned} \sup_{k=0, \dots, p} |v^k|^2 &\leq C|v_0|^2 \\ \Delta t \sum_{k=0}^p |D^- u^k|^2 &\leq C|v_0|^2. \end{aligned}$$

Moreover, if  $\eta < 0$  we have

$$\Delta t \sum_{k=0}^p |D^- D^- u^k|^2 \leq C|u_0|^2.$$

This result means that independently of the discretization of the nonlinear term, the scheme is unconditionally stable. Let us observe that in agreement with the gain of regularity of the (KdVK) equation, we obtain an additional estimation with respect to the analogous numerical scheme of the KdV equation, studied in detail in [7].

### Some numerical results.

We study the re-normalized Korteweg-de Vries-Kawahara equation with initial and boundary condition on  $[0, L]$ , with  $L = 10$  :

$$u_t + \frac{1}{L^5} \eta u_{xxxxx} + \frac{1}{L^3} \eta u_{xxx} + \frac{1}{L} u u_x + \frac{1}{L} u_x = 0.$$

We have taken  $\eta = -1$ ,  $\Delta t = 2.5 \times 10^{-5}$ ,  $\Delta x = 5 \times 10^{-4}$ . We compute the solution during 9600 iterations in time, that is on the time interval  $[0, T]$  with  $T = 0.24$ .

The initial value (Figure 1) is

$$u_0(x) = \frac{\alpha}{\cosh^2(\beta L(x - 1/2))} + \frac{4\alpha}{\cosh^2(2\beta L(x - 1/8))}$$

with  $\alpha = 12\beta^2$  and  $\beta = 2$ . This correspond to the superposition of two solitons with different speeds. Using, Matlab we obtain In Figure 2, we have represented the solution at time  $t_i = iT/10$  for  $i = 1, \dots, 9$ .

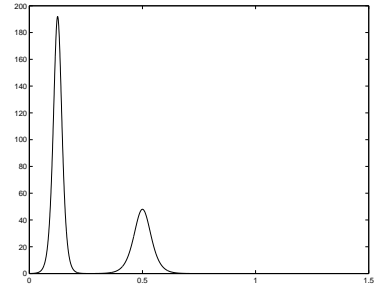


Figure 1 : Initial Value  $u_o$

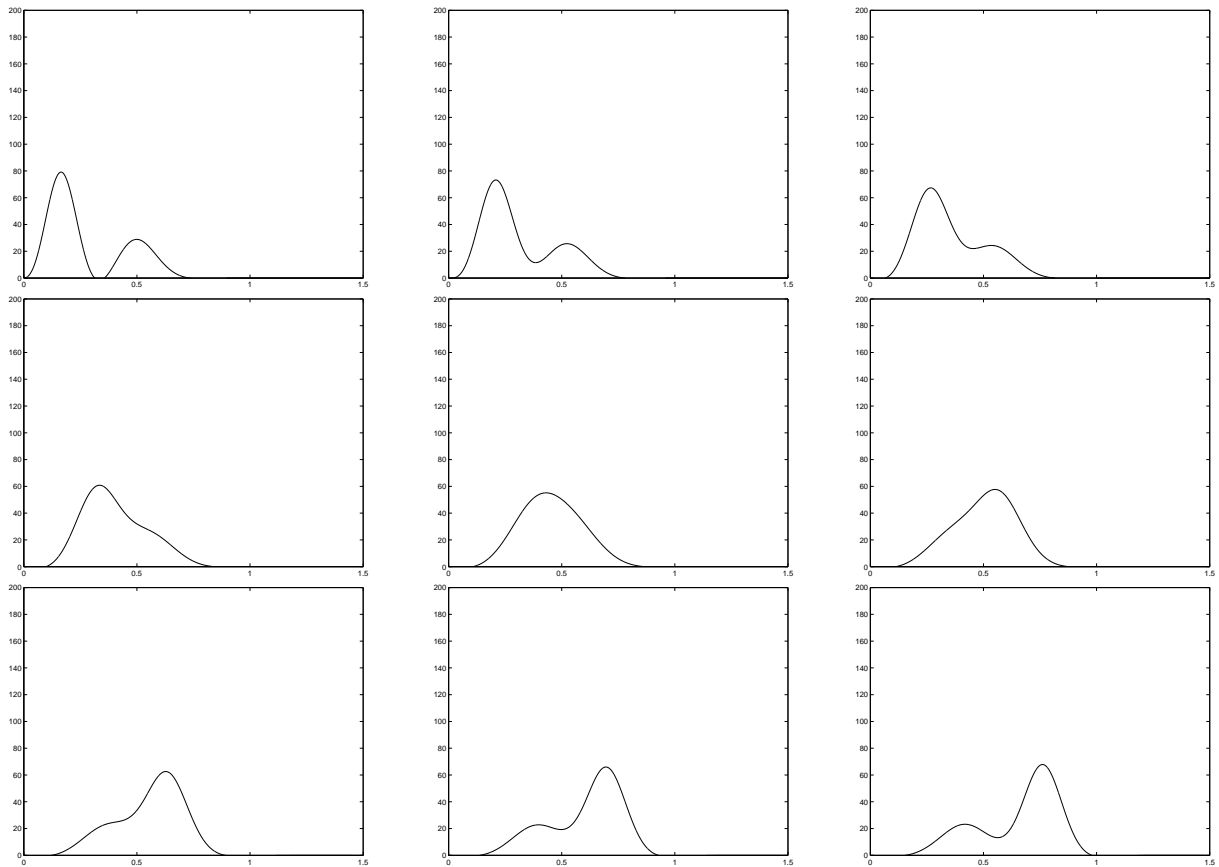


Figure 2 : Interaction of two solitons for the Korteweg-de Vries- Kawahara equation ( $\eta = -1$ ).

In figure 3, we have represented the solution of a KdV equation (with  $\eta = 0$ ), with  $\Delta t = 2.5 \times 10^{-6}$ ,  $\Delta x = 5 \times 10^{-5}$ . We compute the solution during 96000 iterations in time.

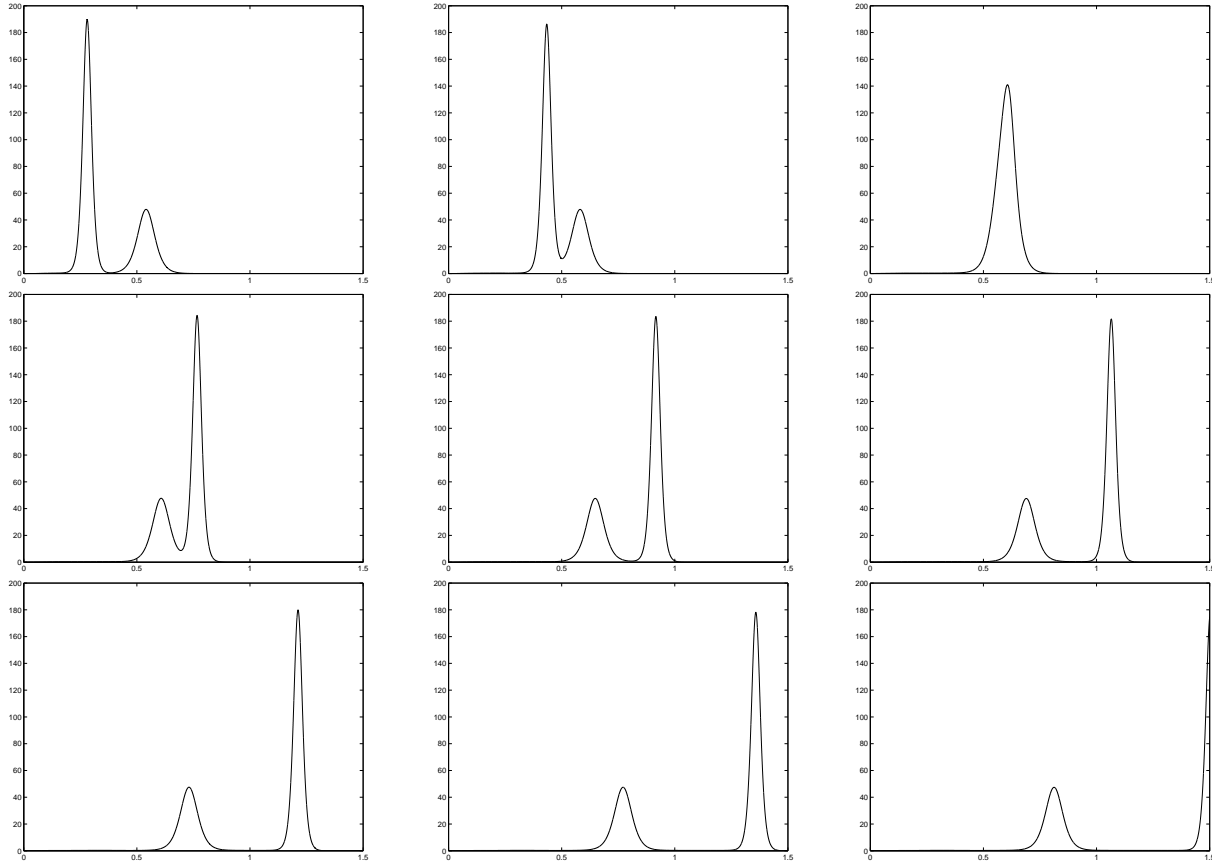


Figure 3 : Interaction of two solitons for the Korteweg-de Vries equation ( $\eta = 0$ ).

The validity of the results can be expressed by the graphic of the function  $t \mapsto \|u_{\Delta}(\cdot, t)\|_{L^2(0,L)}$ , where  $u_{\Delta}$  is the discrete solution of the numerical scheme (see Figure 4). In theory, if the support of the solution stays in the interval  $(0, L)$ , the norm  $L^2$  must be conserved. It is not rigorously the case of our simulations, but watching on the graphs of Figures 3, we can say that it is approximately certain for the KdV equation ( $\eta = 0$ ) : we have 0.06% of lost of the norm  $L^2$  value. In the case of the simulation of the KdV-K equation (Figure 2), we note that the first pic touch the boundary  $x = 0$  and then, we have 0.25% of lost of the norm  $L^2$  value.

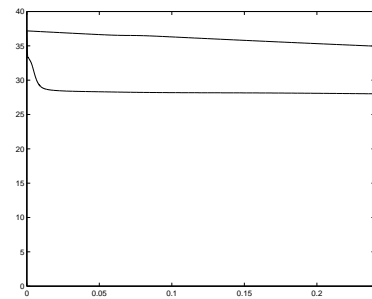


Figure 4 :  $t \mapsto \|u_{\Delta}(\cdot, t)\|_{L^2(0,L)}$  for  $\eta = 0$  and  $\eta = -1$ .

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