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Parametrized Kantorovich-Rubinstein theorem and application to the coupling of random variables

Théorème de Kantorovich-Rubinstein avec paramètre et application au couplage des variables aléatoires

Jérôme DEDECKER,* Clémentine PRIEUR,† Paul RAYNAUD DE FITTE‡

Abstract

We prove a version for random measures of the celebrated Kantorovich-Rubinstein duality theorem and we give an application to the coupling of random variables which extends and unifies known results.

Résumé

Nous démontrons une version du théorème de dualité de Kantorovich-Rubinstein pour les mesures aléatoires, et nous donnons une application au couplage des variables aléatoires qui étend et unifie les résultats antérieurs.

1 Introduction and notations

Let μ and ν be two probability measures on a Polish space (\mathbb{S}, d) . In 1970 Dobrušin [9, page 472] proved that there exists a probability measure λ on $\mathbb{S} \times \mathbb{S}$ with margins μ and ν , such that

$$(1) \quad \lambda(\{x \neq y, (x, y) \in \mathbb{S} \times \mathbb{S}\}) = \frac{1}{2} \|\mu - \nu\|_v,$$

where $\|\cdot\|_v$ is the variation norm. More precisely, Dobrušin gave an explicit solution to (1) defined by

$$(2) \quad \lambda(A \times B) = (\mu - \pi_-)(A \cap B) + \frac{\pi_-(A)\pi_+(B)}{\pi_+(\mathbb{S})} \quad \text{for } A, B \text{ in } \mathcal{B}_{\mathbb{S}},$$

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where $\mu - \nu = \pi_+ - \pi_-$ is the Hahn decomposition of $\pi = \mu - \nu$.

Starting from (2) (see [1, Proposition 4.2.1]), Berbee obtained the following coupling result ([1, Corollary 4.2.5]): let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, let \mathcal{M} be a σ -algebra of \mathcal{A} , and let X be a random variable with values in \mathbb{S} . Denote by P_X the distribution of X and by $P_{X|\mathcal{M}}$ a regular conditional distribution of X given \mathcal{M} . If Ω is rich enough, there exists X^* distributed as X and independent of \mathcal{M} such that

$$(3) \quad \mathbb{P}(X \neq X^*) = \frac{1}{2}E(\|P_{X|\mathcal{M}} - P_X\|_v).$$

To prove (3), Berbee built a couple (X, X^*) whose conditional distribution given \mathcal{M} is the random probability λ_ω defined by (2), with random margins $\mu = P_{X|\mathcal{M}}$ and $\nu = P_X$.

It is by now well known that Dobrušin's result (1) is a particular case of the Kantorovich-Rubinstein duality theorem (which we recall at the beginning of Section 2) applied to the discrete metric $c(x, y) = \mathbb{1}_{x \neq y}$ (see [16, page 93]). Starting from this simple remark, Berbee's proof can be described as follows: one can find a couple (X, X^*) whose conditional distribution with respect to \mathcal{M} solves the duality problem with cost function $c(x, y) = \mathbb{1}_{x \neq y}$ and random margins $\mu = P_{X|\mathcal{M}}$ and $\nu = P_X$.

A reasonable question is then: for what class of cost functions can we obtain the same kind of coupling than Berbee's? Or, equivalently, given two random probabilities μ_ω and ν_ω on a Polish space (\mathbb{S}, d) , for what class of cost functions is there a random probability on $\mathbb{S} \times \mathbb{S}$ solution to the duality problem with margins (μ_ω, ν_ω) ? In 2004, this question has been partially answered in two independent works. In Proposition 1.2 of [5] the authors prove the existence of such a random probability for the cost function $c = d$. From the proof of Theorem 3.4.1 in [3], we see that this result holds in fact for any distance c which is continuous with respect to d . To summarize, we know that there exists a random probability solution to the duality problem with cost function c and given random margins in two distinct situations: on one hand c is the discrete metric, on the other hand c is any continuous distance with respect to d . A general result containing both situations as particular cases would be more satisfactory.

The main result of this paper (point 1 of Theorem 2.1) asserts that there exists a random probability on $\mathbb{S} \times \mathbb{S}$ solution to the duality problem with given random margins provided the cost function c satisfies

$$(4) \quad c(x, y) = \sup_{u \in \text{Lip}_\mathbb{S}^{(c)}} |u(x) - u(y)|,$$

where $\text{Lip}_\mathbb{S}^{(c)}$ is the class of continuous bounded functions u on \mathbb{S} such that $|u(x) - u(y)| \leq c(x, y)$. As in [3, Theorem 3.4.1], the main tool to prove this result is a measurable selection lemma (see Lemma 2.2) for an appropriate multifunction.

Starting from point 1 of Theorem 2.1, we prove in point 2 of Theorem 2.1 that the parametrized Kantorovich–Rubinštein theorem given in [3, Theorem 3.4.1] still holds for any cost function c satisfying (4). Next, we give in Section 3 the application of Theorem 2.1 to the coupling of random variables. In particular, Corollary 3.2 extends Berbee’s coupling in the following way: if $(\Omega, \mathcal{A}, \mathbb{P})$ is rich enough, and if c is a mapping satisfying (4) such that $\int c(X, x_0)d\mathbb{P}$ is finite for some x_0 in \mathbb{S} , then there exists a random variable X^* distributed as X and independent of \mathcal{M} such that

$$(5) \quad E(c(X, X^*)) = \left\| \sup_{f \in \text{Lip}_{\mathbb{S}}^{(c)}} \left| \int f(x)P_{X|\mathcal{M}}(dx) - \int f(x)P_X(dx) \right| \right\|_1.$$

If $c(x, y) = \mathbb{1}_{x \neq y}$ is the discrete metric, (5) is exactly Berbee’s coupling (3). If $c = d$, (5) has been proved in [5, Corollaire 2.2] (see also [7, Section 7.1]). For more details on the coupling property (5) and its applications, see Section 3.2.

Preliminary notations In the sequel, For any topological space \mathfrak{X} , we denote by $\mathcal{B}_{\mathfrak{X}}$ the Borel σ -algebra of \mathfrak{X} and by $\mathcal{P}(\mathfrak{X})$ the space of probability laws on $(\mathfrak{X}, \mathcal{B}_{\mathfrak{X}})$, endowed with the narrow topology, that is, for every mapping $\varphi : \mathfrak{X} \rightarrow [0, 1]$, the mapping $\mu \mapsto \int_{\mathfrak{X}} \varphi d\mu$ is l.s.c. if and only if φ is l.s.c.

Throughout, \mathbb{S} is a given completely regular topological space and $(\Omega, \mathcal{A}, \mathbb{P})$ a given probability space. Our results are new (at least we hope so) even in the setting of Polish spaces or simply the real line. However they are valid in much more general spaces, without significant changes in the proofs. *The reader who is not interested by this level of generality may assume as well in the sequel that all topological spaces we consider are Polish.* On the other hand, we give in appendix some definitions and references which might be useful for a complete reading.

2 Parametrized Kantorovich–Rubinštein theorem

The results of this section draw inspiration from [3, §3.4].

For any $\mu, \nu \in \mathcal{P}(\mathbb{S})$, let $D(\mu, \nu)$ be the set of probability laws π on $(\mathbb{S} \times \mathbb{S}, \mathcal{B}_{\mathbb{S} \times \mathbb{S}})$ with margins μ and ν , that is, $\pi(A \times \mathbb{S}) = \mu(A)$ and $\pi(\mathbb{S} \times A) = \nu(A)$ for every $A \in \mathcal{B}_{\mathbb{S}}$. Let us recall the

Kantorovich–Rubinštein duality theorem [14], [16, Theorem 4.6.6]

Assume that \mathbb{S} is a completely regular pre-Radon space¹, that is, every finite τ -additive Borel measure on \mathbb{S} is inner regular with respect to the compact

¹In [14] and [16, Theorem 4.6.6], the space \mathbb{S} is assumed to be a universally measurable subset of some compact space. But this amounts to assume that it is completely regular and pre-Radon: see [16, Lemma 4.5.17] and [12, Corollary 11.8].

subsets of \mathbb{S} . Let $c : \mathbb{S} \times \mathbb{S} \rightarrow [0, +\infty]$ be a universally measurable mapping. For every $(\mu, \nu) \in \mathcal{P}(\mathbb{S}) \times \mathcal{P}(\mathbb{S})$, let us denote

$$\begin{aligned}\Delta_{\text{KR}}^{(c)}(\mu, \nu) &:= \inf_{\pi \in D(\mu, \nu)} \int_{\mathbb{S} \times \mathbb{S}} c(x, y) d\pi(x, y), \\ \Delta_{\text{L}}^{(c)}(\mu, \nu) &:= \sup_{f \in \text{Lip}_{\mathbb{S}}^{(c)}} (\mu(f) - \nu(f))\end{aligned}$$

where $\text{Lip}_{\mathbb{S}}^{(c)} = \{u \in C_b(\mathbb{S}); \forall x, y \in \mathbb{S} \quad |u(x) - u(y)| \leq c(x, y)\}$. Then the equality $\Delta_{\text{KR}}^{(c)}(\mu, \nu) = \Delta_{\text{L}}^{(c)}(\mu, \nu)$ holds for all $(\mu, \nu) \in \mathcal{P}(\mathbb{S}) \times \mathcal{P}(\mathbb{S})$ if and only if (4) holds.

Note that, if c satisfies (4), it is the supremum of a set of continuous functions, thus it is l.s.c. Every continuous metric c on \mathbb{S} satisfies (4) (see [16, Corollary 4.5.7]), and, if \mathbb{S} is compact, every l.s.c. metric c on \mathbb{S} satisfies (4) (see [16, Remark 4.5.6]).

Now, we denote

$$\mathcal{Y}(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{S}) = \{\mu \in \mathcal{P}(\Omega \times \mathbb{S}, \mathcal{A} \otimes \mathcal{B}_{\mathbb{S}}); \forall A \in \mathcal{A} \quad \mu(A \times \mathbb{S}) = \mathbb{P}(A)\}.$$

When no confusion can arise, we omit some part of the information, and use notations such as $\mathcal{Y}(\mathcal{A})$ or simply \mathcal{Y} (same remark for the set $\mathcal{Y}^{c,1}(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{S})$ defined below). If \mathbb{S} is a Radon space, every $\mu \in \mathcal{Y}$ is *disintegrable*, that is, there exists a (unique, up to P-a.e. equality) \mathcal{A}_{μ}^* -measurable mapping $\omega \mapsto \mu_{\omega}$, $\Omega \rightarrow \mathcal{P}(\mathbb{S})$, such that

$$\mu(f) = \int_{\Omega} \int_{\mathbb{S}} f(\omega, x) d\mu_{\omega}(x) d\mathbb{P}(\omega)$$

for every measurable $f : \Omega \times \mathbb{S} \rightarrow [0, +\infty]$ (see [21]). If furthermore the compact subsets of \mathbb{S} are metrizable, the mapping $\omega \mapsto \mu_{\omega}$ can be chosen \mathcal{A} -measurable, see the Appendix.

Let c satisfy (4). We denote

$$\mathcal{Y}^{c,1}(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{S}) = \{\mu \in \mathcal{Y}; \int_{\Omega \times \mathbb{S}} c(x, x_0) d\mu(\omega, x) < +\infty\}$$

where x_0 is some fixed element of \mathbb{S} (this definition is independent of the choice of x_0). For any $\mu, \nu \in \mathcal{Y}$, let $\underline{D}(\mu, \nu)$ be the set of probability laws π on $\Omega \times \mathbb{S} \times \mathbb{S}$ such that $\pi(\cdot \times \cdot \times \mathbb{S}) = \mu$ and $\pi(\cdot \times \mathbb{S} \times \cdot) = \nu$. We now define the parametrized versions of $\Delta_{\text{KR}}^{(c)}$ and $\Delta_{\text{L}}^{(c)}$. Set, for $\mu, \nu \in \mathcal{Y}^{c,1}$,

$$\underline{\Delta}_{\text{KR}}^{(c)}(\mu, \nu) = \inf_{\pi \in \underline{D}(\mu, \nu)} \int_{\Omega \times \mathbb{S} \times \mathbb{S}} c(x, y) d\pi(\omega, x, y).$$

Let also $\underline{\text{Lip}}^{(c)}$ denote the set of measurable integrands $f : \Omega \times \mathbb{S} \rightarrow \mathbb{R}$ such that $f(\omega, \cdot) \in \text{Lip}_{\mathbb{S}}^{(c)}$ for every $\omega \in \Omega$. We denote

$$\underline{\Delta}_{\text{L}}^{(c)}(\mu, \nu) = \sup_{f \in \underline{\text{Lip}}^{(c)}} (\mu(f) - \nu(f)).$$

Theorem 2.1 (Parametrized Kantorovich–Rubinštejn theorem) *Assume that \mathbb{S} is a completely regular Radon space and that the compact subsets of \mathbb{S} are metrizable (e.g. \mathbb{S} is a regular Suslin space). Let $c : \mathbb{S} \times \mathbb{S} \rightarrow [0, +\infty[$ satisfy (4). Let $\mu, \nu \in \mathcal{Y}^{c,1}$ and let $\omega \mapsto \mu_\omega$ and $\omega \mapsto \nu_\omega$ be disintegrations of μ and ν respectively.*

1. *Let $G : \omega \mapsto \Delta_{\text{KR}}^{(c)}(\mu_\omega, \nu_\omega) = \Delta_{\text{L}}^{(c)}(\mu_\omega, \nu_\omega)$ and let \mathcal{A}^* be the universal completion of \mathcal{A} . There exists an \mathcal{A}^* -measurable mapping $\omega \mapsto \lambda_\omega$ from Ω to $\mathcal{P}(\mathbb{S} \times \mathbb{S})$ such that λ_ω belongs to $D(\mu_\omega, \nu_\omega)$ and*

$$G(\omega) = \int_{\mathbb{S} \times \mathbb{S}} c(x, y) d\lambda_\omega(x, y).$$

2. *The following equalities hold:*

$$\underline{\Delta}_{\text{KR}}^{(c)}(\mu, \nu) = \int_{\Omega \times \mathbb{S} \times \mathbb{S}} c(x, y) d\lambda(\omega, x, y) = \underline{\Delta}_{\text{L}}^{(c)}(\mu, \nu),$$

where λ is the element of $\mathcal{Y}(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{S} \times \mathbb{S})$ defined by $\lambda(A \times B \times C) = \int_A \lambda_\omega(B \times C) d\mathbb{P}(\omega)$ for any A in \mathcal{A} , B and C in $\mathcal{B}_{\mathbb{S}}$. In particular, λ belongs to $\underline{D}(\mu, \nu)$, and the infimum in the definition of $\underline{\Delta}_{\text{KR}}^{(c)}(\mu, \nu)$ is attained for this λ .

Let us first prove the following lemma. The set of compact subsets of a topological space \mathfrak{T} is denoted by $\mathcal{K}(\mathfrak{T})$.

Lemma 2.2 (A measurable selection lemma) *Assume that \mathbb{S} is a Suslin space. Let $c : \mathbb{S} \times \mathbb{S} \rightarrow [0, +\infty]$ be an l.s.c. mapping. Let \mathcal{B}^* be the universal completion of the σ -algebra $\mathcal{B}_{\mathcal{P}(\mathbb{S}) \times \mathcal{P}(\mathbb{S})}$. For any $\mu, \nu \in \mathcal{P}(\mathbb{S})$, let*

$$r(\mu, \nu) = \inf_{\pi \in D(\mu, \nu)} \int c(x, y) d\pi(x, y) \in [0, +\infty].$$

The function r is \mathcal{B}^ -measurable. Furthermore, the multifunction*

$$K : \begin{cases} \mathcal{P}(\mathbb{S}) \times \mathcal{P}(\mathbb{S}) & \rightarrow \mathcal{K}(\mathcal{P}(\mathbb{S} \times \mathbb{S})) \\ (\mu, \nu) & \mapsto \{\pi \in D(\mu, \nu); \int c(x, y) d\pi(x, y) = r(\mu, \nu)\} \end{cases}$$

has a \mathcal{B}^ -measurable selection, that is, there exists a \mathcal{B}^* -measurable mapping $\lambda : (\mu, \nu) \mapsto \lambda_{\mu, \nu}$ defined on $\mathcal{P}(\mathbb{S}) \times \mathcal{P}(\mathbb{S})$ with values in $\mathcal{K}(\mathcal{P}(\mathbb{S} \times \mathbb{S}))$, such that $\lambda_{\mu, \nu} \in K(\mu, \nu)$ for all $\mu, \nu \in \mathcal{P}(\mathbb{S})$.*

Proof. Observe first that the mapping r can be defined as

$$r : (\mu, \nu) \mapsto \inf \{\psi(\pi); \pi \in D(\mu, \nu)\},$$

with

$$\psi : \begin{cases} \mathcal{P}(\mathbb{S} \times \mathbb{S}) & \rightarrow [0, +\infty] \\ \pi & \mapsto \int_{\mathbb{S} \times \mathbb{S}} c(x, y) d\pi(x, y). \end{cases}$$

The mapping ψ is l.s.c. because it is the supremum of the l.s.c. mappings $\pi \mapsto \pi(c \wedge n)$, $n \in \mathbb{N}$ (if c is bounded and continuous, ψ is continuous). Furthermore, we have $D = \Phi^{-1}$, where Φ is the continuous mapping

$$\Phi : \begin{cases} \mathcal{P}(\mathbb{S} \times \mathbb{S}) & \rightarrow \mathcal{P}(\mathbb{S}) \times \mathcal{P}(\mathbb{S}) \\ \lambda & \mapsto (\lambda(\cdot \times \mathbb{S}), \lambda(\mathbb{S} \times \cdot)) \end{cases}$$

(recall that $D(\mu, \nu)$ is the set of probability laws π on $\mathbb{S} \times \mathbb{S}$ with margins μ and ν). Therefore, the graph $\text{gph}(D)$ of D is a closed subset of the Suslin space $\mathbb{X} = (\mathcal{P}(\mathbb{S}) \times \mathcal{P}(\mathbb{S})) \times \mathcal{P}(\mathbb{S} \times \mathbb{S})$. Thus, for every $\alpha \in \mathbb{R}$, the set

$$\{((\mu, \nu), \pi) \in \text{gph}(D); \psi(\pi) < \alpha\}$$

is a Suslin subset of \mathbb{X} . We thus have, by the Projection Theorem (see [4, Lemma III.39]),

$$\begin{aligned} \forall \alpha \in \mathbb{R} \quad & \{(\mu, \nu); r(\mu, \nu) < \alpha\} \\ & = \text{proj}_{\mathcal{P}(\mathbb{S}) \times \mathcal{P}(\mathbb{S})} \{((\mu, \nu), \pi) \in \text{gph}(D); \psi(\pi) < \alpha\} \in \mathcal{B}^*. \end{aligned}$$

Now, for each $(\mu, \nu) \in \mathcal{P}(\mathbb{S}) \times \mathcal{P}(\mathbb{S})$, we have

$$K(\mu, \nu) = \{\pi \in D(\mu, \nu); \psi(\pi) = r(\mu, \nu)\}.$$

The multifunction K has nonempty compact values because D has nonempty compact values and ψ is l.s.c. Let

$$F : \begin{cases} (\mathcal{P}(\mathbb{S}) \times \mathcal{P}(\mathbb{S})) \times \mathcal{P}(\mathbb{S} \times \mathbb{S}) & \rightarrow \mathbb{R} \\ ((\mu, \nu), \pi) & \mapsto \psi(\pi) - r(\mu, \nu). \end{cases}$$

The mapping F is $\mathcal{B}^* \otimes \mathcal{B}_{\mathcal{P}(\mathbb{S} \times \mathbb{S})}$ -measurable. Furthermore, the graph of K is

$$\begin{aligned} \text{gph}(K) &= \{((\mu, \nu), \pi); \mu = \pi(\cdot \times \mathbb{S}), \nu = \pi(\mathbb{S} \times \cdot), F((\mu, \nu), \pi) = 0\} \\ &= \text{gph}(D) \cap F^{-1}(0) \\ &\in \mathcal{B}^* \otimes \mathcal{B}_{\mathcal{P}(\mathbb{S} \times \mathbb{S})}. \end{aligned}$$

As \mathbb{S} is Suslin, this proves that K is \mathcal{B}^* -measurable (see [4, Theorem III.22]). Thus K has a \mathcal{B}^* -measurable selection. \square

Proof of Theorem 2.1. By the Radon property, the probability measures $\mu(\Omega \times \cdot)$ and $\nu(\Omega \times \cdot)$ are tight, that is, for every integer $n \geq 1$, there exists a compact subset K_n of \mathbb{S} such that $\mu(\Omega \times (\mathbb{S} \setminus K_n)) \leq 1/n$ and $\nu(\Omega \times (\mathbb{S} \setminus K_n)) \leq 1/n$. Now, we can clearly replace \mathbb{S} in the statements of Theorem 2.1 by the smaller space $\cup_{n \geq 1} K_n$. But $\cup_{n \geq 1} K_n$ is Suslin (and even Lusin), so we can assume without loss of generality that \mathbb{S} is a regular Suslin space.

We easily have

$$\begin{aligned}
\Delta_{\mathbb{L}}^{(c)}(\mu, \nu) &= \sup_{f \in \underline{\text{Lip}}^{(c)}} \int_{\Omega} \int_{\mathbb{S}} \int_{\mathbb{S}} (f(\omega, x) - f(\omega, y)) d\mu_{\omega}(x) d\nu_{\omega}(y) d\mathbb{P}(\omega) \\
&\leq \int_{\Omega} \int_{\mathbb{S}} \int_{\mathbb{S}} c(x, y) d\mu_{\omega}(x) d\nu_{\omega}(y) d\mathbb{P}(\omega) \\
(6) \quad &\leq \Delta_{\text{KR}}^{(c)}(\mu, \nu).
\end{aligned}$$

So, to prove Theorem 2.1, we only need to prove that $\Delta_{\text{KR}}^{(c)}(\mu, \nu) \leq \Delta_{\mathbb{L}}^{(c)}(\mu, \nu)$ and that the minimum in the definition of $\Delta_{\text{KR}}^{(c)}(\mu, \nu)$ is attained.

Using the notations of Lemma 2.2, we have $G(\omega) = r(\mu_{\omega}, \nu_{\omega})$, thus G is \mathcal{A}^* -measurable (indeed, the mapping $\omega \mapsto (\mu_{\omega}, \nu_{\omega})$ is measurable for \mathcal{A}^* and \mathcal{B}^* because it is measurable for \mathcal{A} and $\mathcal{B}_{\mathcal{P}(\mathbb{S}) \times \mathcal{P}(\mathbb{S})}$). From Lemma 2.2, the multifunction $\omega \mapsto D(\mu_{\omega}, \nu_{\omega})$ has an \mathcal{A}^* -measurable selection $\omega \mapsto \lambda_{\omega}$ such that, for every $\omega \in \Omega$, $G(\omega) = \int_{\mathbb{S} \times \mathbb{S}} c(x, y) d\lambda_{\omega}(x, y)$. We thus have

$$(7) \quad \Delta_{\text{KR}}^{(c)}(\mu, \nu) \leq \int_{\Omega \times \mathbb{S} \times \mathbb{S}} c(x, y) d\lambda(\omega, x, y) = \int_{\Omega} G(\omega) d\mathbb{P}(\omega).$$

Furthermore, since $\mu, \nu \in \mathcal{Y}^{c,1}$, we have $G(\omega) < +\infty$ a.e. Let Ω_0 be the almost sure set on which $G(\omega) < +\infty$. Fix an element x_0 in \mathbb{S} . We have, for every $\omega \in \Omega_0$,

$$G(\omega) = \sup_{g \in \text{Lip}_{\mathbb{S}}^{(c)}} (\mu_{\omega}(g) - \nu_{\omega}(g)) = \sup_{g \in \text{Lip}_{\mathbb{S}}^{(c)}, g(x_0)=0} (\mu_{\omega}(g) - \nu_{\omega}(g)).$$

Let $\epsilon > 0$. Let $\tilde{\mu}$ and $\tilde{\nu}$ be the finite measures on \mathbb{S} defined by

$$\tilde{\mu}(B) = \int_{\Omega \times B} c(x_0, x) d\mu(\omega, x) \quad \text{and} \quad \tilde{\nu}(B) = \int_{\Omega \times B} c(x_0, x) d\nu(\omega, x)$$

for any $B \in \mathcal{B}_{\mathbb{S}}$. Let \mathbb{S}_0 be a compact subset of \mathbb{S} containing x_0 such that $\tilde{\mu}(\mathbb{S} \setminus \mathbb{S}_0) \leq \epsilon$ and $\tilde{\nu}(\mathbb{S} \setminus \mathbb{S}_0) \leq \epsilon$. For any $f \in \underline{\text{Lip}}^{(c)}$, we have

$$\begin{aligned}
(8) \quad &\left| \int_{\Omega} (\mu_{\omega} - \nu_{\omega})(f(\omega, \cdot)) d\mathbb{P}(\omega) - \int_{\Omega} (\mu_{\omega} - \nu_{\omega})(f(\omega, \cdot) \mathbb{1}_{\mathbb{S}_0}) d\mathbb{P}(\omega) \right| \\
&= \left| \int_{\Omega} (\mu_{\omega} - \nu_{\omega})(f(\omega, \cdot) \mathbb{1}_{\mathbb{S} \setminus \mathbb{S}_0}) d\mathbb{P}(\omega) \right| \leq 2\epsilon.
\end{aligned}$$

Set, for all $\omega \in \Omega_0$,

$$G'(\omega) = \sup_{g \in \text{Lip}_{\mathbb{S}}^{(c)}, g(x_0)=0} (\mu_{\omega} - \nu_{\omega})(g \mathbb{1}_{\mathbb{S}_0}).$$

We thus have

$$(9) \quad \left| \int_{\Omega_0} G d\mathbb{P} - \int_{\Omega_0} G' d\mathbb{P} \right| \leq 2\epsilon.$$

Let $\text{Lip}_{\mathbb{S}}^{(c)}|_{\mathbb{S}_0}$ denote the set of restrictions to \mathbb{S}_0 of elements of $\text{Lip}_{\mathbb{S}}^{(c)}$. The set \mathbb{S}_0 is metrizable, thus $C_b(\mathbb{S}_0)$ (endowed with the topology of uniform convergence) is metrizable separable, thus its subspace $\text{Lip}_{\mathbb{S}}^{(c)}|_{\mathbb{S}_0}$ is also metrizable separable. We can thus find a dense countable subset $D = \{u_n; n \in \mathbb{N}\}$ of $\text{Lip}_{\mathbb{S}}^{(c)}$ for the seminorm $\|u\|_{C_b(\mathbb{S}_0)} := \sup_{x \in \mathbb{S}_0} |u(x)|$. Set, for all $(\omega, x) \in \Omega_0 \times \mathbb{S}$,

$$N(\omega) = \min\{n \in \mathbb{N}; \int_{\mathbb{S}} u_n(x) d(\mu_\omega - \nu_\omega)(x) \geq G'(\omega) - \epsilon\},$$

and $f(\omega, x) = u_{N(\omega)}(x)$.

We then have, using (8) and (9),

$$\begin{aligned} \underline{\Delta}_{\text{L}}^{(c)}(\mu, \nu) &\geq \int_{\Omega_0 \times \mathbb{S}} f d(\mu - \nu) \geq \int_{\Omega_0 \times \mathbb{S}_0} f d(\mu - \nu) - 2\epsilon \\ &\geq \int_{\Omega_0} G' d\mathbb{P} - 3\epsilon \geq \int_{\Omega_0} G d\mathbb{P} - 5\epsilon. \end{aligned}$$

Thus, in view of (6) and (7),

$$\underline{\Delta}_{\text{KR}}^{(c)}(\mu, \nu) = \int_{\Omega \times \mathbb{S} \times \mathbb{S}} c(x, y) d\lambda(\omega, x, y) = \underline{\Delta}_{\text{L}}^{(c)}(\mu, \nu).$$

□

3 Application: coupling for the minimal distance

In this section \mathbb{S} is a completely regular Radon space with metrizable compact subsets, $c : \mathbb{S} \times \mathbb{S} \rightarrow [0, +\infty]$ is a mapping satisfying (4) and \mathcal{M} is a sub- σ -algebra of \mathcal{A} . Let X be a random variable with values in \mathbb{S} , let \mathbb{P}_X be the distribution of X , and let $\mathbb{P}_{X|\mathcal{M}}$ be a regular conditional distribution of X given \mathcal{M} (see Section 4 for the existence). We assume that $\int c(x, x_0) \mathbb{P}_X(dx)$ is finite for some (and therefore any) x_0 in \mathbb{S} (which means exactly that the unique measure of $\mathcal{Y}(\mathcal{M})$ with disintegration $\mathbb{P}_{X|\mathcal{M}}(\cdot, \omega)$ belongs to $\mathcal{Y}^{c,1}(\mathcal{M})$).

Theorem 3.1 (general coupling theorem) *Assume that Ω is rich enough, that is, there exists a random variable U from (Ω, \mathcal{A}) to $([0, 1], \mathcal{B}([0, 1]))$, independent of $\sigma(X) \vee \mathcal{M}$ and uniformly distributed over $[0, 1]$. Let Q be any element of $\mathcal{Y}^{c,1}(\mathcal{M})$. There exists a $\sigma(U) \vee \sigma(X) \vee \mathcal{M}$ -measurable random variable Y , such that Q is a regular conditional probability of Y given \mathcal{M} , and*

(10)

$$E(c(X, Y)|\mathcal{M}) = \sup_{f \in \text{Lip}_{\mathbb{S}}^{(c)}} \left| \int f(x) \mathbb{P}_{X|\mathcal{M}}(dx) - \int f(x) Q.(dx) \right| \quad \mathbb{P}\text{-a.s.}$$

Proof. We apply Theorem 2.1 to the probability space $(\Omega, \mathcal{M}, \mathbb{P})$ and to the disintegrated measures $\mu_\omega(\cdot) = \mathbb{P}_{X|\mathcal{M}}(\cdot, \omega)$ and $\nu_\omega = Q_\omega$. As in the proof of Theorem 2.1, we assume without loss of generality that \mathbb{S} is Lusin regular. From point 1 of Theorem 2.1 we infer that there exists a mapping $\omega \mapsto \lambda_\omega$ from Ω to $\mathcal{P}(\mathbb{S} \times \mathbb{S})$, measurable for \mathcal{M}^* and $\mathcal{B}_{\mathcal{P}(\mathbb{S} \times \mathbb{S})}$, such that λ_ω belongs to $D(\mathbb{P}_{X|\mathcal{M}}(\cdot, \omega), Q_\omega)$ and $G(\omega) = \int_{\mathbb{S} \times \mathbb{S}} c(x, y) \lambda_\omega(dx, dy)$.

On the measurable space $(\mathbb{M}, \mathcal{T}) = (\Omega \times \mathbb{S} \times \mathbb{S}, \mathcal{M}^* \otimes \mathcal{B}_{\mathbb{S}} \otimes \mathcal{B}_{\mathbb{S}})$ we put the probability

$$\pi(A \times B \times C) = \int_A \lambda_\omega(B \times C) \mathbb{P}(d\omega).$$

If $I = (I_1, I_2, I_3)$ is the identity on \mathbb{M} , we see that a regular conditional distribution of (I_2, I_3) given I_1 is given by $\mathbb{P}_{(I_2, I_3)|I_1=\omega} = \lambda_\omega$. Since $\mathbb{P}_{X|\mathcal{M}}(\cdot, \omega)$ is the first margin of λ_ω , a regular conditional probability of I_2 given I_1 is given by $\mathbb{P}_{I_2|I_1=\omega}(\cdot) = \mathbb{P}_{X|\mathcal{M}}(\cdot, \omega)$. Let $\lambda_{\omega, x} = \mathbb{P}_{I_3|I_1=\omega, I_2=x}$ be a regular conditional distribution of I_3 given (I_1, I_2) , so that $(\omega, x) \mapsto \lambda_{\omega, x}$ is measurable for $\mathcal{M}^* \otimes \mathcal{B}_{\mathbb{S}}$ and $\mathcal{B}_{\mathcal{P}(\mathbb{S})}$. From the unicity (up to P-a.s. equality) of regular conditional probabilities, it follows that

$$(11) \quad \lambda_\omega(B \times C) = \int_B \lambda_{\omega, x}(C) \mathbb{P}_{X|\mathcal{M}}(dx, \omega) \quad \text{P-a.s.}$$

Assume that we can find a random variable \tilde{Y} from Ω to \mathbb{S} , measurable for $\sigma(U) \vee \sigma(X) \vee \mathcal{M}^*$ and $\mathcal{B}_{\mathbb{S}}$, such that $\mathbb{P}_{\tilde{Y}|\sigma(X) \vee \mathcal{M}^*}(\cdot, \omega) = \lambda_{\omega, X(\omega)}(\cdot)$. Since $\omega \mapsto \mathbb{P}_{X|\mathcal{M}}(\cdot, \omega)$ is measurable for \mathcal{M}^* and $\mathcal{B}_{\mathcal{P}(\mathbb{S})}$, one can check that $\mathbb{P}_{X|\mathcal{M}}$ is a regular conditional probability of X given \mathcal{M}^* . For A in \mathcal{M}^* , B and C in $\mathcal{B}_{\mathbb{S}}$, we thus have

$$\begin{aligned} E(\mathbf{1}_A \mathbf{1}_{X \in B} \mathbf{1}_{\tilde{Y} \in C}) &= E(\mathbf{1}_A E(\mathbf{1}_{X \in B} E(\mathbf{1}_{\tilde{Y} \in C} | \sigma(X) \vee \mathcal{M}^*) | \mathcal{M}^*)) \\ &= \int_A \left(\int_B \lambda_{\omega, x}(C) \mathbb{P}_{X|\mathcal{M}}(dx, \omega) \right) \mathbb{P}(d\omega) \\ &= \int_A \lambda_\omega(B \times C) \mathbb{P}(d\omega). \end{aligned}$$

We infer that λ_ω is a regular conditional probability of (X, \tilde{Y}) given \mathcal{M}^* . By definition of λ_ω , we obtain that

$$(12) \quad E(c(X, \tilde{Y}) | \mathcal{M}^*) = \sup_{f \in \text{Lip}_{\mathbb{S}}^{(c)}} \left| \int f(x) \mathbb{P}_{X|\mathcal{M}}(dx) - \int f(x) Q_\omega(dx) \right| \quad \text{P-a.s.}$$

Since \mathbb{S} is Lusin, it is standard Borel (see Section 4). Applying Lemma 4.1, there exists a $\sigma(U) \vee \sigma(X) \vee \mathcal{M}$ -measurable modification Y of \tilde{Y} , so that (12) still holds for $\mathbb{E}(c(X, Y) | \mathcal{M}^*)$. We obtain (10) by noting that $E(c(X, Y) | \mathcal{M}^*) = E(c(X, Y) | \mathcal{M})$ P-a.s.

It remains to build \tilde{Y} . Since \mathbb{S} is standard Borel, there exists a one to one map f from \mathbb{S} to a Borel subset of $[0, 1]$, such that f and f^{-1} are measurable for $\mathcal{B}([0, 1])$ and $\mathcal{B}_{\mathbb{S}}$. Define $F(t, \omega) = \lambda_{\omega, X(\omega)}(f^{-1}([-\infty, t]))$. The map $F(\cdot, \omega)$ is a distribution function with càdlàg inverse $F^{-1}(\cdot, \omega)$. One can see that the map $(u, \omega) \rightarrow F^{-1}(u, \omega)$ is $\mathcal{B}([0, 1]) \otimes \mathcal{M}^* \vee \sigma(X)$ -measurable. Let $T(\omega) = F^{-1}(U(\omega), \omega)$ and $\tilde{Y} = f^{-1}(T)$. It remains to see that $P_{\tilde{Y}|\sigma(X) \vee \mathcal{M}^*}(\cdot, \omega) = \lambda_{\omega, X(\omega)}(\cdot)$. For any A in \mathcal{M}^* , B in $\mathcal{B}_{\mathbb{S}}$ and t in \mathbb{R} , we have

$$E\left(\mathbf{1}_A \mathbf{1}_{X \in B} \mathbf{1}_{\tilde{Y} \in f^{-1}([-\infty, t])}\right) = \int_A \mathbf{1}_{X(\omega) \in B} \mathbf{1}_{U(\omega) \leq F(t, \omega)} P(d\omega).$$

Since U is independent of $\sigma(X) \vee \mathcal{M}$, it is also independent of $\sigma(X) \vee \mathcal{M}^*$. Hence

$$\begin{aligned} E\left(\mathbf{1}_A \mathbf{1}_{X \in B} \mathbf{1}_{\tilde{Y} \in f^{-1}([-\infty, t])}\right) &= \int_A \mathbf{1}_{X(\omega) \in B} F(t, \omega) P(d\omega) \\ &= \int_A \mathbf{1}_{X(\omega) \in B} \lambda_{\omega, X(\omega)}(f^{-1}([-\infty, t])) P(d\omega). \end{aligned}$$

Since $\{f^{-1}([-\infty, t]), t \in [0, 1]\}$ is a separating class, the result follows.

□

Coupling and dependence coefficients Define the coefficient

$$(13) \quad \tau_c(\mathcal{M}, X) = \left\| \sup_{f \in \text{Lip}_{\mathbb{S}}^{(c)}} \left| \int f(x) P_{X|\mathcal{M}}(dx) - \int f(x) P_X(dx) \right| \right\|_1.$$

If $\text{Lip}_{\mathbb{S}}^{(c)}$ is a separating class, this coefficient measures the dependence between \mathcal{M} and X ($\tau_c(\mathcal{M}, X) = 0$ if and only if X is independent of \mathcal{M}). From point 2 of Theorem 2.1, we see that an equivalent definition is

$$\tau_c(\mathcal{M}, X) = \sup_{f \in \text{Lip}_{\mathbb{S}, \mathcal{M}}^{(c)}} \int f(\omega, X(\omega)) P(d\omega) - \int \left(\int f(\omega, x) P_X(dx) \right) P(d\omega).$$

where $\text{Lip}_{\mathbb{S}, \mathcal{M}}^{(c)}$ is the set of integrands f from $\Omega \times \mathbb{S} \rightarrow \mathbb{R}$, measurable for $\mathcal{M} \otimes \mathcal{B}_{\mathbb{S}}$, such that $f(\omega, \cdot)$ belongs to $\text{Lip}_{\mathbb{S}}^{(c)}$ for any $\omega \in \Omega$.

Let $c(x, y) = \mathbf{1}_{x \neq y}$ be the discrete metric and let $\|\cdot\|_v$ be the variation norm. From the Riesz-Alexandroff representation theorem (see [23, Theorem 5.1]), we infer that for any (μ, ν) in $\mathcal{P}(\mathbb{S}) \times \mathcal{P}(\mathbb{S})$,

$$\sup_{f \in \text{Lip}_{\mathbb{S}}^{(c)}} |\mu(f) - \nu(f)| = \frac{1}{2} \|\mu - \nu\|_v.$$

Hence, for the discrete metric $\tau_c(\mathcal{M}, X) = \beta(\mathcal{M}, \sigma(X))$ is the β -mixing coefficient between \mathcal{M} and $\sigma(X)$ introduced in [18]. If c is a distance for which \mathbb{S} is Polish, $\tau_c(\mathcal{M}, X)$ has been introduced in [5] and [7].

Applying Theorem 3.1 with $Q = P \otimes P_X$, we see that this coefficient has a characteristic property which is often called the *coupling* or *reconstruction* property.

Corollary 3.2 (reconstruction property) *If Ω is rich enough (see Theorem 3.1), there exists a $\sigma(U) \vee \sigma(X) \vee \mathcal{M}$ -measurable random variable X^* , independent of \mathcal{M} and distributed as X , such that*

$$(14) \quad \tau_c(\mathcal{M}, X) = E(c(X, X^*)) .$$

If $c(x, y) = \mathbb{1}_{x \neq y}$, (14) is given in [1, Corollary 4.2.5] (note that in Berbee's corollary, \mathbb{S} is assumed to be standard Borel. For other proofs of Berbee's coupling, see [2] and [17, Section 5.3]). If c is a distance for which \mathbb{S} is a Polish space, (14) has been proved by [5].

Coupling is a very useful property in the area of limit theorems and statistics. Many authors have used Berbee's coupling to prove various limit theorems (see for instance the review paper [15] and the references therein) as well as exponential inequalities (see for instance the paper [10] for Bernstein-type inequalities and applications to empirical central limit theorems). Unfortunately, these results apply only to β -mixing sequences, but this property is very hard to check and many simple processes (such as iterates of maps or many non-irreducible Markov chains) are not β -mixing. In many cases however, this difficulty may be overcome by considering another distance c , more adapted to the problem than the discrete metric (typically c is a norm for which \mathbb{S} is a separable Banach space). The case $\mathbb{S} = \mathbb{R}$ and $c(x, y) = |x - y|$, is studied in the paper [6], where many non β -mixing examples are given. In this paper the authors used the coefficients τ_c to prove Bernstein-type inequalities and a strong invariance principle for partial sums. In the paper [7, Section 4.4] the same authors show that if T is a uniformly expanding map preserving a probability μ on $[0, 1]$, then $\tau_c(\sigma(T^n), T) = O(a^n)$ for $c(x, y) = |x - y|$ and some a in $[0, 1]$.

The following inequality (which can be deduced from [15, page 174]) shows clearly that $\beta(\mathcal{M}, \sigma(X))$ is in some sense the more restrictive coefficient among all the $\tau_c(\mathcal{M}, X)$: for any x in \mathbb{S} , we have that

$$(15) \quad \tau_c(\mathcal{M}, X) \leq 2 \int_0^{\beta(\mathcal{M}, \sigma(X))} Q_{c(X, x)}(u) du ,$$

where $Q_{c(X, x)}$ is the generalized inverse of the function $t \mapsto P(c(X, x) > t)$. In particular, if c is bounded by M , $\tau_c(\mathcal{M}, X) \leq 2M\beta(\mathcal{M}, \sigma(X))$.

4 Appendix: topological and measure-theoretical complements

Topological spaces Let us recall some definitions (see [19, 12] for complements on Radon and Suslin spaces). A topological space \mathbb{S} is said to be

- *regular* if, for any $x \in \mathbb{S}$ and any closed subset F of \mathbb{S} which does not contain x , there exist two disjoint open subsets U and V such that $x \in U$ and $F \subset V$,
- *completely regular* if, for any $x \in \mathbb{S}$ and any closed subset F of \mathbb{S} which does not contain x , there exists a continuous function $f : \mathbb{S} \rightarrow [0, 1]$ such that $f(x) = 0$ and $f = 1$ on F (equivalently, \mathbb{S} is *uniformizable*, that is, the topology of \mathbb{S} can be defined by a set of semidistances),
- *pre-Radon* if every finite τ -additive Borel measure on \mathbb{S} is inner regular with respect to the compact subsets of \mathbb{S} (a Borel measure μ on \mathbb{S} is τ -*additive* if, for any family $(F_\alpha)_{\alpha \in A}$ of closed subsets of \mathbb{S} such that $\forall \alpha, \beta \in A \quad \exists \gamma \in A \quad F_\gamma \subset F_\alpha \cap F_\beta$, we have $\mu(\bigcap_{\alpha \in A} F_\alpha) = \inf_{\alpha \in A} \mu(F_\alpha)$),
- *Radon* if every finite Borel measure on \mathbb{S} is inner regular with respect to the compact subsets of \mathbb{S} ,
- *Suslin*, or *analytic*, if there exists a continuous mapping from some Polish space onto \mathbb{S} ,
- *Lusin* if there exists a continuous injective mapping from some Polish space onto \mathbb{S} . Equivalently, \mathbb{S} is Lusin if there exists a Polish topology on \mathbb{S} which is finer than the given topology of \mathbb{S} .

Obviously, every Lusin space is Suslin and every Radon space is pre-Radon. Much less obviously, every Suslin space is Radon. Every regular Suslin space is completely regular.

Many usual spaces of Analysis are Lusin: besides all separable Banach spaces (e.g. L^p ($1 \leq p < +\infty$), or the Sobolev spaces $W^{s,p}(\Omega)$ ($0 < s < 1$ and $1 \leq p < +\infty$)), the spaces of distributions \mathcal{E}' , \mathcal{S}' , \mathcal{D}' , the space $\mathcal{H}(\mathbb{C})$ of holomorphic functions, or the topological dual of a Banach space, endowed with its weak*-topology are Lusin. See [19, pages 112–117] for many more examples.

Standard Borel spaces A measurable space $(\mathbb{M}, \mathcal{M})$ is said to be *standard Borel* if it is Borel-isomorphic with some Polish space \mathbb{T} , that is, there exists a mapping $f : \mathbb{T} \rightarrow \mathbb{M}$ which is one-one and onto, such that f and

f^{-1} are measurable for $\mathcal{B}_{\mathbb{T}}$ and \mathcal{M} . We say that a topological space \mathbb{S} is *standard Borel* if $(\mathbb{S}, \mathcal{B}_{\mathbb{S}})$ is standard Borel.

If τ_1 and τ_2 are two comparable Suslin topologies on \mathbb{S} , they share the same Borel sets. In particular, every Lusin space is standard Borel.

A useful property of standard Borel spaces is that every standard space \mathbb{S} is Borel-isomorphic with a Borel subset of $[0, 1]$. This is a consequence of e.g. [13, Theorem 15.6 and Corollary 6.5], see also [20] or [8, Théorème III.20]. (Actually, we have more: every standard Borel space is countable or Borel-isomorphic with $[0, 1]$. Thus, for standard Borel spaces, the Continuum Hypothesis holds true!)

Another useful property of standard Borel spaces is that, if \mathbb{S} is a standard Borel space, if $X : \Omega \mapsto \mathbb{S}$ is a measurable mapping, and if \mathcal{M} is a sub- σ -algebra of \mathcal{A} , there exists a regular conditional distribution $P_{X|\mathcal{M}}$ (see e.g. [11, Theorem 10.2.2] for the Polish case, which immediately extends to standard Borel spaces from their definition). Note that, if \mathbb{S} is radon, then the distribution P_X of X is tight, that is, for every integer $n \geq 1$, there exists a compact subset K_n of \mathbb{S} such that $P_X(\mathbb{S} \setminus K_n) \leq 1/n$. Hence one can assume without loss of generality that X takes its values in $\cup_{n \geq 1} K_n$. If moreover \mathbb{S} has metrizable compact subsets, then $\cup_{n \geq 1} K_n$ is Lusin (and hence standard Borel), and there exists a regular conditional distribution $P_{X|\mathcal{M}}$. Thus, if \mathbb{S} is Radon with metrizable compact subsets, every element μ of \mathcal{Y} has an \mathcal{A} -measurable disintegration. Indeed, denoting $\mathcal{A}' = \mathcal{A} \otimes \{\emptyset, \mathbb{S}\}$, one only needs to consider the conditional distribution $P_{X|\mathcal{A}'}$ of the random variable $X : (\omega, x) \mapsto x$ defined on the probability space $(\Omega \times \mathbb{S}, \mathcal{A} \otimes \mathcal{B}_{\mathbb{S}}, \mu)$.

For any σ -algebra \mathcal{M} on a set \mathbb{M} , the *universal completion* of \mathcal{M} is the σ -algebra $\mathcal{M}^* = \cap_{\mu} \mathcal{M}_{\mu}^*$, where μ runs over all finite nonnegative measures on \mathcal{M} and \mathcal{M}_{μ}^* is the μ -completion of \mathcal{M} . A subset of a topological space \mathbb{S} is said to be *universally measurable* if it belongs to $\mathcal{B}_{\mathbb{S}}^*$. The following lemma can be deduced from e.g. [22, Exercise 10 page 14] and the Borel-isomorphism theorem.

Lemma 4.1 *Assume that \mathbb{S} is a standard Borel space. Let $X : \Omega \rightarrow \mathbb{S}$ be \mathcal{A}^* -measurable. Then there exists an \mathcal{A} -measurable modification $Y : \Omega \rightarrow \mathbb{S}$ of X , that is, Y is \mathcal{A} -measurable and satisfies $Y = X$ a.e.*

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