# Edgeworth type expansions for transition densities of Markov chains converging to diffusions

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**JUILLET 2004**

Prépublication n$^o$ 923

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8th July 2004

**Abstract**

We consider triangular arrays of Markov chains that converge weakly to a diffusion process. Edgeworth type expansions of order $o(n^{-1-\delta}), \delta > 0$, for transition densities are proved. For this purpose we represent the transition density as a functional of densities of sums of i.i.d. variables. This will be done by application of the parametrix method. Then we apply Edgeworth expansions to the densities. The resulting series gives our Edgeworth-type expansion for the transition density of Markov chain.

*Key words and phrases:* Markov chains, diffusion processes, transition densities, Edgeworth expansions

*AMS subject classification:* 62G07, 60G60

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*This research was supported by grant 436RUS113/467/81-2 from the Deutsche Forschungsgemeinschaft and by grants 02-01-04001 and 01-01-00649 from the Russian Foundation of Fundamental Researches. The first author worked on the paper during a visit at the Laboratory of Probability Theory and Random Models of the University Paris VI in 2003/2004. He is grateful for the hospitality during his stay.

# 1 Introduction.

In this paper we study triangular arrays of homogeneous Markov chains $X_n(k)$ ($n \geq 1, 0 \leq k \leq n$) that converge weakly to a diffusion process (for $n \to \infty$). Our main result will give Edgeworth type expansions for the transition densities. The order of the expansions is $o(n^{-1-\delta}), \delta > 0$. The theory of Edgeworth expansions is well developped for sums of independent random variables. For more general models approaches have been used where the expansion is reduced to models with sums of independent random variables. This is also the basic idea of our approach. We will make use of the parametrix method. In this approach the transition density is represented as a nested sum of functionals of densities of sums of independent variables. Plugging Edgeworth expansions into this representation will result in an expansion for the transition density.

Weak convergence of the distribution of scaled discrete time Markov processes to diffusions has been extensively studied in the literature [see Skorohod (1965) and Strook and Varadhan (1979)]. Local limit theorems for Markov chains were given in Konovalov (1981), Konakov and Molchanov (1984) and Konakov and Mammen (2000,2001). In Konakov and Mammen (2000) it was shown that the transition density of a Markov chain converges with rate $O(n^{-1/2})$ to the transition density in the diffusion model. For the proof there an analytical approach was chosen that made essential use of the parametrix method. This method permits to obtain tractable representations of transition densities of diffusions that are based on Gaussian densities, see Lemma 3.1 below. Similar representations hold for discrete time Markov chains $X_n$, see Lemma 5.1 below. For a short exposition of the parametrix method, see Section 3 and Konakov and Mammen (2000). The parametrix method for Markov chains developped in Konakov and Mammen (2000) is exposed in Subsection 5.1. Applications to Markov random walks are given in Konakov and Mammen (2001). In Konakov and Mammen (2002) the approach is used to give Edgeworth-type expansions for Euler schemes for differential equations. Standard references for the parametrix method are the books by Friedman (1964) and Ladyženskaja, Solonnikov and Ural´ceva (1968) on parabolic PDEs and for diffusions McKean and Singer (1967).

The paper is organized as follows. In the next section we will present our model for the Markov chain. In Section 3 we will give a short introduction into the parametrix method for diffusions. Our main result that states an Edgeworth-type expansion for Markov chains is given in Section 4. Some auxiliary results are given in Section 5. In particular, in Subsection 5.1 we will recall the parametrix approach developped in Konakov and Mammen (2000) for Markov chains. The proof of our main result is given in Section 6.

## 2 Markov chain model.

We now give a more detailed description of Markov chains and their diffusion limit. For each $n \geq 1$ we consider Markov chains $X_n(k)$ where the time $k$ runs from 0 to $n$. The Markov chain $X_n$ is assumed to take values in $\mathbb{R}^p$. The dynamics of the chain $X_n$ is described by

$$X_n(k+1) = X_n(k) + n^{-1} m\{X_n(k)\} + n^{-1/2}\varepsilon_n(k+1) \tag{1}$$

Here, $m$ is a function $m : \mathbb{R}^p \to \mathbb{R}^p$. We make the Markov assumption that the conditional distribution of the innovation $\varepsilon_n(k+1)$ given the past $X_n(k), X_n(k-1), \ldots$ depends only on the last value $X_n(k)$. Given $X_n(i) = x(i)$ for $i = 0, \ldots, k$ the variable $\varepsilon_n(k+1)$ has a conditional density $q\{x(k), \bullet\}$. The conditional covariance matrix of $\varepsilon_n(k+1)$ is denoted by $\Sigma\{x(k)\}$. Here $q$ is a function mapping $\mathbb{R}^p \times \mathbb{R}^p$ into $\mathbb{R}_+$. Furthermore, $\Sigma$ is a function mapping $\mathbb{R}^p$ into the set of positive definite $p \times p$ matrices. The conditional density of $X_n(n)$, given $X_n(0) = x$, is denoted by $p_n(x, \bullet)$. Study of the transition densities $p_n(x, y)$ is the topic of this paper. Conditions on $m\{x(k)\}, q\{x(k), \bullet\}$ and $\Sigma\{x(k)\}$ will be given below.

By time change the Markov chain $X_n$ defines a process $Y_n$ on $[0, 1]$. More precisely, put $Y_n(t) = X_n(k)$ for $k/n \leq t < (k+1)/n$. Under our assumptions, see below, the process $Y_n$ converges weakly to a diffusion $Y(t)$. This follows for instance from Theorem 1, p. 82 in Skorohod (1987). The diffusion is defined by $Y(0) = x$ and

$$dY(t) = m\{Y(t)\}dt + \Lambda\{Y(t)\}dW(t),$$

where $W$ is a $p$ dimensional Brownian motion. The matrix $\Lambda(z)$ is the symmetric matrix defined by $\Lambda(z)\Lambda(z)^T = \Sigma(z)$. The conditional density of $Y(1)$, given $Y(0) = x$, is denoted by $p(x, \bullet)$. Remind that the conditional density of $Y_n(1)$, given $Y_n(0) = x$, is denoted by $p_n(x, \bullet)$.

For our result we use the following conditions.

(A1) For $x \in \mathbb{R}^p$ let $q\{x, \bullet\}$ be a density in $\mathbb{R}^p$ with $\int q\{x, z\} z dz = 0$ for all $x \in \mathbb{R}^p$, $\int q\{x, z\} z_i z_j dz = \sigma_{ij}(x)$ for all $x \in \mathbb{R}^p$ and $i, j = 1, ..., p$. The matrix with elements $\sigma_{ij}(x)$ is denoted by $\Sigma(x)$.

(A2) There exist a positive integer $S'$, a constant $\gamma > 0$ and a function $\psi : \mathbb{R}^p \to \mathbb{R}$ with $\sup_{x \in \mathbb{R}^p} \psi(x) < \infty$ and $\int_{\mathbb{R}^p} \|x\|^S \psi(x) dx < \infty$ for $S = 2pS' + 4$ such that $|\mathrm{D}_z^\nu q\{x, z\}| \leq \psi(z)$ for all $x, z \in \mathbb{R}^p$ , and $|\nu| = 0, ..., 6$, $|\mathrm{D}_x^\nu q\{x, z\}| \leq \psi(z)$ for all $x, z \in \mathbb{R}^p$ , and $|\nu| = 0, ..., 6$, $|\mathrm{D}_x^\nu q^{(k)}\{x, z\}| \leq k^\gamma \psi(z)$ for all $x, z \in \mathbb{R}^p$ , $k \geq 1$, and $|\nu| = 0, 1$.

Here $q^{(k)}(x, z)$ denotes the $k$ - fold convolution of $q$ for fixed $x$ as a function of $z$.

(A3) There exist positive constants $c$ and $C$ such that

$$c \leq \theta^T \Sigma(x)\theta \leq C$$

for all $\theta$, $\|\theta\| = 1$ and $x$.

(A4) The functions $m(x)$ and $\Sigma(x)$ and their derivatives up to order six are bounded (uniformly in $x$) and Lipschitz continuous with respect to $x$.

# 3 The parametrix method.

Our approach makes use of the parametrix method. This approach allows to state series expansions for the transition densities of the limiting diffusion and for the Markov chain. The series only depend on transition densities of "frozen" processes. The "frozen" diffusion is a Gaussian process that has a Gaussian density as transition density. For the "frozen" Markov chain we get transition densities that are densities of sums of independent variables. In this section we will give an overview on the method for diffusions and Markov chains.

We now discuss the parametrix method for diffusions. This gives an infinite series expansion of the transition density $p$ of the limiting diffusion process $Y$, see Lemma 3.1. We will give a similar expansion for the Markov chain in the next subsection, see Lemma 3.3. Our proof of Theorem 4.1 will be based on the comparison of these two series. The series for the transition densities will be derived by the parametrix method. We will give a description of the parametrix method below.

For the statement of the expansion of $p$ in Lemma 3.1 we have to introduce additional diffusion processes. For $0 < s < 1$ and $x, y \in I\!R^p$ we define diffusions $\tilde{Y} = \tilde{Y}_{s,x,y}$ that are defined for $s \leq t \leq 1$ by

$$\tilde{Y}(s) = x$$

and

$$d\tilde{Y}(t) = m\{y\}dt + \Lambda\{y\}dW(t).$$

The processes $\tilde{Y}$ are called "frozen" diffusions. We define $\tilde{p}(s,t,x,y)$ as the conditional density of $\tilde{Y}(t)[= \tilde{Y}_{s,x,y}(t)]$ at the point $y$, given $\tilde{Y}(s) = x$. Note that the variable $y$ acts here twice: as the argument of the density and as a defining quantity of the process $\tilde{Y} = \tilde{Y}_{s,x,y}$. Furthermore, we denote by $\tilde{p}_j^y(x,z)$ the conditional density of $\tilde{Y}((j+1)/n)[= \tilde{Y}_{j/n,x,y}((j+1)/n)]$ at the point $z$, given $\tilde{Y}(j/n) = x$. The process $\tilde{Y}$ is a simple Gaussian process. Its transition densities $\tilde{p}$ are given explicitely. By definition, we have that

$$\tilde{p}(s,t,x,y) = (2\pi(t-s))^{-p/2} (\det \Sigma(y))^{-1/2}$$

$$\exp[-\frac{1}{2}(t-s)^{-1}\{y-x-(t-s)m(y)\}' \Sigma(y)^{-1}\{y-x-(t-s)m(y)\}]. \tag{2}$$

Let us introduce the following differential operators $L$ and $\tilde{L}$:

$$Lf(s,t,x,y) = m(x)^T \frac{\partial f(s,t,x,y)}{\partial x} + \frac{1}{2} tr[\Lambda(x)^T \frac{\partial^2 f(s,t,x,y)}{(\partial x)^2} \Lambda(x)],$$

$$\tilde{L}f(s,t,x,y) = m(y)^T \frac{\partial f(s,t,x,y)}{\partial x} + \frac{1}{2} tr[\Lambda(y)^T \frac{\partial^2 f(s,t,x,y)}{(\partial x)^2} \Lambda(y)].$$

Note that $L$ and $\tilde{L}$ corresponds to the infinitesimal operators of $Y$ or of the frozen process $\tilde{Y}_{s,x,y}$, respectively, i.e.

$$Lf(s,t,x,y) = \lim_{h \to 0} h^{-1} \{E[f(s,t,Y(s+h),y) \mid Y(s) = x] - f(s,t,x,y)\}, \tag{3}$$

$$\tilde{L}f(s,t,x,y) = \lim_{h \to 0} h^{-1} \{E[f(s,t,\tilde{Y}_{s,x,y}(s+h),y)] - f(s,t,x,y)\}. \tag{4}$$

We put

$$H = (L - \tilde{L})\tilde{p}.$$

Then

$$H(s,t,x,y) = \frac{1}{2} \sum_{i,j=1}^{p} (\sigma_{ij}(x) - \sigma_{ij}(y)) \frac{\partial^2 \tilde{p}(s,t,x,y)}{\partial x_i \partial x_j} \tag{5}$$

$$+ \sum_{i=1}^{p} (m_i(x) - m_i(y)) \frac{\partial \tilde{p}(s,t,x,y)}{\partial x_i}.$$

Now we define the following convolution type binary operation $\otimes$:

$$(f \otimes g)(s,t,x,y) = \int_s^t du \int_{\mathbb{R}^p} f(s,u,x,z) g(u,t,z,y)\, dz.$$

We write $g \otimes H^{(0)}$ for $g$ and for $r = 1, 2, \ldots$ we denote the $r$-fold "convolution" $(g \otimes H^{(r-1)}) \otimes H$ by $g \otimes H^{(r)}$. With these notations we can state our expansion for $p$.

**Lemma 3.1** *For $0 \leq s < t \leq 1$ the following formula holds:*

$$p(s,t,x,y) = \sum_{r=0}^{\infty} \left(\tilde{p} \otimes H^{(r)}\right)(s,t,x,y).$$

A proof of Lemma 3.1 can be found in McKean and Singer (1967). We will make use of the bounds on $H$ and $\tilde{p} \otimes H^{(r)}$ that are stated in the following lemma. Proofs of these bounds can be found again in McKean and Singer (1967). For a more detailed proof of Lemma 3.2 see also Ladyženskaja, Solonnikov and Ural'ceva (1968).

**Lemma 3.2** *There exist constants $C$ and $C_1$ (that do not depend on $x$ and $y$ ) such that the following inequalities hold:*

$$|H(s,t,x,y)| \leq C_1 \rho^{-1} \phi_{C,\rho}(y - x),$$

*and*

$$\left|\widetilde{p} \otimes H^{(r)}(s,t,x,y)\right| \leq C_1^{r+1} \frac{\rho^r}{\Gamma(1+\frac{r}{2})} \phi_{C,\rho}(y-x),$$

*where* $\rho^2 = t-s$, $\phi_{C,\rho}(u) = \rho^{-p}\phi_C(u/\rho)$ *and*

$$\phi_C(u) = \frac{\exp(-C\left\|u\right\|^2)}{\int \exp(-C\left\|v\right\|^2)dv}.$$

# 4 Edgeworth type expansions for Markov chains.

The following theorem contains our main result. It gives Edgeworth type expansions for $p_n$. For the statement of the theorem we introduce the following differential operators

$$\mathcal{F}_1[f](s,t,x,y) = \sum_{|\nu|=3} \frac{\mu_\nu(x)}{\nu!} D_x^\nu f(s,t,x,y),$$

$$\mathcal{F}_2[f](s,t,x,y) = \sum_{|\nu|=4} \frac{\chi_\nu(x)}{\nu!} D_x^\nu f(s,t,x,y).$$

Furthermore,

$$\mu_\nu(x) = \int z^\nu q(x,z)dz,$$

$$\widetilde{\pi}_1(s,t,x,y) = (t-s)\sum_{|\nu|=3} \frac{\chi_\nu(y)}{\nu!} D_x^\nu \widetilde{p}(s,t,x,y),$$

$$\widetilde{\pi}_2(s,t,x,y) = (t-s)\sum_{|\nu|=4} \frac{\chi_\nu(y)}{\nu!} D_x^\nu \widetilde{p}(s,t,x,y)$$

$$+\frac{1}{2}(t-s)^2 \left\{ \sum_{|\nu|=3} \frac{\chi_\nu(y)}{\nu!} D_x^\nu \right\}^2 \widetilde{p}(s,t,x,y),$$

where $\chi_\nu(x)$ are the cumulants of the density $q(x,\bullet)$.

**Theorem 4.1.** *Assume (A1)-(A4). Then there exists a constant* $\delta > 0$ *such that the following expansion holds:*

$$\sup_{x,y\in\mathbb{R}^p} \left(1+\left\|y-x\right\|^{2(S'-1)}\right) \left|p_n(x,y) - p(x,y) - n^{-1/2}\pi_1(x,y) - n^{-1}\pi_2(x,y)\right| = O(n^{-1-\delta}),$$

*where* $S'$ *is defined in Assumption (A2) and where*

$$\pi_1(x,y) = (p \otimes \mathcal{F}_1[p])(0,1,x,y), \tag{6}$$

$$\pi_2(x,y) = \frac{1}{2}\left(p \otimes (L_\star^2 - L^2)p\right)(0,1,x,y) + (p \otimes \mathcal{F}_2[p])(0,1,x,y) \tag{7}$$

$$+(p \otimes \mathcal{F}_1[p \otimes \mathcal{F}_1[p]])(0,1,x,y).$$

*Here $p(s,t,x,y)$ is the transition density of the diffusion $Y(t)$, $L_\star$ is defined analogously to $\widetilde{L}$ but with the coefficients "frozen" at the point $x$. The norm $\|\bullet\|$ is the usual Euclidean norm.*

The proof of Theorem 4.1 will be given in Section 6. Now we make some remarks concerning the approximating terms $\pi_1(x,y)$ and $\pi_2(x,y)$.

**Discussion and remarks.**

- It can be shown that the term $\pi_1(x,y)$ and each term on the right hand side of (7) have subgaussian tails. This means that these terms can be bounded from above by $C_1 \exp[-C_2(y-x)^2]$ with some positive constants $C_1$ and $C_2$.
- If the innovation densitiy $q(x,\bullet)$ does not depend on $x$ then one gets that $L_\star = L$ and that $p(s,t,x,y) = \widetilde{p}(s,t,x,y)$ where $\widetilde{p}$ is defined in (2) with $\Sigma(y) = \Sigma$ and $m(y) = m$. This gives

$$\pi_1(x,y) = \int_0^1 ds \int \widetilde{p}(0,s,x,v) \sum_{|\nu|=3} \frac{\mu_\nu}{\nu!} D_v^\nu \widetilde{p}(s,1,v,y) dv$$

$$= -\sum_{|\nu|=3} \frac{\mu_\nu}{\nu!} D_y^\nu \int_0^1 ds \int \widetilde{p}(0,s,x,v) \widetilde{p}(s,1,v,y) dv$$

$$= \sum_{|\nu|=3} \frac{\mu_\nu}{\nu!} D_x^\nu \widetilde{p}(0,1,x,y)$$

$$= \widetilde{\pi}_1(0,1,x,y),$$

$$\widetilde{p} \otimes \mathcal{F}_1[\widetilde{p}])(s,1,z,y) = \int_s^1 du \int \widetilde{p}(s,u,z,w) dy \sum_{|\nu|=3} \frac{\mu_\nu}{\nu!} D_w^\nu \widetilde{p}(u,1,w,y)\ dw$$

$$= -\sum_{|\nu|=3} \frac{\mu_\nu}{\nu!} D_y^\nu \int_s^1 du \widetilde{p}(s,u,z,w) \widetilde{p}(u,1,w,y) dw$$

$$= (1-s) \sum_{|\nu|=3} \frac{\mu_\nu}{\nu!} D_z^\nu \widetilde{p}(s,1,z,y),$$

$$\mathcal{F}_1[\widetilde{p} \otimes \mathcal{F}_1[\widetilde{p}]])(s,1,z,y) = (1-s) \left\{ \sum_{|\nu|=3} \frac{\mu_\nu}{\nu!} D_z^\nu \right\}^2 \widetilde{p}(s,1,z,y),$$

$$(\widetilde{p} \otimes \mathcal{F}_2[\widetilde{p}])(0,1,x,y) + (\widetilde{p} \otimes \mathcal{F}_1[\widetilde{p} \otimes \mathcal{F}_1[\widetilde{p}]])(0,1,x,y)$$

$$= \int_0^1 ds \int \widetilde{p}(0,s,x,v) \left( \sum_{|\nu|=4} \frac{\chi_\nu}{\nu!} D_v^\nu \widetilde{p}(s,1,v,y) \right.$$

$$+(1-s)\left\{\sum_{|\nu|=3}\frac{\mu_\nu}{\nu!}D_v^\nu\right\}^2\widetilde{p}(s,1,v,y)\Bigg)\,dv$$

$$\sum_{|\nu|=4}\frac{\chi_\nu}{\nu!}D_y^\nu\int_0^1 ds\int\widetilde{p}(0,s,x,v)\widetilde{p}(s,1,v,y)dv$$

$$+\left\{\sum_{|\nu|=3}\frac{\mu_\nu}{\nu!}D_y^\nu\right\}^2\int_0^1(1-s)ds\int\widetilde{p}(0,s,x,v)\widetilde{p}(s,1,v,y)dv$$

$$=\widetilde{\pi}_2(0,1,x,y).$$

Thus from Theorem 4.1 for this case we just get the first two terms of the classical Edgeworth expansion $n^{-1/2}\widetilde{\pi}_1(0,1,x,y)+n^{-1}\widetilde{\pi}_2(0,1,x,y)$.

- If $\mu_\nu(x)=0$ for $|\nu|=3$ and for $x\in I\!R^p$ then it holds that $\mathcal{F}_1=0$. This gives that the expansion of Theorem 4.1 holds with

$$\pi_1(x,y)=0,$$

$$\pi_2(x,y)=(p\otimes\mathcal{F}_2)[p](0,1,x,y)+\frac{1}{2}\left(p\otimes(L_\star^2-L^2)p\right)(0,1,x,y).$$

If in addition we have that $\chi_\nu(x)=0$ for $|\nu|=4$ then the first four moments of the innovations coincide with the first four moments of a normal distribution with zero mean and covariance matrix $\Sigma(x)$. In this case we have $\mathcal{F}_2=0$ and

$$\pi_1(x,y)=0, \tag{8}$$

$$\pi_2(x,y)=\frac{1}{2}\left(p\otimes(L_\star^2-L^2)p\right)(0,1,x,y). \tag{9}$$

- Our expansion can be applied to study the performance of discrete approximations of diffusions. An Euler approximating scheme is defined by putting $Y_n([k+1]/n)=Y_n(k/n)+n^{-1}m(Y_n(k/n))+\Lambda(Y_n(k/n))[W([k+1]/n)-W(k/n)]$. It has been shown that $Y_n(1)=Y(1)+O_P(n^{-1/2})$. For a discussion of Euler approximations, see Kloeden and Platen (1992). For this scheme it holds that $\mathcal{F}_1=\mathcal{F}_2=0$. Thus the expansion of Theorem 4.1 holds with (8) and (9). This result was obtained by Bally and Talay (1996). Higher order asymptotic expansions for Euler schemes are given in Konakov and Mammen (2002).

- A more refined approximating scheme for stochastic differential equations was introduced by Mil'shtein (1974). Mil'shtein's scheme is based on higher order stochastic approximations of the stochastic differential equation. Mil'shtein (1974) proved that for his scheme $Y_n(1)=Y(1)+O_P(n^{-1})$. Thus this scheme has a better strong approximation rate than Euler schemes. We now apply Theorem 4.1 to this approximating scheme. We will compare the approximations

of the transition densities for these two schemes. It will turn out that the rate is not improved for Mil'shtein's schemes, in contrast to the mentioned rates of strong approximation. However, we will argue that Mil'shtein schemes lead to more stable approximations. For simplicity we consider only the one dimensional case. For Mil'shtein schemes the innovation density $q_n(x, \bullet)$ depends on $n$ but asymptotically this dependence vanishes. For Mil'shtein schemes it holds that

$$\mu_{2,n}(x) = \sigma^2(x) + \frac{1}{2n}(\sigma(x)\sigma'(x))^2, \mu_{3,n}(x) = \frac{3\sigma(x)\sigma'(x)}{\sqrt{n}},$$

$$\chi_{4,n}(x) = \mu_{4,n}(x) - 3\sigma^4(x) = \frac{15}{2n}\sigma^5(x)(\sigma'(x))^3.$$

Hence,

$$\pi_1(x, y) = 0,$$

$$\pi_1(x, y) = 0, \pi_2(x, y) = \frac{1}{2}\left(p \otimes (L_\star^2 - L^2)p\right)(0, 1, x, y) + (p \otimes M)(0, 1, x, y),$$

$$\pi_1(x, y) = 0,$$

$$\pi_2(x, y) = \frac{1}{2}\left(p \otimes (L_\star^2 - L^2)p\right)(0, 1, x, y) + (p \otimes M)(0, 1, x, y),$$

where

$$M(s, t, x, y) = \frac{1}{2}\sigma(x)\sigma'(x)\frac{\partial^3 p(s, t, x, y)}{\partial x^3}.$$

The last expression for $\pi_2(x, y)$ allows to compare Mil'shtein and Euler schemes. In the one dimensional case the function $\frac{1}{2}(L_\star^2 - L^2)p(s, 1, z, y)$ is equal to

$$\frac{1}{2}(L_\star^2 - L^2)p(s, 1, z, y) = R(s, 1, z, y) - M(s, 1, z, y),$$

where

$$R(s, 1, z, y) = -\left[\frac{1}{2}m(z)m'(z) + \frac{1}{4}m''(z)\sigma^2(z)\right]\frac{\partial p(s, 1, z, y)}{\partial z}$$

$$-\left[\frac{1}{2}m(z)\sigma(z)\sigma'(z) + \frac{1}{2}m'(z)\sigma^2(z) + \frac{1}{4}(\sigma(z)\sigma'(z))^2 + \frac{1}{4}\sigma^3(z)\sigma''(z)\right]$$

$$\times\frac{\partial^2 p(s, 1, z, y)}{\partial z^2}.$$

By linearity of $\otimes$ we get that for Mil'shtein schemes

$$\pi_2(x, y) = (p \otimes R)(0, 1, x, y).$$

Thus Mil'shtein schemes are constructed such that in the expansion the third derivative of the diffusion density $p$ is eliminated from the expansion of the Euler scheme. This derivative is the most unstable and singular summand near point $s = 1$. This suggests that Mil'shtein schemes lead to more stable approximations of transition densities of diffusions.

# 5 Some auxiliary results.

This section contains some auxiliary results that will be used in the proof of Theorem 4.1. In Section 3 we represented the transition densities of the diffusion by nested sums of functionals of densities of 'frozen' processes. The difference between the densities of the 'frozen' Markov chains and the Gaussian densities can be treated by Edgeworth expansions. This is done in Subsection 5.2. In contrast to Konakov and Mammen (2000a) now we use higher order Edgeworth expansions. These are the main steps of the proof of Theorem 4.1. In Section 5.3 we will give some bounds for the kernels and their differences used in the expansions of the parametrix methods.

## 5.1 Application of the parametrix method to Markov chains.

In this subsection we derive a finite series expansion of the transition density $p_n(s,t,x,y)$ of the Markov chain, see Lemma 5.1. Here, $p_n(s,t,x,\bullet)$ denotes the conditional density of $Y_n(t)$, given $Y_n(s) = x$ (in particular, $p_n(0,1,x,y) = p_n(x,y)$). We proceed similarly as in Section 3. Again we apply the parametrix method and for this purpose we introduce additional "frozen" Markov chains. These are defined as follows. For all $0 \le j \le n$ and $x, y \in I\!R^p$ we define the Markov chains $\tilde{X}_n = \tilde{X}_{n,j,x,y}$. For fixed $j, x$ and $y$, the chain is defined for $i$ with $j \le i \le n$. The dynamics of the chain is described by

$$\tilde{X}_n(j) = x$$

and

$$\tilde{X}_n(i+1) = \tilde{X}_n(i) + n^{-1} m\{y\} + n^{-1/2}\tilde{\varepsilon}_n(i+1).$$

The stochastic structure of the $I\!R^p$ valued innovations $\tilde{\varepsilon}_n(i)$ is described as follows. Given $\tilde{X}_n(l) = x(l)$ for $l = j, \ldots, i$ the variable $\tilde{\varepsilon}_n(i+1)$ has a conditional density $q\{y, \bullet\}$. Note that the conditional distribution of $\tilde{X}_n(i+1) - \tilde{X}_n(i)$ does not depend on the past $\tilde{X}_n(l)$ for $l = j, \ldots, i$. Let us call $\tilde{X}_n$ the Markov chain *frozen at* $y$. We put $\tilde{Y}_n(t) = \tilde{X}_n\{k\}$ for $k/n \le t < (k+1)/n$ and we write $\tilde{p}_n(j/n, k/n, x, y)$ for the conditional density of $\tilde{X}_n(k)[= \tilde{X}_{n,j,x,y}(k)]$ at the point $y$, given $\tilde{X}_n(j) = x$. Note that, as in the case of a "frozen" diffusion the variable $y$ acts here twice: as the argument of the density and as a defining quantity of the process $\tilde{X}_n = \tilde{X}_{n,j,x,y}$. Let us introduce the following infinitesimal operators $L_n$ and $\tilde{L}_n$:

$$\mathrm{L_n f(j/n,k/n,x,y)}$$
$$= \mathrm{n}\left[\int p_{n,j}(x,z) f((j+1)/n, k/n, z, y)dz - f((j+1)/n, k/n, x, y)\right],$$
$$\tilde{L}_n \mathrm{f(j/n,k/n,x,y)}$$
$$= \mathrm{n}\left[\int \tilde{p}^{\,y}_{n,j}(x,z) f((j+1)/n, k/n, z, y)dz - f((j+1)/n, k/n, x, y)\right],$$

where we write

$$p_{n,j}(x,z) = p_n(j/n,(j+1)/n,x,z)$$

and where $\tilde{p}^y_{n,j}(x,\bullet)$ denotes the conditional density of $\tilde{X}_n(j+1)[=\tilde{X}_{n,j,x,y}(j+1)]$ given $\tilde{X}_n(j)=x$. Note that $L_n$ and $\tilde{L}_n$ are defined in analogy with the definition of $L$ and $\tilde{L}$, see (3)-(4). We remark that for some technical reasons on the right hand side of the definitions of $L_nf$ and $\tilde{L}_nf$ the terms $f((j+1)/n,\ldots)$ appear instead of $f(j/n,\ldots)$. The reasons will become apparent in the development of the proof of Theorem 4.1. For $k>j$ we put in analogy with the definition $H$

$$H_n = \{L_n - \tilde{L}_n\}\tilde{p}_n.$$

The next lemma from Konakov and Mammen (2000) gives the "parametrix" expansion of $p_n$.

**Lemma 5.1** *For $0\leq j<k\leq n$ the following formula holds:*

$$p_n(j/n,k/n,x,y) = \sum_{r=0}^{k-j}\left(\widetilde{p_n}\otimes_n H_n^{(r)}\right)(j/n,k/n,x),$$

*where in the calculation of $\widetilde{p_n}\otimes_n H_n^{(r)}$ we define*

$$p_n(k/n,k/n,x,y) = \widetilde{p}_n(k/n,k/n,x,y) = \delta(x-y).$$

*Here $\delta$ denotes the Dirac function.*

## 5.2 Bounds on $\tilde{p}_n - \tilde{p}$ based on Edgeworth expansions.

In this subsection we will develop some tools that are helpful for the comparison of the expansion of $p$ (see Lemma 3.1) and the expansion of $p_n$ ( see Lemma 5.1). These expansions are simple expressions in $\tilde{p}$ or $\tilde{p}_n$, respectively. Recall that $\tilde{p}$ is a Gaussian density, see (2), and that $\tilde{p}_n$ is the density of a sum of independent variables. The densities $\tilde{p}$ and $\tilde{p}_n$ can be compared by application of Edgeworth expansions. This is done in Lemma 5.2. This is the essential step for the comparison of the expansions of $p$ and $p_n$. In the next lemma bounds will be given for derivatives of $\tilde{p}_n$. The proof of this lemma also makes essential use of Edgeworth expansions. The following lemma is a higher order extension of the results in Section 3.3 in Konakov and Mammen (2000).

**Lemma 5.2** *The following bound holds with a constant $C$ for $\nu=(\nu_1,\ldots\nu_p)^T$ with $0\leq|\nu|\leq 6$*

$$\begin{aligned}&|D_z^\nu\widetilde{p}_n(j/n,k/n,x,y) - D_z^\nu\widetilde{p}(j/n,k/n,x,y)\\ &-n^{-1/2}D_z^\nu\widetilde{\pi}_1(j/n,k/n,x,y) - n^{-1}D_z^\nu\widetilde{\pi}_2(j/n,k/n,x,y)\big|\\ &\leq Cn^{-3/2}\rho^{-3}\zeta_\rho^{S-|\nu|}(y-x)\end{aligned}$$

*for all $j < k, x$ and $y$. Here $D_z^\nu$ denotes the partial differential operator of order $\nu$ with respect to $z = \rho^{-1}\Sigma(y)^{-1/2}(y - x - \rho^2 m(y))$. The quantity $\rho$ denotes again the term $\rho = [(k-j)/n]^{1/2}$. We write $\zeta_\rho^k(\bullet) = \rho^{-p}\zeta^k(\bullet/\rho)$ where*

$$\zeta^k(z) = \frac{[1 + \|z\|^k]^{-1}}{\int [1 + \|z'\|^k]^{-1} dz'}.$$

PROOF OF LEMMA 5.2. We note first that $\tilde{p}_n(j/n, k/n, x, \bullet)$ is the density of the vector

$$x + \rho^2 m(y) + n^{-1/2} \sum_{i=j}^{k-1} \tilde{\varepsilon}_n(i+1),$$

where, as above in the definition of the "frozen" Markov chain $\tilde{Y}_n$, $\tilde{\varepsilon}_n(i+1)$ is a sequence of independent variables with densities $q(y, \bullet)$. Let $f_n(\bullet)$ be the density of the normalized sum

$$n^{-1/2}\left[n^{-1}(k-j)\Sigma(y)\right]^{-1/2} \sum_{i=j}^{k-1} \tilde{\varepsilon}_n(i+1) = n^{-1/2}\rho^{-1}\Sigma(y)^{-1/2} \sum_{i=j}^{k-1} \tilde{\varepsilon}_n(i+1).$$

Clearly, we have

$$\tilde{p}_n(j/n, k/n, x, \bullet) = \rho^{-p} \det \Sigma(y)^{-1/2} f_n\{\rho^{-1}\Sigma(y)^{-1/2}[\bullet - x - \rho^2 m(y)]\}.$$

We now argue that an Edgeworth expansion holds for $f_n$. This implies the following expansion for $\tilde{p}_n(j/n, k/n, x, \bullet)$

$$\tilde{p}_n(j/n, k/n, x, \bullet) \tag{10}$$

$$= \rho^{-p} \det \Sigma(y)^{-1/2}[\sum_{r=0}^{S-3}(k-j)^{-r/2} P_r(-\phi : \{\bar{\chi}_{\beta,r}\})\{\rho^{-1}\Sigma(y)^{-1/2}[\bullet - x - \rho^2 m(y)]\}$$

$$+O([k-j]^{-(S-2)/2}[1 + \left\|\{\rho^{-1}\Sigma(y)^{-1/2}[\bullet - x - \rho^2 m(y)]\}\right\|^S]^{-1})]$$

with standard notations, see Bhattacharya and Rao (1976), p. 53. In particular, $P_r$ denotes a product of a standard normal density with a polynomial that has coefficients depending only on cumulants of order $\leq r+2$. Expansion (10) follows from Theorem 19.3 in Bhattacharya and Rao (1976). This can be seen as in the proof of Lemma 3.7 in Konakov and Mammen (2000a).

It follows from (10) and Condition (A3) that

$$|\tilde{p}_n(j/n, k/n, x, y) - \tilde{p}(j/n, k/n, x, y) \tag{11}$$

$$-n^{-1/2}\widetilde{\pi}_1(j/n, k/n, x, y) - n^{-1}\widetilde{\pi}_2(j/n, k/n, x, y)|$$

$$\leq Cn^{-3/2}\rho^{-3}\zeta_\rho^{S-|\nu|}(y - x),$$

where

$$\tilde{p}(j/n,k/n,x,y) = \rho^{-p}\det\Sigma(y)^{-1/2}(2\pi)^{-p/2}$$
$$\exp\{-\frac{1}{2}(y-x-\rho^2 m(y))^T\rho^{-2}\Sigma(y)^{-1}(y-x-\rho^2 m(y))\},$$

$$\tilde{\pi}_1(j/n,k/n,x,y) = -\rho^{-1-p}\det\Sigma(y)^{-1/2}\sum_{|\nu|=3}\frac{\tilde{\chi}_\nu(y)}{\nu!}D_z^\nu\phi\left\{\rho^{-1}\Sigma(y)^{-1/2}(y-x-\rho^2 m(y))\right\},$$

$$\tilde{\pi}_2(j/n,k/n,x,y) = \rho^{-2-p}\det\Sigma(y)^{-1/2}\left[\sum_{|\nu|=4}\frac{\tilde{\chi}_\nu(y)}{\nu!}D_z^\nu\phi\left\{\rho^{-1}\Sigma(y)^{-1/2}(y-x-\rho^2 m(y))\right\}\right.$$
$$\left.+\frac{1}{2}\left\{\sum_{|\nu|=3}\frac{\tilde{\chi}_\nu(y)}{\nu!}D_z^\nu\right\}^2\phi\left\{\rho^{-1}\Sigma(y)^{-1/2}(y-x-\rho^2 m(y))\right\}\right],$$

where $\tilde{\chi}_\nu(y)$ are the cumulants of $\Sigma(y)^{-1/2}\tilde{\varepsilon}_n(i+1)$ and $D_z^\nu\phi$ denotes the derivative of $\phi$ of order $\nu$. The definitions of $\tilde{\pi}_1$ and $\tilde{\pi}_2$ coincide with the definitions given at the beginning of Section 4. This follows by replacing the differential operator $D_z^\nu$ by $D_x^\nu$. For $\nu = 0$ the statement of the lemma immediately follows from (11). For $\nu > 0$ one proceeds similarly. See the remark at the end of the proof of Lemma 3.7 in Konakov and Mammen (2000).

From the last lemma we get the following corollary. The statement of this lemma is an extension of Lemma 3.7 in Mammen and Konakov (2000) where the result has been shown for $0 \le |b| \le 2, a = 0$.

**Lemma 5.3** *The following bounds hold:*

$$\left|D_y^a D_x^b\tilde{p}_n(j/n,k/n,x,y)\right| \le C\rho^{-|a|-|b|}\zeta_\rho^{S-|a|}(y-x)$$

*for all $j < k$ , for all $x$ and $y$ and for all $a$ ,$b$ with $0 \le |a| + |b| \le 6$. Here, $\rho = [(k-j)/n]^{1/2}$ [for simplicity the indices $n$, $j$ and $k$ are suppressed in the notation]. The constant $S$ has been defined in Assumption (A2).*

## 5.3 Bounds on operator kernels used in the parametrix expansions.

In this subsection we will present bounds for operator kernels appearing in the expansions based on the parametrix method. In Lemma 5.4 we compare the infinitesimal operators $L_n$ and $\tilde{L}_n$ with the differential operators $L$ and $\tilde{L}$. We give an approximation for the error if, in the definition of $H_n = (L_n - \tilde{L}_n)\tilde{p}_n$, the terms $L_n$ and $\tilde{L}_n$ are replaced by $L$ or $\tilde{L}$, respectively. We show that this term can be approximated by $K_n + M_n$, where $K_n = (L - \tilde{L})\tilde{p}_n$ and where $M_n$ is defined in Lemma 5.4 . Bounds on $H_n$, $K_n$, and $M_n$ are given in Lemma 5.5. These bounds will be used in the proof of

our theorem to show that in the expansion of $p_n$ the terms $\tilde{p}_n \otimes_n H_n^{(r)}$ can be replaced by $\tilde{p}_n \otimes_n (M_n + K_n)^{(r)}$.

**Lemma 5.4.** *The following bound holds with a constant $C$*

$$\left| H_n(\frac{j}{n}, \frac{k}{n}, x, y) - K_n(\frac{j}{n}, \frac{k}{n}, x, y) - M_n(\frac{j}{n}, \frac{k}{n}, x, y) \right|$$

$$\leq C n^{-3/2} \rho^{-1} \zeta_\rho^S(y - x),$$

*where*

$$K_n(\frac{j}{n}, \frac{k}{n}, x, y) = (L - \widetilde{L}) \widetilde{p}_n(\frac{j}{n}, \frac{k}{n}, x, y),$$

$$M_n(\frac{j}{n}, \frac{k}{n}, x, y) = \sum_{l=1}^{3} M_{n,l}(\frac{j}{n}, \frac{k}{n}, x, y),$$

$$M_{n,1}(\frac{j}{n}, \frac{k}{n}, x, y) = M_{n,11}(\frac{j}{n}, \frac{k}{n}, x, y) + M_{n,12}(\frac{j}{n}, \frac{k}{n}, x, y) + M_{n,13}(\frac{j}{n}, \frac{k}{n}, x, y),$$

$$M_{n,11}(\frac{j}{n}, \frac{k}{n}, x, y) = n^{-1/2} \sum_{|\nu|=3} \frac{D_x^\nu \widetilde{p}_n(\frac{j}{n}, \frac{k}{n}, x, y)}{\nu!} (\mu_\nu(x) - \mu_\nu(y)),$$

$$M_{n,12}(\frac{j}{n}, \frac{k}{n}, x, y) = n^{-1} \sum_{|\nu|=4} \frac{D_x^\nu \widetilde{p}_n(\frac{j}{n}, \frac{k}{n}, x, y)}{\nu!} \{\mu_\nu(x) - \mu_\nu(y)$$

$$- \sum_{|\nu'|=2} \nu! N(\nu, \nu') \mu_{\nu'}(y) [\mu_{\nu - \nu'}(x) - \mu_{\nu - \nu'}(y)]\},$$

$$M_{n,13}(\frac{j}{n}, \frac{k}{n}, x, y) =$$

$$n^{-3/2} \left\{ \sum_{|\nu|=5} \frac{5}{\nu!} \int \int_0^1 [q(x, \theta) - q(y, \theta)] \, \theta^\nu D^\nu \lambda(x + \delta \widetilde{h}(\theta)) [1 - \delta]^4 d\delta d\theta \right.$$

$$+ \frac{1}{2} \sum_{i,l} \sigma_{il}(y) \sum_{|\nu|=1} \int \int_0^1 q(y, \theta) (\theta + n^{-1/2} m(y))^\nu (L - \widetilde{L}) D^{\nu + e_i + e_l} \lambda(x + \delta \widetilde{h}(\theta)) d\delta d\theta$$

$$- 3 \sum_{|\nu|=3} \frac{1}{\nu!} \int \int_0^1 q(y, \theta) (\theta + n^{-1/2} m(y))^\nu (1 - \delta)^2 (L - \widetilde{L}) D^\nu \lambda(x + \delta \widetilde{h}(\theta)) d\delta d\theta$$

$$- 2 \sum_{|\nu|=3} \frac{\mu_\nu(x) - \mu_\nu(y)}{\nu!} \sum_{|\nu'|=2} \frac{1}{\nu'!} \int \int_0^1 q(y, \theta) (\theta + n^{-1/2} m(y))^{\nu'} (1 - \delta)$$

$$D^{\nu + \nu'} \lambda(x + \delta \widetilde{h}(\theta)) d\delta d\theta$$

$$\left. - \sum_{|\nu|=4} \frac{\mu_\nu(x) - \mu_\nu(y)}{\nu!} \sum_{|\nu'|=1} \int \int_0^1 q(y, \theta) (\theta + n^{-1/2} m(y))^{\nu'} D^{\nu + \nu'} \lambda(x + \delta \widetilde{h}(\theta)) d\delta d\theta \right\},$$

$$M_{n,2}(\frac{j}{n}, \frac{k}{n}, x, y) = M_{n,21}(\frac{j}{n}, \frac{k}{n}, x, y) + ... + M_{n,25}(\frac{j}{n}, \frac{k}{n}, x, y),$$

$$M_{n,21}(\frac{j}{n},\frac{k}{n},x,y)=$$

$$n^{-1}\sum_{|\nu|=3}\frac{D_x^\nu\widetilde{p}_n(\frac{j}{n},\frac{k}{n},x,y)}{\nu!}\sum_{i=1}^p\nu_i(m_i(x)-m_i(y))\cdot(\mu_{\nu-e_i}(x)-\mu_{\nu-e_i}(y)),$$

$$M_{n,22}(\frac{j}{n},\frac{k}{n},x,y)=n^{-3/2}\left\{\sum_{|\nu|=4}\frac{(D^\nu\lambda)(x)}{\nu!}\sum_{r=1}^p\nu_r[m_r(x)\mu_{\nu-e_r}(x)-m_r(y)\mu_{\nu-e_r}(y)]\right.$$

$$+\sum_{i=1}^p m_i(y)\sum_{|\nu'|=1}\int\int_0^1 q(y,\theta)(\theta+n^{-1/2}m(y))^{\nu'}(L-\widetilde{L})D^{\nu'+e_i}\lambda(x+\delta\widetilde{h}(\theta))d\delta d\theta$$

$$-\sum_{|\nu|=2}\frac{m(x)^\nu-m(y)^\nu}{\nu!}\sum_{|\nu'|=1}\int\int_0^1 q(y,\theta)(\theta+n^{-1/2}m(y))^{\nu'}D_x^{\nu+\nu'}\lambda(x+\delta\widetilde{h}(\theta))d\delta d\theta$$

$$-\sum_{|\nu|=3}\frac{1}{\nu!}\sum_{i=1}^p\nu_i[m_i(x)\mu_{\nu-e_i}(x)-m_i(y)\mu_{\nu-e_i}(y)]$$

$$\cdot\sum_{|\nu'|=1}\int\int_0^1 q(y,\theta)(\theta+n^{-1/2}m(y))^{\nu'}D_x^{\nu+\nu'}\lambda(x+\delta\widetilde{h}(\theta))d\delta d\theta$$

$$\left.-\sum_{|\nu|=3}\frac{\mu_\nu(x)-\mu_\nu(y)}{\nu!}\sum_{|\nu'|=1}m(y)^{\nu'}D_x^{\nu+\nu'}\lambda(x)\right\}$$

$$M_{n,23}(\frac{j}{n},\frac{k}{n},x,y)=5n^{-2}\sum_{|\nu|=5}\frac{1}{\nu!}\sum_{i=1}^p(m_i(x)-m_i(y))\int\int_0^1 q(x,\theta)$$

$$\theta^{\nu-e_i}D_x^\nu\lambda(x+\delta\widetilde{h}(\theta))[1-\delta]^4d\delta d\theta,$$

$$M_{n,24}(\frac{j}{n},\frac{k}{n},x,y)=5n^{-2}\sum_{|\nu|=5}\frac{1}{\nu!}\sum_{i=1}^p m_i(y))\int\int_0^1[q(x,\theta)-q(y,\theta)]$$

$$\cdot\theta^{\nu-e_i}D^\nu\lambda(x+\delta\widetilde{h}(\theta))[1-\delta]^4d\delta d\theta,$$

$$M_{n,25}(\frac{j}{n},\frac{k}{n},x,y)=5n^{-5/2}\sum_{|\nu|=5}\frac{1}{\nu!}\sum_{i=1}^p\int\int_0^1 q(x,\theta)\theta^\nu\sum_{|\mu|=1}D^{\nu+\mu}\lambda(x+\delta\widetilde{h}(\theta))$$

$$(m_i(x)-m_i(y))^\mu\delta[1-\delta]^4d\delta d\theta,$$

$$M_{n,3}(\frac{j}{n},\frac{k}{n},x,y)=M_{n,31}(\frac{j}{n},\frac{k}{n},x,y)+...+M_{n,34}(\frac{j}{n},\frac{k}{n},x,y),$$

$$M_{n,34}(\frac{j}{n},\frac{k}{n},x,y)=5n^{-3}\sum_{|\nu|=5}\frac{1}{\nu!}\sum_{i=1}^p m_i(x)\int\int_0^1 q(x,\theta)\theta^{\nu-e_i}$$

$$\cdot\sum_{|\mu|=1}D^{\nu+\mu}\lambda(x+\delta\widetilde{h}(\theta))(m_i(x)-m_i(y))^\mu\delta[1-\delta]^4d\delta d\theta.$$

$$M_{n,32}(\frac{j}{n},\frac{k}{n},x,y)=5n^{-5/2}\sum_{|\nu|=5}\frac{1}{\nu!}\sum_{i,i'=1}^{p}[m_i(x)m_{i'}(x)-m_i(y)m_{i'}(y)]$$

$$\int\int_0^1 q(x,\theta)\theta^{\nu-e_i-e_{i'}}D^{\nu}\lambda(x+\delta\widetilde{h}(\theta))(1-\delta)^4 d\delta d\theta,$$

$$M_{n,33}(\frac{j}{n},\frac{k}{n},x,y)=5n^{-5/2}\sum_{|\nu|=5}\frac{1}{\nu!}\sum_{i,i'=1}^{p}m_i(y)m_{i'}(y)\int\int_0^1[q(x,\theta)-q(y,\theta)]$$

$$\theta^{\nu-e_i-e_{i'}}D^{\nu}\lambda(x+\delta\widetilde{h}(\theta))(1-\delta)^4 d\delta d\theta,$$

$$M_{n,34}(\frac{j}{n},\frac{k}{n},x,y)=5n^{-3}\sum_{|\nu|=5}\frac{1}{\nu!}\sum_{i=1}^{p}m_i(x)\int\int_0^1 q(x,\theta)\theta^{\nu-e_i}$$

$$\cdot\sum_{|\mu|=1}D^{\nu+\mu}\lambda(x+\delta\widetilde{h}(\theta))(m_i(x)-m_i(y))^{\mu}\delta[1-\delta]^4 d\delta d\theta.$$

*Here $e_r$ denotes a $p$ - dimensional vector with $r-th$ element equal to 1 and with all other elements equal to 0. Furthermore, for $|\nu|=4, |\nu'|=2$ we define*

$$N(\nu,\nu')=2^{\chi[\nu'!=1]+\chi[(\nu-\nu')!=1]-2}$$

*where $\chi(\bullet)$ means an indicator function. We put $m(x)^{\nu}=m_1(x)^{\nu_1}\cdot...\cdot m_p(x)^{\nu_p}$ and $m(x)^{\nu}=0$ and $\mu_{\nu}(x)=0$ if at least one of the coordinates of $\nu$ is negative. We define also the following functions*

$$\lambda(z)=\widetilde{p}_n(\frac{j+1}{n},\frac{k}{n},z,y),$$

$$\widetilde{h}(\theta)=n^{-1}m(y)+n^{-1/2}\theta.$$

*For all $j<k$, $x$ and $y$ the function $\zeta_{\rho}$ is defined as in Lemma 5.2. Here again $\rho$ denotes the term $\rho=\left[\frac{k-j}{n}\right]^{1/2}$. For $j=k-1$ and $l=1,...,3$ we define*

$$M_{n,l}(\frac{j}{n},\frac{k}{n},x,y)=0.$$

The proof of Lemma 5.4 is based on some lengthy calculations. It follows the lines of the proof of Lemma 3.9 in Konakov and Mammen (2000). The difference is that this time we use higher order Taylor expansion. Then we replace $\lambda(x)=\widetilde{p}_n(\frac{j+1}{n},\frac{k}{n},x,y)$ by $\widetilde{p}_n(\frac{j}{n},\frac{k}{n},x,y)$ in $(L-\widetilde{L})\lambda(x)$ and in the expressions for $M_{n,11}, M_{n,12}, M_{n,21}$ and $M_{n,31}$. To this end we use the Taylor expansion for $\lambda$ in the following formula

$$D_x^{\nu}\widetilde{p}_n(\frac{j}{n},\frac{k}{n},x,y)=\int q(y,\theta)D_x^{\nu}\lambda(x+\widetilde{h}(\theta))d\theta.$$

**Lemma 5.5** *The following bound holds with a constant $C$*

$$|M_{n,l}(j/n,k/n,x,y)|\le Cn^{-(l-1)/2}\rho^{-1}\zeta_{\rho}^{S-1}(y-x) \qquad for\ l=1,...,3, \tag{12}$$

$$(13) \qquad \left|D_x^a D_y^b M_{n,l}(j/n, k/n, x, y)\right| \leq C\rho^{-|a|-|b|-1}\zeta_\rho^{S-|a|-1}(y-x) \qquad for\ l = 1, ..., 3,$$

$$(14) \qquad \left|D_x^a D_y^b H_n(j/n, k/n, x, y)\right| \leq C\rho^{-|a|-|b|-1}\zeta_\rho^{S-|a|-1}(y-x),$$

$$(15) \qquad \left|D_x^a D_y^b K_n(j/n, k/n, x, y)\right| \leq C\rho^{-|a|-|b|-1}\zeta_\rho^{S-|a|-1}(y-x),$$

$$(16) \qquad \left|D_x^a D_y^b [\tilde{p} \otimes_n (K_n + M_n)^{(r)}](j/n, k/n, x, y)\right|$$

$$\leq C^r B(\frac{1}{2}, \frac{1}{2}) \cdot ... \cdot B(\frac{r}{2}, \frac{1}{2})\rho^{r-1-|a|-|b|}\zeta_\rho^{S-|a|-2}(y-x),$$

*for all $j < k$ and $y$. Here again $\rho = \left[\frac{k-j}{n}\right]^{1/2}$.*

PROOF OF LEMMA 5.5. For $a, b = 0$ claims (14) and (15) have been shown in Konakov and Mammen (2000), Lemma 3.10. For a proof of these claims for $|a| > 0$ or $|b| > 0$ and for the proofs of (12) and (16) one proceeds similarly.

# 6 Proof of Theorem 4.1.

We now come to the proof of Theorem 4.1. Main tools for the proof have been given in Subsections 5.1 - 5.3. From Lemmas 3.1 and 3.2 we get that

$$p(0, 1, x, y) = \sum_{r=0}^{n} \tilde{p} \otimes H^{(r)}(0, 1, x, y) + \frac{1}{n^2} R_n(x, y),$$

where $R_n(x, y)$ is a function with subgaussian tails, i.e. for constants $C, C'$ it holds that

$$|R_n(x, y)| \quad \leq \quad C \exp[-C'(x-y)^2].$$

With Lemma 5.1 this gives

$$(17) \qquad p(0, 1, x, y) - p_n(0, 1, x, y) = T_1 + ... + T_5 + n^{-2} R_n(x, y),$$

where

$$T_1 = \sum_{r=0}^{n} \tilde{p} \otimes H^{(r)}(0, 1, x, y) - \sum_{r=0}^{n} \tilde{p} \otimes_n H^{(r)}(0, 1, x, y),$$

$$T_2 = \sum_{r=0}^{n} \tilde{p} \otimes_n H^{(r)}(0, 1, x, y) - \sum_{r=0}^{n} \tilde{p} \otimes_n (H + M_n + n^{-1/2} N_1)^{(r)}(0, 1, x, y),$$

$$T_3 = \sum_{r=0}^{n} \tilde{p} \otimes_n (H + M_n + n^{-1/2} N_1)^{(r)}(0, 1, x, y) - \sum_{r=0}^{n} \tilde{p} \otimes_n (K_n + M_n)^{(r)}(0, 1, x, y),$$

$$T_4 = \sum_{r=0}^{n} \tilde{p} \otimes_n (K_n + M_n)^{(r)}(0,1,x,y) - \sum_{r=0}^{n} \tilde{p}_n \otimes_n (K_n + M_n)^{(r)}(0,1,x,y),$$

$$T_5 = \sum_{r=0}^{n} \tilde{p}_n \otimes_n (K_n + M_n)^{(r)}(0,1,x,y) - \sum_{r=0}^{n} \tilde{p}_n \otimes_n H_n^{(r)}(0,1,x,y).$$

Here we put $N_1(s,t,x,y) = (L - \widetilde{L})\widetilde{\pi}_1(s,t,x,y)$.

We now discuss the asymptotic behaviour of the terms $T_1, ..., T_5$.

*Asymptotic treatment of the term $T_1$.* Using Theorem 2.1 and the remark following Theorem 1.1 in Konakov and Mammen (2002) we get that

$$T_1 \;=\; \frac{1}{2n} p \otimes_n (L - \widetilde{L})^2 \widetilde{p} \otimes_n \Phi(0,1,x,y) + \frac{1}{n^2} R_n(x,y),$$

where $R_n(x,y)$ is a function with subgaussian tails, i.e. for constants $C, C'$ it holds that

$$|R_n(x,y)| \;\leq\; C \exp[-C'(x-y)^2]$$

and where $\Phi(s,t,x,y) = \sum_{r=0}^{\infty} H^{(r)}(s,t,x,y)$.

*Asymptotic treatment of the term $T_2$.* We will show the following expansion of $T_2$ for a constant $C > 0$ and for $\delta > 0$ small enough.

$$\begin{aligned} (18) \qquad & \Big| T_2 - 4 \sum_{r=0}^{\infty} \widetilde{p} \otimes_n H^{(r)}(0,1,x,y) \\ & + \sum_{r=0}^{\infty} \widetilde{p} \otimes_n (H + M_{n,11} + n^{-1/2} N_1)^{(r)}(0,1,x,y) \\ & + \sum_{r=0}^{\infty} \widetilde{p} \otimes_n (H + M_{n,12})^{(r)}(0,1,x,y) \\ & + \sum_{r=0}^{\infty} \widetilde{p} \otimes_n (H + M_{n,21})^{(r)}(0,1,x,y) \\ & + \sum_{r=0}^{\infty} \widetilde{p} \otimes_n (H + M_{n,31})^{(r)}(0,1,x,y) \Big| \leq C n^{-1-\delta} \zeta(y-x). \end{aligned}$$

For the terms on the left hand side of (18) we will show the following bounds with a constant $C > 0$.

$$\begin{aligned} (19) \qquad & |\widetilde{p} \otimes_n (M_{n,11} + n^{-1/2} N_1 + H)^{(r)}(0, k/n, x, y) \\ & - \widetilde{p} \otimes_n H^{(r)}(0, k/n, x, y)| \\ & \leq \frac{C^r}{\sqrt{n}} B(\frac{1}{2}, \frac{1}{2}) \cdot \ldots \cdot B(\frac{r}{2}, \frac{1}{2}) (\frac{k}{n})^{(r-1)/2} \zeta^{r+1,0,k}(y-x), \end{aligned}$$

$$|\widetilde{p} \otimes_n (M_{n,12} + H)^{(r)}(0, k/n, x, y) - \widetilde{p} \otimes_n H^{(r)}(0, k/n, x, y)| \tag{20}$$

$$\leq \frac{C^r}{n} B(\frac{1}{2}, \frac{1}{2}) \cdot \ldots \cdot B(\frac{r}{2}, \frac{1}{2}) (\frac{k}{n})^{(r-2)/2} \zeta^{r+1,0,k}(y - x),$$

$$|\widetilde{p} \otimes_n (H + M_{n,21})^{(r)}(0, k/n, x, y) - \widetilde{p} \otimes_n H^{(r)}(0, k/n, x, y)| \tag{21}$$

$$\leq \frac{C^r}{n} B(\frac{1}{2}, \frac{1}{2}) \cdot \ldots \cdot B(\frac{r}{2}, \frac{1}{2}) (\frac{k}{n})^{(r-1)/2} \zeta^{r+1,0,k}(y - x),$$

$$|\widetilde{p} \otimes_n (H + M_{n,31})^{(r)}(0, k/n, x, y) - \widetilde{p} \otimes_n H^{r}(0, k/n, x, y)| \tag{22}$$

$$\leq \frac{C^r}{n} B(\frac{1}{2}, \frac{1}{2}) \cdot \ldots \cdot B(\frac{r}{2}, \frac{1}{2}) (\frac{k}{n})^{(r-1)/2} \zeta^{r+1,0,k}(y - x),$$

where

$$\zeta^{l,j,k}(x) = \max\{\zeta_{\rho_1} \star \ldots \star \zeta_{\rho_l}(x) : \rho_1 \geq 0, \ldots, \rho_l \geq 0,$$
$$\rho_1^2 + \ldots + \rho_l^2 = (k - j)/n\},$$
$$\zeta(x) = [1 + \|x\|^{2s'}]^{-1} \left\{ \int [1 + \|u\|^{2s'}]^{-1} du \right\}^{-1}.$$

We now give the proofs of (18), (19) and (20). The proofs of claims (21)-(22) are omitted. These claims follow by similar arguments as (19)-(20). All claims are proved iteratively by induction. In the induction steps the bounds given in (19)-(22) are used for $1 \leq k \leq n - 1$. Note that in (18) the terms only appear for $k = n$. We now start by proving (19) and (20). The proof of (18) will be given afterwards.

*Proof of (19).* We first prove (19) for $r = 1$. For this purpose we write

$$\widetilde{p} \otimes_n (M_{n,11} + n^{-1/2} N_1)(0, k/n, x, y) = n^{-1/2} \sum_{|\nu|=3} \frac{1}{\nu!} S_\nu,$$

where

$$S_\nu = \frac{1}{n} \sum_{j=0}^{k-1} \int \widetilde{p}(0, j/n, x, u) D_u^\nu \widetilde{p}_n(j/n, k/n, u, y)[\mu_\nu(u) - \mu_\nu(y)] \tag{23}$$

$$+\chi_\nu(y)\rho^2 \int \widetilde{p}(0, j/n, x, u) \left[ \sum_{i=1}^{p} D_u^{\nu+e_i} \tilde{p}(j/n, k/n, u, y)( m_i(u) - m_i(y)) \right.$$

$$\left. + \frac{1}{2} \sum_{i,l=1}^{p} D_u^{\nu+e_i+e_l} \widetilde{p}(j/n, k/n, u, y)(\sigma_{il}(u) - \sigma_{il}(y)) \right]$$

We now decompose the summand $S_\nu$

$$S_\nu = \frac{1}{n} \sum_{j \in J_1} \int \ldots du + \frac{1}{n} \sum_{j \in J_2} \int \ldots du,$$

where $J_1 = \{j : 0 \le j \le (k-1)/2\}$ and $J_2 = \{j : (k-1)/2 < j \le k-1\}$. With $\rho = \sqrt{k/n - j/n}$ and $\kappa = \sqrt{k/n}$ we get from Lemma 5.3 , (23) and (A2) with some constants $C$ and $C_1$

$$\begin{aligned}
|n^{-1} \sum_{j \in J_1} \int \dots du| &\le C n^{-1} \sum_{j \in J_1} \rho^{\cdot -2} \zeta_\kappa(y-x) \\
&\le C \zeta_\kappa(y-x) \int_0^{k/(2n)} \frac{dt}{(k/n-t)} \\
&= C \zeta_\kappa(y-x) \ln(2) \\
&\le C_1 \zeta^{2,0,k}(y-x) B(\frac{1}{2}, \frac{1}{2}).
\end{aligned}$$

Furthermore, with $e_r + e_s \le \nu$ (componentwise)

$$\begin{aligned}
\left| n^{-1} \sum_{j \in J_2} \int \dots du \right| = \Big| n^{-1} \sum_{j \in J_2} \Big\{ \int & D_u^{e_r+e_s} [\widetilde{p}(0, j/n, x, u)\{\mu_\nu(u) - \mu_\nu(y)\}] \\
& D_u^{\nu - e_r - e_s} \widetilde{p}_n(j/n, k/n, u, y) du \\
+ \chi_\nu(y) \rho^2 \sum_{i=1}^p \int & D_u^{e_r+e_s} [\widetilde{p}(0, j/n, x, u)\{m_i(u) - m_i(y)\}] \\
& D_u^{\nu + e_i - e_r - e_s} \widetilde{p}(j/n, k/n, u, y) du \\
+ \chi_\nu(y) \rho^2 \frac{1}{2} \sum_{i,l=1}^p \int & D_u^{e_r+e_s} [\widetilde{p}(0, j/n, x, u)\{\sigma_{il}(u) - \sigma_{il}(y)\}] \\
& D_u^{\nu + e_i + e_l - e_r - e_s} \widetilde{p}(j/n, k/n, u, y) du \Big\} \Big| \\
\le C_2 n^{-1} \sum_{j \in J_2} & [\frac{1}{j/n} + \frac{1}{\sqrt{j/n}\sqrt{k/n - j/n}}] \zeta^{2,0,k}(y-x) \\
& \le C_3 \zeta^{2,0,k}(y-x)
\end{aligned}$$

with some constants $C_2, C_3 > 0$. Combining the last two estimates we get that

$$|S_\nu| \le C_4 \zeta^{2,0,k}(y-x) B(\frac{1}{2}, \frac{1}{2})$$

for some constant $C_4$.

This shows claim (19) for $r = 1$. We now check the claim for $r = 2$. We have

$$\begin{aligned}
& \widetilde{p} \otimes_n (M_{n,11} + n^{-1/2} N_1 + H)^{(2)} - \widetilde{p} \otimes_n H^{(2)} \\
&= \widetilde{p} \otimes_n (M_{n,11} + n^{-1/2} N_1) \otimes_n (M_{n,11} + n^{-1/2} N_1 + H) \\
&\quad + \widetilde{p} \otimes_n H \otimes_n (M_{n,11} + n^{-1/2} N_1).
\end{aligned}$$

The first term on the right hand side can be bounded as follows

$$\left| \widetilde{p} \otimes_n (M_{n,11} + n^{-1/2} N_1) \otimes_n (M_{n,11} + n^{-1/2} N_1 + H)(0, k/n, x, y) \right|$$

$$\leq \frac{C}{\sqrt{n}}\frac{1}{n}\sum_{i=0}^{k-1}\int \zeta^{2,0,i}(z-x)B(\frac{1}{2},\frac{1}{2})\frac{1}{\sqrt{\frac{k}{n}-\frac{i}{n}}}\zeta^{1,i,k}(y-z)dz$$

$$\leq \frac{C}{\sqrt{n}}\zeta^{3,0,k}(y-x)B(\frac{1}{2},\frac{1}{2})\frac{1}{n}\sum_{i=0}^{k-1}\frac{1}{\sqrt{\frac{k}{n}-\frac{i}{n}}}$$

$$\leq \frac{C}{\sqrt{n}}(\frac{k}{n})^{1/2}\zeta^{3,0,k}(y-x)B(\frac{1}{2},\frac{1}{2})B(1,\frac{1}{2}).$$

For the second term we get

$$\left|\widetilde{p}\otimes_n H\otimes_n (M_{n,11}+n^{-1/2}N_1)(0,k/n,x,y)\right|$$

$$= \Bigg| n^{-1/2}\sum_{|\nu|=3}\frac{1}{\nu!}\frac{1}{n}\sum_{j=0}^{k-1}\Bigg\{\int (\widetilde{p}\otimes_n H)(0,j/n,x,u)D_u^{\nu}\widetilde{p}_n(\frac{j}{n},\frac{k}{n},u,y)[\mu_\nu(u)-\mu_\nu(y)]$$

$$+\chi_\nu(y)\rho^2\int(\widetilde{p}\otimes_n H)(0,j/n,x,u)\Bigg[\sum_{i=1}^{p}D_u^{\nu+e_i}\tilde{p}(j/n,k/n,u,y)(\,m_i(u)-m_i(y))$$

$$+\frac{1}{2}\sum_{i,l=1}^{p}D_u^{\nu+e_i+e_l}\tilde{p}(j/n,k/n,u,y)(\sigma_{il}(u)-\sigma_{il}(y))\Bigg]du\Bigg\}\Bigg|$$

$$\leq n^{-1/2}\sum_{|\nu|=3}\frac{1}{\nu!}\frac{1}{n}\sum_{j\in J_1}|\int\ldots du|+n^{-1/2}\sum_{|\nu|=3}\frac{1}{\nu!}\frac{1}{n}\sum_{j\in J_2}|\int\ldots du|.$$

These two terms can be treated similar as in the proof for $r=1$. The first term can be bounded by use of direct estimates. The second term can be easily bounded after two applications of partial integrations.

The proof for $r\geq 2$ follows by iteration and use of similar methods.

*Proof of (20).* The claim follows similarly as in the proof of (19). Again the region of the summation is splitted into two regions $J_1$ and $J_2$. Again, for the treatment of the second sum partial integration is used.

*Proof of (18).* This expansion immediately follows from the following bounds

$$\left|\widetilde{p}\otimes_n (M_{n,11}+n^{-1/2}N_1+M_{n,12}+M_{n,13}+H)^r(0,k/n,x,y)\right. \tag{24}$$

$$\left.-\widetilde{p}\otimes_n (M_{n,11}+n^{-1/2}N_1+M_{n,12}+H)^{(r)}(0,k/n,x,y)\right|$$

$$\leq \frac{C^r}{n^{3/2}}\log(n)B(\frac{1}{2},\frac{1}{2})\cdot\ldots\cdot B(\frac{r}{2},\frac{1}{2})(\frac{k}{n})^{(r-3)/2}\zeta^{r+1,0,k}(y-x),$$

$$|\widetilde{p}\otimes_n (H+M_{n,1}+n^{-1/2}N_1+M_{n,2})^{(r)}(0,k/n,x,y) \tag{25}$$

$$-\widetilde{p}\otimes_n (H+M_{n,1}+n^{-1/2}N_1+M_{n,21})^{(r)}(0,k/n,x,y)|$$

$$\leq \frac{C^r}{n^{3/2}}B(\frac{1}{2},\frac{1}{2})\cdot\ldots\cdot B(\frac{r}{2},\frac{1}{2})(\frac{k}{n})^{(r-1)/2}\zeta^{r+1,0,k}(y-x),$$

$$(26)\qquad |\widetilde{p}\otimes_n (H+M_{n,1}+n^{-1/2}N_1+M_{n,2}+M_{n,3})^{(r)}(0,k/n,x,y)$$
$$-\widetilde{p}\otimes_n (H+M_{n,1}+n^{-1/2}N_1+M_{n,2}+M_{n,31})^{(r)}(0,k/n,x,y)|$$
$$\le \frac{C^r}{n^{3/2}}B(\frac{1}{2},\frac{1}{2})\cdot\ldots\cdot B(\frac{r}{2},\frac{1}{2})(\frac{k}{n})^{(r-1)/2}\zeta^{r+1,0,k}(y-x),$$

$$(27)\qquad \big|\widetilde{p}\otimes_n (M_{n,11}+n^{-1/2}N_1+M_{n,12}+H)^{(r)}(0,1,x,y)$$
$$-\widetilde{p}\otimes_n (M_{n,11}+n^{-1/2}N_1+H)^{(r)}(0,1,x,y)$$
$$-[\widetilde{p}\otimes_n (M_{n,12}+H)^{(r)}-\widetilde{p}\otimes_n H^{(r)}](0,1,x,y)\big|$$
$$\le \frac{C^r}{n^{3/2-\varepsilon}}B(\varepsilon,\varepsilon)\cdot\ldots\cdot B(\frac{1}{2}+r\varepsilon,\varepsilon)\zeta(y-x),$$

$$(28)\qquad \big|\widetilde{p}\otimes_n (M_{n,1}+n^{-1/2}N_1+M_{n,21}+H)^{(r)}(0,1,x,y)$$
$$-\widetilde{p}\otimes_n (M_{n,1}+n^{-1/2}N_1+H)^{(r)}(0,1,x,y)$$
$$-[\widetilde{p}\otimes_n (H+M_{n,21})^{(r)}-\widetilde{p}\otimes_n H^{(r)}](0,1,x,y)\big|$$
$$\le \frac{C^r}{n^{3/2-\varepsilon}}B(1,\varepsilon)\cdot\ldots\cdot B(1+r\varepsilon,\varepsilon)\zeta(y-x),$$

$$(29)\qquad \big|\widetilde{p}\otimes_n (M_{n,1}+n^{-1/2}N_1+M_{n,2}+M_{n,31}+H)^{(r)}(0,1,x,y)$$
$$-\widetilde{p}\otimes_n (M_{n,1}+n^{-1/2}N_1+M_{n,2}+H)^{(r)}(0,1,x,y)$$
$$-[\widetilde{p}\otimes_n (H+M_{n,31})^{(r)}-\widetilde{p}\otimes_n H^{(r)}](0,1,x,y)\big|$$
$$\le \frac{C^r}{n^{3/2-\varepsilon}}B(1,\varepsilon)\cdot\ldots\cdot B(1+r\varepsilon,\varepsilon)\zeta(y-x).$$

These estimates are valid for any $\varepsilon\in(0,\frac{1}{2})$ with a constant $C(\varepsilon)<\infty$ depending on $\varepsilon$.

We will prove (24) and (27). The proofs of the other claims are quite similar to the proofs of (19), (24) and (27) and will be omitted.

*Proof of (24).* For $r=1$ we get by use of direct bounds

$$|\widetilde{p}\otimes_n M_{n,13}(0,k/n,x,y)|\le\frac{C}{n^{3/2}}(\frac{k}{n})^{-1}B(\frac{1}{2},\frac{1}{2})\zeta^{2,0,k}(y-x).$$

In the next step ($r=2$) we use the bound

$$\sum_{j=1}^{k/2}\frac{1}{n}(j/n)^{-1}\le C\ \log n.$$

This gives the additional log factor in (24). Besides this the claim can be proved along the lines of the proof of (19).

*Proof of (27).* Denote the expression under the sign of the absolute value in (27) by $\Gamma_r$. Then we have the following reccurrence formula

$$\begin{aligned}(30)\qquad \Gamma_r &= \Gamma_{r-1} \otimes_n H + \big[\widetilde{p} \otimes_n (M_{n,11} + n^{-1/2}N_1 + M_{n,12} + H)^{(r-1)} \\ &\quad - \widetilde{p} \otimes_n (M_{n,11} + n^{-1/2}N_1 + H)^{(r-1)}\big] \otimes_n (M_{n,11} + n^{-1/2}N_1) \\ &\quad + \big[\widetilde{p} \otimes_n (M_{n,11} + n^{-1/2}N_1 + M_{n,12} + H)^{(r-1)} \\ &\quad - \widetilde{p} \otimes_n (M_{n,12} + H)^{(r-1)}\big] \otimes_n M_{n,12} = I + II + III.\end{aligned}$$

Note that $\Gamma_0 = \Gamma_1 = 0$. We start from the estimation of the second summand $II$ in (30). Similarly as in the proof of (19) we decompose the sum into two sums

$$II = \frac{1}{n}\sum_{j\in J_1}\int \ldots du + \frac{1}{n}\sum_{j\in J_2}\int \ldots du.$$

The following bound is a modification of (20).

$$\begin{aligned}(31)\qquad &\big|\widetilde{p} \otimes_n (M_{n,11} + n^{-1/2}N_1 + M_{n,12} + H)^{(r)}(0, k/n, x, y) \\ &\qquad - \widetilde{p} \otimes_n (M_{n,11} + n^{-1/2}N_1 + H)^{(r)}(0, k/n, x, y)\big| \\ &\le \frac{C^r}{n} B(\frac{1}{2}, \frac{1}{2}) \cdot \ldots \cdot B(\frac{r}{2}, \frac{1}{2})(\frac{k}{n})^{(r-2)/2} \zeta^{r+1,0,k}(y - x).\end{aligned}$$

*Proof of (31).* The claim can be shown similarly as in the proof of (19). Again the sum is splitted into two regions $J_1$ and $J_2$ and for the second sum partial integration is used. For the partial integration we make use of the following bounds that easily follow from the definition of $M_{n,11}$

$$\big|D_y^a D_x^b (M_{n,11} + n^{-1/2}N_1)(j/n, k/n, x, y)\big| \le \frac{\mathrm{C}_1}{n^{1/2}} (\frac{k-j}{n})^{-(2+|a|+|b|)/2} \zeta_\rho(y - x),$$

$$\big|D_x^b (M_{n,11} + n^{-1/2}N_1)(j/n, k/n, x, x + v)\big| \le \frac{C_2}{n^{1/2}} (\frac{k-j}{n})^{-1} \zeta_\rho(v),$$

for some constants $C_1, C_2 > 0$. We also have that for some constants $C_3$ and $C_4$

$$\begin{aligned}(32)\qquad &\big|D_y^a D_x^b (\widetilde{p} \otimes_n (M_{n,11} + n^{-1/2}N_1)^{(r)})(0, k/n, x, y)\big| \\ &\le \frac{C_3^r}{n^{1/2}} B(\frac{1}{2}, \frac{1}{2}) \cdot \ldots \cdot B(\frac{r}{2}, \frac{1}{2})(\frac{k}{n})^{-(|a|+|b|+1-r)/2} \zeta_\rho(y - x),\end{aligned}$$

$$\begin{aligned}(33)\qquad &\big|D_x^b (\widetilde{p} \otimes_n (M_{n,11} + n^{-1/2}N_1)^{(r)})(0, k/n, x, x + v)\big| \\ &\le \frac{C_4^r}{n^{1/2}} B(\frac{1}{2}, \frac{1}{2}) \cdot \ldots \cdot B(\frac{r}{2}, \frac{1}{2})(\frac{k}{n})^{(r-1)/2} \zeta_\rho(v).\end{aligned}$$

The inequalities (32), (33) can be shown as in the proof of (5.7) and (5.8) in Konakov and Mammen (2002).

Using (31) we get for $r \geq 2$

$$\left|\frac{1}{n}\sum_{j\in J_1}\int \ldots du\right| \leq \frac{C^{r-1}}{n}B(\frac{1}{2},\frac{1}{2})\cdot\ldots\cdot B(\frac{r-1}{2},\frac{1}{2}) \tag{34}$$

$$\times\frac{1}{n}\sum_{j\in J_1}(\frac{j}{n})^{(r-3)/2}\frac{1}{n^{1/2}(\frac{k}{n}-\frac{j}{n})}\int \zeta^{r,0,j}(u-x)\zeta(y-u)du$$

$$\leq \frac{C_1^r}{n^{3/2}}B(\frac{1}{2},\frac{1}{2})\cdot\ldots\cdot B(\frac{r}{2},\frac{1}{2})(\frac{k}{n})^{(r-3)/2}\zeta^{r+1,0,k}(y-x).$$

For $j \in J_2$ we have with $\varepsilon \in (0,\frac{1}{2})$

$$\left|\frac{1}{n}\sum_{j\in J_2}\int \ldots du\right| \leq \frac{C^{r-1}}{n}B(\varepsilon,\varepsilon)B(\frac{1}{2}+\varepsilon,\varepsilon)\cdot\ldots\cdot B(\frac{1}{2}+(r-2)\varepsilon,\varepsilon) \tag{35}$$

$$\times\frac{1}{n}\sum_{j\in J_2}(\frac{j}{n})^{(r-2)\varepsilon}\frac{1}{n^{\frac{1}{2}-\varepsilon}(\frac{k}{n}-\frac{j}{n})^{1-\varepsilon}}\int \zeta^{r,0,j}(u-x)\zeta(y-u)du$$

$$\leq \frac{C_1^r}{n^{\frac{3}{2}-\varepsilon}}B(\varepsilon,\varepsilon)B(\frac{1}{2}+\varepsilon,\varepsilon)\cdot\ldots\cdot B(\frac{1}{2}+(r-1)\varepsilon,\varepsilon)(\frac{k}{n})^{(r-1)\varepsilon}\zeta^{r+1,0,k}(y-x).$$

To estimate the third term $III$ in (30) we use the following estimate for the derivatives

$$\left|D_y^{e_i}\left[\widetilde{p}\otimes_n (M_{n,11}+n^{-1/2}N_1+M_{n,12}+H)^{(r)}\right.\right. \tag{36}$$

$$\left.\left.-\widetilde{p}\otimes_n (M_{n,12}+H)^{(r)}\right](0,k/n,x,y)\right|$$

$$\leq \frac{C^r}{n^{1/2}}B(\frac{1}{2},\frac{1}{2})\cdot\ldots\cdot B(\frac{r}{2},\frac{1}{2})(\frac{k}{n})^{(r-2)/2}\zeta^{r+1,0,k}(y-x).$$

Inequality (36) can be shown by induction on $r$. The basic tools are integration by parts and the following estimates for $H$ and $\widetilde{p}_n$

$$\left|D_y^{e_i}H(\frac{j}{n},\frac{k}{n},x,y)+D_x^{e_i}H(\frac{j}{n},\frac{k}{n},x,y)\right| \leq \frac{C}{\rho}\phi_{C,\rho}(y-x), \tag{37}$$

$$\left|D_y^{e_i}D_x^{\nu}\widetilde{p}_n(\frac{j}{n},\frac{k}{n},x,y)+D_x^{e_i}D_x^{\nu}\widetilde{p}_n(\frac{j}{n},\frac{k}{n},x,y)\right| \leq \frac{C}{\rho}\zeta_\rho(y-x). \tag{38}$$

Inequality (37) is contained in Lemma 3.4 in Konakov and Mammen (2000). Inequality (42) can be shown by direct calculations. The proof uses the representation of $\widetilde{p}_n$ and of its derivatives (with respect to covariance and mean) from Lemma 3.7 in Konakov and Mammen (2000). To estimate the derivatives we also use Lemma 4 from Konakov and Molchanov (1985). We omit the details. For estimating the third term $III$ in (30) we again split the summation region

$$III = \frac{1}{n}\sum_{j\in J_1}\int \ldots du + \frac{1}{n}\sum_{j\in J_2}\int \ldots du.$$

For an estimation of $\frac{1}{n}\sum_{j\in J_1}\int\ldots du$ we use the direct estimate

$$\Big|\widetilde{p}\otimes_n (M_{n,11}+n^{-1/2}N_1+M_{n,12}+H)^{(r-1)}(0,\frac{k}{n},x,y)$$

$$-\widetilde{p}\otimes_n (M_{n,12}+H)^{(r-1)}(0,\frac{k}{n},x,y)\Big|$$

$$\leq \frac{C^{r-1}}{n^{1/2}}B(\frac{1}{2},\frac{1}{2})\cdot\ldots\cdot B(\frac{r-1}{2},\frac{1}{2})(\frac{k}{n})^{(r-2)/2}\zeta^{r,0,k}(y-x).$$

For an estimation of $\frac{1}{n}\sum_{j\in J_2}\int\ldots du$ we apply integration by parts and (36) several times. This completes the proof of (27).

*Asymptotic treatment of* $T_3$. We will show that

$$T_3=\sum_{r=1}^{n}\widetilde{p}\otimes_n H^{(r)}(0,1,x,y) \tag{39}$$

$$-\sum_{r=1}^{n}\widetilde{p}\otimes_n [H+\frac{1}{n}N_2]^{(r)}(0,1,x,y)+R_n^{\star}(x,y),$$

with $N_2(s,t,x,y)=(L-\widetilde{L})\widetilde{\pi}_2(s,t,x,y)$, $|R_n^{\star}(x,y)|\leq Cn^{-1-\delta}\xi(y-x)$ for $\delta>0$ small enough, and a constant $C$ depending on $\delta$. For the proof of (39) it suffices to show that for $\delta$ small enough:

$$\Big|\sum_{r=1}^{n}\widetilde{p}\otimes_n (H+M_n+n^{-1/2}N_1+n^{-1}N_2)^{(r)}(0,1,x,y) \tag{40}$$

$$-\sum_{r=1}^{n}\widetilde{p}\otimes_n (K_n+M_n)^{(r)}(0,1,x,y)\Big|$$

$$\leq Cn^{-1-\delta}\xi(y-x),$$

$$\Big|\sum_{r=1}^{n}\widetilde{p}\otimes_n (H+M_n+n^{-1/2}N_1)^{(r)}(0,1,x,y) \tag{41}$$

$$-\sum_{r=1}^{n}\widetilde{p}\otimes_n (H+M_n+n^{-1/2}N_1+n^{-1}N_2)^{(r)}(0,1,x,y)$$

$$-[\sum_{r=1}^{n}\widetilde{p}\otimes_n H^{(r)}(0,1,x,y)-\sum_{r=1}^{n}\widetilde{p}\otimes_n (H+n^{-1}N_2)^{(r)}(0,1,x,y)]\Big|$$

$$\leq Cn^{-1-\delta}\xi(y-x).$$

*Proof of (44).* Denote for $0\leq m\leq n$

$$D_{3,m}(0,\frac{j}{n},x,y)=\sum_{r=0}^{m}[\widetilde{p}\otimes_n (K_n+M_n)^{(r)}(0,\frac{j}{n},x,y)$$

$$-\widetilde{p} \otimes_n (H + M_n + n^{-1/2}N_1 + n^{-1}N_2)^{(r)}(0, \frac{j}{n}, x, y)].$$

Then we have to show

$$|D_{3,n}(0,1,x,y)| \leq Cn^{-1-\delta}\xi(y-x).$$

We now make iterative use of

$$D_{3,m} = D_{3,m-1} \otimes_n (H + M_n + n^{-1/2}N_1 + n^{-1}N_2) + h_{m-1},$$

for $m = 1, 2, ...$, where

$$h_m(0, \frac{j}{n}, x, y) = -\sum_{r=0}^{m} \widetilde{p} \otimes_n (K_n + M_n)^{(r)} \otimes_n (H - K_n + n^{-1/2}N_1 + n^{-1}N_2)(0, \frac{j}{n}, x, y)$$

$$= S_{n,m} \otimes_n (L - \widetilde{L})d_n(0, \frac{j}{n}, x, y)$$

with

$$d_n = \widetilde{p}_n - \widetilde{p} - n^{-1/2}\widetilde{\pi}_1 - n^{-1}\widetilde{\pi}_2,$$

$$S_{n,m}(0, \frac{i}{n}, x, y) = \sum_{r=0}^{m} \widetilde{p} \otimes_n (K_n + M_n)^{(r)}(0, \frac{i}{n}, x, y).$$

Iterative application of this equation gives

$$D_{3,n}(0,1,x,y) = \sum_{r=0}^{n-1} h_r \otimes_n (H + M_n + n^{-1/2}N_1 + n^{-1}N_2)^{(n-r-1)}(0,1,x,y). \tag{42}$$

To prove (40) we will show that

$$\left|h_r \otimes_n (H + M_n + n^{-1/2}N_1 + n^{-1}N_2)^{(n-r-1)}(0,1,x,y)\right| \tag{43}$$

$$\leq n^{-1-\delta}C^{n-r-1}B(1, \frac{1}{2}) \cdot \ldots \cdot B(\frac{n-r-1}{2}, \frac{1}{2})\zeta(y-x).$$

For this purpose we decompose the left handside of (43) into four terms

$$a_{r,1} = \sum_{0 \leq i \leq n/2} \frac{1}{n} \int h_r(0, \frac{i}{n}, x, u)(H + M_n + n^{-1/2}N_1 + n^{-1}N_2)^{(n-r-1)}(\frac{i}{n}, 1, u, y)du,$$

$$a_{r,2} = \sum_{n/2 \leq i \leq n} \frac{1}{n^2} \sum_{0 \leq k \leq i/2} \int S_{n,r}(0, \frac{k}{n}, x, v)$$

$$\times (L - \widetilde{L})d_n(\frac{k}{n}, \frac{i}{n}, v, u)(H + M_n + n^{-1/2}N_1 + n^{-1}N_2)^{(n-r-1)}(\frac{i}{n}, 1, u, y)dvdu,$$

$$a_{r,3} = \sum_{n/2 < i \leq n} \frac{1}{n^2} \sum_{i/2 < k \leq i - n^{\delta'}} \int (L^t - \widetilde{L}^t)S_{n,r}(0, \frac{k}{n}, x, v)$$

$$\times d_n(\frac{k}{n}, \frac{i}{n}, v, u)(H + M_n + n^{-1/2}N_1 + n^{-1}N_2)^{(n-r-1)}(\frac{i}{n}, 1, u, y)dvdu,$$

$$a_{r,4} = \sum_{n/2<i\le n} \frac{1}{n^2} \sum_{i-n^{\delta'}<k\le i-1} \int (L^t - \widetilde{L}^t) S_{n,r}(0, \frac{k}{n}, x, v)$$

$$\times d_n(\frac{k}{n}, \frac{i}{n}, v, u)(H + M_n + n^{-1/2}N_1 + n^{-1}N_2)^{(n-r-1)}(\frac{i}{n}, 1, v, y)dvdu.$$

Here $L^t$ and $\widetilde{L}^t$ denote the adjoint operators of $L$ and $\widetilde{L}$. Note that

$$(L^t - \widetilde{L}^t)f(s,t,x,u) = \frac{1}{2}\sum_{i,j} \frac{\partial^2}{\partial u_i \partial u_j}[f(s,t,x,u)$$

$$\times(\sigma_{ij}(u) - \sigma_{ij}(y))] - \sum_i \partial/\partial u_i[f(s,t,x,u)(m_i(u) - m_i(y))].$$

In particular, we have

$$h_r = (L^t - \widetilde{L}^t)S_{n,r} \otimes_n d_n.$$

For the proof of (43) it suffices to show for j=1,...,4

$$|a_{r,j}| \le n^{-1-\delta} C^{n-r-1} B(1, \frac{1}{2}) \cdot \ldots \cdot B(\frac{n-r-1}{2}, \frac{1}{2})\zeta(y-x). \tag{44}$$

*Proof of (44) for $j = 2$.* The claim follows from (12) and (15) by noting that for $k \le i/2, n/2 < i$

$$\left| S_{n,r}(0, \frac{k}{n}, x, v) \right| \le C\zeta^{S-2}_{\sqrt{k/n}}(v-x), \tag{45}$$

$$\left| (H + M_n + n^{-1/2}N_1 + n^{-1}N_2)^{(n-r-1)}(\frac{i}{n}, 1, u, y) \right|$$

$$\le C^{n-r-1}(1 - \frac{i}{n})^{-1/2} B(1, \frac{1}{2}) \cdot \ldots \cdot B(\frac{n-r-1}{2}, \frac{1}{2})\zeta_{\sqrt{(n-i)/n}}(y-u),$$

$$|(L - \widetilde{L})d_n(\frac{k}{n}, \frac{i}{n}, v, u)| \le Cn^{-3/2}\zeta^{S-8}_{\sqrt{\frac{i-k}{n}}}(v-u).$$

*Proof of ( 44) for $j = 3$.* We now apply that $i/2 < k \le i - n^{\delta'}, n/2 < i$. It follows from (11), (13), (15) and (45) that

$$\left| (L^t - \widetilde{L}^t) S_{n,r}(0, \frac{k}{n}, x, v) \right| \le C\zeta^{S-4}_{\sqrt{k/n}}(v-x),$$

$$\left| d_n(\frac{k}{n}, \frac{i}{n}, v, u) \right| \le Cn^{-3/2}\rho^{-3}\zeta^{S-6}_{\rho}(u-v)$$

with $\rho = \sqrt{(i-k)/n}$. Note that

$$\left| \frac{1}{n} \sum_{i/2<k\le i-n^{\delta'}} \int (L^t - \widetilde{L}^t) S_{n,r}(0, \frac{k}{n}, x, v) d_n(\frac{k}{n}, \frac{i}{n}, v, u) dv \right|$$

$$\leq \frac{1}{n} \sum_{i/2<k\leq i-n^{\delta'}} n^{-3/2}\rho^{-3}\zeta(u-x)$$

$$\leq n^{-1-(\delta')^2/2} \cdot \frac{1}{n} \sum_{i/2<k\leq i-n^{\delta'}} \rho^{-2+\delta'}\zeta(u-x)$$

$$\leq Cn^{-1-\delta''}\zeta(u-x)$$

for $\delta''$ small enough.

*Proof of (44) for $j=4$.* For $i-n^{\delta'} < k \leq i-1, n/2 < i$ we have

$$\int (L^t - \widetilde{L}^t) S_{n,r}(0, \frac{k}{n}, x, v) \widetilde{p}_n(\frac{k}{n}, \frac{i}{n}, v, u) dv$$

$$= \int (L^t - \widetilde{L}^t) S_{n,r}(0, \frac{k}{n}, x, u - \frac{w}{\sqrt{n}} - \frac{m(u)}{n}) q^{(i-k)}(u, w) dw$$

$$= (L^t - \widetilde{L}^t) S_{n,r}(0, \frac{k}{n}, x, u - \frac{m(u)}{n})$$

$$+\frac{1}{2} \sum_{j,l=1}^{p} D_{w_j w_l} \left[ (L^t - \widetilde{L}^t) S_{n,r}(0, \frac{k}{n}, x, u - \frac{m(u)}{n}) \right] \frac{i-k}{n} \sigma_{j,l}(u)$$

$$+O(n^{-1-\delta})\zeta^{S-2}(u-x)$$

for $\delta$ small enough. Here ( A2) has been applied and it has been used that

$$\int q^{(i-k)}(u,w)dw = 1, \int w_l q^{(i-k)}(u,w)dw = 0, \int w_l w_j q^{(i-k)}(u,w)dw = (i-k)\sigma_{j,l}(u).$$

The same expansion holds with $\widetilde{p}_n$ replaced by $\widetilde{p}$. Furthermore, one can show by partial integration that

$$\int (L^t - \widetilde{L}^t) S_{n,j}(0, \frac{k}{n}, x, v) [\frac{1}{\sqrt{n}} \widetilde{\pi}_1 + \frac{1}{n} \widetilde{\pi}_2](\frac{k}{n}, \frac{i}{n}, v, u) dv$$

$$= O(n^{-1-\delta}) \zeta_{\sqrt{i/n}}(u-x)$$

for $\delta$ small enough. Hence, (44) holds for $j=4$.

*Proof of (44) for $j=1$.* We define

$$a_{r,5} = \sum_{0\leq i\leq n/2} \frac{1}{n^2} \sum_{i-n^{\delta'}<k\leq i-1} \int (L^t - \widetilde{L}^t) S_{n,r}(0, \frac{k}{n}, x, v)$$

$$d_n(\frac{k}{n}, \frac{i}{n}, v, u)(H + M_n + n^{-1/2}N_1 + n^{-1}N_2)^{(n-r-1)}(\frac{i}{n}, 1, u, y) dv du.$$

By integrating by parts with respect to $v$ and by using (45) and arguments as in the proof of (44) for $j=4$ one can show that

$$|a_{r,5}| \leq O(n^{-1-\delta}) B(1, \frac{1}{2}) \cdot \ldots \cdot B(\frac{n-r-1}{2}, \frac{1}{2}) \zeta(y-x)$$

for $\delta$ small enough. Now, by using arguments as in the proof of (44) for $j = 3$ one can show that

$$|a_{r,4} - a_{r,5}| \leq O(n^{-1-\delta})B(1, \frac{1}{2}) \cdot \ldots \cdot B(\frac{n-r-1}{2}, \frac{1}{2})\xi(y-x)$$

for $\delta$ small enough. This shows (44) for $j = 1$. So for (39) it remains to show (41). This can be done by arguments as in the proof of (27).

*Asymptotic treatment of* $T_4$. We will show that

$$T_4 = -n^{-1/2} \sum_{r=0}^{\infty} \widetilde{\pi}_1 \otimes_n (H + M_{n,11} + n^{-1/2} N_1)^{(r)}(0, 1, x, y) \tag{46}$$

$$-n^{-1} \sum_{r=0}^{\infty} \widetilde{\pi}_2 \otimes_n H^{(r)}(0, 1, x, y) + R_n(x, y)$$

with $|R_n(x, y)| = o(n^{-1-\delta}) \cdot \zeta(y - x)$.

*Proof of (46).* Note first that with $S_n(s, t, x, y) = \sum_{r=1}^{n} (K_n + M_n)^{(r)}(s, t, x, y)$ the term $T_4$ can be rewritten as

$$T_4 = (\widetilde{p} - \widetilde{p}_n)(0, 1, x, y) + (\widetilde{p} - \widetilde{p}_n) \otimes_n S_n(0, 1, x, y).$$

We start by showing that for $\delta > 0$ small enough (uniformly for $x, y, \in I\!R^p$)

$$\left| \frac{1}{n} \sum_{1 \leq j \leq n^{\delta}} \int (\widetilde{p}_n - \widetilde{p})(0, \frac{j}{n}, x, u) S_n(\frac{j}{n}, 1, u, y)\, du \right| \leq C n^{-1-\delta'} \zeta(y - x) \tag{47}$$

for $\delta'$ small enough. For the proof of (47) we will show that uniformly for $1 \leq j \leq n^{\delta}$ and for $x, y \in I\!R^p$

$$\int \widetilde{p}_n(0, j/n, x, u) S_n(\frac{j}{n}, 1, u, y) du = S_n(\frac{j}{n}, 1, x, y) + o(n^{-\delta} \zeta(y - x)), \tag{48}$$

$$\int \widetilde{p}(0, j/n, x, u) S_n(\frac{j}{n}, 1, u, y) du = S_n(\frac{j}{n}, 1, x, y) + o(n^{-\delta} \zeta(y - x)). \tag{49}$$

Claim (47) immediately follows from (48) - (49). We now show (48) for $j = 1$. The proof for $j \geq 1$ and for (49) follows along the same lines and, in particular, makes use of the last condition in (A2). For the proof we will make use of the fact that for all $1 \leq j \leq n$ and all $x, y \in I\!R^p$ and $|\nu| = 1$

$$|D_x^{\nu} S_n(\frac{j}{n}, 1, x, y)| \leq C(1 - \frac{j}{n})^{-1} \zeta_{\rho}(y - x) \tag{50}$$

for some constant $C > 0$. Claim (50) can be shown with the same arguments as the proof of (5.7) in Konakov and Mammen (2002). Note that the function $\Phi$ in that paper

has a similar structure as $S_n$. For $1 \le j \le n^\delta$ the bound (50) immediately implies for a constant $C' > 0$

$$|D_x^\nu S_n(\frac{j}{n}, 1, x, y)| \le C'\zeta(y-x). \tag{51}$$

We have $\widetilde{p}_n(0, 1/n, x, u) = n^{p/2}q[u, \sqrt{n}(u - x - \frac{1}{n}m(u))]$. Denote the determinant of the Jacobian matrix of $u - n^{-1}m(u)$ by $\Delta_n$. So, because of ( A2) and (51), it holds with the substitution $w = \sqrt{n}(u - x - \frac{1}{n}m(u))$

$$\int \widetilde{p}_n(0, 1/n, x, u) S_n(\frac{1}{n}, 1, u, y) du$$

$$= \int n^{p/2} q(u, \sqrt{n}(u - x - \frac{1}{n}m(u))) S_n(\frac{1}{n}, 1, u, y) du$$

$$= \int q(x + \frac{w}{\sqrt{n}} + \frac{m(u)}{n}, w) \Delta_n^{-1} S_n(\frac{1}{n}, 1, x + \frac{w}{\sqrt{n}} + \frac{m(u)}{n}, y) dw$$

$$= \int [q(x, w) + o(n^{-\delta})\psi(w)][1 + o(n^{-\delta})] S_n(\frac{1}{n}, 1, x + \frac{w}{\sqrt{n}} + \frac{m(u)}{n}, y) dw$$

$$= S_n(\frac{1}{n}, 1, x, y) + o(n^{-\delta})\zeta(y - x).$$

From (47) we get that for a $\delta > 0$

$$T_4 = (\widetilde{p} - \widetilde{p}_n)(0, 1, x, y) + \frac{1}{n} \sum_{n^\delta < j \le n} \int (\widetilde{p} - \widetilde{p}_n)(0, j/n, x, u) S_n(\frac{j}{n}, 1, u, y) du + R_n(x, y)$$

with $|R_n(x, y)| \le C n^{-1-\delta'} \zeta(y - x)$.

We now make use of the expansion of $\widetilde{p}_n - \widetilde{p}$, given in Lemma 5.2 . We have with $\rho = (j/n)^{1/2} \ge n^{\delta/2 - 1/2}$

$$\left| \frac{1}{n} \sum_{j=n^\delta}^{n} n^{-3/2} \rho^{-3} \int \zeta_\rho(u - x) S_n(\frac{j}{n}, 1, u, y) du \right|$$

$$\le C n^{-1-\delta''} n^{-1} \sum_{j=n^\delta}^{n} \rho^{-2+\delta'} \int \left| \zeta_\rho(u - x) S_n(\frac{j}{n}, 1, u, y) du \right|,$$

where $\delta' < \delta(1 - \delta)^{-1}$, $\delta'' > 0$ is small enough. Now using similar arguments as in the proof of (19) we get that

$$n^{-1} \sum_{j=1}^{n} \rho^{-2+\delta'} \int \zeta_\rho(u - x) S_n(\frac{j}{n}, 1, u, y) du \le C\zeta(y - x)$$

for a constant $C$. This shows that

$$T_4 = -\left[n^{-1/2}\widetilde{\pi}_1 + n^{-1}\widetilde{\pi}_2\right](0, 1, x, y)$$

$$-\frac{1}{n}\sum_{j=n^\delta}^{n}\int \left[n^{-1/2}\widetilde{\pi}_1 + n^{-1}\widetilde{\pi}_2\right](0, j/n, x, u)S_n(\frac{j}{n}, 1, u, y)\ du + R'_n(x, y)$$

with $|R'_n(x,y)| \leq Cn^{-1-\delta''}\zeta(y-x)$.

Claim (46) now follows from (47) by application of the expansions of $K_n$, used above.

*Asymptotic treatment of* $T_5$. From Lemma 5.4 we immediately get that

$$|T_5| \quad \leq \quad O(n^{-3/2}\zeta(y-x)).$$

*Plugging in the asymptotic expansions of* $T_1$,..., $T_5$. We now plug the asymptotic expansions of $T_1$,..., $T_5$ into (17). This gives

$$\begin{aligned}(52)\qquad p_n(x,y) - p(x,y) \cong\ & n^{-1/2}\sum_{r=0}^{\infty}\widetilde{\pi}_1 \otimes_n (H + M_{n,11} + n^{-1/2}N_1)^{(r)}(0,1,x,y)\\
&+\sum_{r=0}^{\infty}\widetilde{p}\otimes_n [(H + M_{n,11} + n^{-1/2}N_1)^{(r)} - H^{(r)}](0,1,x,y)\\
&+\sum_{r=0}^{\infty}\widetilde{p}\otimes_n [(H + M_{n,12})^{(r)} - H^{(r)}](0,1,x,y)\\
&+\sum_{r=0}^{\infty}\widetilde{p}\otimes_n [(H + M_{n,21})^{(r)} - H^{(r)}](0,1,x,y)\\
&+\sum_{r=0}^{\infty}\widetilde{p}\otimes_n [(H + M_{n,31})^{(r)} - H^{(r)}](0,1,x,y)\\
&+\sum_{r=0}^{\infty}\widetilde{p}\otimes_n [(H + n^{-1}N_2)^{(r)} - H^{(r)}](0,1,x,y)\\
&+\frac{1}{n}\sum_{r=0}^{\infty}\widetilde{\pi}_2 \otimes_n H^{(r)}(0,1,x,y) - \frac{1}{2n}p\otimes_n (L-\widetilde{L})^2\widetilde{p}\otimes_n \Phi(0,1,x,y),\end{aligned}$$

where $\cong$ means an equality up to terms that are absolutely smaller than $Cn^{-1-\delta}$ $\exp[-C'(x-y)^2]$ for positive constants $C, C'$ and $\delta$.

For the proof of Theorem 4.1 it remains to show that the right hand side of (52) can be approximated by $n^{-1/2}\pi_1(x,y) + n^{-1}\pi_2(x,y)$. We will prove this claim in three steps. In a first step we will prove that $\widetilde{p}_n(\frac{j}{n}, \frac{k}{n}, x, y)$ can be replaced by $\widetilde{p}(\frac{j}{n}, \frac{k}{n}, x, y)$ in $M_{n,11}$, $M_{n,12}$, $M_{n,21}$ and $M_{n,31}$. Then in a second, we show that the convolution operator $\otimes_n$ can be replaced by the operator $\otimes$ in (52). In a third step we will show that the resulting expression is asymptotically equivalent to to $n^{-1/2}\pi_1(x,y) + n^{-1}\pi_2(x,y)$.

*Asymptotic replacement of* $\widetilde{p}_n$ *by* $\widetilde{p}$. We now show that

$$(53)\qquad p_n(x,y) - p(x,y) \cong n^{-1/2}[\widetilde{\pi}_1 + p^d \otimes_n \Re_1]\otimes_n \Phi(0,1,x,y)$$

$$+n^{-1}\left\{[\widetilde{\pi}_2+\widetilde{\pi}_1\otimes_n\Phi\otimes_n\Re_1+p^d\otimes_n\Re_2+p^d\otimes_n\Re_3]\otimes_n\Phi(0,1,x,y)\right.$$
$$\left.+p^d\otimes_n(\Re_1\otimes_n\Phi)^{(2)}(0,1,x,y)+\frac{1}{2}\,p^d\otimes_n(L_\star^2-L^2)p^d(0,1,x,y)\right\},$$

where

$$\Re_1(s,t,x,y)=(L-\widetilde{L})\widetilde{\pi}_1(s,t,x,y)+M_3(s,t,x,y)-\widetilde{M_3}(s,t,x,y),$$

$$\Re_2(s,t,x,y)=(L-\widetilde{L})\widetilde{\pi}_2(s,t,x,y)+M_4(s,t,x,y)-\widetilde{M_4}(s,t,x,y),$$

$$\Re_3(s,t,x,y)=\sum_{|\nu|=4}\frac{1}{\nu!}D_x^\nu\widetilde{p}(s,t,x,y)(\chi_\nu(x)-\chi_\nu(y)),$$

$$M_3(s,t,x,y)=\sum_{|\nu|=3}\frac{\mu_\nu(x)}{\nu!}D_x^\nu\widetilde{p}(s,t,x,y),$$

$$\widetilde{M_3}(s,t,x,y)=\sum_{|\nu|=3}\frac{\mu_\nu(y)}{\nu!}D_x^\nu\widetilde{p}(s,t,x,y),$$

$$M_4(s,t,x,y)=\sum_{|\nu|=3}\frac{\mu_\nu(x)}{\nu!}D_x^\nu\widetilde{\pi}_1(s,t,x,y),$$

$$\widetilde{M_4}(s,t,x,y)=\sum_{|\nu|=3}\frac{\mu_\nu(y)}{\nu!}D_x^\nu\widetilde{\pi}_1(s,t,x,y).$$

*Proof of (62).* We will make use of the following approximation

(54)
$$p_n(x,y)-p(x,y)$$
$$\cong n^{-1/2}\sum_{r=0}^{\infty}\widetilde{\pi}_1\otimes_n(H+N_{n,11}+n^{-1/2}N_1)^{(r)}(0,1,x,y)$$
$$+\sum_{r=0}^{\infty}\widetilde{p}\otimes_n[(H+N_{n,11}+n^{-1/2}N_1)^{(r)}-H^{(r)}](0,1,x,y)$$
$$+\sum_{r=0}^{\infty}\widetilde{p}\otimes_n[(H+N_{n,12})^{(r)}-H^{(r)}](0,1,x,y)$$
$$+\sum_{r=0}^{\infty}\widetilde{p}\otimes_n[(H+N_{n,21})^{(r)}-H^{(r)}](0,1,x,y)$$
$$+\sum_{r=0}^{\infty}\widetilde{p}\otimes_n[(H+N_{n,31})^{(r)}-H^{(r)}](0,1,x,y)$$
$$+\sum_{r=0}^{\infty}\widetilde{p}\otimes_n[(H+n^{-1}N_2)^{(r)}-H^{(r)}](0,1,x,y)$$
$$+\sum_{r=0}^{\infty}\widetilde{p}\otimes_n[(H+n^{-1}N_3)^{(r)}-H^{(r)}](0,1,x,y)$$

$$+\frac{1}{n}\sum_{r=0}^{\infty}\widetilde{\pi}_2 \otimes_n H^{(r)}(0,1,x,y) - \frac{1}{2n}p \otimes_n (L-\widetilde{L})^2\widetilde{p} \otimes_n \Phi(0,1,x,y),$$

where $N_3(s,t,x,y) = M_4(s,t,x,y) - \widetilde{M}_4(s,t,x,y)$ and where $N_{n,ij}$ is defined as $M_{n,ij}$ but with $\widetilde{p}_n$ replaced by $\widetilde{p}$.

A proof of (54) will be given below. To simplify (54) we make use of the following identities

$$\sum_{r=0}^{\infty}\widetilde{p} \otimes_n [(H+n^{-1}N)^{(r)} - H^{(r)}](0,1,x,y) \tag{55}$$

$$= \sum_{r=1}^{\infty}\frac{1}{n^r}p^d \otimes_n (N \otimes_n \Phi)^{(r)}(0,1,x,y),$$

$$\sum_{r=0}^{\infty}\widetilde{p} \otimes_n [(H+n^{-1/2}N')^{(r)} - H^{(r)}](0,1,x,y) \tag{56}$$

$$= \sum_{r=1}^{\infty}\frac{1}{n^{r/2}}p^d \otimes_n (N' \otimes_n \Phi)^{(r)}(0,1,x,y),$$

where $p^d(s,t,x,y) = (\widetilde{p} \otimes_n \Phi)(s,t,x,y)$ and $N$ is one of the functions $nN_{n,12}$ , $nN_{n,21}$, $nN_{n,31}$, $N_2$ or $N_3$ and where $N' = n^{1/2}N_{n,11} + N_1$. These identities follow from linearity of $\otimes_n$ by simple calculations. We will show the following approximations for the right hand sides of (55) and (56)

$$\sum_{r=1}^{\infty}\frac{1}{n^r}p^d \otimes_n (N \otimes_n \Phi)^{(r)}(0,1,x,y) \tag{57}$$

$$\cong \frac{1}{n}p^d \otimes_n N \otimes_n \Phi(0,1,x,y),$$

$$\sum_{r=1}^{\infty}\frac{1}{n^{r/2}}p^d \otimes_n (N' \otimes_n \Phi)^{(r)}(0,1,x,y) \tag{58}$$

$$\cong \frac{1}{n^{1/2}}p^d \otimes_n N \otimes_n \Phi(0,1,x,y) + \frac{1}{n}p^d \otimes_n (N' \otimes_n \Phi)^{(2)}(0,1,x,y).$$

We prove (57) for $N = nN_{n,12}$. The proofs for the other cases are essentially the same. For $r=1$ and $|\nu| = 4$ it is sufficient to estimate

$$\sum_{j=0}^{k-1}\frac{1}{n}\int p^d(0,\frac{j}{n},x,z)D_z^{\nu}\widetilde{p}(\frac{j}{n},\frac{k}{n},z,y)(\mu_\nu(z)-\mu_\nu(y))dz.$$

Splitting the last sum into two sums

$$\sum_{j=0}^{k-1}\ldots = \sum_{\{j:\frac{j}{n}\le\frac{k}{2n}\}}\ldots + \sum_{\{j:\frac{j}{n}>\frac{k}{2n}\}}\ldots$$

we get by applying integration by parts and by using Theorem 2.3 in Konakov and Mammen (2002)that

$$\left|(p^d \otimes_n N \otimes_n \Phi)(0, \frac{k}{n}, x, y)\right| \leq C \left(\frac{k}{n}\right)^{-1/2} \phi_{C,\sqrt{\frac{k}{n}}}(y-x).$$

This bound implies

$$\begin{aligned}
&\left|\frac{1}{n^2} p^d \otimes_n (N \otimes_n \Phi)^{(2)}(0, \frac{k}{n}, x, y)\right| \\
&= \left|\frac{1}{n^{1+\delta}} (p^d \otimes_n N \otimes_n \Phi) \otimes_n (n^{-1+\delta} N \otimes_n \Phi)\right| \\
&\leq \frac{C^2}{n^{1+\delta}} B(\frac{1}{2}, \varepsilon) \left(\frac{k}{n}\right)^{-1/2+\varepsilon} \phi_{C,\sqrt{\frac{k}{n}}}(y-x)
\end{aligned}$$

for $\varepsilon$ small enough. Iterative application of similar arguments for $r \geq 2$ gives

$$\begin{aligned}
&\left|\frac{1}{n^r} p^d \otimes_n (N \otimes_n \Phi)^{(r)}(0, \frac{k}{n}, x, y)\right| \\
&\leq \frac{C^r}{n^{1+\delta}} B(\frac{1}{2}, \varepsilon) B(\frac{1}{2}+\varepsilon, \varepsilon) ... B(\frac{1}{2}+(r-2)\varepsilon, \varepsilon) \\
&\qquad \left(\frac{k}{n}\right)^{-1/2+(r-1)\varepsilon} \phi_{C,\sqrt{\frac{k}{n}}}(y-x).
\end{aligned} \tag{59}$$

The bound (59) immediately implies (57) for $N = nN_{n,12}$.

By plugging (57) and (58) into (54) and taking into account the relation

$$\begin{aligned}
&\frac{1}{2n}(L_\star - \widetilde{L})^2 \widetilde{p}(\frac{j}{n}, \frac{k}{n}, x, y) + \frac{1}{n} \Re_3(\frac{j}{n}, \frac{k}{n}, x, y) \\
&= (N_{n,12} + N_{n,21} + N_{n,31})(\frac{j}{n}, \frac{k}{n}, x, y)
\end{aligned}$$

we get (53) by collecting similar terms. So it remains to show (54).

*Proof of (54).* We shall use the following recurrence relation

$$\begin{aligned}
&\left[\sum_{r=0}^{n} \widetilde{p} \otimes_n (H + M_{n,12})^{(r)} - \sum_{r=0}^{n} \widetilde{p} \otimes_n (H + N_{n,12})^{(r)}\right](0, 1, x, y) \\
&= \left[\sum_{r=0}^{n-1} \widetilde{p} \otimes_n (H + M_{n,12})^{(r)} - \sum_{r=0}^{n-1} \widetilde{p} \otimes_n (H + N_{n,12})^{(r)}\right] \\
&\otimes_n (H + M_{n,12})(0, 1, x, y) + S_n \otimes_n (M_{n,12} - N_{n,12})(0, 1, x, y),
\end{aligned} \tag{60}$$

where $S_n(s,t,x,y) = \sum_{r=0}^{n-1} \widetilde{p} \otimes_n (H + N_{n,12})^{(r)}(s,t,x,y)$. For fixed $\nu$, $|\nu| = 4$, we consider

$$\frac{1}{n^2} \sum_{j=1}^{n-1} \int S_n(0, \frac{j}{n}, x, u) G_\nu(u, y) (D_u^\nu \widetilde{p}_n(\frac{j}{n}, 1, u, y) - D_u^\nu \widetilde{p}(\frac{j}{n}, 1, u, y)) du \tag{61}$$

$$= \frac{1}{n^2} \sum_{n-n^\delta \le j < n} ... + \frac{1}{n^2} \sum_{0 \le j < n-n^\delta} ...$$
$$= I + II,$$

where

$$G_\nu(u,y) = \mu_\nu(u) - \mu_\nu(y) - \sum_{|\nu'|=2} \nu! N(\nu,\nu') \mu_{\nu'}(y)(\mu_{\nu-\nu'}(u) - \mu_{\nu-\nu'}(y))$$

with $N(\nu,\nu')$ defined as in the statement of Lemma 5.4. We start with estimating $I$. For the summand in $I$ for $j = n-1$ we have to consider $\widetilde{p}_n(\frac{n-1}{n}, 1, u, y) = n^{p/2} q[y, \sqrt{n}(y - u - \frac{1}{n} m(y))]$. With the substitution $w = \sqrt{n}(y - u - \frac{1}{n} m(y))$ and by using integration by parts one gets

$$\int S_n(0, \frac{n-1}{n}, x, u) G_\nu(u,y) D_u^\nu \widetilde{p}_n(\frac{n-1}{n}, 1, u, y) du$$

$$= n^{p/2} \int D_u^\nu [S_n(0, \frac{n-1}{n}, x, u) G_\nu(u,y)] q(y, \sqrt{n}(y - u - \frac{1}{n} m(y))) du$$

$$= \int D_u^\nu [S_n(0, \frac{n-1}{n}, x, y - \frac{w}{\sqrt{n}} - \frac{m(y)}{n}) G_\nu(y - \frac{w}{\sqrt{n}} - \frac{m(y)}{n}, y)] q(y,w) dw$$

$$= D_u^\nu [S_n(0, \frac{n-1}{n}, x, u) G_\nu(u,y)] \mid_{u=y} + o(n^{-\delta}) \zeta(y-x).$$

Analogously,

$$\int S_n(0, \frac{n-1}{n}, x, u) G_\nu(u,y) D_u^\nu \widetilde{p}(\frac{n-1}{n}, 1, u, y) du$$

$$= D_u^\nu [S_n(0, \frac{n-1}{n}, x, u) G_\nu(u,y)] \mid_{u=y} + o(n^{-\delta}) \zeta(y-x).$$

These two expansions imply that the summand in $I$ for $j = n-1$ is smaller than $Cn^{-2-\delta'} \zeta(y-x)$ for $\delta'$ small enough. For $n - n^\delta \le j \le n-2$ one can show the same bound. This implies that

$$|I| \le Cn^{-1-\delta'} \zeta(y-x) \tag{62}$$

for $\delta'$ small enough.

We now treat the sum $II$. For $0 \le j < n - n^\delta$ we have that $\rho_2 = \sqrt{1 - \frac{j}{n}} \ge n^{(\delta-1)/2}$. By applying integration by parts we get

$$II = \frac{1}{n^2} \sum_{0 \le j < n-n^\delta} \int D_u^{e_l} [S_n(0, \frac{j}{n}, x, u) G_\nu(u,y)] D_u^{\nu - e_l} [\widetilde{p}_n(\frac{j}{n}, 1, u, y) - \widetilde{p}(\frac{j}{n}, 1, u, y)] du$$

$$= \frac{1}{n^2} \sum_{0 \le j < n-n^\delta} \int D_u^{e_l} [S_n(0, \frac{j}{n}, x, u)] G_\nu(u,y) D_u^{\nu - e_l} [\widetilde{p}_n(\frac{j}{n}, 1, u, y) - \widetilde{p}(\frac{j}{n}, 1, u, y)] du$$

$$+ \frac{1}{n^2} \sum_{0 \le j < n-n^\delta} \int D_u^{e_q} [S_n(0, \frac{j}{n}, x, u)] D_u^{e_l} G_\nu(u,y) D_u^{\nu - e_l - e_q} [\widetilde{p}_n(\frac{j}{n}, 1, u, y) - \widetilde{p}(\frac{j}{n}, 1, u, y)] du$$

$$+\frac{1}{n^2}\sum_{0\le j<n-n^\delta}\int S_n(0,\frac{j}{n},x,u)D_u^{e_l+e_q}G_\nu(u,y)D_u^{\nu-e_l-e_q}[\widetilde{p}_n(\frac{j}{n},1,u,y)-\widetilde{p}(\frac{j}{n},1,u,y)]du$$

$$= I' + II' + III'.$$

For $I'$ one can show that $|I'| \le Cn^{-1-\delta}\zeta(y-x)$ with $\delta$ small enough. The summands $II'$ and $III'$ can be bounded similarly. Because of the expansion given in Lemma 5.2 this only requires application of the following estimate

$$\frac{1}{n^2}\sum_{0\le j<n-n^\delta} n^{-3/2}\rho_2^{-5}\int \left|D_u^{e_l}[S_n(0,\frac{j}{n},x,u)]\right|\zeta_{\rho_2}(y-u)du$$

$$\le Cn^{-1-\delta'}\frac{1}{n}\sum_{j=1}^{n}\rho_1^{-1}\rho_2^{-2+\delta''}\int \zeta_{\rho_1}(u-x)\zeta_{\rho_2}(y-u)du$$

$$\le C_1 n^{-1-\delta'}\zeta(y-x)$$

with $\rho_1 = \sqrt{\frac{j}{n}}$ and $\delta'$ and $\delta''$ small enough. With the resulting bound on $II$ and with (61) and (62) we get

$$|S_n \otimes_n (M_{n,12} - N_{n,12})(0,1,x,y)| \le C_n n^{-1-\delta'}\zeta(y-x), \tag{63}$$

where $\sum_{n=1}^{\infty} C_n < \infty$. From iterations of (60) and (63) we obtain

$$\left|\left[\sum_{r=0}^{\infty}\widetilde{p}\otimes_n (H+M_{n,12})^{(r)} - \sum_{r=0}^{\infty}\widetilde{p}\otimes_n (H+N_{n,12})^{(r)}\right](0,1,x,y)\right|$$

$$\le Cn^{-1-\delta'}\zeta(y-x).$$

For the terms in (52) that contain $M_{n,21}$ or $M_{n,31}$ analogous estimates can be obtained for the errors if $M_{n,21}$ and $M_{n,31}$ are replaced by $N_{n,21}$ or $N_{n,31}$, respectively. In $M_{n,11}$ we replace $\widetilde{p}_n$ by $\widetilde{p}+\frac{1}{\sqrt{n}}\widetilde{\pi}_1$ and we get a similar bound for the resulting error. By collecting these bounds we get (54).

In the next step of the proof of Theorem 4.1 we will replace $p^d$ by $p$ in our expansion of $p_n - p$.

*Asymptotic replacement of $p^d$ by $p$.* We now show that in (53) $p^d$ can be replaced by $p$. This gives the following expansion

$$p_n(x,y) - p(x,y) \cong n^{-1/2}[\widetilde{\pi}_1 + p\otimes_n \Re_1]\otimes_n \Phi(0,1,x,y) \tag{64}$$

$$+n^{-1}\left\{[\widetilde{\pi}_2 + \widetilde{\pi}_1\otimes_n\Phi\otimes_n\Re_1 + p\otimes_n\Re_2 + p\otimes_n\Re_3]\otimes_n\Phi(0,1,x,y)\right.$$

$$\left.+p\otimes_n(\Re_1\otimes_n\Phi)^{(2)}(0,1,x,y) + \frac{1}{2}\,p\otimes_n(L_\star^2 - L^2)p(0,1,x,y)\right\}.$$

*Proof of (64).* The claim immediately follows from the following formula

$$\left|p(0,\frac{l}{n},x,y) - p^d(0,\frac{l}{n},x,y)\right| \tag{65}$$

$$\leq \frac{C}{n^{1-\varepsilon}} B(\frac{1}{2},\frac{\varepsilon}{2}) B(\frac{\varepsilon+1}{2},\frac{\varepsilon}{2}) \left(\frac{l}{n}\right)^{\varepsilon-1/2} \phi_{C,\sqrt{\frac{l}{n}}}(y-x) \quad \varepsilon \in (0,\frac{1}{2}).$$

To prove (65) we proceed as in the proof of Theorem 2.1 in Konakov and Mammen (2002). This gives the following relation

$$p(0,\frac{l}{n},x,y) - p^d(0,\frac{l}{n},x,y) \tag{66}$$

$$= \frac{1}{2n} p \otimes_n (L-\widetilde{L})^2 \widetilde{p} \otimes_n \Phi(0,\frac{l}{n},x,y) + \frac{1}{n^2} R(0,\frac{l}{n},x,y),$$

where

$$R(0,\frac{l}{n},x,y) = \frac{1}{2} \sum_{i=0}^{l-1} \frac{1}{n} \sum_{j=0}^{i-1} \int_{j/n}^{(j+1)/n} [n(u-\frac{j}{n})]^2 du \int_0^1 (1-\delta) d\delta \int dz \tag{67}$$

$$[\int p(0,s_j,x,v)(L-\widetilde{L})^3 \widetilde{p}(s_j,\frac{i}{n},v,z) dv \Phi(\frac{i}{n},\frac{l}{n},z,y)]$$

with $s_j = s_j(u,\delta) = \frac{j}{n} + \delta(u-\frac{j}{n})$ . By iteratively using integration by parts in (67) a derivative operator of order 3 can be transferred from $\widetilde{p}$ to $\Phi$ and a derivative operator of order 1 can be transferred from $\widetilde{p}$ to $p$. We also make use of the inequality

$$\left| D_\xi^b \widetilde{p}(s,t,\xi,\xi+x) \right| \leq C_1 (t-s)^{-p/2} \exp\left\{ -C_2 \frac{\|x\|^2}{(t-s)} \right\}.$$

This enables us to pass from derivatives with respect to $v$ to derivatives with respect to $z$. Using Beta functions to bound the integrals appearing in the definition of $R(0,\frac{l}{n},x,y)$ one can show that $n^{-2}R(0,\frac{l}{n},x,y)$ is bounded by the right hand side of (65). The first summand in the right hand side of (66) can be estimated analogously. For the proof of this claim with the help of integration by parts a derivative operator of order 1 is transferred from $\widetilde{p}$ to $\Phi$ and a derivative operator of order 2 from $\widetilde{p}$ to $p$. By using linearity of $\otimes_n$ and by applying (65) we easily get that $p^d$ can be replaced by $p$. For proving that $\otimes_n$ can be replaced by $\otimes$ in the last summand of (53) we proceed as in the proof of Theorem 2.1 in Konakov and Mammen (2002). This gives the following inequality

$$\left| p \otimes_n (L_\star^2 - L^2) p(0,1,x,y) - p \otimes (L_\star^2 - L^2) p(0,1,x,y) \right|$$

$$\leq \frac{C}{n} \phi_{C,1}(y-x).$$

We now come to the next modification of our expansion of $p_n - p$.

*Asymptotic replacement of* $p \otimes_n (\Re_1 \otimes_n \Phi)^{(2)}(0,1,x,y)$ *by* $p \otimes (\Re_1 \otimes_n \Phi)^{(2)}(0,1,x,y)$ *and* $p \otimes_n \Re_i \otimes_n \Phi(0,1,x,y)$ *by* $p \otimes \Re_i \otimes_n \Phi(0,1,x,y)$, *i=1,2,3.*

We now show the following expansion

$$p_n(x,y) - p(x,y) \cong n^{-1/2} [\widetilde{\pi}_1 + p \otimes \Re_1] \otimes_n \Phi(0,1,x,y) \tag{68}$$

$$+n^{-1}\left\{[\widetilde{\pi}_2+\widetilde{\pi}_1\otimes_n\Phi\otimes_n\Re_1+p\otimes\Re_2+p\otimes\Re_3]\otimes_n\Phi(0,1,x,y)\right.$$
$$\left.+p\otimes(\Re_1\otimes_n\Phi)^{(2)}(0,1,x,y)+\frac{1}{2}\,p\otimes(L_\star^2-L^2)p(0,1,x,y)\right\}.$$

*Proof of (68).* The claim follows from the following estimates

$$|p\otimes\Re_i\otimes_n\Phi(0,1,x,y)-p\otimes_n\Re_i\otimes_n\Phi(0,1,x,y)| \tag{69}$$
$$\leq C(\varepsilon)n^{-1+\varepsilon}\phi_{C,1}(y-x)$$

for $i=1,2,3$ and for $\varepsilon\in(0,\frac{1}{2})$. Thus it remains to show (69). We will prove it for $i=1$. The proof for $i=2,3$ is quite similar. Because of linearity of $\otimes$ it is sufficient to consider the differences corresponding to the four summands in the definition of $\Re_1(s,t,x,y)$. The proof for the four summands is quite similar. We only consider the difference $p\otimes\widetilde{L}\widetilde{\pi}_1\otimes_n\Phi(0,1,x,y)-p\otimes_n\widetilde{L}\widetilde{\pi}_1\otimes_n\Phi(0,1,x,y)$. As in the proof of Theorem 2.1 in Konakov and Mammen (2002) (with $H$ replaced by $\widetilde{L}\widetilde{\pi}_1\otimes_n\Phi$) we get

$$p\otimes\widetilde{L}\widetilde{\pi}_1\otimes_n\Phi(0,1,x,y)-p\otimes_n\widetilde{L}\widetilde{\pi}_1\otimes_n\Phi(0,1,x,y)$$
$$=\frac{1}{2n}p\otimes_n(L\widetilde{L}-\widetilde{L}L)\widetilde{\pi}_1\otimes_n\Phi(0,1,x,y)+\frac{1}{n^2}R(0,1,x,y),$$

where now

$$R(0,1,x,y)=\frac{1}{2}\sum_{l=0}^{n-1}\frac{1}{n}\sum_{j=0}^{l-1}\int_{j/n}^{(j+1)/n}[n(u-\frac{j}{n})]^2du\int_0^1(1-\delta)d\delta\int dz$$
$$\left[\int p(0,s_j,x,v)(L\widetilde{L}-\widetilde{L}L)^2\widetilde{\pi}_1(s_j,\frac{l}{n},v,z)dv\Phi(\frac{l}{n},1,z,y)\right].$$

These terms can be bounded by using integration by parts and dividing the sums in the definition of $R$ into appropriate partial sums. This completes the proof of (68).

Now we further simplify our expansion of $p_n-p$. We now show the following expansion

$$p_n(x,y)-p(x,y)\simeq n^{-1/2}(p\otimes\mathcal{F}_1[p_\Delta])(0,1,x,y) \tag{70}$$
$$+n^{-1}\left\{(p\otimes\mathcal{F}_2[p_\Delta])(0,1,x,y)+(p\otimes\mathcal{F}_1[p\otimes\mathcal{F}_1[p_\Delta]])(0,1,x,y)\right.$$
$$\left.+\frac{1}{2}p\otimes(L_\star^2-L^2)p(0,1,x,y)\right\},$$

where for $t\in\left\{\frac{1}{n},...,1\right\}$, $s\in\left[0,t-\frac{1}{n}\right]$

$$p_\Delta(s,t,v,y)=(\widetilde{p}\otimes_n'\Phi)(s,t,v,y)$$
$$=\widetilde{p}(s,t,v,y)+\sum_{j\geq ns}^{nt-1}\frac{1}{n}\int\widetilde{p}(s,\frac{j}{n},v,z)\Phi_1(\frac{j}{n},t,z,y)\,dz.$$

Here $\Phi_1 = \sum_{r=1}^{\infty} H^{(r)}$, $H^{(r)} = H^{(r-1)} \otimes_n H$, and the binary type operation $\otimes_n'$ is defined as follows

$$(f \otimes_n' g)(s,t,x,y) = \sum_{j \geq ns}^{nt-1} \frac{1}{n} \int f(s, \frac{j}{n}, x, z) g(\frac{j}{n}, t, z, y)\, dz.$$

Note that for $s \in \left\{\frac{1}{n}, ..., 1\right\}$ these operations coincide, that is $\otimes_n' \equiv \otimes_n$.

*Proof of (70).* Linearity of $\otimes$ implies

$$(71) \qquad (p \otimes \Re_1)(s.t,x,y) = (p \otimes L\widetilde{\pi}_1)(s,t,x,y) - (p \otimes \widetilde{L}\widetilde{\pi}_1)(s,t,x,y)$$
$$+(p \otimes M_3)(s,t,x,y) - (p \otimes \widetilde{M_3})(s,t,x,y).$$

We now consider the second summand on the right hand side of (71).

$$(p \otimes \widetilde{L}\widetilde{\pi}_1)(s,t,x,y) = \sum_{|\nu|=3} \frac{\mu_\nu(y)}{\nu!} \int_s^s d\tau \int p(s,\tau,x,v)(t-\tau) D_v^\nu (\widetilde{L}\widetilde{p}(\tau,t,v,y) dv$$

$$= \sum_{|\nu|=3} \frac{\mu_\nu(y)}{\nu!} \int_s^{(s+t)/2} ... + \sum_{|\nu|=3} \frac{\mu_\nu(y)}{\nu!} \int_{(s+t)/2}^t ...$$

$$= I + II.$$

By application of the Kolmogorov backward and forward equations and by using integration by parts with respect to the time variable we get that

$$(72) \qquad I = -\sum_{|\nu|=3} \frac{\mu_\nu(y)}{\nu!} \int dv \int_s^{(s+t)/2} d\tau (t-\tau) p(s,\tau,x,v) \frac{\partial}{\partial \tau} \left(D_v^\nu \widetilde{p}(\tau,t,v,y)\right)$$

$$- \sum_{|\nu|=3} \frac{\mu_\nu(y)}{\nu!} \int dv \left[ p(s,\tau,x,v)(t-\tau) D_v^\nu \widetilde{p}(\tau,t,v,y) \mid_s^{(s+t)/2} \right.$$

$$\left. - \int_s^{(s+t)/2} D_v^\nu \widetilde{p}(\tau,t,v,y) \left( \frac{\partial p(s,\tau,x,v)}{\partial \tau}(t-\tau) - p(s,\tau,x,v) \right) d\tau \right]$$

$$= -\sum_{|\nu|=3} \frac{\mu_\nu(y)}{\nu!} \int \frac{(t-s)}{2} p(s, \frac{s+t}{2}, x, v) D_v^\nu \widetilde{p}(\frac{s+t}{2}, t, v, y) dv + \widetilde{\pi}_1(s,t,x,y)$$

$$+ \sum_{|\nu|=3} \frac{\mu_\nu(y)}{\nu!} \int_s^{(s+t)/2} d\tau (t-\tau) \int L^t p(s,\tau,x,v) D_v^\nu \widetilde{p}(\tau,t,v,y) dv$$

$$- \sum_{|\nu|=3} \frac{\mu_\nu(y)}{\nu!} \int_s^{(s+t)/2} d\tau \int p(s,\tau,x,v) D_v^\nu \widetilde{p}(\tau,t,v,y) dv.$$

Analogously we get

$$(73) \qquad II = \sum_{|\nu|=3} \frac{\mu_\nu(y)}{\nu!} \int \frac{(t-s)}{2} p(s, \frac{s+t}{2}, x, v) D_v^\nu \widetilde{p}(\frac{s+t}{2}, t, v, y)$$

$$+\sum_{|\nu|=3}\frac{\mu_\nu(y)}{\nu!}\int_{(s+t)/2}^{1}d\tau(t-\tau)\int L^t p(s,\tau,x,v)D_v^\nu\widetilde{p}(\tau,t,v,y)dv$$

$$-\sum_{|\nu|=3}\frac{\mu_\nu(y)}{\nu!}\int_{(s+t)/2}^{t}d\tau\int p(s,\tau,x,v)D_v^\nu\widetilde{p}(\tau,t,v,y)dv.$$

Substituting $(p\otimes\widetilde{L}\widetilde{\pi}_1)(0,1,x,y)=I+II$ into (71) we get after cancellation of some terms

$$\widetilde{\pi}_1(s,t,x,y)+(p\otimes\Re_1)(s,t,x,y)=(p\otimes M_3)(s,t,x,y). \tag{74}$$

Similarly, by using integration by parts with respect to the time variable we obtain

$$(\widetilde{\pi}_1\otimes_n'\Phi_1)(s,t,x,y)+(p\otimes\Re_1\otimes_n'\Phi_1)(s,t,x,y)=(p\otimes M_3\otimes_n'\Phi_1)(s,t,x,y), \tag{75}$$

where $t\in\left\{\frac{1}{n},...,1\right\}$, $s\in\left[0,t-\frac{1}{n}\right]$. From (74) and (75) we have

$$\begin{aligned}&(\widetilde{\pi}_1\otimes_n'\Phi)(s,t,x,y)+(p\otimes\Re_1\otimes_n'\Phi)(s,t,x,y)\\&=(p\otimes M_3\otimes_n'\Phi)(s,t,x,y)=(p\otimes\mathcal{F}_1[p_\Delta])(s,t,x,y).\end{aligned} \tag{76}$$

Using similar arguments as in the proof of (76) one can show that

$$\begin{aligned}&(\widetilde{\pi}_2\otimes_n'\Phi+p\otimes\Re_2\otimes_n'\Phi+p\otimes\Re_3\otimes_n'\Phi)(s,t,x,y)\\&=(p\otimes M_4\otimes_n'\Phi+p\otimes\mathcal{F}_2[\widetilde{p}]\otimes_n'\Phi)(s,t,x,y)\\&=(p\otimes M_4\otimes_n'\Phi)(s,t,x,y)+(p\otimes\mathcal{F}_2[p_\Delta])(s,t,x,y)\end{aligned} \tag{77}$$

By plugging (76) and (77) into (68) we obtain that the right hand side of (70) is equal to

$$\begin{aligned}&n^{-1/2}(p\otimes\mathcal{F}_1[p_\Delta])(0,1,x,y)+n^{-1}\left\{\left(p\otimes\mathcal{F}_2[p_\Delta]+\frac{1}{2}p\otimes(L_\star^2-L^2)p\right.\right.\\&\left.\left.+p\otimes M_4\otimes_n'\Phi+\widetilde{\pi}_1\otimes_n\Phi\otimes_n\Re_1\otimes_n\Phi+p\otimes(\Re_1\otimes_n\Phi)^{(2)}\right)(0,1,x,y)\right\}\end{aligned} \tag{78}$$

For the sum of the two last terms in (78) we get from (76)

$$[\widetilde{\pi}_1\otimes_n\Phi+p\otimes\Re_1\otimes_n\Phi]\otimes_n(\Re_1\otimes_n\Phi)(0,1,x,y)$$

$$=\{(p\otimes\mathcal{F}_1[p_\Delta])\otimes_n\Re_1\otimes_n\Phi\}(0,1,x,y)=p\otimes\mathcal{F}_1[p_\Delta\otimes_n'(\Re_1\otimes_n\Phi)](0,1,x,y).$$

Moreover,

$$\begin{aligned}&(p\otimes M_4\otimes_n'\Phi)(0,1,x,y)\\&=\int_0^1du\int p(0,u,x,v)\sum_{|\nu|=3}\frac{\mu_\nu(v)}{\nu!}D_v^\nu[(\widetilde{\pi}_1\otimes_n'\Phi)(u,1,x,y)]dv,\end{aligned}$$

and, hence, the sum of the last three terms in (78) is equal to

$$(p \otimes \mathcal{F}_1[\widetilde{\pi}_1 \otimes_n' \Phi + p_\Delta \otimes_n' (\Re_1 \otimes_n \Phi)])(0,1,x,y).$$

For the proof of (70) it remains to show that

$$(p \otimes \mathcal{F}_1[\widetilde{\pi}_1 \otimes_n' \Phi + p_\Delta \otimes_n' (\Re_1 \otimes_n \Phi)])(0,1,x,y) \tag{79}$$
$$\simeq (p \otimes \mathcal{F}_1[p \otimes \mathcal{F}_1[p_\Delta]])(0,1,x,y).$$

We shall show that

$$n^{-1}(p \otimes \mathcal{F}_1[(p - p_\Delta) \otimes_n' (\Re_1 \otimes_n \Phi)])(0,1,x,y) \simeq 0 \tag{80}$$

and

$$n^{-1}(p \otimes \mathcal{F}_1[p \otimes_n' (\Re_1 \otimes_n \Phi)](0,1,x,y) \simeq n^{-1}(p \otimes \mathcal{F}_1[p \otimes (\Re_1 \otimes_n \Phi)])(0,1,x,y). \tag{81}$$

Then (79) will follow from (80), (81) and (76). We now make use of the following representation

$$p(u,\frac{j}{n},x,y) - p_\Delta(u,\frac{j}{n},x,y) = (p \otimes H - p \otimes_n' H)(u,\frac{j}{n},x,y) \tag{82}$$
$$+ \{(p \otimes H - p \otimes_n' H) \otimes_n' \Phi_1\}(u,\frac{j}{n},x,y),$$

where

$$(p \otimes H - p \otimes_n' H)(u,\frac{j}{n},x,y) = \int_u^{\frac{j^\star(u)}{n}} d\tau \int p(u,\tau,x,z) H(\tau,\frac{j}{n},z,y) dz + R(u,\frac{j}{n},x,y),$$

$$R(u,\frac{j}{n},x,y) = \sum_{i=j^\star(u)}^{j-1} \int_{i/n}^{(i+1)/n} (\tau - \frac{i}{n}) \int_0^1 \int [p(u,\tau,x,z) \times$$
$$\left(L - \widetilde{L}\right)^2 \widetilde{p}(\tau,\frac{j}{n},z,y)] \mid_{\tau=\tau^\star} dz d\delta d\tau,$$

where $j^\star(u) = [un] + 1$ (with a convention $[x] = x - 1$ for $x \in \mathbb{N}$) and $\tau^\star = \tau^\star(i,\delta,\tau) = \frac{i}{n} + \delta(\tau - \frac{i}{n})$ . Representation (82) was obtained in the proof of Theorem 2.1 in Konakov and Mammen (2002). For the remainder term $R$ the following estimate holds uniformly in $\delta \in [0,1]$ and for $j \geq j^\star(u) + 2$

$$\left| R(u,\frac{j}{n},x,y) \right| \leq \frac{C}{n} \sum_{i=j^\star(u)}^{j-2} \frac{1}{n} \cdot \frac{1}{\left(\frac{j}{n} - \frac{i+1}{n}\right)^{3/2}} \cdot \phi_\rho(y-x) \tag{83}$$

$$+ \int_{(j-1)/n}^{j/n} \frac{d\tau}{\sqrt{\frac{j}{n} - \tau}} \cdot \phi_\rho(y-x) \leq \left\{ \frac{C}{n^{1/2-\varepsilon}} \int_u^{(j-1)/n} \frac{d\tau}{\left(\frac{j-1}{n} - \tau\right)^{1-\varepsilon}} + \frac{C}{\sqrt{n}} \right\} \phi_\rho(y-x)$$

$$\leq \frac{C}{n^{1/2-\varepsilon}} B(\varepsilon,1) \phi_\rho(y-x),$$

where $\rho = \sqrt{\frac{j}{n} - u}$. For $j = j^\star(u) + 1$ the estimate (83) follows directlty from the definitions of $p$ and $p_\Delta$. Moreover,

(84)
$$\left| \int_u^{\frac{j^\star(u)}{n}} d\tau \int p(u,\tau,x,z) H(\tau, \frac{j}{n}, z, y) dz \right| \le C \sqrt{\frac{j^\star(u)}{n} - u} \cdot \phi_\rho(y-x) \le \frac{C}{\sqrt{n}} \cdot \phi_\rho(y-x)$$

and, hence, the estimate (83) holds for the first summand in (82). It is easy to obtain that the same estimate (83) remains true for the second summand in (82). It follows from the smoothing properties of the operation $... \otimes'_n \Phi_1$. Hence, we get an estimate

$$\left| p(u, \frac{j}{n}, x, y) - p_\Delta(u, \frac{j}{n}, x, y) \right| \le \frac{C}{n^{1/2-\varepsilon}} B(\varepsilon, 1) \phi_\rho(y-x). \tag{85}$$

We give only the sketch of the proofs of (80) and (81). From the definitions of $\Re_1$ and $\Phi_1$ we have

$$\left| (\Re_1 \otimes_n \Phi_1)(\frac{j}{n}, 1, z, y) \right| \le C n^\varepsilon (1 - \frac{j}{n})^{\varepsilon - 1/2} B(\varepsilon, \frac{1}{2}) \cdot \phi_{\sqrt{1-j/n}}(y-z) \tag{86}$$

and from (85) and (86)

$$|(p - p_\Delta) \otimes'_n \Re_1 \otimes_n \Phi_1(u, 1, v, y)| \le \frac{C(\varepsilon)}{n^{\frac{1}{2}-2\varepsilon}} \cdot \phi_{\sqrt{1-u}}(y-v) \tag{87}$$

Now for each summand in (80) we split the integral in two integrals

(88)
$$n^{-1} \int_0^1 du \int p(0,u,x,v) \frac{\mu_\nu(v)}{\nu!} D_v^\nu [ \sum_{j=j^\star(u)}^{n-1} \frac{1}{n} \int (p - p_\Delta)(u, \frac{j}{n}, v, z) \cdot (\Re_1 \otimes_n \Phi_1)(\frac{j}{n}, 1, z, y)] dz$$

$$= n^{-1} \int_0^{1/2} du ... + n^{-1} \int_{1/2}^1 du = I + II.$$

By integration by parts we obtain from (87) that $II \simeq 0$. To estimate $I$ we consider two cases, namely, $a)$ $\frac{j}{n} - u \ge \frac{1}{4}$ and $b)$ $\frac{j}{n} - u < \frac{1}{4}$, $1 - \frac{j}{n} \ge \frac{1}{4}$. In the case a) we differentiate with respect to $v$ in (88) and use (82). With a substitution $v + w' = w$ we have

$$\left| D_v^\nu \left[ \int_u^{\frac{j^\star(u)}{n}} d\tau \int p(u,\tau,v,w) H(\tau, \frac{j}{n}, w, z) dw \right] \right|$$

$$\left| D_v^\nu \left[ \int_u^{\frac{j^\star(u)}{n}} d\tau \int p(u,\tau,v,v+w') H(\tau, \frac{j}{n}, v+w', z) dw' \right] \right|$$

$$\le \frac{C}{n} \cdot \phi_{\sqrt{\frac{j}{n}-u}}(z-v),$$

where we used that $\frac{j}{n} - \tau > \frac{1}{5}$ for $\tau \in [u, \frac{j^\star(u)}{n}]$ for $n$ large enough and the following inequality

$$|D_v^\nu p(u,\tau,v,v+w')| \le \frac{C_1}{(\tau - u)^{p/2}} \cdot \exp[-C_2 \frac{|w'|^2}{\tau - u}]. \tag{89}$$

This inequality is proved in Freedman (1964) , p. 260. The other terms in (82) can be estimated analogously. At last, in the case b) with similar substitutions we make use of (89) and the following inequality (see Konakov and Mammen (2002))

$$|D_v^\nu H(u,\tau,v,v+w')| \le \frac{C_1}{(\tau-u)^{(p+1)/2}} \cdot \exp[-C_2\frac{|w'|^2}{\tau-u}].$$

The proof of (81) is similar to the proof of Theorem 2.1 in Konakov and Mammen (2002) where $\Re_1$ plays now role of $H$ in Theorem 2.1. We omit the details. This completes the proof of (70).

*Asymptotic replacement of* $p_\Delta$ by $p$. We start from a comparison of $n^{-1}(p\otimes\mathcal{F}_2[p_\Delta])(0,1,x,y)$ and $n^{-1}(p\otimes\mathcal{F}_2[p])(0,1,x,y)$. From simple estimates

$$\left|\int_0^{n^{-\delta}} du \int p(0,u,x,z)\cdot\chi_\nu(z)D_v^\nu p(u,1,z,y)\right| \le Cn^{-\delta}\phi(y-x),$$

$$\left|\int_{1-n^{-\delta}}^1 du \int D_v^\nu[p(0,u,x,z)\cdot\chi_\nu(z)]\cdot p(u,1,z,y)\right| \le Cn^{-\delta}\phi(y-x),$$

$$\left|\int_0^{n^{-\delta}} du \int p(0,u,x,z)\cdot\chi_\nu(z)D_v^\nu p_\Delta(u,1,z,y)\right| \le Cn^{-\delta}\phi(y-x),$$

$$\left|\int_{1-n^{-\delta}}^1 du \int D_v^\nu[p(0,u,x,z)\cdot\chi_\nu(z)]\cdot p_\Delta(u,1,z,y)\right| \le Cn^{-\delta}\phi(y-x),$$

we obtain that it is enough to consider $u\in[n^{-\delta},1-n^{-\delta}]$. Then we get

$$\int_{n^{-\delta}}^{1-n^{-\delta}} du \int p(0,u,x,z)\cdot\chi_\nu(z)\cdot D_v^\nu(p-p_\Delta)(u,1,z,y)dz$$

$$= \int_{n^{-\delta}}^{1/2} du... + \int_{1/2}^{1-n^{-\delta}} du... = I + II.$$

The relation $n^{-1}\cdot II \simeq 0$ for the second term $II$ follows from (84) and from the following estimates

$$(90)\qquad \left|\int_{1/2}^{1-n^{-\delta}} du \int D_z^\nu[p(0,u,x,z)\cdot\chi_\nu(z)]\cdot\sqrt{\frac{j^\star(u)}{n}-u}\cdot\phi_{\sqrt{1-u}}(y-z)dz\right|$$

$$\le C\phi(y-x)\cdot\int_{1/2}^{1-n^{-\delta}}\sqrt{\frac{j^\star(u)}{n}-u}\,du$$

$$\le C\phi(y-x)\cdot\sum_{i=0}^{n-1}\int_{i/n}^{(i+1)/n}\sqrt{\frac{i+1}{n}-u}\,du \le \frac{C}{\sqrt{n}}\phi(y-x),$$

$$(91)\qquad \sum_{i=j^\star(u)}^{n-1}\int_{i/n}^{(i+1)/n}(\tau-\frac{i}{n})\int_0^1\int L_v^t p(u,\tau,z,v)H(\tau,1,v,y)\mid_{\tau=\tau^\star} dvd\delta d\tau$$

$$+\sum_{i=j^\star(u)}^{n-1}\int_{i/n}^{(i+1)/n}(\tau-\frac{i}{n})\int_0^1\int(\widetilde{L}_v-L_v)^t p(u,\tau,z,v)\widetilde{L_v}\widetilde{p}(\tau,1,v,y)]\mid_{\tau=\tau^\star} dvd\delta d\tau$$

$$=I'+II'.$$

Taking into account that $u\in[n^{-\delta},1-n^{-\delta}]$ we obtain

$$|I'|\le\frac{C}{n^{1-\delta}}(1-u)^{\delta-1/2}B(\delta,\frac{1}{2})\cdot\phi_{\sqrt{1-u}}(y-z)$$

and an analogous estimate holds for $II'$. Thus, $n^{-1}\cdot II\simeq 0$. For $u\in[n^{-\delta},\frac{1}{2}]$

$$n^{-1}\cdot\int_{n^{-\delta}}^{1/2}du\int D_z^\nu[p(0,u,x,z)\cdot\chi_\nu(z)](p-p_\Delta)(u,1,z,y)dz\simeq 0$$

because

$$|D_z^\nu[p(0,u,x,z)|\le\frac{C}{u^2}\cdot\phi_{\sqrt{u}}(z-x)\le C\cdot n^{2\delta}\,\phi_{\sqrt{u}}(z-x)$$

and the only difference from the previous estimate is an additional factor $n^{2\delta}$ where $\delta>0$ can be choosen arbitrary small. Finally we obtain

$$n^{-1}(p\otimes\mathcal{F}_2[p_\Delta])(0,1,x,y)\simeq n^{-1}(p\otimes\mathcal{F}_2[p])(0,1,x,y). \tag{92}$$

To prove that

$$n^{-1}(p\otimes\mathcal{F}_1[p\otimes\mathcal{F}_1[p]])(0,1,x,y)-n^{-1}(p\otimes\mathcal{F}_1[p\otimes\mathcal{F}_1[p_\Delta]])(0,1,x,y) \tag{93}$$

$$=n^{-1}(p\otimes\mathcal{F}_1[p\otimes\mathcal{F}_1[p-p_\Delta]])(0,1,x,y)\simeq 0$$

we proceed as above. We consider a typical summand in (93)

$$n^{-1}\int_0^1 du\int p(0,u,x,z)\mu_\nu(z)\cdot D_z^\nu\left[\int_u^1 d\tau\int p(u,\tau,z,v)\mu_\nu(v)\cdot\right. \tag{94}$$

$$\left.\times\, D_v^\nu(p-p_\Delta)(\tau,1,v,y)dv\right]dz$$

As in the proof of (92) it is enough to consider $u\in[n^{-\delta},1-n^{-\delta}]$. Now (94) is a sum of 4 integrals

$$1)\ I_1=\ n^{-1}\int_{n^{-\delta}}^{1/2}du\int\ ...\ D_z^\nu\int_u^{(1+u)/2}d\tau\int ...$$

$$2)\ I_2=\ n^{-1}\int_{n^{-\delta}}^{1/2}du\int\ ...\ D_z^\nu\int_{(1+u)/2}^{1}d\tau\int ...$$

$$3)\ I_3=\ n^{-1}\int_{1/2}^{1-n^{-\delta}}du\int\ ...\ D_z^\nu\int_u^{(1+u)/2}d\tau\int ...$$

$$4)\ I_4=\ n^{-1}\int_{1/2}^{1-n^{-\delta}}du\int\ ...\ D_z^\nu\int_{(1+u)/2}^{1}d\tau\int ...$$

We show that $I_i \simeq 0$, $i = 1, 2, 3, 4$. The proofs for all cases are similar. They use an integrationtha by parts and estimates for the derivatives of $p$ or $p - p_\Delta$. We consider only the case $I_2$. For this case $\tau - u \geq \frac{1}{4}$ and we get from (85) that

$$n^{-1} \int_{n^{-\delta}}^{1/2} du \int p(0,u,x,z)\mu_\nu(z) \cdot \left\{ \int_{(1+u)/2}^{1} d\tau \int D_z^\nu \left[ D_v^\nu p(u,\tau,z,v)\mu_\nu(v)\cdot \right] \right.$$

$$\left. \times (p - p_\Delta)(\tau, 1, v, y) dv \right\} dz \leq \frac{C(\varepsilon)}{n^{3/2-\varepsilon}} \phi(y - x) \simeq 0$$

for $\varepsilon \in (0, \frac{1}{2})$.

Now we consider the first term in (70)

$$n^{-1/2}(p \otimes \mathcal{F}_1[p_\Delta])(0,1,x,y) = n^{-1/2}(p \otimes \mathcal{F}_1[p])(0,1,x,y) \tag{95}$$

$$+ n^{-1/2}(p \otimes \mathcal{F}_1[p_\Delta - p])(0,1,x,y).$$

By (82) the last term in (95) is equal to

$$n^{-1/2}(p \otimes \mathcal{F}_1[S_1])(0,1,x,y) + n^{-1/2}(p \otimes \mathcal{F}_1[S_2])(0,1,x,y) \tag{96}$$

$$n^{-1/2}(p \otimes \mathcal{F}_1[S_3])(0,1,x,y),$$

where

$$S_1(u,1,z,y) = \int_u^{\frac{j^\star(u)}{n}} d\tau \int p(u,\tau,z,v) H(\tau,1,v,y) dv, \; S_2(u,1,z,y) = R(u,1,z,y),$$

$$S_3 = \left\{ (p \otimes H - p \otimes_n' H) \otimes_n' \Phi_1 \right\}(u,1,z,y).$$

From (83)

$$\left| n^{-1/2} \int_{1-n^{-\delta}}^{1} du \int D_z^\nu [p(0,u,x,z)\mu_\nu(z)] R(u,1,z,y) dz \right| \leq \frac{C(\varepsilon)}{n^{1+(\delta-\varepsilon)}} \phi(y-x) \simeq 0$$

for $0 < \varepsilon < \delta$. Analogously,

$$\left| n^{-1/2} \int_0^{n^{-\delta}} du \int p(0,u,x,z)\mu_\nu(z) D_z^\nu R(u,1,z,y) dz \right| \leq \frac{C(\varepsilon)}{n^{1+(\delta-\varepsilon)}} \phi(y-x) \simeq 0.$$

For $u \in [n^{-\delta}, 1 - n^{-\delta}]$ we obtain

$$n^{-1/2} \int_{n^{-\delta}}^{1-n^{-\delta}} du \int p(0,u,x,z)\mu_\nu(z) D_z^\nu R(u,1,z,y) dz$$

$$= n^{-1/2} \int_{n^{-\delta}}^{1/2} du \int p(0,u,x,z)\mu_\nu(z) D_z^\nu R(u,1,z,y) dz$$

$$+ n^{-1/2} \int_{1/2}^{1-n^{-\delta}} du \int D_z^\nu [p(0,u,x,z)\mu_\nu(z)] R(u,1,z,y) dz = I + II.$$

We have $I \simeq 0$ and $II \simeq 0$. This follows from simple estimates and from the following estimate for $n^{-1/2}R(u,1,z,y)$ with $u \in [n^{-\delta}, 1-n^{-\delta}]$

$$\left|n^{-1/2}R(u,1,z,y)\right| \leq n^{-1/2}\left|\sum_{i=j^\star(u)+1}^{n-2}\int_{i/n}^{(i+1)/n}(\tau-\frac{i}{n})\right.$$

$$\left.\times\int_0^1\int[p(u,\tau,z,v)\left(L-\widetilde{L}\right)^2\widetilde{p}(\tau,1,v,y)]\mid_{\tau=\tau^\star}dvd\delta d\tau\right|$$

$$+n^{-1/2}\left|\int_{j^\star(u)/n}^{(j^\star(u)+1)/n}(\tau-\frac{i}{n})\int_0^1\int[p(u,\tau,z,v)\left(L-\widetilde{L}\right)^2\widetilde{p}(\tau,1,v,y)]\mid_{\tau=\tau^\star}dvd\delta d\tau\right|$$

$$+n^{-1/2}\left|\int_{1-\frac{1}{n}}^1(\tau-\frac{i}{n})\int_0^1\int[p(u,\tau,z,v)\left(L-\widetilde{L}\right)^2\widetilde{p}(\tau,1,v,y)]\mid_{\tau=\tau^\star}dvd\delta d\tau\right|$$

$$\leq\frac{C}{n^{3/2}}\sum_{i=j^\star(u)+1}^{n-2}\frac{1}{n}\cdot\frac{1}{(1-\frac{i+1}{n})^{3/2}}\phi_{\sqrt{1-u}}(y-z)+n^{-1/2}\cdot\int_{j^\star(u)/n}^{(j^\star(u)+1)/n}(\tau-\frac{j^\star(u)}{n})$$

$$\times\frac{d\tau}{(1-\tau)^{3/2}}+n^{-1/2}\int_{1-\frac{1}{n}}^1(\tau-\frac{n-1}{n})\int_0^1\int[L^tp(u,\tau,z,v)\left(L-\widetilde{L}\right)\widetilde{p}(\tau,1,v,y)$$

$$+(\widetilde{L})^t(\widetilde{L}-L)^tp(u,\tau,z,v)\widetilde{p}(\tau,1,v,y)]dvd\delta d\tau$$

$$\leq\left[\frac{C}{n^{3/2-\varepsilon}}\int_u^1\frac{dt}{(1-t)^{1-\varepsilon}}+\frac{C}{n^{5/2-3\delta/2}}\right]\phi_{\sqrt{1-u}}(y-z).$$

Thus, we get $n^{-1/2}(p\otimes\mathcal{F}_1[S_2])(0,1,x,y)\simeq 0$. The proof that $n^{-1/2}(p\otimes\mathcal{F}_1[S_1])(0,1,x,y)\simeq 0$ is quite similar. First, we show that it is enough to consider $u\in[n^{-\delta},1-n^{-\delta}]$. Then the assertion follows from the following estimates

$$\left|n^{-1/2}\int_{n^{-\delta}}^{1/2}du\int p(0,u,x,z)\mu_\nu(z)D_z^\nu S_1(u,1,z,y)dz\right|$$

$$=\left|n^{-1/2}\int_{n^{-\delta}}^{1/2}du\int p(0,u,x,z)\mu_\nu(z)D_z^\nu[\int_u^{\frac{j^\star(u)}{n}}d\tau\int p(u,\tau,z,z+v')\right.$$

$$\left.\times H(\tau,1,z+v',y)dv']\right|\leq Cn^{-1/2}\int_{n^{-\delta}}^{1/2}\left(\frac{j^\star(u)}{n}-u\right)du\cdot\phi(y-x)\simeq 0,$$

$$\left|n^{-1/2}\int_{1/2}^{1-n^{-\delta}}du\int D_z^\nu[p(0,u,x,z)\mu_\nu(z)]S_1(u,1,z,y)dz\right|$$

$$\leq\frac{C}{n^{1/2-\delta}}\int_{1/2}^{1-n^{-\delta}}\left(\frac{j^\star(u)}{n}-u\right)du\cdot\phi(y-x)\simeq 0.$$

The same estimate holds true for the last summand in (96) that is

$$n^{-1/2}(p\otimes\mathcal{F}_1[S_3])(0,1,x,y)\simeq 0.$$

It follows from the smoothing properties of the operation $\otimes_n'\Phi_1$and can be shown by the same methods. This completes the proof of Theorem 4.1.

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