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On the Hedging of American Options in Discrete Time Markets with Proportional Transaction Costs

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Abstract
In this note, we consider a general discrete time financial market with proportional transaction costs as in [4], [5], [6] and [10]. We provide a dual formulation for the set of initial endowments which allow to super-hedge some American claim. This extends the results of [1] which was obtained in a model with constant transaction costs and risky assets which evolve on a finite dimensional tree. We also provide fairly general conditions under which the expected formulation in terms of stopping times does not work.

Key words : Transaction cost, American option.

MSC Classification (2000): 91B28, 60G40.

1 Introduction and main result

Set $T = \{0, \ldots, T\}$ for some $T \in \mathbb{N} \setminus \{0\}$ and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space endowed with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in T}$. We assume that $\mathcal{F}_T = \mathcal{F}$ and that $\mathcal{F}_0$ is trivial. Given an integer $d \geq 1$, we denote by $\mathcal{K}$ the set of $\mathcal{C}$-valued processes $K$ such that $\mathbb{R}_+^d \setminus \{0\} \subset \text{int}(K_t)$ $\mathbb{P}$-a.s. for all $t \in T$.

Here, we follow [6] and say that a sequence of set-valued mappings $(K_t)_{t \in T}$ is a $\mathcal{C}$-valued process if there is a countable sequence of $\mathbb{R}^d$-valued $\mathbb{F}$-adapted processes $X^n_t = (X^n_t)_{t \in T}$ such that, for every $t \in T$, $\mathbb{P}$--a.s. only a finite but non-zero number of $X^n_t$ is different from zero and $K_t = \text{cone}\{X^n_t, n \in \mathbb{N}\}$. This means that $K_t$ is the polyhedral cone generated by the $\mathbb{P}$--a.s. finite set $\{X^n_t, n \in \mathbb{N} \text{ and } X^n_t \neq 0\}$.
Following the modelization of [5], for a given $K \in \mathcal{K}$ and $x \in \mathbb{R}^d$, we define the process $V^{x,\xi}$ by

$$V_t^{x,\xi} := x + \sum_{s=0}^{t} \xi_s , \quad t \in T ,$$

where $\xi$ belongs to

$$\mathcal{A}(K) = \{ \xi \in L^0(\mathbb{R}^d; \mathbb{F}) \text{ s.t. } \xi_t \in -K_t \quad \mathbb{P} - \text{a.s. for all } t \in T \} ,$$

and, for a random set $E \subset \mathbb{R}^d \mathbb{P} - \text{a.s.}$ and $\mathcal{G} \subset \mathcal{F}$, $L^0(E; \mathbb{F})$ (resp. $L^0(E; \mathcal{G})$) is the collection of $\mathbb{F}$-adapted processes (resp. $\mathcal{G}$-measurable variables) with values in $E \mathbb{P}$-a.s.

The financial interpretation is the following: $x$ is the initial endowment in units of the financial assets, $\xi_t$ is the amount of units of assets which is exchanged at $t$ and $-K_t$ is the set of affordable exchanges. We refer to [5] and [6] for a more detailed description.

Therefore,

$$A(x; K) := \left\{ V^{x,\xi}, \xi \in \mathcal{A}(K) \right\}$$

stands for the set of all portfolio processes with initial endowment $x$, and

$$A_t(x; K) := \{ V_t, V \in A(x; K) \}$$

corresponds to the collection of their values at time $t \in T$.

It is known from the work of [4], [5] [6] and [10], see also the references therein, that, under mild assumptions, the set $A_T(x; K)$ can be written as

$$\left\{ g \in L^0(\mathbb{R}^d; \mathcal{F}) : \mathbb{E}[Z_T \cdot g - Z_0 \cdot x] \leq 0, \text{ for all } Z \in \mathcal{Z}(K), (Z \cdot g)^- \in L^1(\mathbb{R}; \mathbb{P}) \right\}$$

where $\mathcal{Z}(K)$ is the set of $(\mathbb{F}, \mathbb{P})$-martingales $Z$ such that $Z_t \in K^*_t \quad \mathbb{P} - \text{a.s. for all } t \in T ,$

and $K^*_t(\omega)$ denotes the positive polar of $K_t(\omega)$, i.e.

$$K^*_t(\omega) := \left\{ y \in \mathbb{R}^d : x \cdot y \geq 0, \text{ for all } x \in K_t(\omega) \right\} .$$

The operator $\cdot$ denotes the natural scalar product on $\mathbb{R}^d$ and $L^1(E; \mathbb{P})$ (resp. $L^1(E; \mathcal{F}, \mathbb{P})$) is the subset of $\mathbb{P}$-integrable elements of $L^0(E; \mathcal{F})$ (resp. $L^0(E; \mathcal{F})$).

In this paper, we are interested in

$$A^*(x; K) := \left\{ \vartheta \in L^0(\mathbb{R}^d; \mathcal{F}) : V - \vartheta \in -\mathcal{A}(K) \text{ for some } V \in A(x; K) \right\} ,$$

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the set of processes which are dominated by a portfolio in the sense of $K$: $V_t - \vartheta_t \in K_t$, for all $t \in \mathbb{T}$, $\mathbb{P}$ a.s. More precisely, our aim is to provide a dual formulation for

$$
\Gamma(\vartheta; K) := \left\{ x \in \mathbb{R}^d : \vartheta \in A^s(x; K) \right\}.
$$

In analogy with the standard result for markets without transaction cost, one could expect that $(\vartheta; K)$ can admit the dual formulation

$$
\Theta(\vartheta; K) = \left\{ x \in \mathbb{R}^d : \sup_{\tau \in \mathcal{T}(\mathbb{T})} \mathbb{E}[Z_{\tau} \cdot \vartheta_{\tau} - Z_0 \cdot x] \leq 0, \text{ for all } Z \in \mathcal{Z}(K) \right\}
$$

(1.1)

where $\mathcal{T}(\mathbb{T})$ is the set of all $\mathbb{F}$-stopping times with values in $\mathbb{T}$. However, this characterization does not hold true in general, as shown in the following section. This phenomenon was already pointed out in [1] in a model consisting of one bank account and one risky asset evolving on a finite dimensional tree. In [1], the authors show that a correct dual formulation can be obtained if we replace stopping times by randomized stopping times.

In our general framework, this amounts to introduce a new set of dual variables. For $\tilde{\mathbb{P}} \sim \mathbb{P}$, the associated set of dual variables, $\mathcal{D}(K, \tilde{\mathbb{P}})$, is defined as the collection of process $Z \in L^1(\mathbb{R}^d; \mathbb{F}, \tilde{\mathbb{P}})$ such that

$$
Z_t \in K^*_t \text{ and } \tilde{Z}_t := \mathbb{E}^{\tilde{\mathbb{P}}}\left[ \sum_{s=t}^{T} Z_s \mid \mathcal{F}_t \right] \in K^*_t \text{ a.s. for all } t \in \mathbb{T}.
$$

In the following, we shall say that a subset $B$ of $L^0(\mathbb{R}^d; \mathbb{F})$ is closed in measure if it is closed in probability when identified as a subset of $L^0(\mathbb{R}^{d \times (T+1)}; \mathcal{F})$, i.e.

$$
v^n \in B \text{ and } \forall \varepsilon > 0 \lim_{n \to \infty} \mathbb{P}\left[ \sum_{t \in \mathbb{T}} \|v^n_t - v_t\| > \varepsilon \right] = 0 \implies v \in B.
$$

We then have the following characterization of $A^s(K) := A^s(0; K)$.

**Theorem 1.1** Assume that $A^s(K)$ is closed in measure and that the no-arbitrage condition

$$
NA(K) : A_T(0; K) \cap L^0(\mathbb{R}^d_+; \mathcal{F}) = \{0\}
$$

holds. Then, the following assertions are equivalent:

(i) $\vartheta \in A^s(K)$
(ii) for all $\tilde{\mathbb{P}} \sim \mathbb{P}$ and $Z \in \mathcal{D}(K,\tilde{\mathbb{P}})$ such that $(\vartheta \cdot Z)^{-} \in L^{1}(\mathbb{R}; \mathbb{F}, \tilde{\mathbb{P}})$ we have
\[
\mathbb{E}^{\tilde{\mathbb{P}}} \left[ \sum_{t=0}^{T} \vartheta_{t} \cdot Z_{t} \right] \leq 0.
\]

(iii) for some $\tilde{\mathbb{P}} \sim \mathbb{P}$ we have
\[
\mathbb{E}^{\tilde{\mathbb{P}}} \left[ \sum_{t=0}^{T} \vartheta_{t} \cdot Z_{t} \right] \leq 0
\]
for all $Z \in \mathcal{D}(K,\tilde{\mathbb{P}})$ such that $(\vartheta \cdot Z)^{-} \in L^{1}(\mathbb{R}; \mathbb{F}, \tilde{\mathbb{P}})$.

Since $A^{s}(0; K) = A^{s}(x; K) - x$, this immediately provides a dual formulation for $\Gamma(\vartheta; K)$.

**Corollary 1.1** Let the conditions of Theorem 1.1 hold. Then, for all $\vartheta \in L^{0}(\mathbb{R}^{d}; \mathbb{F})$,

\[
\Gamma(\vartheta; K) = \left\{ x \in \mathbb{R}^{d} : \mathbb{E} \left[ \sum_{t=0}^{T} \vartheta_{t} \cdot Z_{t} \right] \leq \tilde{Z}_{0} \cdot x \ \forall \ Z \in \mathcal{D}(K; \mathbb{P}) , (Z \cdot \vartheta)^{-} \in L^{1}(\mathbb{R}; \mathbb{F}, \mathbb{P}) \right\} .
\]

**Remark 1.1** The integrability conditions on $(Z \cdot \vartheta)^{-}$ are trivially satisfied if there is some $\mathbb{R}^{d}$-valued constant $c$ such that $\vartheta_{t} + c \in K_{t} \mathbb{P}$ - a.s. for all $t \in \mathbb{T}$, i.e. the liquidation value of $\vartheta$ is uniformly bounded from below. Indeed, in that case $Z_{t} \cdot (\vartheta_{t} + c) \geq 0 \ \mathbb{P}$ - a.s. for all $Z_{t} \in L^{0}(K_{t}^{*}; \mathbb{F})$.

Following the approach of [5] and [6] the closure property of $A^{s}(0; K)$ can be obtained under the general assumption

$\xi \in \mathcal{A}(K)$ and $\sum_{t \in \mathbb{T}} \xi_{t} = 0 \ \mathbb{P}$ - a.s. $\implies \xi_{t} \in K_{t}^{0} \ \mathbb{P}$ - a.s. for all $t \in \mathbb{T}$ (1.2)

where $K^{0} = (K_{i}^{0})_{i \in \mathbb{T}}$ is defined by $K_{i}^{0} = K_{i} \cap (-K_{i})$ for $t \in \mathbb{T}$.

**Proposition 1.1** Assume that (1.2) holds, then $A^{s}(K)$ is closed in measure.

**Remark 1.2** 1. In the case of efficient frictions, i.e. $K_{t}^{0} = \{0\}$, $\forall t \in \mathbb{T}$, it is shown in [5] that the assumption (1.2) is a consequence of the strict no-arbitrage property

$NA^{s}(K) : A_{t}(0; K) \cap L^{0}(K_{t}; \mathbb{F}_{t}) \subset L^{0}(K_{t}^{0}; \mathbb{F}_{t})$ for all $t \in \mathbb{T}$.

2. In the case where $K_{t}^{0}$ may not be trivial, (1.2) holds under the robust no-arbitrage condition introduced by [10] and further studied by [6],

$NA^{s}(K) : NA(\tilde{K})$ holds for some $\tilde{K} \in \mathbb{K}$ which dominates $K$,
where $\tilde{K}$ dominates $K$ if $K_{t} \setminus K_{t}^{0} \subset \text{ri}(\tilde{K}_{t})$ $\mathbb{P}$ - a.s. for all $t \in \mathbb{T}$.

3. It is shown in [7] that the condition $K_{t}^{0} = \{0\}$ in 1. can be replaced by the weaker one: $L^{0}(K_{t}^{0}; \mathbb{F}_{t-1}) \subset L^{0}(K_{t-1}^{0}; \mathbb{F}_{t-1})$ for all $1 \leq t \leq T$. See also [8].
2 Counter examples

In this section, we first show that the duality relation

\[ D(K) : \Gamma(\vartheta; K) = \Theta(\vartheta; K) \text{ for all } \vartheta \in L^0(\mathbb{R}^d; \mathbb{F}) \]

does not hold for a large class of \( C \)-valued process \( K \in \mathcal{K} \). For \( x \in \mathbb{R}^d \), let us define

\[ c_t(x) := \min \{ c \in \mathbb{R} : c1_1 - x \in K_t \} . \quad (2.1) \]

In financial terms, \( c_t(x) \) is the minimal amount, in terms of the first asset, necessary to dominate \( x \) in the sense of \( K_t \) at time \( t \). If the first asset is interpreted as a numeraire, it corresponds to the constitution value of \( x \) in terms of this numeraire. Here, \( 1_1 \) stands for the \( \mathbb{R}^d \) vector \((1,0,\ldots,0)\).

**Proposition 2.1** If there exists \( x \in \mathbb{R}^d \) such that

(i) \( y - c_0(x)1_1 \in K_0^0 \Rightarrow y - x \in K_0^0 \text{ or } \mathbb{P}[y - x \in K_1] < 1 \)

(ii) \( x - c_0(x)1_1 \notin K_0 \).

Then, there exists \( \vartheta \) such that \( \Theta(\vartheta; K) \neq \Gamma(\vartheta; K) \), i.e. \( D(K) \) is not satisfied.

The proof is postponed to the end of the section.

**Remark 2.1** Condition (ii) means that there are directions with efficient frictions at time 0. Condition (i) has the following interpretation. If a portfolio \( y \) is equivalent to the constitution value of \( x \) then it dominates \( x \) in the sense of \( K_0 \). However, since \( x \) and \( y \) have the same constitution value, \( c_0(x) = c_0(y) \), it can not be too large. In particular, if it is not equivalent to \( x \), then it can not dominate \( x \) component by component. In that case, we assume that there is randomness enough so that the probability that \( y \) still dominates \( x \) at time 1 is less than 1.

**Remark 2.2** 1. If \( K_0^0 = \{0\} \) and \( x \neq c_0(x)1_1 \) then (ii) holds since by definition we already have \( c_0(x)1_1 - x \in K_0 \). If we also assume that \( \mathbb{P}[c_1(x) > c_0(x)] > 0 \) then (i) is satisfied too.

**Example 2.1** 1. Efficient frictions: consider the following cones

\[ K_t = \{ (x^1, x^2) \in \mathbb{R}^2 : x^1 + (1 + \lambda_t)x^2 \geq 0, \ x^1 + (1 - \mu_t)x^2 \geq 0 \} , \]

where \( t \in \mathbb{T} := \{0,1\} \), \( \lambda_0 < \lambda_1 \) and \( \mu_0, \mu_1 \in (0,1) \). Observe that \( K_0^0 = \{0\} \). For \( x = (0,1) \), \( c_0(x) = 1 + \lambda_0 < c_1(x) = 1 + \lambda_1 \). Then, the conditions of the remark above hold so that \( D(K) \) is not true.
2. Partial frictions: consider the preceding case where we add an asset which has no transaction cost with the first one, i.e.

\[ K_t = \{(x^1, x^2, x^3) \in \mathbb{R}^3 : x^1 + (1 + \lambda_t)x^2 + x^3 \geq 0, x^1 + (1 - \mu_t)x^2 + x^3 \geq 0\}. \]

We put \( x = (0, 1, 0) \) so that assumption (ii) holds. We now check (i). It is clear that if \( y - c_0(x)1 \in K^0_t \) then \( y - x \notin K^0_t \). Observe that \( y = (y^1, 0, y^3) \) with \( y^1 + y^3 = c_0(x) \), so \( y^1 + (-1)(1 + \lambda_1) + y^3 < 0 \) which implies that \( y - x \notin K_1 \).

On the contrary, we can also show that \( \mathbf{D}(K) \) does not only hold in the case where \( K = K^0_t + \mathbb{R}^d_+ \) a.s., i.e. there is no transaction costs.

**Proposition 2.2** There exists \((\Omega, \mathcal{F}, \mathbb{P})\) and \( K \in \mathcal{R} \) such that \( NA(K) \) holds, \( K^0_t = \{0\} \) for all \( t \), and such that for all \( \vartheta \in L^0(\mathbb{R}^d; \mathbb{F}) \) we have \( \Theta(\vartheta; K) = \Gamma(\vartheta; K) \).

**Proof.** We take \( \Omega \) trivial, i.e. \(|\Omega| = 1\) with \( \mathcal{F}_0 = \mathcal{F}_T = \{\emptyset, \Omega\} \), and put \( K = K_0 \) constant. Then, \( x \in \Theta(\vartheta; K) \) reads \( \sup_{Z_t \in K^*_t} Z_t \cdot (\vartheta_t - x) \leq 0 \), i.e. \( x - \vartheta \in K_t \) for all \( t \in T \). \( \square \)

**Proof of Proposition 2.1:** Let \( x \) be such that \((i)-(ii)\) are satisfied. We consider the asset \( \vartheta \) defined by \( \vartheta_t = c_0(x)1_11_1 + x1_11_1 \). From the martingale property of \( Z \),

\[
\sup_{\tau \in T(\mathbb{F})} \mathbb{E}[Z_{\tau} \cdot \vartheta_{\tau} - Z_0 \cdot (c_0(x)1_1)] = \sup_{\tau \in T(\mathbb{F})} \mathbb{E}[Z_{\tau} \cdot (x - c_0(x)1_1)1_11_1] = \max \{0 ; Z_0 \cdot (x - c_0(x)1_1)\}
\]

which is non positive by (2.1). Hence, \( c_0(x)1_1 \in \Theta(\vartheta; K) \). If \( \mathbf{D}(K) \) holds, then there exists a portfolio \( V \in A(c_0(x)1_1; K) \) such that \( V_0 - c_0(x)1_1 \in K_0 \) and therefore \( V_0 - c_0(x)1_1 \in K^0_0 \). By (i) there is two cases. If \( V_0 - x \in K^0_0 \), then \( x - c_0(x)1_1 \in K^0_0 \cap K_0 \) which is a contradiction of (ii). If \( \mathbb{P}[V_0 - x \in K_1] < 1 \), we can not have \( V_1 - x = V_0 + \xi_1 - x \in K_1 \) a.s. with \( \xi_1 \in -K_1 \) a.s. \( \square \)

### 3 Proofs

In this section, we first provide the proof of Theorem 1.1. It follows from standard arguments based on the Hahn-Banach separation theorem. For ease of notations, we simply write \( A(K) \) and \( A^s(K) \) in place of \( A(0; K) \) and \( A^s(0; K) \). We denote by \( \mathcal{L}^0 \) the set of \( \mathbb{F} \)-adapted processes with values in \( \mathbb{R}^d \) and by \( \mathcal{L}^1(\mathbb{F}) \) (resp. \( \mathcal{L}^\infty \)) the subset of these elements which are \( \mathbb{P} \)-integrable, \( \mathbb{P} \sim \mathbb{P} \) (resp. bounded). Observe that \( \mathcal{L}^0 \) (resp. \( \mathcal{L}^\infty \)) can be identified as a subset of \( L^0(\mathbb{R}^{d \times (T+1)}; \mathbb{F}) \) (resp. \( L^\infty(\mathbb{R}^{d \times (T+1)}; \mathbb{F}) \)), the set of bounded elements of \( L^0(\mathbb{R}^{d \times (T+1)}; \mathbb{F}) \).
Proposition 3.1 Let the conditions of Theorem 1.1 hold. Then, for all \( \tilde{\mathbb{P}} \sim \mathbb{P} \), there is some \( Z \in \mathcal{D}(K; \tilde{\mathbb{P}}) \cap \mathcal{L}^\infty \) such that

\[
\sup_{\vartheta \in A^s(K) \cap \mathcal{L}^1(\tilde{\mathbb{P}})} \mathbb{E}_{\tilde{\mathbb{P}}} \left[ \sum_{t \in T} Z_t \cdot \vartheta_t \right] \leq 0 .
\]

Proof. Since \( A^s(K) \cap \mathcal{L}^1(\tilde{\mathbb{P}}) \) is closed in \( \mathcal{L}^1(\mathbb{R}^{d \times (T+1)}; \mathcal{F}, \tilde{\mathbb{P}}) \) (when identified with a subset of \( \mathcal{L}^1(\mathbb{R}^{d \times (T+1)}; \mathcal{F}, \tilde{\mathbb{P}}) \)) and convex, it follows from the Hahn-Banach separation theorem, \( NA(K) \) and the fact that \( A^s(K) \cap \mathcal{L}^1(\tilde{\mathbb{P}}) \) is a cone, that there is some \( \eta = (\eta_t)_{t \in T} \in \mathcal{L}^\infty(\mathbb{R}^d \times (T+1); \mathcal{F}) \) such that

\[
\sup_{\vartheta \in A^s(K) \cap \mathcal{L}^1(\tilde{\mathbb{P}})} \mathbb{E}_{\tilde{\mathbb{P}}} \left[ \sum_{t \in T} \eta_t \cdot \vartheta_t \right] \leq 0 . \tag{3.1}
\]

By possibly replacing \( \eta_t \) by \( \mathbb{E} [\eta_t | \mathcal{F}_t] \), we can assume that \( \eta \) is \( \mathbb{F} \)-adapted. Fix an arbitrary \( \xi \in A(K) \cap \mathcal{L}^\infty \), so that \( V^0,\xi \in A^s(K) \cap \mathcal{L}^1(\tilde{\mathbb{P}}) \). Since

\[
\sum_{t \in T} \eta_t \cdot V^0,\xi_t = \sum_{t \in T} \xi_t \cdot \left( \sum_{s=t}^T \eta_s \right)
\]

we deduce from the above inequality that

\[
\sup_{\xi \in A(K) \cap \mathcal{L}^\infty} \mathbb{E}_{\tilde{\mathbb{P}}} \left[ \sum_{t \in T} \bar{\eta}_t \cdot \xi_t \right] \leq 0 ,
\]

where we defined

\[
\bar{\eta}_t := \mathbb{E}_{\tilde{\mathbb{P}}} \left[ \sum_{s=t}^T \eta_s | \mathcal{F}_t \right] \quad t \in T .
\]

This shows that \( \bar{\eta}_t \in K^*_t \mathbb{P} \) a.s. for all \( t \in T \). For an arbitrary bounded element \( \xi_t \in L^0(K_t; \mathcal{F}_t) \), the process \( V^0,\xi = -\mathbb{I}_{s=t} \xi_t, s \in T, \) belongs to \( A^s(K) \). In view of (3.1), this implies that \( \eta_t \in K^*_t \mathbb{P} \) a.s. \( \square \)

Proposition 3.2 Let the conditions of Theorem 1.1 hold. Fix \( \tilde{\mathbb{P}} \sim \mathbb{P} \) and \( \vartheta \in \mathcal{L}^1(\tilde{\mathbb{P}}) \). If

\[
\mathbb{E}_{\tilde{\mathbb{P}}} \left[ \sum_{t=0}^T \vartheta_t \cdot Z_t \right] \leq 0
\]

for all \( Z \in \mathcal{D}(K; \tilde{\mathbb{P}}) \) such that \( \vartheta \cdot Z \in \mathcal{L}^1(\tilde{\mathbb{P}}) \), then \( \vartheta \in A^s(K) \).
Proof. Since \( A^s(K) \cap L^1(\bar{\mathbb{P}}) \) is closed and convex, if \( \vartheta \notin A^s(K) \), we can find some \( \eta = (\eta_t)_{t \in T} \in L^\infty(\mathbb{R}^d \times (T+1); \mathcal{F}) \) such that

\[
\sup_{\vartheta \in A^s(K) \cap L^1(\bar{\mathbb{P}})} \mathbb{E}^{\bar{\mathbb{P}}}_{\vartheta} \left[ \sum_{t=0}^{T} \vartheta_t \cdot \eta_t \right] < \mathbb{E}^{\bar{\mathbb{P}}}_{\vartheta} \left[ \sum_{t=0}^{T} \vartheta_t \cdot \eta_t \right].
\]

The same arguments as in the proof of Proposition 3.1 then shows that we can choose \( \eta \in \mathcal{D}(K, \bar{\mathbb{P}}) \) which leads to a contradiction. \( \Box \)

Proof of Theorem 1.1. 1. In view Proposition 3.2, the implication (ii) \( \Rightarrow \) (i) is obtained by considering \( \bar{\mathbb{P}} \) with density with respect to \( \mathbb{P} \) defined by \( H := \exp(- \sum_{t \in T} L_t \| \theta_t \|) \).

2. It is clear that (ii) implies (iii) while the reverse implication follows from the fact that \( Z \in \mathcal{D}(K, \mathbb{P}) \) if and only if \( HZ \in \mathcal{D}(K, \bar{\mathbb{P}}) \) where \( \bar{H}_t := \mathbb{E} \left[ \frac{d\bar{\mathbb{P}}}{d\mathbb{P}} \mid \mathcal{F}_t \right] \).

3. The last implication (i) \( \Rightarrow \) (ii) is trivial. Indeed, recall that, for \( \xi \in A(K) \),

\[
\mathbb{E} \left[ \sum_{t \in T} Z_t \cdot V_{t}^{0, \xi} \right] = \mathbb{E} \left[ \sum_{t \in T} \bar{Z}_t \cdot \xi_t \right].
\]

Since \( \bar{Z}_t \in L^0(K_t^*; \mathcal{F}_t) \) and \( \xi_t \in L^0(-K_t; \mathcal{F}_t) \), the last term is non-positive. Moreover, \( V_{t}^{0, \xi} - \vartheta_t \in L^0(K_t; \mathcal{F}_t) \) implies \( Z_t \cdot V_{t}^{0, \xi} \geq Z_t \cdot \vartheta_t \). \( \Box \)

We now provide the proof of Proposition 1.1. The following Lemma can be found in [3].

Lemma 3.1 Set \( G \subset \mathcal{F} \) and \( E \subset \mathbb{R}^d \). Let \((\eta^n)_{n \geq 1}\) be a sequence in \( L^0(E; G) \). Set \( \bar{\Omega} := \{ \liminf_{n \to \infty} \| \eta^n \| < \infty \} \). Then, there is an increasing sequence of random variables \((\tau(n))_{n \geq 1}\) in \( L^0(\bar{\Omega}; G) \) such that \( \tau(n) \to \infty \mathbb{P} \) a.s. and, for each \( \omega \in \bar{\Omega} \), \( \eta^{\tau(n)}(\omega) \) converges to some \( \eta^*(\omega) \) with \( \eta^* \in L^0(E; G) \).

Proof of Lemma 1.1. We use an inductive argument. For \( t \in \mathbb{T} \), we denote by \( \Sigma_t \) the set of processes \( \vartheta \in L^0 \) such that

\[
\exists \xi \in \mathcal{A}(K) \ \text{s.t.} \ \sum_{s=t}^{\tau} \xi_s - \vartheta_{\tau} \in K_{\tau}, \ \mathbb{P} \text{-a.s. for all } t \leq \tau \leq T.
\]

Clearly, \( \Sigma_{T+1} \) is closed in measure. Assume that \( \Sigma_{t+1} \) is closed and let \( \vartheta^n \) be a sequence in \( \Sigma_t \) such that \( \vartheta^n_t \to \vartheta_t \mathbb{P} \) a.s. for \( t \leq s \leq T \). Let \( \xi^n \in \mathcal{A}(K) \) be such that

\[
\sum_{s=t}^{\tau} \xi^n_s - \vartheta^n_{\tau} \in K_{\tau}, \ \mathbb{P} \text{-a.s. for all } t \leq \tau \leq T.
\]
Set \( \tilde{\Omega} := \{ \liminf_{n \to \infty} \| \xi^n_t \| < \infty \} \). Since \( \tilde{\Omega} \in \mathcal{F}_t \), we can work separately on \( \tilde{\Omega} \) and \( \tilde{\Omega}^c \).

1. If \( \mathbb{P} \left[ \tilde{\Omega} \right] = 1 \), after possibly passing to a random sequence (see Lemma 3.1), we can assume that \( \xi^n_t \) converges \( \mathbb{P} \)-a.s. to some \( \xi_t \in L^0(-K_t; \mathcal{F}_t) \). Since 
\[
\sum_{s=t+1}^\tau \xi^n_s - (\vartheta^n_t - \xi^n_t) \in K_\tau \mathbb{P} \text{-a.s. for all } t+1 \leq \tau \leq T,
\]
and \( \Sigma_{\tau+1} \) is closed, we can find some \( \tilde{\xi} \in A(K) \) such that 
\[
\sum_{s=t+1}^\tau \tilde{\xi}_s - (\vartheta_t - \xi_t) \in K_\tau \mathbb{P} \text{-a.s. for all } t+1 \leq \tau \leq T.
\]
This shows that \( \vartheta \in \Sigma_t \).

2. If \( \mathbb{P} \left[ \tilde{\Omega} \right] < 1 \), then we can assume without loss of generality that \( \mathbb{P} \left[ \tilde{\Omega} \right] = 0 \).
Following line by line the proof of Lemma 2 in \cite{6} and using the \( K_s \)'s closure property, we can find some \( \hat{\xi} \in A(K) \) with \( \| \hat{\xi}_t \| = 1 \) such that 
\[
\kappa_\tau := \sum_{s=t}^\tau \hat{\xi}_s \in K_\tau \mathbb{P} \text{-a.s. for all } t \leq \tau \leq T.
\]
By (1.2), we must have \( \hat{\xi}_\tau - \kappa_\tau \in K^0_\tau \mathbb{P} \text{-a.s. } \forall t \leq \tau \leq T \). Since \( \hat{\xi}_\tau \) and \( -\kappa_\tau \in -K_\tau \mathbb{P} \text{-a.s.} \), we deduce that 
\[
\hat{\xi}_\tau \in K^0_\tau \text{ and } \kappa_\tau = \sum_{s=t}^\tau \hat{\xi}_s \in K^0_\tau \mathbb{P} \text{-a.s. for all } t \leq \tau \leq T. \quad (3.2)
\]
Since \( \| \hat{\xi}_t \| = 1 \), there is a partition of \( \tilde{\Omega} \) into disjoint subsets \( \Gamma_i \subseteq \mathcal{F}_t \) such that \( \Gamma_i \subseteq \{ (\xi_t)^i \neq 0 \} \) for \( i = 1, \ldots, d \). We then define 
\[
\hat{\xi}^n_s = \sum_{i=1}^d \left( \xi^n_s - \beta^n_t \hat{\xi}_t \right) \mathbbm{1}_{\Gamma_i}, \quad s \in \mathbb{T}
\]
with \( \beta^n_t = (\xi^n_t)^i / (\hat{\xi}_t)^i \) on \( \Gamma_i, i = 1, \ldots, d \). In view of (3.2) and definition of \( \xi^n \), we have 
\[
\sum_{s=t}^\tau \hat{\xi}^n_s - \vartheta^n_t \in K_\tau \mathbb{P} \text{-a.s. for all } t \leq \tau \leq T,
\]
since \( K_\tau - K^0_\tau \subseteq K_\tau, \tau \in \mathbb{T} \). We can then proceed as in \cite{6} and obtain the required result by repeating the above argument with \( (\hat{\xi}^n)_{n \geq 1} \) instead of \( (\xi^n)_{n \geq 1} \) and by iterating this procedure a finite number of times. \[\square\]
References


