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**Formulas and equations for finding scattering data
from the Dirichlet-to-Neumann map with nonzero background potential**

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Abstract

For the Schrödinger equation at fixed energy with a potential supported in a bounded domain we give formulas and equations for finding scattering data from the Dirichlet-to-Neumann map with nonzero background potential. For the case of zero background potential these results were obtained in [R.G.Novikov, Multidimensional inverse spectral problem for the equation $-\Delta\psi + (v(x) - Eu(x))\psi = 0$, Funkt. Anal. i Ego Prilozhen 22(4), pp.11-22, (1988)].

1.Introduction

Consider the Schrödinger equation

$$-\Delta\psi + v(x)\psi = E\psi, \quad x \in D, \quad (1.1)$$

where

$$D \text{ is an open bounded domain in } \mathbb{R}^d, \quad d \geq 2, \quad (1.2a)$$
$$\text{with } \partial D \in C^2,$$

$$v \in L^\infty(D). \quad (1.2b)$$

We also assume that

$$E \text{ is not a Dirichlet eigenvalue for} \quad (1.3)$$
$$\text{the operator } -\Delta + v \text{ in } D.$$

Consider the map $\Phi(E)$ such that

$$\frac{\partial\psi}{\partial\nu}\Big|_{\partial D} = \Phi(E)(\psi|_{\partial D}) \quad (1.4)$$

for all sufficiently regular solutions of (1.1) in $\bar{D} = D \cup \partial D$, for example, for all $\psi \in H^1(D)$ satisfying (1.1), where ν is the outward normal to ∂D . The map $\Phi(E)$ is called the Dirichlet-to-Neumann map for equation (1.1).

Consider the Schrödinger equation

$$-\Delta\psi + v(x)\psi = E\psi, \quad x \in \mathbb{R}^d, \quad (1.5)$$

where

$$\rho^{d+\varepsilon}v \in L^\infty(\mathbb{R}^d), \quad d \geq 2, \quad \text{for some } \varepsilon > 0, \quad (1.6)$$

where ρ denotes the multiplication operator by the function $\rho(x) = 1 + |x|$. For equation (1.5) we consider the functions ψ^+ and f of the classical scattering theory and the Faddeev functions ψ , h , ψ_γ , h_γ (see, for example, [F1], [F2], [F3], [HN], [Ne]).

The functions ψ^+ and f are defined as follows:

$$\psi^+(x, k) = e^{ikx} + \int_{\mathbb{R}^d} G^+(x - y, k)v(y)\psi^+(y, k)dy, \quad (1.7)$$

$$G^+(x, k) = -\left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} \frac{e^{i\xi x} d\xi}{\xi^2 - k^2 - i0}, \quad (1.8)$$

where $x, k \in \mathbb{R}^d$, $k^2 > 0$ (and at fixed k the formula (1.7) is an equation for ψ^+ in $L^\infty(\mathbb{R}^d)$);

$$f(k, l) = \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} e^{-ilx} v(x)\psi^+(x, k)dx, \quad (1.9)$$

where $k, l \in \mathbb{R}^d$, $k^2 > 0$. Here $\psi^+(x, k)$ satisfies (1.5) for $E = k^2$ and describes scattering of the plane waves e^{ikx} ; $f(k, l)$, $k^2 = l^2$, is the scattering amplitude for equation (1.5) for $E = k^2$. The equation (1.7) is called the Lippman-Schwinger integral equation.

The functions ψ and h are defined as follows:

$$\psi(x, k) = e^{ikx} + \int_{\mathbb{R}^d} G(x - y, k)v(y)\psi(y, k)dy, \quad (1.10)$$

$$G(x, k) = e^{ikx}g(x, k), \quad g(x, k) = -\left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} \frac{e^{i\xi x} d\xi}{\xi^2 + 2k\xi}, \quad (1.11)$$

where $x \in \mathbb{R}^d$, $k \in \mathbb{C}^d \setminus \mathbb{R}^d$ (and at fixed k the formula (1.10) is an equation for $\psi = e^{ikx}\mu(x, k)$, where μ is sought in $L^\infty(\mathbb{R}^d)$);

$$h(k, l) = \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} e^{-ilx} v(x)\psi(x, k)dx, \quad (1.12)$$

where $k, l \in \mathbb{C}^d \setminus \mathbb{R}^d$, $Im k = Im l$. Here $\psi(x, k)$ satisfies (1.5) for $E = k^2$, and ψ , G and h are (nonanalytic) continuations of ψ^+ , G^+ and f to the complex domain. In particular, $h(k, l)$ for $k^2 = l^2$ can be considered as the "scattering" amplitude in the complex domain for equation (1.5) for $E = k^2$. The functions ψ_γ and h_γ are defined as follows:

$$\psi_\gamma(x, k) = \psi(x, k + i0\gamma), \quad h_\gamma(k, l) = h(k + i0\gamma, l + i0\gamma), \quad (1.13)$$

where $x, k, l, \gamma \in \mathbb{R}^d$, $|\gamma| = 1$. Note that

$$\psi^+(x, k) = \psi_{k/|k|}(x, k), \quad f(k, l) = h_{k/|k|}(k, l), \quad (1.14)$$

where $x, k, l \in \mathbb{R}^d$, $|k| > 0$.

We consider $f(k, l)$ and $h_\gamma(k, l)$, where $k, l, \gamma \in \mathbb{R}^d$, $k^2 = l^2 = E$, $\gamma^2 = 1$, and $h(k, l)$, where $k, l \in \mathbb{C}^d \setminus \mathbb{R}^d$, $Im k = Im l$, $k^2 = l^2 = E$, as scattering data S_E for equation (1.5) at

fixed $E \in]0, +\infty[$. We consider $h(k, l)$, where $k, l \in \mathbb{C}^d \setminus \mathbb{R}^d$, $\text{Im } k = \text{Im } l$, $k^2 = l^2 = E$, as scattering data S_E for equation (1.5) at fixed $E \in \mathbb{C} \setminus]0, +\infty[$.

Let D be a fixed domain satisfying (1.2a). Let

$$v \in L^\infty(D) \quad \text{and} \quad v \equiv 0 \quad \text{on} \quad \mathbb{R}^d \setminus \bar{D}. \quad (1.15)$$

For v of (1.15) we consider the Dirichlet-to-Neumann map $\Phi(E)$ for equation (1.1) and the scattering data S_E for equation (1.5).

In the present work we continue studies of [No1] on the following inverse boundary value problem for equation (1.1):

Problem 1. Find v (in (1.1)) from $\Phi(E)$ (where E is fixed or belongs to some set).

More precisely, we develop formulas and equations of [No1] which reduce Problem 1 to the following inverse scattering problem for equation (1.5):

Problem 2. Find v (in (1.5)) from $S(E)$ (where E is fixed or belongs to some set).

Concerning results given in the literature on Problem 1, see [SU], [No1], [A], [NSU], [Na1], [Na2], [M] and references therein. Concerning results given in the literature on Problem 2, see [BC], [HN], [No2], [No3], [IS], [GN], [No4], [No5], [E], [Ch], [BBMRS], [BMR] and references therein.

The main results of the present work consist of Theorem 1 and Propositions 1 and 2 of Section 2. In these results we consider for fixed E two potentials v_0 and v satisfying (1.15) and (1.3). In Theorem 1 and Proposition 1 we give formulas and equations for finding S_E from $\Phi(E) - \Phi_0(E)$ and from (some functions found from) v_0 , where S_E and $\Phi(E)$ correspond to v and $\Phi_0(E)$ corresponds to v_0 . In Proposition 2 we give a result on the solvability of equations of Theorem 1.

For the case when $v_0 \equiv 0$, Theorem 1 and Propositions 1 and 2 were obtained for the first time in [No1] (see also [HN], [Na1], [Na2]), where using these results Problem 1 was reduced to Problem 2. For the case when the best known approximation v_0 to v of Problem 1 is not identically zero (and especially when v_0 is really close to v), results of the present work reduce Problem 1 to Problem 2 in a more stable way than it was done in [No1].

Note that generalizing results of [No1] to the case of nonzero background potential v_0 we used essentially results of [No4]. The main results of the present work are presented in detail in Section 2.

Note, finally, that by the present work we start a development of new exact reconstruction algorithms for Problem 1.

2. Main results

To formulate our results we need to introduce some additional notations. Consider (under the assumption (1.6)) the sets \mathcal{E} , \mathcal{E}_γ , \mathcal{E}^+ defined as follows:

$$\begin{aligned} \mathcal{E} = \{ \zeta \in \mathbb{C}^d \setminus \mathbb{R}^d : \text{equation (1.10) for } k = \zeta \text{ is not} \\ \text{uniquely solvable for } \psi = e^{ikx} \mu \text{ with } \mu \in L^\infty(\mathbb{R}^d) \}, \end{aligned} \quad (2.1a)$$

$$\begin{aligned} \mathcal{E}_\gamma = \{ \zeta \in \mathbb{R}^d \setminus 0 : \text{equation (1.10) for } k = \zeta + i0\gamma \text{ is not} \\ \text{uniquely solvable for } \psi \in L^\infty(\mathbb{R}^d) \}, \quad \gamma \in \mathbb{S}^{d-1}, \end{aligned} \quad (2.1b)$$

$$\mathcal{E}^+ = \{\zeta \in \mathbb{R}^d \setminus 0 : \text{equation (1.7) for } k = \zeta \text{ is not uniquely solvable for } \psi^+ \in L^\infty(\mathbb{R}^d)\}. \quad (2.1c)$$

Note that \mathcal{E}^+ is a well-known set of the classical scattering theory for equation (1.5) and that $\mathcal{E}^+ = \emptyset$ for real-valued v satisfying (1.6) (see, for example, [Ne]). The sets \mathcal{E} and \mathcal{E}_γ were considered for the first time in [F1], [F2], [F3]. Concerning the properties of \mathcal{E} and \mathcal{E}_γ , see [F3], [HN], [LN], [Ne], [We], [Na2], [No4], [No6].

Consider (under the assumptions (1.6)) the functions R , R_γ , R^+ defined as follows:

$$R(x, y, k) = G(x - y, k) + \int_{\mathbb{R}^d} G(x - z, k)v(z)R(z, y, k)dz, \quad (2.2)$$

where $x, y \in \mathbb{R}^d$, $k \in \mathbb{C}^d \setminus \mathbb{R}^d$, G is defined by (1.11), and at fixed y and k the formula (2.2) is an equation for

$$R(x, y, k) = e^{ik(x-y)}r(x, y, k), \quad (2.3)$$

where r is sought with the properties

$$r(\cdot, y, k) \text{ is continuous on } \mathbb{R}^d \setminus y, \quad (2.4a)$$

$$r(x, y, k) \rightarrow 0 \text{ as } |x| \rightarrow \infty, \quad (2.4b)$$

$$r(x, y, k) = O(|x - y|^{2-d}) \text{ as } x \rightarrow y \text{ for } d \geq 3, \quad (2.4c)$$

$$r(x, y, k) = O(|\ln |x - y||) \text{ as } x \rightarrow y \text{ for } d = 2;$$

$$R_\gamma(x, y, k) = R(x, y, k + i0\gamma), \quad (2.5)$$

where $x, y \in \mathbb{R}^d$, $k \in \mathbb{R}^d \setminus 0$, $\gamma \in \mathbb{S}^{d-1}$;

$$R^+(x, y, k) = R_{k/|k|}(x, y, k), \quad (2.6)$$

where $x, y \in \mathbb{R}^d$, $k \in \mathbb{R}^d \setminus 0$. Note that $R(x, y, k)$, $R_\gamma(x, y, k)$ and $R^+(x, y, k)$ (for their domains of definition in k and γ) satisfy the equation

$$(\Delta + E - v(x))R(x, y, k) = \delta(x - y), \quad x \in \mathbb{R}^d, \quad y \in \mathbb{R}^d, \quad E = k^2. \quad (2.7)$$

The function $R^+(x, y, k)$ (defined by means of (2.2) for $k \in \mathbb{R}^d \setminus 0$ with G replaced by G^+ of (1.8)) is well-known in the scattering theory for equations (1.5), (2.7). In particular, this function describes scattering of the spherical waves $G^+(x - y, k)$ generated by a source at y . Apparently, the functions R and R_γ were considered for the first time in [No5].

Note that under the assumption (1.6): equation (2.2) at fixed y and k is uniquely solvable for R with the properties (2.3), (2.4) if and only if $k \in \mathbb{C}^d \setminus (\mathbb{R}^d \cup \mathcal{E})$; equation (2.2) with $k = \zeta + i0\gamma$, $\zeta \in \mathbb{R}^d \setminus 0$, $\gamma \in \mathbb{S}^{d-1}$, at fixed y , ζ and γ is uniquely solvable for R_γ if and only if $\zeta \in \mathbb{R}^d \setminus (0 \cup \mathcal{E}_\gamma)$; equation (2.2) with $k = \zeta + i0\zeta/|\zeta|$, $\zeta \in \mathbb{R}^d \setminus 0$, at fixed y and ζ is uniquely solvable for R^+ if and only if $\zeta \in \mathbb{R}^d \setminus (0 \cup \mathcal{E}^+)$.

For v of (1.15) we consider the map $\Phi(E)$ defined by means of (1.4), the functions ψ^+ , f , ψ , h , ψ_γ , h_γ and R^+ , R , R_γ defined by means of (1.7)-(1.13) and (2.2)-(2.6) and

the sets \mathcal{E} , \mathcal{E}_γ , \mathcal{E}^+ defined by (2.1). The Schwartz kernel of the integral operator $\Phi(E)$ will be denoted by $\Phi(x, y, E)$, where $x, y \in \partial D$.

Theorem 1. *Let D satisfying (1.2a) and E be fixed. Let v^0 and v be two potentials satisfying (1.15), (1.3). Let Φ , ψ^+ , f , ψ , h , ψ_γ , h_γ , R^+ , R , R_γ , \mathcal{E} , \mathcal{E}_γ , \mathcal{E}^+ correspond to v (as defined above) and Φ^0 , $\psi^{+,0}$, f^0 , ψ^0 , h^0 , ψ_γ^0 , h_γ^0 , $R^{+,0}$, R^0 , R_γ^0 , \mathcal{E}^0 , \mathcal{E}_γ^0 , $\mathcal{E}^{+,0}$ correspond to v^0 (as defined above with $v = v^0$). Then the following formulas hold:*

$$h(k, l) = h^0(k, l) + \left(\frac{1}{2\pi}\right)^d \int_{\partial D} \int_{\partial D} \psi^0(x, -l)(\Phi - \Phi^0)(x, y, E)\psi(y, k)dydx \quad (2.8)$$

for $k, l \in \mathbb{C}^d \setminus (\mathbb{R}^d \cup \mathcal{E}^0 \cup \mathcal{E})$, $\text{Im } k = \text{Im } l$, $k^2 = l^2 = E$,

$$\psi(x, k) = \psi^0(x, k) + \int_{\partial D} A(x, y, k)\psi(y, k)dy, \quad x \in \partial D, \quad (2.9a)$$

$$A(x, y, k) = \int_{\partial D} R^0(x, z, k)(\Phi - \Phi^0)(z, y, E)dz, \quad x, y \in \partial D, \quad (2.9b)$$

for $k \in \mathbb{C}^d \setminus (\mathbb{R}^d \cup \mathcal{E}^0 \cup \mathcal{E})$, $k^2 = E$;

$$h_\gamma(k, l) = h_\gamma^0(k, l) + \left(\frac{1}{2\pi}\right)^d \int_{\partial D} \int_{\partial D} \psi_{-\gamma}^0(x, -l)(\Phi - \Phi^0)(x, y, E)\psi_\gamma(y, k)dydx \quad (2.10)$$

for $k, l \in \mathbb{R}^d \setminus (0 \cup \mathcal{E}_\gamma^0 \cup \mathcal{E}_\gamma)$, $\gamma \in \mathbb{S}^{d-1}$, $k^2 = l^2 = E$, $k\gamma = l\gamma$,

$$\psi_\gamma(x, k) = \psi_\gamma^0(x, k) + \int_{\partial D} A_\gamma(x, y, k)\psi_\gamma(y, k)dy, \quad x \in \partial D, \quad (2.11a)$$

$$A_\gamma(x, y, k) = \int_{\partial D} R_\gamma^0(x, z, k)(\Phi - \Phi^0)(z, y, E)dz, \quad x, y \in \partial D, \quad (2.11b)$$

for $k \in \mathbb{R}^d \setminus (0 \cup \mathcal{E}_\gamma^0 \cup \mathcal{E}_\gamma)$, $\gamma \in \mathbb{S}^{d-1}$, $k^2 = E$;

$$f(k, l) = f^0(k, l) + \left(\frac{1}{2\pi}\right)^d \int_{\partial D} \int_{\partial D} \psi^{+,0}(x, -l)(\Phi - \Phi^0)(x, y, E)\psi^+(y, k)dydx \quad (2.12)$$

for $k, l \in \mathbb{R}^d \setminus (0 \cup \mathcal{E}^{+,0} \cup \mathcal{E}^+)$, $k^2 = l^2 = E$,

$$\psi^+(x, k) = \psi^{+,0}(x, k) + \int_{\partial D} A^+(x, y, k)\psi^+(y, k)dy, \quad x \in \partial D, \quad (2.13a)$$

$$A^+(x, y, k) = \int_{\partial D} R^+(x, z, k)(\Phi - \Phi^0)(z, y, E)dz, \quad x, y \in \partial D, \quad (2.13b)$$

for $k \in \mathbb{R}^d \setminus (0 \cup \mathcal{E}^{+,0} \cup \mathcal{E}^+)$, $k^2 = E$.

Note that in Theorem 1 dx and dy denote the standard measure on ∂D in \mathbb{R}^d .

Note that in the formula (2.10) for $h_\gamma(k, l)$ there is the additional restriction: $k\gamma = l\gamma$. To extend (2.10) to the general case, consider $\psi_\gamma(x, k, l)$ defined as follows:

$$\psi_\gamma(x, k, l) = e^{ilx} + \int_{\mathbb{R}^d} G_\gamma(x - y, k)v(y)\psi_\gamma(y, k, l)dy, \quad (2.14a)$$

$$G_\gamma(x, k) = G(x, k + i0\gamma), \quad (2.14b)$$

where $x, k, l \in \mathbb{R}^d$, $k^2 = l^2 > 0$, $\gamma \in \mathbb{S}^{d-1}$ and (2.14) at fixed γ , k, l is an equation for $\psi_\gamma(\cdot, k, l)$ in $L^\infty(\mathbb{R}^d)$.

Proposition 1. *Let the assumptions of Theorem 1 be valid. In addition, let $\psi_\gamma(x, k, l)$ correspond to v and $\psi_\gamma^0(x, k, l)$ correspond to v^0 . Then*

$$h_\gamma(k, l) = h_\gamma^0(k, l) + \left(\frac{1}{2\pi}\right)^d \int_{\partial D} \int_{\partial D} \psi_{-\gamma}^0(x, -k, -l)(\Phi - \Phi^0)(x, y, E)\psi_\gamma(y, k)dydx \quad (2.15)$$

for $\gamma \in \mathbb{S}^{d-1}$, $k \in \mathbb{R}^d \setminus (0 \cup \mathcal{E}_\gamma^0 \cup \mathcal{E}_\gamma)$, $l \in \mathbb{R}^d$, $k^2 = l^2 = E$.

Note that (see [F3], [No4])

$$G_\gamma(x, k) = G_\gamma(x, l) \text{ for } x, k, l \in \mathbb{R}^d, \quad \gamma \in \mathbb{S}^{d-1}, \quad k^2 = l^2 > 0, \quad k\gamma = l\gamma. \quad (2.16)$$

Therefore,

$$\psi_\gamma(x, k, l) = \psi_\gamma(x, l, l) = \psi_\gamma(x, l) \text{ for } x, k, l \in \mathbb{R}^d, \quad \gamma \in \mathbb{S}^{d-1}, \quad k^2 = l^2 > 0, \quad k\gamma = l\gamma. \quad (2.17)$$

Therefore, (2.15), under the additional restriction $k\gamma = l\gamma$ is reduced to (2.10).

Suppose that v is unknown, but v^0 and $\Phi(E) - \Phi^0(E)$ are known (v^0 is considered as the best known approximation to v). Then Theorem 1 and Proposition 1 (and equations and formulas (1.7)-(1.13), (2.2)-(2.6), (2.14) for finding $\psi^{+,0}$, f^0 , ψ^0 , h^0 , ψ_γ^0 , h_γ^0 , $R^{+,0}$, R^0 , R_γ^0 from v^0) give a method for finding the scattering data S_E (defined in the introduction) for v from the background potential v^0 and the difference $\Phi(E) - \Phi^0(E)$. In addition, (2.9a), (2.11a), (2.13a) at fixed k and γ are linear integral equations for finding ψ , ψ_γ , ψ^+ on ∂D from ψ^0 , ψ_γ^0 , $\psi^{+,0}$ on ∂D and A , A_γ , A^+ on $\partial D \times \partial D$ (where A , A_γ , A^+ are given by (2.9b), (2.11b), (2.13b)).

Proposition 2. *Under the assumptions of Theorem 1, equation (2.9a) at fixed $k \in \mathbb{C}^d \setminus (\mathbb{R}^d \cup \mathcal{E}^0)$, $k^2 = E$, equation (2.11a) at fixed $\gamma \in \mathbb{S}^{d-1}$ and $k \in \mathbb{R}^d \setminus (0 \cup \mathcal{E}_\gamma^0)$, $k^2 = E$, and equation (2.13a) at fixed $k \in \mathbb{R}^d \setminus (0 \cup \mathcal{E}^{+,0})$ are Fredholm linear integral equations of the second kind for ψ , ψ_γ and ψ^+ (respectively) in $L^\infty(\partial D)$ and are uniquely solvable (in this space) if and only if $k \notin \mathcal{E}$ for (2.9a), $k \notin \mathcal{E}_\gamma$ for (2.11a) and $k \notin \mathcal{E}^+$ for (2.13a).*

Note that

$$\psi(x, k) = e^{ikx} \mu(x, k), \quad \psi^0(x, k) = e^{ikx} \mu^0(x, k), \quad x \in \mathbb{R}^d, \quad k \in \mathbb{C}^d \setminus \mathbb{R}^d, \quad (2.18)$$

where e^{ikx} is an exponentially increasing factor and

$$\mu(x, k) \rightarrow 1, \quad \mu^0(x, k) \rightarrow 1 \text{ as } |k| \rightarrow \infty \quad (2.19)$$

for $k^2 = E$ at fixed E , where $|k| = \sqrt{(\operatorname{Re} k)^2 + (\operatorname{Im} k)^2}$. Therefore, it is convenient to write equation (2.9) of Theorem 1 as follows:

$$\mu(x, k) = \mu^0(x, k) + \int_{\partial D} B(x, y, k) \mu(y, k) dy, \quad x \in \partial D, \quad (2.20a)$$

$$B(x, y, k) = \int_{\partial D} r^0(x, z, k) e^{-ikz} (\Phi - \Phi^0)(z, y, E) dz e^{iky}, \quad x, y \in \partial D, \quad (2.20b)$$

for $k \in \mathbb{C}^d \setminus (\mathbb{R}^d \cup \mathcal{E}^0 \cup \mathcal{E})$, where r^0 and R^0 are related by (2.3).

Theorem 1 and Proposition 1 reduce Problem 1 to Problem 2 (these problems are formulated in the introduction).

For the case when $v^0 \equiv 0$, Theorem 1 and Propositions 1 and 2 were obtained in [No1] (see also [Na1], [Na2]). Note that the basic results of [No1], in particular formula (2.8) and equation (2.9) for $v^0 \equiv 0$ and $d = 3$, were presented already in the survey given in [HN].

For the case when the best known approximation v^0 to v is not identically zero (and especially when v^0 is really close to v) Theorem 1 and Proposition 1 give a more convenient (in particular, for the stability analysis) method for reducing Problem 1 to Problem 2 than in [No1]. To explain this more precisely, consider, in particular, the integral operators $B(k)$, $A_\gamma(k)$, $A^+(k)$ (with the Schwartz kernels $B(x, y, k)$, $A_\gamma(x, y, k)$, $A^+(x, y, k)$) of equations (2.20a), (2.11a), (2.13a). To have a simple and stable (with respect to small errors in $\Phi(E)$) method for solving equations (2.20a) (for fixed $k \in \mathbb{C}^d \setminus (\mathbb{R}^d \cup \mathcal{E}^0)$, $k^2 = E$), (2.11a) (for fixed $\gamma \in \mathbb{S}^{d-1}$ and $k \in \mathbb{R}^d \setminus (0 \cup \mathcal{E}_\gamma^0)$, $k^2 = E$) and (2.13a) (for fixed $k \in \mathbb{R}^d \setminus (0 \cup \mathcal{E}^{+,0})$, $k^2 = E$) it is important to have that

$$\|B(k)\| < \eta, \quad \|A_\gamma(k)\| < \eta, \quad \|A^+(k)\| < \eta, \quad (2.21)$$

respectively, for some $\eta < 1$, where $\|A\|$ is the norm of an operator A (for example) in $L^\infty(\partial D)$. In this case equations (2.20a), (2.11a), (2.13a) are uniquely solvable by the method of successive approximations. In addition, if $\eta \ll 1$, then (2.20a), (2.11a), (2.13a) can be solved in the first approximation as

$$\mu(x, k) \approx \mu^0(x, k), \quad \psi_\gamma(x, k) \approx \psi_\gamma^0(x, k), \quad \psi^+(x, k) \approx \psi^{+,0}(x, k) \quad (2.22)$$

and h , h_γ , f can be determined in the first (nontrivial) approximation as

$$h(k, l) \approx h^0(k, l) + \left(\frac{1}{2\pi}\right)^d \int_{\partial D} \int_{\partial D} \psi^0(x, -l) (\Phi - \Phi^0)(x, y, E) \psi^0(y, k) dy dx \quad (2.23a)$$

for $l \in \mathbb{C}^d \setminus \mathbb{R}^d$, $\operatorname{Im} l = \operatorname{Im} k$, $l^2 = k^2 = E$ (and for k of $B(k)$ of (2.21)),

$$h_\gamma(k, l) \approx h_\gamma^0(k, l) + \left(\frac{1}{2\pi}\right)^d \int_{\partial D} \int_{\partial D} \psi_{-\gamma}^0(x, -k, -l) (\Phi - \Phi^0)(x, y, E) \psi_\gamma^0(y, k) dy dx \quad (2.23b)$$

for $l \in \mathbb{R}^d \setminus \mathcal{E}_\gamma^0$, $l^2 = k^2 = E$ (and for k , γ of $A_\gamma(k)$ of (2.21)),

$$f(k, l) \approx f^0(k, l) + \left(\frac{1}{2\pi}\right)^d \int_{\partial D} \int_{\partial D} \psi^{+,0}(x, -l) (\Phi - \Phi^0)(x, y, E) \psi^{+,0}(y, k) dy dx \quad (2.23c)$$

for $l \in \mathbb{R}^d$, $l^2 = k^2 = E$ (and for k of $A^+(k)$ of (2.21)).

Note that the direct problem of finding $\psi^{+,0}$, f^0 , ψ^0 , h^0 , ψ_γ^0 , h_γ^0 , $R^{+,0}$, R^0 , R_γ^0 (involved into (2.8)-(2.13), (2.15), (2.20)) from v^0 is (relatively) well understood (in comparison with the problem of solving (2.20a), (2.11a), (2.13a) without the assumptions (2.21)) and is sufficiently stable.

Note that in (2.20a), apparently, unfortunately, almost always

$$\|B(k)\| \rightarrow \infty \quad (\text{exponentially fast}) \quad \text{as } |k| \rightarrow \infty \quad (2.24)$$

for $k \in \mathbb{C}^d$, $k^2 = E$ at fixed E , in spite of (2.19).

To have (2.21) for $B(k)$ for maximally large domain in $k \in \mathbb{C}^d \setminus \mathbb{R}^d$, $k^2 = E$, and when $E > 0$ for $A_\gamma(k)$ for maximally large domain in $\gamma \in \mathbb{S}^{d-1}$ and $k \in \mathbb{R}^d$, $k^2 = E$, and for $A^+(k)$ for $k \in \mathbb{R}^d$, $k^2 = E$, it is important to have that $\|\Phi(E) - \Phi^0(E)\|$ is as small as possible. The smallness of $\|\Phi(E) - \Phi^0(E)\|$ follows from the closeness of v^0 to v (for example) in $L^\infty(D)$ for fixed D , v and E , under the conditions (1.2), (1.3).

As soon as Problem 1 is reduced to Problem 2, one can use for solving Problem 1 methods of [HN], [No2], [Na2], [IS], [No4], [No5], [E], [BBMRS], [BMR] (and further references given therein).

3. Proofs of Theorem 1 and Propositions 1 and 2

For the case when $v^0 \equiv 0$, Theorem 1 and Propositions 1 and 2 were proved in [No1]. In this section we generalize these proofs of [No1] to the case of nonzero background potential v^0 . To this end we use, in particular, some results of [A] and [No4].

Proof of Theorem 1. We proceed from the following formulas and equations (being valid under the assumption (1.6) on v^0 and v):

$$h(k, l) = h^0(k, l) + \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} \psi^0(x, -l)(v(x) - v^0(x))\psi(x, k)dx \quad (3.1)$$

for $k, l \in \mathbb{C}^d \setminus (\mathbb{R}^d \cup \mathcal{E}^0 \cup \mathcal{E})$, $Im k = Im l$, $k^2 = l^2$,

$$\psi(x, k) = \psi^0(x, k) + \int_{\mathbb{R}^d} R^0(x, y, k)(v(y) - v^0(y))\psi(y, k)dy, \quad (3.2)$$

where $x \in \mathbb{R}^d$, $k \in \mathbb{C}^d \setminus (\mathbb{R}^d \cup \mathcal{E}^0)$ (and (3.2) at fixed k is an equation for $\psi = e^{ikx}\mu(x, k)$, where μ is sought in $L^\infty(\mathbb{R}^d)$),

$$h_\gamma(k, l) = h_\gamma^0(k, l) + \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} \psi_{-\gamma}^0(x, -l)(v(x) - v^0(x))\psi_\gamma(x, k)dx \quad (3.3)$$

for $\gamma \in \mathbb{S}^{d-1}$, $k, l \in \mathbb{R}^d \setminus (0 \cup \mathcal{E}_\gamma^0 \cup \mathcal{E}_\gamma)$, $k^2 = l^2$, $k\gamma = l\gamma$.

$$\psi_\gamma(x, k) = \psi_\gamma^0(x, k) + \int_{\mathbb{R}^d} R_\gamma^0(x, y, k)(v(y) - v^0(y))\psi_\gamma(y, k)dy, \quad (3.4)$$

where $x \in \mathbb{R}^d$, $\gamma \in \mathbb{S}^{d-1}$, $k \in \mathbb{R}^d \setminus (0 \cup \mathcal{E}_\gamma^0)$ (and (3.4) at fixed γ and k is an equation for ψ_γ in $L^\infty(\mathbb{R}^d)$),

$$f(k, l) = f^0(k, l) + \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} \psi^{+,0}(x, -l)(v(x) - v^0(x))\psi^+(x, k)dx \quad (3.5)$$

for $k, l \in \mathbb{R}^d \setminus (0 \cup \mathcal{E}^{+,0} \cup \mathcal{E}^+)$, $k^2 = l^2$,

$$\psi^+(x, k) = \psi^{+,0}(x, k) + \int_{\mathbb{R}^d} R^{+,0}(x, y, k)(v(y) - v^0(y))\psi^+(y, k)dy, \quad (3.6)$$

where $x \in \mathbb{R}^d$, $k \in \mathbb{R}^d \setminus (0 \cup \mathcal{E}^{+,0})$ (and (3.6) at fixed k is an equation for ψ^+ in $L^\infty(\mathbb{R}^d)$). (We remind that ψ^+ , f , ψ , h , ψ_γ , h_γ were defined in the introduction by means of (1.7)-(1.13).) Equation (3.6) is well-known in the classical scattering theory for the Schrödinger equation (1.5). Formula (3.5) was given, in particular, in [St]. To our knowledge formulas and equations (3.1)-(3.4) were given for the first time in [No4].

Note that, under the assumption (1.6):

$$\begin{aligned} (3.2) \text{ at fixed } k \in \mathbb{C}^d \setminus (\mathbb{R}^d \cup \mathcal{E}^0) \text{ is uniquely solvable} \\ \text{for } \psi = e^{ikx}\mu(x, k) \text{ with } \mu \in L^\infty(\mathbb{R}^d) \text{ if and only if } k \notin \mathcal{E}; \\ (3.4) \text{ at fixed } \gamma \in \mathbb{S}^{d-1}, k \in \mathbb{R}^d \setminus (0 \cup \mathcal{E}_\gamma^0) \text{ is uniquely solvable} \\ \text{for } \psi_\gamma \in L^\infty(\mathbb{R}^d) \text{ if and only if } k \notin \mathcal{E}_\gamma; \\ (3.6) \text{ at fixed } k \in \mathbb{R}^d \setminus (0 \cup \mathcal{E}^{+,0}) \text{ is uniquely solvable} \\ \text{for } \psi^+ \in L^\infty(\mathbb{R}^d) \text{ if and only if } k \notin \mathcal{E}^+. \end{aligned} \quad (3.7)$$

In a similar way with [No1], (under the assumptions of Theorem 1) formulas and equations (3.1)-(3.6) can be transformed into (2.8)-(2.13) by means of the following Green's formula

$$\begin{aligned} \int_D (u_1(x)\Delta u_2(x) - u_2(x)\Delta u_1(x))dx = \\ \int_{\partial D} \left(u_1(x) \frac{\partial u_2(x)}{\partial \nu} - u_2(x) \frac{\partial u_1(x)}{\partial \nu} \right) dx, \end{aligned} \quad (3.8)$$

(where dx in the right-hand side of (3.8) denotes the standard measure on ∂D in \mathbb{R}^d). (To start these transformations, we use that $\psi(x, k)$, $\psi_\gamma(x, k)$ and $\psi^+(x, k)$ satisfy (1.5) for $E = k^2$ and replace $v\psi$ in (3.1), (3.2) by $(\Delta + E)\psi$, $v\psi_\gamma$ in (3.3), (3.4) by $(\Delta + E)\psi_\gamma$ and $v\psi^+$ in (3.5), (3.6) by $(\Delta + E)\psi^+$.) However, these calculations can be shortened by means of the following Alessandrini identity (being valid under the assumptions of Theorem 1):

$$\int_D (v(x) - v^0(x))\psi(x)\psi^0(x)dx = \int_{\partial D} \int_{\partial D} \psi^0(x)(\Phi - \Phi^0)(x, y, E)\psi(y)dydx \quad (3.9)$$

for any ψ and ψ^0 such that ψ satisfies (1.1), ψ^0 satisfies (1.1) with v replaced by v^0 and where ψ and ψ^0 are sufficiently regular in D , for example, $\psi, \psi^0 \in H^1(D)$. In a slightly

different from the identity (3.9) was given in Lemma 1 of [A]. The proof of (3.9) is based on (3.8).

Formula (2.8) follows from (3.1), (3.9) and the fact that $\psi(x, k)$ satisfies (1.5) for $E = k^2$ and $\psi^0(x, -l)$ satisfies (1.5) with v replaced by v^0 , for $E = l^2$. Formulas (2.9) with $x \in \mathbb{R}^d \setminus \bar{D}$ follow from (3.2) with $x \in \mathbb{R}^d \setminus \bar{D}$, (3.9) and the fact that $\psi(y, k)$ satisfies (1.5) in y for $E = k^2$ and that $R^0(x, y, k)$ for $x \in \mathbb{R}^d \setminus \bar{D}$ satisfies (1.5) in y in an open neighborhood of \bar{D} , with v replaced by v^0 , for $E = k^2$. The latter statement about R^0 follows from (2.7) and the symmetry

$$R(x, y, k) = R(y, x, -k), \quad (3.10)$$

where $x, y \in \mathbb{R}^d$, $k \in \mathbb{C}^d \setminus \mathbb{R}^d$ (and R is defined by means of (2.2)-(2.4)). The symmetry (3.10) was found in [No4]. Finally, formulas (2.9) for $x \in \partial D$ arise as a limit of (2.9) with $x \in \mathbb{R}^d \setminus \bar{D}$.

The proof of (2.10)-(2.13) is similar to the proof of (2.8), (2.9).

Theorem 1 is proved.

Proof of Proposition 1. In this proof we obtain and use the following formula (being valid under the assumption (1.6) on v^0 and v):

$$h_\gamma(k, l) = h_\gamma^0(k, l) + \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} \psi_{-\gamma}^0(x, -k, -l)(v(x) - v^0(x))\psi_\gamma(x, k)dx \quad (3.11)$$

for $\gamma \in \mathbb{S}^{d-1}$, $k \in \mathbb{R}^d \setminus (0 \cup \mathcal{E}_\gamma^0 \cup \mathcal{E}_\gamma)$, $l \in \mathbb{R}^d$, $k^2 = l^2$. Formula (2.15) follows from (3.11), (3.9) and the fact that $\psi_\gamma(x, k)$ satisfies (1.5) for $E = k^2$ and $\psi_{-\gamma}^0(x, -k, -l)$ satisfies (1.5) with v replaced by v^0 , for $E = k^2 = l^2$. Thus, to prove Proposition 1, it remains to prove (3.11).

To prove (3.11) we use, in particular, that

$$\psi_\gamma(x, k, l) = \psi_\gamma^0(x, k, l) + \int_{\mathbb{R}^d} R_\gamma^0(x, y, k)(v(y) - v^0(y))\psi_\gamma(y, k, l)dy \quad (3.12)$$

for $x \in \mathbb{R}^d$, $\gamma \in \mathbb{S}^{d-1}$, $k \in \mathbb{R}^d \setminus (0 \cup \mathcal{E}_\gamma^0 \cup \mathcal{E}_\gamma)$, $l \in \mathbb{R}^d$, $k^2 = l^2$. To obtain (3.12) we write (2.14a) as

$$\begin{aligned} \psi_\gamma(x, k, l) - \int_{\mathbb{R}^d} G_\gamma(x - y, k)v^0(y)\psi_\gamma(y, k, l)dy - e^{ilx} = \\ \int_{\mathbb{R}^d} G_\gamma(x - y, k)(v(y) - v^0(y))\psi_\gamma(y, k, l)dy, \end{aligned} \quad (3.13)$$

where $x \in \mathbb{R}^d$, $\gamma \in \mathbb{S}^{d-1}$, $k \in \mathbb{R}^d \setminus (0 \cup \mathcal{E}_\gamma)$, $l \in \mathbb{R}^d$, $k^2 = l^2$. Replacing e^{ilx} in (3.13) by its expression from (2.14a) with v and ψ_γ replaced by v^0 and ψ_γ^0 we have

$$\begin{aligned} \psi_\gamma(x, k, l) - \psi_\gamma^0(x, k, l) - \int_{\mathbb{R}^d} G_\gamma(x - y, k)v^0(y)(\psi_\gamma(y, k, l) - \psi_\gamma^0(y, k, l))dy = \\ \int_{\mathbb{R}^d} G_\gamma(x - y, k)(v(y) - v^0(y))\psi_\gamma(y, k, l)dy, \end{aligned} \quad (3.14)$$

where $x \in \mathbb{R}^d$, $\gamma \in \mathbb{S}^{d-1}$, $k \in \mathbb{R}^d \setminus (0 \cup \mathcal{E}_\gamma^0 \cup \mathcal{E}_\gamma)$, $l \in \mathbb{R}^d$, $k^2 = l^2$.

Comparing (3.14) (as an equation for $\psi_\gamma - \psi_\gamma^0$) with the following equation (arising from (2.2), (2.5), (2.14b)) for $R_\gamma^0(x, y, k)$:

$$R_\gamma^0(x, y, k) - \int_{\mathbb{R}^d} G_\gamma(x - z, k) v^0(z) R_\gamma^0(z, y, k) dz = G_\gamma(x - y, k), \quad (3.15)$$

$x, y \in \mathbb{R}^d$, $\gamma \in \mathbb{S}^{d-1}$, $k \in \mathbb{R}^d \setminus (0 \cup \mathcal{E}_\gamma^0 \cup \mathcal{E}_\gamma)$, we obtain (3.12).

Note that for $v \equiv 0$ equation (3.12) with k, γ and l replaced by $-k, -\gamma$ and $-l$ takes the form

$$e^{-ilx} = \psi_{-\gamma}^0(x, -k, -l) - \int_{\mathbb{R}^d} R_{-\gamma}^0(x, y, -k) v^0(y) e^{-ily} dy, \quad (3.16)$$

$x \in \mathbb{R}^d$, $\gamma \in \mathbb{S}^{d-1}$, $k \in \mathbb{R}^d \setminus (0 \cup \mathcal{E}_\gamma^0)$. To prove (3.11) we will use also the following symmetry (following from (3.10))

$$R_\gamma(x, y, k) = R_{-\gamma}(y, x, -k), \quad x, y \in \mathbb{R}^d, \quad \gamma \in \mathbb{S}^{d-1}, \quad k \in \mathbb{R}^d \setminus 0. \quad (3.17)$$

The following sequences of equalities proves (3.11):

$$\begin{aligned} (2\pi)^d h_\gamma(k, l) &= \int_{\mathbb{R}^d} e^{-ilx} v(x) \psi_\gamma(x, k) dx = \int_{\mathbb{R}^d} e^{-ilx} v^0(x) \psi_\gamma(x, k) dx + \\ &\int_{\mathbb{R}^d} e^{-ilx} (v(x) - v^0(x)) \psi_\gamma(x, k) dx \stackrel{(3.16)}{=} \int_{\mathbb{R}^d} e^{-ilx} v^0(x) \psi_\gamma(x, k) dx + \\ &\int_{\mathbb{R}^d} \psi_{-\gamma}^0(x, -k, -l) (v(x) - v^0(x)) \psi_\gamma(x, k) dx - \\ &\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} R_{-\gamma}^0(x, y, -k) v^0(y) e^{-ily} (v(x) - v^0(x)) \psi_\gamma(x, k) dy dx \stackrel{(3.17)}{=} \\ &\int_{\mathbb{R}^d} e^{-ilx} v^0(x) \psi_\gamma(x, k) dx + \int_{\mathbb{R}^d} \psi_{-\gamma}^0(x, -k, -l) (v(x) - v^0(x)) \psi_\gamma(x, k) dx - \\ &\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} R_\gamma^0(y, x, k) v^0(y) e^{-ily} (v(x) - v^0(x)) \psi_\gamma(x, k) dx dy \stackrel{(3.4)}{=} \\ &\int_{\mathbb{R}^d} e^{-ilx} v^0(x) \psi_\gamma(x, k) dx + \int_{\mathbb{R}^d} \psi_{-\gamma}^0(x, -k, -l) (v(x) - v^0(x)) \psi_\gamma(x, k) dx + \\ &\int_{\mathbb{R}^d} e^{-ily} v^0(y) (\psi_\gamma^0(y, k) - \psi_\gamma(y, k)) dy = \\ &(2\pi)^d h_\gamma^0(k, l) + \int_{\mathbb{R}^d} \psi_{-\gamma}^0(x, -k, -l) (v(x) - v^0(x)) \psi_\gamma(x, k) dx \end{aligned}$$

for $\gamma \in \mathbb{S}^{d-1}$, $k \in \mathbb{R}^d \setminus (0 \cup \mathcal{E}_\gamma^0 \cup \mathcal{E}_\gamma)$, $l \in \mathbb{R}^d$, $k^2 = l^2$.

Proposition 1 is proved.

Proof of Proposition 2. Let us prove Proposition 2 for the case of equation (2.9a). Consider the operator $A(k)$ of (2.9a):

$$A(k) = R^0(k) (\Phi(E) - \Phi^0(E)), \quad (3.18)$$

where $k \in \mathbb{C}^d \setminus (\mathbb{R}^d \cup \mathcal{E}^0)$, $k^2 = E$, and where the operator $R^0(k)$ is defined by

$$R^0(k)\varphi(x) = \int_{\partial D} R^0(x, y, k)\varphi(y)dy, \quad x \in \partial D, \quad (3.19)$$

where φ is a test function.

Under the assumptions of Theorem 1, the operator $\Phi(E) - \Phi^0(E)$ is compact in $L^\infty(\partial D)$. This follows from the following properties of the Schwartz kernel of $\Phi(E) - \Phi^0(E)$:

$$(\Phi - \Phi^0)(x, y, E) \text{ is continuous for } x, y \in \partial D, \quad x \neq y, \quad (3.20)$$

$$\begin{aligned} |(\Phi - \Phi^0)(x, y, E)| &\leq C_1|x - y|^{2-d}, \quad x, y \in \partial D, \quad \text{for } d \geq 3, \\ |(\Phi - \Phi^0)(x, y, E)| &\leq C_1|\ln|x - y||, \quad x, y \in \partial D, \quad \text{for } d = 2, \end{aligned} \quad (3.21)$$

where C_1 is some constant (dependent on D , v , v^0 , E and d). Note that for $v^0 \equiv 0$ the result that $\Phi(E) - \Phi^0(E)$ is compact in $L^\infty(\partial D)$ (under the assumptions of Theorem 1) was given in [No1].

If (1.2a) is fulfilled, v^0 satisfies (1.15) and $k \in \mathbb{C}^d \setminus (\mathbb{R}^d \cup \mathcal{E}^0)$, then $R^0(k)$ is a compact operator in $L^\infty(\partial D)$. This follows already from the following properties of $R^0(x, y, k)$:

$$R^0(x, y, k) \text{ is continuous for } x, y \in \bar{D}, \quad x \neq y, \quad (3.22)$$

$$\begin{aligned} |R^0(x, y, k)| &\leq C_2|x - y|^{2-d}, \quad x, y \in \bar{D}, \quad \text{for } d \geq 3, \\ |R^0(x, y, k)| &\leq C_2|\ln|x - y||, \quad x, y \in \bar{D}, \quad \text{for } d = 2, \end{aligned} \quad (3.23)$$

where C_2 is some constant (dependent on D , v^0 , k and d).

Actually, under the aforementioned assumptions on D , v^0 and k , $R^0(k)$ is a bounded operator from $L^\infty(\partial D)$ to $C^\alpha(\partial D)$ for any $\alpha \in [0, 1[$ (where C^α denotes the Hölder space). This result in the general case follows from this result for $v^0 \equiv 0$ (when $R^0(x, y, k) = G(x - y, k)$), the relation (2.2), the estimate (3.23) and the property that

$$\begin{aligned} \int_D G(x - z, k)u(z)dz &\in C^\alpha(\mathbb{R}^d), \quad \text{at least, for any } \alpha \in [0, 1] \\ &\text{(as a function of } x) \text{ for } u \in L^\infty(D) \text{ (and } k \in \mathbb{C}^d \setminus \mathbb{R}^d). \end{aligned}$$

The formula (3.18) and the aforementioned properties of the operators $\Phi(E) - \Phi^0(E)$ and $R^0(k)$ imply that, under the assumptions of Theorem 1, for fixed $k \in \mathbb{C}^d \setminus (\mathbb{R}^d \cup \mathcal{E}^0)$, $k^2 = E$, $A(k)$ is a compact operator in $L^\infty(\partial D)$ and, thus, (2.9a) is a Fredholm linear integral equation of the second kind for ψ in $L^\infty(\partial D)$.

Under the assumptions of Theorem 1, for $k \in \mathbb{C}^d \setminus (\mathbb{R}^d \cup \mathcal{E}^0)$, $k^2 = E$, the aforementioned properties of $\Phi(E) - \Phi^0(E)$ and $R^0(k)$ and the property that $\psi^0 \in C^\alpha(\partial D)$, $\alpha \in [0, 1[$, imply also that

$$\text{if } \psi \in L^\infty(\partial D) \text{ satisfies (2.9a), then } \psi \in C^\alpha(\partial D) \text{ for any } \alpha \in [0, 1[. \quad (3.24)$$

Note also that if (1.2a) is fulfilled, v^0 satisfies (1.15) and $k \in \mathbb{C}^d \setminus (\mathbb{R}^d \cup \mathcal{E}^0)$, then $R^0(k)$ is a bounded operator from $L^2(\partial D)$ to $H^1(\partial D)$. One can prove this result in the general case proceeding from this result for $v^0 \equiv 0$ (when $R^0(x, y, k) = G(x - y, k)$), equation (2.2) with its iterations for $R(x, y, k)$, $R(x, y, k) - G(x - y, k)$ and so on, estimate (3.23) and the property that

$$\int_D G(x - z, k)u(z)dz \in H^2(D) \quad (\text{as a function of } x) \quad \text{for } u \in L^2(D) \quad (\text{and } k \in \mathbb{C}^d \setminus \mathbb{R}^d).$$

(For $v^0 \equiv 0$ this result was given, for example, in [Na1].) Therefore, under the assumptions of Theorem 1, for $k \in \mathbb{C}^d \setminus (\mathbb{R}^d \cup \mathcal{E}^0)$, $k^2 = E$, one can see that

$$\text{if } \psi \in L^\infty(\partial D) \text{ satisfies (2.9a), then also } \psi \in H^1(\partial D). \quad (3.25)$$

To prove Proposition 2 for the case of equation (2.9a) it remains to show that, under the assumptions of Theorem 1, equation (2.9a) at fixed $k \in \mathbb{C}^d \setminus (\mathbb{R}^d \cup \mathcal{E}^0)$, $k^2 = E$, is uniquely solvable for $\psi \in L^\infty(\partial D)$ iff $k \notin \mathcal{E}$.

For $k \in \mathbb{C}^d \setminus (\mathbb{R}^d \cup \mathcal{E}^0)$, under the assumption (1.6) for v and v^0 , due to (2.1a) and (3.7), $k \notin \mathcal{E}$ iff equation (3.2) is uniquely solvable for $\psi = e^{ikx}\mu(x, k)$ with $\mu \in L^\infty(\mathbb{R}^d)$. In turn, for $k \in \mathbb{C}^d \setminus (\mathbb{R}^d \cup \mathcal{E}^0)$, under the assumption (1.15) for v and v^0 , equation (3.2) is uniquely solvable for $\psi = e^{ikx}\mu(x, k)$ with $\mu \in L^\infty(\mathbb{R}^d)$ iff (3.2) is uniquely solvable for $\psi \in C(\mathbb{R}^d)$. Thus, it remains to show that, under the assumptions of Theorem 1, equation (2.9a) at fixed $k \in \mathbb{C}^d \setminus (\mathbb{R}^d \cup \mathcal{E}^0)$, $k^2 = E$, is uniquely solvable for $\psi \in L^\infty(\partial D)$ iff (3.2) is uniquely solvable for $\psi \in C(\mathbb{R}^d)$. This proof consists of the following two parts.

Part 1. Suppose that (under the assumptions of Theorem 1) at fixed $k \in \mathbb{C}^d \setminus (\mathbb{R}^d \cup \mathcal{E}^0)$, $k^2 = E$, equation (3.2) has several solutions in $C(\mathbb{R}^d)$. Then, repeating the proof of (2.9) separately for each solution, we find that the restriction to ∂D of each of these solutions satisfies (2.9a). In addition, different solutions ψ have different restrictions to ∂D . This follows from (1.3). Thus at fixed $k \in \mathbb{C}^d \setminus (\mathbb{R}^d \cup \mathcal{E}^0)$, $k^2 = E$, equation (2.9a) has, at least, as many solutions as equation (3.2).

Part 2. To prove the converse, we use the following identities:

$$\begin{aligned} & \int_D R^0(x, y, k)(v(y) - v^0(y))\psi(y)dy \stackrel{(1.1)}{=} \int_D R^0(x, y, k)(\Delta + E)\psi(y)dy - \\ & \int_D R^0(x, y, k)v^0(y)\psi(y)dy \stackrel{(3.8)}{=} \int_D \psi(y)(\Delta_y + E - v^0(y))R^0(x, y, k)dy + \\ & \int_{\partial D} (R^0(x, y, k)\frac{\partial}{\partial \nu_y}\psi(y) - \psi(y)\frac{\partial}{\partial \nu_y}R^0(x, y, k))dy \stackrel{(2.7), (3.10)}{=} \\ & \int_D \psi(y)\delta(x - y)dy + \int_{\partial D} (R^0(x, y, k)\frac{\partial}{\partial \nu_y}\psi(y) - \psi(y)\frac{\partial}{\partial \nu_y}R^0(x, y, k))dy \\ & \text{for } x \in \mathbb{R}^d \setminus \partial D, \end{aligned} \quad (3.26)$$

$$\int_D R^0(x, y, k)(v(y) - v^0(y))\psi(y)dy \stackrel{(3.9), (2.9b)}{=} \int_{\partial D} A(x, y, k)\psi(y)dy \quad \text{for } x \in \mathbb{R}^d \setminus \bar{D}, \quad (3.27)$$

where $k \in \mathbb{C}^d \setminus (\mathbb{R}^d \cup \mathcal{E}^0)$, $k^2 = E$, ψ satisfies (1.1) and is sufficiently regular in D , for example, $\psi \in H^1(D)$.

Let $k \in \mathbb{C}^d \setminus (\mathbb{R}^d \cup \mathcal{E}^0)$, $k^2 = E$, be fixed. Suppose that $\psi \in L^\infty(\partial D)$ solves (2.9a). Due to (3.24), (3.25), we have also that $\psi \in C^\alpha(\partial D)$, $\alpha \in [0, 1[$, and $\psi \in H^1(\partial D)$. Consider this ψ as Dirichlet data for equation (1.1) and consider the solution ψ (of (1.1)) corresponding to these data. We have that $\psi \in C^\alpha(\bar{D})$, $\alpha \in [0, 1[$, and $\psi \in H^{3/2}(D)$.

Let ψ be also defined on $\mathbb{R}^d \setminus \bar{D}$ by (2.9) with $x \in \mathbb{R}^d \setminus \bar{D}$ (in terms of $\psi|_{\partial D}$). Let us prove that ψ defined in such a way on $\mathbb{R}^d = \partial D \cup D \cup (\mathbb{R}^d \setminus \bar{D})$ satisfies (3.2) and belongs to $C(\mathbb{R}^d)$. As a particular case of the aforementioned property $\psi \in C^\alpha(\bar{D})$, $\alpha \in [0, 1[$, we have that $\psi \in C(\bar{D})$. Proceeding from the definition of ψ on $\mathbb{R}^d \setminus \bar{D}$, one can easily show that, at least, $\psi \in C(\mathbb{R}^d \setminus D)$. The properties $\psi \in C(\bar{D})$ and $\psi \in C(\mathbb{R}^d \setminus D)$ imply that $\psi \in C(\mathbb{R}^d)$. The proof that ψ satisfies (3.2) consists in the following. First, from (2.9) with $x \in \mathbb{R}^d \setminus \bar{D}$ and (3.27) (and the continuity of ψ) we obtain that ψ satisfies (3.2) for $x \in \mathbb{R}^d \setminus D$. In addition, taking into account (3.26) we have also that

$$\psi(x) = \psi^0(x, k) + \int_{\partial D} (R^0(x, y, k) \frac{\partial}{\partial \nu_y} \psi(y) - \psi(y) \frac{\partial}{\partial \nu_y} R^0(x, y, k)) dy \quad (3.28)$$

for $x \in \mathbb{R}^d \setminus \bar{D}$, where $\frac{\partial}{\partial \nu} \psi$ is taken for ψ defined on \bar{D} . Further, (as well as for $v^0 \equiv 0$, see [No1],[Na2]) proceeding from (3.28) we obtain that

$$\psi(x) = \psi^0(x, k) + \psi(x) + \int_{\partial D} (R^0(x, y, k) \frac{\partial}{\partial \nu_y} \psi(y) - \psi(y) \frac{\partial}{\partial \nu_y} R^0(x, y, k)) dy \quad (3.29)$$

for $x \in D$, where $\frac{\partial}{\partial \nu} \psi$ is taken for ψ defined in \bar{D} . Note that proceeding from (3.28) and using that $\psi \in H^1(\partial D)$, $\frac{\partial}{\partial \nu} \psi \in L^2(\partial D)$ and the jump property of the double layer potential $\frac{\partial}{\partial \nu_y} R^0(x, y, k)$, we obtain, first, (3.29) in the limit for $x = \xi - 0\nu_\xi$, $\xi \in \partial D$ (where ν_ξ is the outward normal to ∂D at ξ). Further, using that $\psi^0(x, k)$ and $R^0(x, y, k)$ with $\frac{\partial}{\partial \nu_y} R^0(x, y, k)$, $y \in \partial D$, satisfy (1.1) with v^0 in place of v , for $E = k^2$, we obtain (3.29) for $x \in D$. Finally, from (3.29) and (3.26) we obtain that ψ satisfies (3.2) also for $x \in D$. Thus, any solution ψ of (2.9a) can be continued to a continuous solution of (3.2). This completes the (part 2 of) proof that, under the assumptions of Theorem 1, at fixed $k \in \mathbb{C}^d \setminus (\mathbb{R}^d \cup \mathcal{E}^0)$, $k^2 = E$, equation (2.9a) is uniquely solvable for $\psi \in L^\infty(\partial D)$ iff (3.2) is uniquely solvable for $\psi \in C(\mathbb{R}^d)$.

The proof of Proposition 2 for the case of equation (2.9a) is completed. The proof of Proposition 2 for the cases of equations (2.11a) and (2.13a) is similar.

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