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RD-FLATNESS AND RD-INJECTIVITY

F. COUCHOT

ABSTRACT. It is proved that every commutative ring whose RD-injective modules are Σ -RD-injective is the product of a pure semi-simple ring and a finite ring. A complete characterization of commutative rings for which each artinian (respectively simple) module is RD-injective, is given. These results can be obtained by using the properties of RD-flat modules and RD-coflat modules which are respectively the RD-relativization of flat modules and fp-injective modules. It is also shown that a commutative ring is perfect if and only if each RD-flat module is RD-projective.

RD-purity is an important example of relative purity. It is the first notion of purity (for submodules) that appeared in the mathematical literature. Another reason of this importance is that for some classes of rings, not necessarily commutative, RD-purity coincides with purity. In [12] G. Puninski, M. Prest and P. Rothmaler studied these rings that they called RD-rings. In particular they proved that the classe of commutative RD-rings is exactly the classe of arithmetic rings, i.e the rings with a distributive lattice of ideals. The first section of this paper is devoted to the study of RD-flat modules and RD-coflat modules which are respectively the RD-relativization of flat modules and fp-injective modules. It is interesting to note that the RD-flat modules form an example of additive accessible category by Proposition 1.1.5. Such categories were studied by Crawley-Boevey in [5], using the term of locally finitely presented additive categories. This allows us to prove that a commutative ring is perfect if and only if each RD-flat module is RD-projective (Corollary 3.3). On the other hand, it is shown (Theorem 1.4) that a ring is RD if and only if any right module is RD-flat (or RD-coflat). So, we can say that each RD-ring is absolutely RD-flat.

Recall that a ring is said to be pure semi-simple if each right module is pure-injective. It is known that a ring R is pure semi-simple if and only if each right pure-injective R -module is Σ -pure-injective. In section 2 (Theorem 2.1) we proved the RD-variation of this result: any right RD-coflat R -module is RD-injective if and only if any right RD-injective R -module is Σ -RD-injective. Moreover, any commutative ring which enjoys these properties, is the product of a pure semi-simple ring and a finite ring.

By [9, Observation 3(4) and Theorem 6] every artinian module over a commutative ring is Σ -pure-injective. However, [4, Example 4.6] is an example of simple module over a noetherian domain that fails to be RD-injective. If R is a commutative artinian ring, it is proved (Proposition 1.2) that R is an RD-injective module if and only if R is quasi-Frobenius. In section 4 we give a characterization of commutative rings for which any simple module is RD-injective (Theorem 4.1) and a characterization of commutative rings whose artinian modules are RD-injective

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(Theorem 4.2). We begin by proving these two theorems for local rings. We show that the family of proper principal ideals of a local ring R is directed if and only if R/P is RD-injective, where P is the maximal ideal of R . In this case every noetherian R -module is RD-coflat and each artinian R -module is RD-flat. When R satisfies the ascending chain condition on principal ideals, then R is a valuation ring if and only if R/P is RD-injective. However this is not generally true and we give examples of local rings R which are not valuation rings and such that R/P is RD-injective.

All rings in this paper are associative with unity, and all modules are unital. A left R -module is said to be **cyclically presented** if it is of the form R/Rr , where $r \in R$. We say that a left module is **uniserial** if its set of submodules is totally ordered by inclusion. Recall that a commutative ring R is a **valuation ring** if it is uniserial as R -module and that R is **arithmetical** if R_P is a valuation ring for every maximal ideal P .

An exact sequence of left R -modules $0 \rightarrow F \rightarrow E \rightarrow G \rightarrow 0$ is **pure-exact** if it remains exact when tensoring it with any right R -module. In this case we say that F is a **pure submodule** of E . When $rE \cap F = rF$ for every $r \in R$, we say that F is an **RD-submodule** of E (relatively divisible) and that the sequence is **RD-exact**.

An R -module F is **pure-injective** (respectively **RD-injective**) if for every pure (respectively RD-) exact sequence $0 \rightarrow N \rightarrow M \rightarrow L \rightarrow 0$ of R -modules, the following sequence $0 \rightarrow \text{Hom}_R(L, F) \rightarrow \text{Hom}_R(M, F) \rightarrow \text{Hom}_R(N, F) \rightarrow 0$ is exact.

An R -module F is **pure-projective** (respectively **RD-projective**) if for every pure (respectively RD-) exact sequence $0 \rightarrow N \rightarrow M \rightarrow L \rightarrow 0$ of R -modules, the following sequence $0 \rightarrow \text{Hom}_R(F, N) \rightarrow \text{Hom}_R(F, M) \rightarrow \text{Hom}_R(F, L) \rightarrow 0$ is exact.

1. RD-FLATNESS AND RD-COFLATNESS.

We begin with some preliminary results. As in [6] we set $M^b = \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ the **character** module of M .

As in [14] we say that a right module M is **RD-flat** if for every RD-exact sequence of left modules $0 \rightarrow H \rightarrow F \rightarrow L \rightarrow 0$, the sequence

$$0 \rightarrow M \otimes_R H \rightarrow M \otimes_R F \rightarrow M \otimes_R L \rightarrow 0 \text{ is exact.}$$

The next proposition and its proof is similiary to that we know for flat modules. In particular, to prove that (1) \Rightarrow (5), we do a similar proof as in [10, Theorem I.1.2].

Proposition 1.1. *Let R be a ring and M a right module. Then the following assertions are equivalent.*

- (1) M is RD-flat
- (2) M^b is RD-injective
- (3) $M \cong P/Q$, where P is RD-projective and Q is a pure submodule of P
- (4) Every RD-exact sequence $0 \rightarrow Q \rightarrow P \rightarrow M \rightarrow 0$ is pure exact
- (5) M is direct limit of finite direct sums of cyclically presented modules.

When R is commutative and E an injective cogenerator, these conditions are equivalent to the following conditions:

- (2') $\text{Hom}_R(M, E)$ is RD-injective.
- (6) M_P is RD-flat over R_P for each maximal ideal P .

Moreover, each direct limit of right RD-flat modules is RD-flat.

Some examples of commutative artinian rings R , such that R is not an RD-injective module are known: [2, Example 1] and [11, Example 1]. From Proposition 1.1 we deduce the following:

Proposition 1.2. *Let R be an artinian commutative ring. Then R is an RD-injective module if and only if R is a quasi-Frobenius ring.*

Proof. R is a finite product of local rings. So, we may assume that R is local. Let E be the injective hull of its residue field. We know that $R \cong \text{Hom}_R(E, E)$. By Proposition 1.1 R is an RD-injective module if and only if E is RD-flat. But it is also known that E is finitely presented. It follows that E is RD-projective because a module is RD-projective if and only if it is RD-flat and pure-projective. By [15, Corollary 2] E is a direct sum of cyclically presented modules. Hence $E \cong R$ since E is indecomposable and faithful. We get that R is quasi-Frobenius. \square

We say that a right R -module M is **RD-coflat** if every RD-exact sequence $0 \rightarrow M \rightarrow P \rightarrow Q \rightarrow 0$ is pure exact. **Pure-essential extension**, **RD-essential extension**, **pure-injective hull** and **RD-injective hull** are defined as in [15] or [6, Chapter XIII]. Observe that the RD-injective hull and the pure-injective hull coincide for any RD-coflat module. So one gets a partial answer to [6, Problem 47].

Proposition 1.3. *Let R be a ring. Then:*

- (1) *Pure submodules of RD-coflat modules are RD-coflat. Direct products and direct sums of RD-coflat right R -modules are RD-coflat.*
- (2) *for any right module M the following conditions are equivalent:*
 - *M is RD-coflat.*
 - *For each RD-exact sequence $0 \rightarrow K \xrightarrow{u} L \xrightarrow{p} F \rightarrow 0$ of right modules, where F is pure-projective, the following sequence is exact:*

$$0 \rightarrow \text{Hom}_R(F, M) \rightarrow \text{Hom}_R(L, M) \rightarrow \text{Hom}_R(K, M) \rightarrow 0.$$
- (3) *A right R -module M is RD-injective if and only if it is RD-coflat and satisfies the following condition: for any family of subgroups $(N_i)_{i \in I}$, where $N_i = (Mr_i : s_i)$, for some elements r_i and s_i of R , and any family $(x_i)_{i \in I}$ of elements of M , if the sets $x_i + N_i$ have the finite intersection property, then their total intersection is non-empty.*

Proof.

(1). Let E be a right RD-coflat module, F a right module and M a pure submodule of E which is an RD-submodule of F . Let H be the module defined by the following pushout diagram:

$$\begin{array}{ccc} M & \rightarrow & E \\ \downarrow & & \downarrow \\ F & \rightarrow & H \end{array}$$

It is easy to prove that E is an RD-submodule of H . It successively follows that E is a pure submodule of H , M a pure submodule of H and M a pure submodule of F .

Let $(M_i)_{i \in I}$ be a family of RD-coflat modules and for each $i \in I$ let E_i be the RD-injective hull of M_i . Then $E = \prod_{i \in I} E_i$ is RD-injective and $M = \prod_{i \in I} M_i$ is a pure submodule of E . It follows that M is RD-coflat. Whence $\bigoplus_{i \in I} M_i$ is also RD-coflat since it is a pure submodule of M .

(2). Assume that M is RD-coflat. Let $M \xrightarrow{v} \widehat{M}$ be the RD-injective hull of M and $f : K \rightarrow M$ be a homomorphism. There exists a morphism $g : L \rightarrow \widehat{M}$ such that $v \circ f = g \circ u$. Then, if $q : \widehat{M} \rightarrow \widehat{M}/M$ is the natural map, there exists a morphism $h : F \rightarrow \widehat{M}/M$ such that $h \circ p = q \circ g$. Since v is a pure monomorphism, there exists a morphism $j : F \rightarrow \widehat{M}$ such that $h = q \circ j$. It is trivial to verify that $q \circ (g - j \circ p) = 0$. It follows that there exists a morphism $l : L \rightarrow M$ such that $v \circ l = g - j \circ p$. We easily get that $v \circ f = v \circ l \circ u$. Since v is a monomorphism, thus $f = l \circ u$.

Conversely, let $x_1, \dots, x_n \in \widehat{M}$, and $\{a_{i,j} \mid 1 \leq i \leq n, 1 \leq j \leq m\}$ be a family of elements of R such that $\sum_{i=1}^{i=n} x_i a_{i,j} = y_j \in M$, for each j , $1 \leq j \leq m$. Let F be a finitely presented module generated by e_1, \dots, e_n with the relations $\sum_{i=1}^{i=n} e_i a_{i,j} = 0$ for each j , $1 \leq j \leq m$ and $h : F \rightarrow \widehat{M}/M$ be the morphism defined by $h(e_i) = q(x_i)$, $\forall i$, $1 \leq i \leq n$. We consider the following pullback diagram:

$$\begin{array}{ccc} L & \xrightarrow{p} & F \\ \downarrow & & \downarrow \\ \widehat{M} & \xrightarrow{q} & \widehat{M}/M \end{array}$$

Let g be the left vertical map. Then M is isomorphic to $\ker p$. It follows that M is an RD-submodule of L . Therefore there exists a morphism $l : L \rightarrow M$ such that $l(x) = x$, for each $x \in M$. Let z_1, \dots, z_n be elements of L such that $g(z_i) = x_i$ and $p(z_i) = e_i$ for each i , $1 \leq i \leq n$. Then we get that $\sum_{i=1}^{i=n} l(z_i) a_{i,j} = y_j$ for each j , $1 \leq j \leq m$. Hence M is a pure submodule of \widehat{M} .

(3). We do the same proof as in [15, Theorem 4]. \square

From these previous propositions and [12, Theorem 2.5] we deduce the following:

Theorem 1.4. *Let R be a ring. The following assertions are equivalent:*

- (1) *Every right R -module is RD-flat*
- (2) *Every left R -module is RD-coflat.*
- (3) *Every left pure-injective module is RD-injective*
- (4) *Every right pure-projective module is RD-projective*
- (5) *Every right finitely presented module is a summand of a direct sum of cyclically presented modules.*
- (6) *Every RD-exact sequence of right modules is pure-exact.*
- (7) *The left-right symmetry of (1) – (6).*

As in [12] we will say that R is an **RD-ring** if it satisfies the equivalent conditions of Theorem 1.4. By [12, Proposition 4.5] a commutative ring R is RD if and only if it is an arithmetic ring.

2. RINGS WHOSE RD-INJECTIVE RIGHT MODULES ARE Σ -RD-INJECTIVE.

In this section we will prove the following Theorem 2.1. In the sequel, for every right R -module M , we denote its RD-injective hull by \widehat{M} .

As in [9, Observation 3(2)], if N is a subgroup of a right module M over a ring R , we say that N is a **finite matrix subgroup** if it is the kernel of the following map $M \rightarrow M \otimes_R X$, defined by $m \rightarrow m \otimes x$, where X is a finitely presented left module, $x \in X$ and $m \in M$. For instance, for any $r, s \in R$, $(Mr : s)$ is a finite matrix subgroup: it is the kernel of the following map: $M \rightarrow M \otimes_R R/Rr$ defined by $m \rightarrow m \otimes s$. If s is a unit then $(Mr : s) = Mr$.

Recall that a right R -module M is Σ -**pure-injective** if $M^{(I)}$ is pure-injective for each index set I . A ring R is said to be **right perfect** if every flat right R -module is projective.

Recall that a right module M is **fp-injective** (or **absolutely pure**) if it is a pure submodule of each overmodule.

Theorem 2.1. *Let R be a ring. Consider the following conditions:*

- (1) *Every RD-coflat right R -module is RD-injective.*
- (2) *Each direct sum of right RD-injective modules is RD-injective.*
- (3) *Every RD-injective right R -module is Σ -pure-injective.*
- (4) *Every RD-injective right R -module is a direct sum of indecomposable submodules.*
- (5) *R is right artinian and RD-injective hulls of finitely generated right modules are finite direct sums of indecomposable submodules.*
- (6) *R is a finite product of artinian valuation rings or finite rings.*

Then:

- *Conditions (1), (2), (3) and (4) are equivalent and imply condition (5).*
- *When R is commutative, the six conditions are equivalent.*

Proof. (1) \Rightarrow (2) follows from Proposition 1.3.

(2) \Rightarrow (3) is obvious.

(3) \Rightarrow (1). Let M be a right RD-coflat module. Then \widehat{M} is Σ -pure-injective. We conclude by [9, Corollary 8].

(4) \Leftrightarrow (3) follows from [9, Proposition 9 and Theorem 10].

(2) \Rightarrow (5). Each direct sum of right injective modules is fp-injective and RD-coflat, hence injective. We deduce that R is right noetherian. Since \widehat{R} is Σ -pure-injective then it satisfies the descending chain condition on its finite matrix subgroups by [9, Observation 3(4) and Theorem 6]. Consequently the family $(\widehat{R}r)_{r \in R}$ satisfies the descending chain condition. Since $Rr = \widehat{R}r \cap R$ for each $r \in R$, it follows that R verifies this descending chain condition on principal left ideals. Hence R is also right perfect by [13, Théorème 6.2.5]. So R is right artinian by [13, Proposition 6.2.10].

Let M be a finitely generated right R -module. Since (2) \Rightarrow (4), $\widehat{M} = \bigoplus_{i \in I} M_i$ where M_i is indecomposable for each $i \in I$. There is a finite subset J of I such that M is relatively divisible in $\bigoplus_{i \in J} M_i$. Since $M \cap \bigoplus_{i \in I \setminus J} M_i = 0$ and $\bigoplus_{i \in J} M_i \cong \widehat{M} / \bigoplus_{i \in I \setminus J} M_i$, we conclude that $\bigoplus_{i \in I \setminus J} M_i = 0$ and $I = J$. \square

To prove the second assertion of this theorem some preliminary results are needed.

Proposition 2.2. *Let R be a finite ring. Then every RD-coflat right (or left) module is RD-injective.*

Proof. For every right R -module M the family of subgroups which are finite intersections of subgroups of the form $(Mr : s)$, where $r, s \in R$, is finite. We conclude by Proposition 1.3. \square

Now we consider a commutative local ring R of maximal ideal P , satisfying the ascending chain condition on principal ideals. If A is a proper ideal we denote A^* the set of principal ideals contained in A , \mathcal{A}^* the subset of maximal elements of

A^* . We put $S = R/P$ and $E = E_R(S)$. For any $r^* \in A^*$ let $\phi_{r^*} : R/r^* \rightarrow R/A$ be the homomorphism defined by $\phi_{r^*}(1 + r^*) = 1 + A$. Let $\psi : \bigoplus_{r^* \in A^*} R/r^* \rightarrow R/A$ be the homomorphism defined by the family $(\phi_{r^*})_{r^* \in A^*}$. We put $\Psi = \text{Hom}_R(\psi, E)$ and $E[A] = \{e \in E \mid A \subseteq (0 : e)\}$. Recall that $\text{Hom}_R(R/A, E) \cong E[A]$. It is easy to check that Ψ is the diagonal homomorphism induced by the inclusion maps $E[A] \rightarrow E[r^*]$. The following lemma holds.

Lemma 2.3. *Then:*

- (1) $\ker \psi$ is an RD-submodule of $\bigoplus_{r^* \in A^*} R/r^*$.
- (2) $\Psi(E[A])$ is an RD-submodule of $\prod_{r^* \in A^*} E[r^*]$.

Proof. (1) Let $f : R/s^* \rightarrow R/A$ be a non-zero homomorphism where s^* is a proper principal ideal of R . We have $f(1 + s^*) = a + A$ where $a \in R \setminus A$. Then $as^* \subseteq A$. Let $t^* \in A^*$ such that $as^* \subseteq t^*$. Let $g : R/s^* \rightarrow \bigoplus_{r^* \in A^*} R/r^*$ be the composition of $\alpha : R/s^* \rightarrow R/t^*$, defined by $\alpha(1 + s^*) = a + t^*$, with the inclusion map $R/t^* \rightarrow \bigoplus_{r^* \in A^*} R/r^*$. Then $f = \phi \circ g$. We conclude by [15, Proposition 2].

(2) is a consequence of the fact that $\text{Hom}_R(M/N, E)$ is isomorphic to an RD-submodule of $\text{Hom}_R(M, E)$ if N is an RD-submodule of M . \square

Since its maximal ideal is T-nilpotent, it is easy to prove that every commutative local perfect ring satisfies the ascending chain condition on principal ideals.

Lemma 2.4. *Let R be a commutative local perfect ring and A a proper ideal. Then $\widehat{E[A]} \cong \prod_{r^* \in A^*} E[r^*]$.*

Proof. We put $G(A) = \prod_{r^* \in A^*} E[r^*]$. By [6, Lemma XIII.1.2] $E[r^*]$ is RD-injective, so $G(A)$ is RD-injective too. If N is a submodule of $G(A)$ such that $N \cap \Psi(E[A]) = 0$ and the image of $\Psi(E[A])$ relatively divisible in $G(A)/N$, then $\forall 0 \neq y \in N$, $Ry \cap \Psi(E[A]) = 0$ and the image of $\Psi(E[A])$ is relatively divisible in $G(A)/Ry$. Consequently it is sufficient to prove that for every $y \in G(A)$ such that $Ry \cap \Psi(E[A]) = 0$, the image of $\Psi(E[A])$ is not relatively divisible in $G(A)/Ry$. Let $0 \neq y = (y_{r^*})_{r^* \in A^*}$ such that $Ry \cap \Psi(E[A]) = 0$. Let F be the submodule of E generated by $\{y_{r^*} \mid r^* \in A^*\}$, $B = \text{ann}(F)$ and $t \in R$ such that $t + B$ generates a minimal non-zero submodule of R/B . It follows that $tF = S$. Hence there exists $s^* \in A^*$ such that $ty_{s^*} \neq 0$. We set $e = ty_{s^*}$ and $z_{s^*} = 0$. If $s^* \neq r^* \in A^*$ and $s^* = Rs$ then $s \notin r^*$. It follows that $(r^* : s) \subseteq P \subseteq (0 : e - ty_{r^*})$. Since $E[r^*]$ is injective over R/r^* there exists $z_{r^*} \in E[r^*]$ such that $sz_{r^*} = e - ty_{r^*}$. We put $z = (z_{r^*})_{r^* \in A^*}$. Then we get the following equality: $sz + ty = \Psi(e)$. Since $Ry \cap \Psi(E[A]) = 0$ we have $sz \neq 0$. If $e = sx$ for some $x \in E[A]$ then $e = 0$ since $s \in A$. It follows that the image of $\Psi(E[A])$ is not relatively divisible in $G(A)/Ry$. \square

Remark 2.5. We know that $E[A]$ is indecomposable and it is a module of finite length if R is a commutative local artinian ring. If A is not principal then $\widehat{E[A]}$ is decomposable, and if A^* is not finite then $\widehat{E[A]}$ is neither artinian nor noetherian.

Remark 2.6. Let $\alpha : F \rightarrow M$ be an epimorphism of right modules where F is RD-projective. Let $K = \ker \alpha$. We say that $\alpha : F \rightarrow M$ is an **RD-projective cover** of M if K is an RD-pure submodule of F and F is the only submodule N of F which verifies $K + N = F$ and $K \cap N$ is relatively divisible in N . In a similar way we define the **pure-projective cover** of M . If R is a commutative local perfect

ring, then Lemma 2.4 implies that, for each ideal A , $\psi : \bigoplus_{r^* \in \mathcal{A}^*} R/r^* \rightarrow R/A$ is an RD-projective cover of R/A .

Now we prove the last assertion of Theorem 2.1.

Proof of Theorem 2.1.

(5) \Rightarrow (6). R is a finite product of local rings. We may assume that R is local. There exists a finite family of indecomposable RD-injective modules $(S_i)_{1 \leq i \leq n}$ such that $\widehat{S} = \bigoplus_{i=1}^n S_i$. By [9, Proposition 9(3)] the endomorphism ring of each indecomposable pure-injective module is local. For every $r^* \in \mathcal{P}^*$, $E[r^*]$ is a summand of \widehat{S} by Lemma 2.4. Then there exists i , $1 \leq i \leq n$ such that $E[r^*] \cong S_i$ by Krull-Schmidt Theorem. Whence \mathcal{P}^* is a finite set. This implies that the set \mathcal{P}'^* of maximal principal proper ideals of R/P^2 is finite. Clearly \mathcal{P}'^* is the set of vector lines of the vector space P/P^2 over R/P . This last set is finite if and only if P is principal or R/P is a finite field.

(6) \Rightarrow (1). We may assume that R is local. If R is an artinian valuation ring then each R -module is RD-injective by [8, Theorem 4.3]. We conclude by Proposition 2.2 if R is finite. \square

3. RINGS WHOSE RD-FLAT MODULES ARE RD-PROJECTIVE.

By using a result of Crawley-Boevey in [5] we get the next theorem. If R is a ring, we denote by J its Jacobson radical and by \mathcal{RD} the family of proper right ideals A such that R/A is isomorphic to an indecomposable summand of a cyclically presented right module.

Theorem 3.1. *Let R be a ring. The following conditions are equivalent:*

- (1) *each RD-flat right R -module is RD-projective*
- (2) *R is right perfect and \mathcal{RD} satisfies the ascending chain condition.*

Proof. The condition (1) implies that each flat right module is projective. Hence R is right perfect. Let \mathcal{A} be the category of RD-flat right R -modules. By Proposition 1.1.5 each object of \mathcal{A} is direct limit of finitely presented objects of \mathcal{A} . So, \mathcal{A} is an accessible additive category (or a locally finitely presented additive category). Then each object of \mathcal{A} is pure-projective. By [5, (3.2) Theorem] we get:

- (a) Every finitely presented object of \mathcal{A} is a finite direct sum of indecomposable objects, each with local endomorphism ring.
- (b) Given a sequence

$$(S) \quad X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} X_4 \xrightarrow{f_4} \dots$$

of non-isomorphism between finitely presented indecomposable objects of \mathcal{A} , the composition $f_N \circ \dots \circ f_2 \circ f_1$ is zero for N sufficiently large.

Suppose that \mathcal{RD} contains a strictly ascending chain $(A_n)_{n \in \mathbb{N}^*}$. Let $X_n = R/A_n$ and let $f_n : X_n \rightarrow X_{n+1}$ be the natural map, $\forall n \in \mathbb{N}^*$. Then we get a contradiction since $f_n \circ \dots \circ f_2 \circ f_1 \neq 0$, $\forall n \in \mathbb{N}^*$.

Conversely we must prove that \mathcal{A} satisfies the above conditions (a) and (b). If we show that $\text{End}_R(X)$ is local for each indecomposable summand of a cyclically presented right module then the condition (a) is an immediate consequence of Krull-Schmidt Theorem. Let x be a generator of X and $f \in \text{End}_R(X)$. First we show that f is an isomorphism if f is onto. For each integer $n \geq 1$ we put $Y_n = \text{Ker } f^n$ and $A_n = \{r \in R \mid xr \in Y_n\}$. Since $X/Y_n \cong X$, $A_n \in \mathcal{RD}$ for each integer $n \geq 1$.

Thus there exists an integer N such that $A_n = A_{n+1}$, $\forall n \geq N$. Consequently $Y_n = Y_{n+1}$, $\forall n \geq N$. We easily deduce that $Y_n \cap f^n(X) = 0$ if $n \geq N$. Since f is surjective we get that f is an isomorphism. Now suppose that f is not surjective. Then $x - f(x)$ is a generator of X because $f(x) \in XJ$. It follows that $\mathbf{1}_X - f$ is onto and consequently it is an isomorphism. So, $\text{End}_R(X)$ is local.

Now we consider the sequence (S). Let x_n be a generator of X_n and let $t_n \in R$ such that $f_n(x_n) = x_{n+1}t_n$, for each integer $n \geq 1$. Then the set of integers n for which $t_n \in J$ is infinite. Otherwise there exists an integer m such that $t_n \notin J$, $\forall n \geq m$. By induction on n we define $y_n \in X_n$ for each integer $n \geq m$ in the following way: $y_m = x_m$ and $y_{n+1} = f_n(y_n)$. Let A_n be the annihilator of y_n . Then y_n is a generator of X_n for each integer $n \geq m$ and we have $A_n \subset A_{n+1}$ since f_n is not injective. This contradicts that \mathcal{RD} satisfies the ascending chain condition. Now, from the T-nilpotence of J we deduce that there exists an integer N such that $t_N \dots t_2 t_1 = 0$, whence $f_N \circ \dots \circ f_2 \circ f_1$ is zero. The proof is now complete. \square

We easily deduce the two following corollaries.

Corollary 3.2. *Let R be a local ring. Then R is right perfect if and only if each RD-flat right module is RD-projective.*

Proof. In this case, \mathcal{RD} is the family of right principal ideals. From the right T-nilpotence of J we deduce that this family satisfies the ascending chain condition. \square

Corollary 3.3. *Let R be a commutative ring. Then R is perfect if and only if each RD-flat module is RD-projective.*

Proof. If R is perfect then R is a finite product of local rings. \square

Remark 3.4. Observe that each right artinian ring satisfies the equivalent conditions of Theorem 3.1. From Theorem 2.1 it follows that the condition "each RD-coflat right R -module is RD-injective" implies the condition "each RD-flat right R -module is RD-projective". But the converse doesn't hold.

If R is a local right perfect ring which is not left perfect then each right RD-flat module is RD-projective but it is not true for left modules. An example of a such ring is given in [13, Exemple 3, p.132].

4. RD-INJECTIVE ARTINIAN MODULES.

The aim of this section is to characterize commutative rings for which every artinian module is RD-injective. Throughout this section all rings are commutative. Recall that a family \mathcal{F} of submodules of an R -module M is **directed** if, $\forall (U, V) \in \mathcal{F}^2$ there exists $W \in \mathcal{F}$ such that $U + V \subseteq W$. We will prove the two following theorems.

Theorem 4.1. *Let R be a ring. Then the following conditions are equivalent:*

- (1) *Every simple R -module is RD-flat.*
- (2) *Every simple R -module is RD-injective.*
- (3) *Every R -module of finite length is RD-flat.*
- (4) *Every R -module of finite length is RD-injective.*
- (5) *Every artinian R -module is RD-flat.*
- (6) *Every noetherian R -module is RD-coflat.*

- (7) For each maximal ideal P the family of proper principal ideals of R_P is directed.

Moreover, if R satisfies these conditions, then every artinian module is a (finite) direct sum of uniserial modules and every noetherian module is a direct sum of 2-generated submodules. However the converse is not true.

Proof. It is obvious that (5) \Rightarrow (3) \Rightarrow (1) and (6) \Rightarrow (4) \Rightarrow (2).

(1) \Leftrightarrow (2). We set $E = \prod_{P \in \text{Max}(R)} E_R(R/P)$ where $\text{Max}(R)$ is the set of maximal ideals of R . Then E is an injective cogenerator and for each simple module S , $S \cong \text{Hom}_R(S, E)$. We conclude by Proposition 1.1.

(3) \Rightarrow (5) since each artinian module is direct limit of modules of finite length. \square

Let R be a ring. For every maximal ideal P we denote by $L(P)$ the sum of all submodules of finite length of $E(R/P)$ and we set $J(P) = \bigcap_{n \in \mathbb{N}} (PR_P)^n$.

Theorem 4.2. *Let R be a ring. Then the following conditions are equivalent:*

- (1) Every artinian R -module is RD-injective.
- (2) Every noetherian R -module is RD-flat.
- (3) R/P and $L(P)$ are RD-injective for every maximal ideal P .
- (4) For each maximal ideal P , the family of proper principal ideals of R_P and the family of principal ideals of R_P contained in $J(P)$ are directed.

Moreover, if $P = P^2$ for each maximal ideal P , then these conditions are equivalent to conditions of Theorem 4.1. In this case, every noetherian module and every artinian module is semi-simple.

To prove this theorem and complete the proof of Theorem 4.1 we study our problem in the local case.

In the sequel we assume that R be a local ring. We denote P its maximal ideal, E the injective hull of R/P and $J = \bigcap_{n \in \mathbb{N}} P^n$. Let A be a proper ideal of R and A^* the set of non-zero principal ideals contained in A . Let $\phi : \bigoplus_{r^* \in A^*} R/r^* \rightarrow R/A$ be the homomorphism defined by the family $(\phi_{r^*})_{r^* \in A^*}$. The two following lemmas hold. The first is similar to Lemma 2.3.

Lemma 4.3. *Then $\ker \phi$ is an RD-submodule of $\bigoplus_{r^* \in A^*} R/r^*$.*

Lemma 4.4. *Let R be a local ring, and A a proper ideal. Then:*

- (1) $E[A]$ is RD-injective if and only if R/A is RD-flat
- (2) R/A is RD-flat if and only if A^* is directed.
- (3) If A is finitely generated then $E[A]$ is RD-injective if and only if A is principal.

Proof. (1) is an immediate consequence of Proposition 1.1.

Assume that R/A is RD-flat. By Proposition 1.1 $\ker \phi$ is a pure submodule of $\bigoplus_{r^* \in A^*} R/r^*$. Let $I \subseteq A$ be a finitely generated ideal and $f : R/I \rightarrow R/A$ be the homomorphism defined by $f(1 + I) = 1 + A$. By [15, Proposition 3], for some morphism $g : R/I \rightarrow \bigoplus_{r^* \in A^*} R/r^*$, $f = \phi \circ g$. Then $g(1 + I) = \sum_{r^* \in F} a_{r^*} + r^*$ where F is a finite family of A^* . We deduce that $1 + A = \sum_{r^* \in F} a_{r^*}(1 + A)$. Since R is local there exists $r^* \in F$ such that a_{r^*} is a unit of R . It follows that $I = (0 : 1 + I) \subseteq (0 : a_{r^*} + r^*) = r^*$.

Conversely if $R/A = \varinjlim_{r^* \in A^*} R/r^*$ then R/A is RD-flat by Proposition 1.1. \square

The next theorem characterizes local rings for which every simple module is RD-injective (and RD-flat). It is an immediate consequence of Lemma 4.4.

Theorem 4.5. *Let R be a local ring. Then R/P is RD-flat if and only if P^* is directed.*

The following corollaries will be useful to provide examples of rings for which every simple module is RD-flat.

Corollary 4.6. *Let R be a local ring. Consider the two following conditions:*

- (1) R/P is RD-flat.
- (2) There exists an ideal $I \subset P$ such that R/I is a valuation ring and for each $r \in R \setminus I$, $I \subset Rr$.

Then (2) \Rightarrow (1). When P is finitely generated the two conditions are equivalent.

Proof. First we assume that R/P is RD-flat and P is finitely generated. By Theorem 4.5 P is principal. We put $P = Rp$ and $I = J$. Let $a \in P \setminus J$. There exists $n \in \mathbb{N}$ such that $a \notin Rp^n$. We may assume that n is minimal. Then $a = bp^{n-1}$ and since $a \notin Rp^n$, b is a unit of R . Hence $I \subset Rp^{n-1} = Ra$ and R/I is a valuation ring. Conversely, if a and b are elements of $P \setminus I$ it is easy to prove that $Ra \subseteq Rb$ or $Rb \subseteq Ra$. By Theorem 4.5 R/P is RD-flat. \square

Corollary 4.7. *Let R be a local ring. Assume that R satisfies the ascending chain condition on principal ideals. Then R is a valuation ring if and only if R/P is RD-injective.*

Proof. Assume that R/P is RD-injective. We claim that there exists only one maximal principal ideal pR and that $pR = P$. If there exist at least two maximal principal ideals p^* and q^* then $p^* + q^*$ is strictly contained in a proper principal ideal. Consequently we obtain a contradiction. If $J \ni a \neq 0$ then, by induction, we get a strictly ascending chain of ideals $(Rc_n)_{n \in \mathbb{N}^*}$ such that $a = pc_1$ and $c_n = pc_{n+1}$, $\forall n \in \mathbb{N}^*$. Hence $J = 0$. We complete the proof by using Corollary 4.6. \square

We give two examples of local rings R which are not valuation rings and such that R/P is RD-injective.

Example 4.8. Let D a valuation domain, Q its field of fractions, $H = Q^2$.

We put $R = \left\{ \begin{pmatrix} d & h \\ 0 & d \end{pmatrix} \mid d \in D, h \in H \right\}$ the trivial extension of D by H .

If $I = \left\{ \begin{pmatrix} 0 & h \\ 0 & 0 \end{pmatrix} \mid h \in H \right\}$ then $R/I \cong D$. By using that H is a divisible D -module, it is easy to prove that $I \subset Ra$ for every $a \in R \setminus I$.

Example 4.9. Let K be a field and T the factor ring of the polynomial ring $K[X_{p,n} \mid p, n \in \mathbb{N}]$ modulo the ideal generated by $\{X_{p,n} - X_{p,n+1}X_{p+1,n} \mid p, n \in \mathbb{N}\}$. We denote $x_{p,n}$ the image of $X_{p,n}$ in T and P the maximal ideal of T generated by $\{x_{p,n} \mid p, n \in \mathbb{N}\}$. We put $R = T_P$. Let $p, q, m, n \in \mathbb{N}$. Clearly $x_{p,n} \in Rx_{p+q,n} \subset Rx_{p+q,n+m}$ and $x_{q,m} \in Rx_{p+q,m} \subset Rx_{p+q,n+m}$. If $a \in R$ then there exists $b \in T$ and $s \in T \setminus P$ such that $a = \frac{b}{s}$. Then b is a linear combination with coefficients in K of finite products of elements of $\{x_{p,n} \mid p, n \in \mathbb{N}\}$. From above we deduce that $a \in Rx_{p,n}$ for some $(p, n) \in \mathbb{N}^2$. It follows from this property that every proper finitely generated ideal is contained in a proper principal ideal.

If we prove that $Rx_{p+1,n}$ and $Rx_{p,n+1}$ are not comparable for every $(p, n) \in \mathbb{N}^2$ then R doesn't satisfy the condition (2) of Corollary 4.6. Else there exists $p, n \in \mathbb{N}$ such that $x_{p,n} \notin I$. It follows that $x_{p+1,n}$ and $x_{p,n+1}$ are not elements of I , whence $Rx_{p+1,n}$ and $Rx_{p,n+1}$ are comparable. We get a contradiction.

For every $\ell \in \mathbb{N}$, let $T_\ell = K[X_{p,\ell-p} \mid 0 \leq p \leq \ell]$, $\varphi_\ell : T_\ell \rightarrow T_{\ell+1}$ be the homomorphism defined by $\varphi_\ell(X_{p,\ell-p}) = X_{p+1,\ell-p}X_{p,\ell-p+1}$, $\forall p, 0 \leq p \leq \ell$ and P_ℓ be the ideal of T_ℓ generated by $\{X_{p,\ell-p} \mid 0 \leq p \leq \ell\}$. Clearly T is the direct limit of $(T_\ell, \varphi_\ell)_{\ell \in \mathbb{N}}$. Now assume that $x_{p,n+1} \in Rx_{p+1,n}$. Then there exist $s \in T \setminus P$ and $c \in T$ such that $sx_{p,n+1} = cx_{p+1,n}$. There exists an integer $\ell \geq p+n+1$ such that c and s are elements of T_ℓ . The following formulæ hold: $x_{p,n+1} = \prod_{j=0}^{j=\ell-n-p-1} X_{p+j,\ell-p-j}^{(\ell-n-j-p-1)}$

and $x_{p+1,n} = \prod_{j=0}^{j=\ell-n-p-1} X_{p+j+1,\ell-p-j-1}^{(\ell-n-j-p-1)}$.

Clearly $x_{p,n+1}$ and $x_{p+1,n}$ are two different monomials of T_ℓ . We have $s = k + t$ and $c = a + b$ where $k, a \in K$, $t, b \in P_\ell$ and $k \neq 0$. We get the following equality: $ax_{p+1,n} - kx_{p,n+1} + bx_{p+1,n} - tx_{p,n+1} = 0$. It is obvious that the degree (in T_ℓ) of $x_{p+1,n}$ and $x_{p,n+1}$ is less than the degree of each monomial of $bx_{p+1,n}$ and $tx_{p,n+1}$. It follows that $k = a = 0$. Hence we get a contradiction. In the same way we prove that $x_{p+1,n} \notin Rx_{p,n+1}$.

Now we study the RD-injectivity of artinian modules over local rings.

Theorem 4.10. *The following conditions are equivalent for a local ring R :*

- (1) *Every artinian R -module is RD-injective.*
- (2) *R/P and R/J are RD-flat.*
- (3) *R/P and $L(P)$ are RD-injective.*
- (4) *P^* and J^* are directed.*

Proof. Assume that P is not finitely generated and R/P is RD-injective. Let $a \in P$. Then there exists $b \in P \setminus Ra$. We deduce that $Ra \subset Ra + Rb \subseteq Rc$ for some $c \in P$. It follows that $a = cd$ for some $d \in P$, whence $P = P^2$. In this case every artinian module is semi-simple of finite length and $L(P)$ is simple.

Assume that P is finitely generated and R/P is RD-injective. By Corollary 4.6 and its proof, $P = pR$ for some $p \in P$, and $R' = R/J$ is a noetherian valuation ring. Let M be an indecomposable artinian R -module. For every $x \in M$, Rx is a module of finite length. It follows that there exists $n \in \mathbb{N}$ such that $p^n \in (0 : x)$. We deduce that M is an R' -module. If $U = E_{R'}(M)$ then $U \cong E_{R'}(R/P)^m = E[J]^m$ for some integer m . We put $U = U_1 \oplus \cdots \oplus U_m$ where $U_j \cong E[J]$ for each j , $1 \leq j \leq m$. Thus $M \cap U_j \neq 0$ for every j , $1 \leq j \leq m$. By [6, Proposition XII.2.1] there exists i , $1 \leq i \leq m$, such that $M \cap U_i$ is a pure R' -submodule of M . It follows that $M \cap U_i$ is a summand of M . Hence M is isomorphic to a submodule of $E[J]$ which is uniserial by [7, Theorem]. If $M \neq E[J]$ there exists a positive integer n such that $Rp^n = \text{ann}(M)$. In this case we easily deduce that M is RD-injective. If $M = E[J]$, by Proposition 1.1 M is RD-injective if and only if R/J is RD-flat. Let us observe that $E[J] = L(P)$. The proof is now complete. \square

The following corollary will be useful to provide examples of rings for which every artinian module is RD-injective.

Corollary 4.11. *Let R be a local ring. Assume that each artinian R -module is RD-injective and P is finitely generated. Suppose that $J \neq J^2$. Let I be the inverse image of $I' = \bigcap_{n \in \mathbb{N}} (JR_J)^n$ by the natural map $h : R \rightarrow R_J$. Then R/I is a discrete*

valuation ring and for every $r \in R \setminus I$, $I \subset rR$. Moreover, if J is nilpotent, then R is a discrete valuation ring.

Proof. Let us observe that $J \neq 0$ and it is a prime ideal. By Corollary 4.6 R/J is a discrete rank one valuation domain and $J \subset rR$ for each $r \in R \setminus J$. Let $q \in J \setminus J^2$ and $r \in J$. Then $Rq + Rr \subseteq Rc$ for some $c \in J \setminus J^2$. There exist $s \in R \setminus J$ and $t \in R$ such that $q = sc$ and $r = tc$. It follows that $\frac{r}{1} = \frac{tq}{s}$. Consequently JR_J is principal over R_J and R_J/I' is a valuation ring by Corollary 4.6. We may assume that $I = 0$, $I' = 0$ and h is a monomorphism. For every $0 \neq a \in J$ there exist a unit u and two integers $m \geq 1$ and $\alpha \geq 0$ such that $\frac{a}{1} = \frac{uq^m}{p^\alpha}$. There exists $c \in J \setminus J^2$ such that $q = cp^\alpha$. It follows that $a = ucq^{m-1}$. If $0 \neq b \in J$, in the same way there exist two integers $n \geq 1$ and $\beta \geq 0$, a unit v and $d \in J \setminus J^2$ such that $b = vdq^{n-1}$ and $q = dp^\beta$. Let $\gamma = \max(\alpha, \beta)$. There exist c' and d' in $J \setminus J^2$ such that $c = c'p^{\gamma-\alpha}$ and $d = d'p^{\gamma-\beta}$. It follows that $q = c'p^\gamma = d'p^\gamma$. Since h is a monomorphism, we deduce that $c' = d'$. Assume that $m = n$. It follows that $b \in Ra$ if $\alpha \geq \beta$. Now assume that $n > m$. If $\beta \leq \alpha$ then $b \in Ra$. If $\beta > \alpha$, let $q' \in J$ such that $q = q'p^{\beta-\alpha}$. Then $b = vp^{\gamma-\alpha}c'q'q^{n-2}$ and $b = vu^{-1}q'q^{n-m-1}a$.

Now suppose that J is nilpotent. Let n be the least integer such that $(JR_J)^n = 0$. We easily deduce that $J^{n+1} = 0$. We claim that $J^n = \ker h$. This is obvious if $n = 1$. Assume that $n > 1$. The inclusion $J^n \subseteq \ker h$ is easy. Conversely let $0 \neq r \in \ker h$. There exists $c \in J \setminus J^2$ such that $Rr + Rq \subseteq Rc$. The following equality $r = cd$ holds for some $d \in R$. Since $n > 1$, $d \in J$. If $d \in \ker h$ then there exists $s \in R \setminus J$ such that $sd = 0$. But there exists $c' \in J$ such that $c = sc'$. It follows that $r = dsc' = 0$. Hence a contradiction. Consequently $d \notin J^n$. Let m be the greatest integer such that $d \in J^m$. Then $m < n$, $\frac{d}{1} = \frac{q^m}{s'}$ and $\frac{c}{1} = \frac{q}{t}$ for some $s', t \in R \setminus J$. It follows that $\frac{r}{1} = \frac{q^{m+1}}{s't}$. We deduce that $m = n - 1$ and $r \in J^n$. Let a and b be two non-zero elements of J . Then there exists $c \in J$ such that $Ra + Rb + Rq = Rc$. It follows that $a = ca'$ and $b = cb'$ for some a' and b' in R . Since $a \neq 0$ and $b \neq 0$, a' and b' are not in J^n . From the first part of the proof we deduce that R/J^n is a valuation ring, so there exist $r \in R$ and $s \in J^n$ such that $b' = ra' + s$. It follows that $b = ra$. \square

Example 4.12 (Examples). If, in our Example 4.8, D is a discrete valuation domain of Krull dimension ≥ 2 , then every artinian R -module is RD-injective. But this property is not satisfied if D is a discrete rank one valuation domain. However, if R is the ring defined in our Example 4.9, then every artinian R -module is RD-injective.

Let R be the ring defined by the following pullback diagram of ring maps:

$$\begin{array}{ccc} R & \rightarrow & T \\ \downarrow & & \downarrow \\ V & \rightarrow & K \end{array}$$

where V is a discrete rank one valuation domain, K its field of fractions and T a local ring of residual class field K . Then R is local and its maximal ideal P is the inverse image of the maximal ideal of V . Clearly R/P is RD-injective. If $J = R \cap Q$, where Q is the maximal ideal of T , then $T = R_J$ and $V = R/J$. It is easy to prove that R/J is RD-flat if and only if T/Q is RD-flat over T .

We complete the proofs of Theorem 4.1 and Theorem 4.2.

Proof of Theorem 4.1.

(1) \Leftrightarrow (7) by Theorem 4.5.

Assume that PR_P is not finitely generated over R_P for every maximal ideal P . If the equivalent conditions (1) and (2) are satisfied, then $P^2R_P = PR_P$ and $P = P^2$ for every maximal ideal P . It follows that each artinian R -module is semi-simple of finite length. Hence each artinian module is RD-flat and RD-injective. Let M be a noetherian R -module and $R' = R/\text{ann}(M)$. Then the equality $P^2R_P = PR_P$ implies that $R'_P = R/P$. It follows that R' and M are semi-simple.

Now we assume that there exists a maximal ideal P such that $P \neq P^2$.

(1) \Rightarrow (3) and (2) \Rightarrow (4). Let M be a module of finite length. By [1, théorème p.368] there exists a finite family F of maximal ideals such that $M \cong \prod_{P \in F} M_P$. Consequently we may assume that M is indecomposable, R is local and its maximal ideal $P = pR$. There exists an integer n such that p^n annihilates M . Then it is easy to show that M is RD-flat and RD-injective.

(5) \Rightarrow (6). Let M be a noetherian module. By [3, lemme 1.3] the diagonal map $M \rightarrow \prod_{P \in \text{Max}(R)} M_P$ is a pure monomorphism. For every maximal ideal P we put $F_P(M) = \text{Hom}_{R_P}(\text{Hom}_{R_P}(M_P, E(R/P)), E(R/P))$. On the other hand, by [6, Lemma XIII.2.3], M_P is a pure submodule of $F_P(M)$. Then M is a pure submodule of $\prod_{P \in \text{Max}(R)} F_P(M)$. It is enough to prove that $F_P(M)$ is RD-injective. Consequently we may assume that R is local, P is its maximal ideal and $E = E(R/P)$. Let $J = \bigcap_{n \in \mathbb{N}} R p^n$, $E' = E[J]$ and $R' = R/J$. Thus E' and $\text{Hom}_R(M, E) \cong \text{Hom}_{R'}(M, E')$ are artinian. It follows that $\text{Hom}_R(\text{Hom}_R(M, E), E)$ is RD-injective.

Let M be an indecomposable artinian module. By [1, Théorème p.368], there exists a maximal ideal P such that M is an R_P -module and every R -submodule of M is also an R_P -submodule. As in the proof of Theorem 4.10 we prove that M is uniserial.

If M is a noetherian module then $R' = R/\text{ann}(M)$ is a noetherian RD-ring. Consequently R' is a finite product of Dedekind domains and artinian valuation rings. It follows that M is a direct sum of 2-generated submodules. \square

The following example shows that the converse of the last assertion of Theorem 4.1 is not true.

Example 4.13. Let K be a field and for every $n \in \mathbb{N}$, let R_n be the localization of the polynomial ring $K[X_n, Y_n]$ at the maximal ideal generated by $\{X_n, Y_n\}$ and P_n the maximal ideal of R_n . For each $n \in \mathbb{N}$, let $\delta_n : R_n \rightarrow R_{n+1}$ be the ring homomorphism defined by $\delta_n(X_n) = X_{n+1}^2$ and $\delta_n(Y_n) = Y_{n+1}^2$. Let R be the direct limit of the system $(R_n, \delta_n)_{n \in \mathbb{N}}$. Then R is local and its maximal ideal P is the direct limit of $(P_n)_{n \in \mathbb{N}}$. It is obvious that $P^2 = P$. Hence every artinian R -module is semi-simple of finite length. Let $r \in R$ such that X_0 and Y_0 are in Rr . We may assume that there exist $n \in \mathbb{N}$, $s, t \in R_n$ such that $X_n^{2^n} = rs$ and $Y_n^{2^n} = rt$. It follows that $tX_n^{2^n} = sY_n^{2^n}$. Since R_n is a unique factorization domain there exists $u \in R_n$ such that $s = uX_n^{2^n}$ and $t = uY_n^{2^n}$. We deduce that $(1 - ru)X_n^{2^n} = 0$. Hence $r \notin P$ since R is local. Consequently R doesn't satisfy the equivalent conditions of Theorem 4.1.

Proof of Theorem 4.2.

(3) \Leftrightarrow (1). Let M be an artinian module. By [1, Théorème p.368] we may assume that R is local. Hence it is an immediate consequence of Theorem 4.10.

(1) \Rightarrow (2). By Proposition 1.1 RD-flatness is a local property, so we may assume that R is local. Since R/P is RD-flat, R/J is noetherian and $E[J] = E_{R/J}(R/P)$ is artinian. Hence, if M is a noetherian module then $\text{Hom}_R(M, E_R(R/P)) = \text{Hom}_{R/J}(M, E[J])$ is artinian. We conclude by Proposition 1.1.

(2) \Rightarrow (3). We may assume that R is local. Since R/P is RD-flat, we may suppose that P is principal. By Corollary 4.6, R/J is noetherian. By Proposition 1.1 it follows that $L(P) = E[J]$ is RD-injective.

(3) \Leftrightarrow (4) by Theorem 4.10. □

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