



**HAL**  
open science

## RD-flatness and RD-injectivity

Francois Couchot

► **To cite this version:**

| Francois Couchot. RD-flatness and RD-injectivity. 2004. hal-00002962v1

**HAL Id: hal-00002962**

**<https://hal.science/hal-00002962v1>**

Preprint submitted on 28 Sep 2004 (v1), last revised 17 Feb 2014 (v3)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

## RD-FLATNESS AND RD-INJECTIVITY

F. COUCHOT

ABSTRACT. RD-flat modules and RD-coflat modules over arbitrary associative rings with unity are studied. If  $R$  is a ring, then every  $R$ -module is RD-flat (resp. RD-coflat) if and only if  $R$  is an RD-ring. It is proved that each commutative ring  $R$  for which every RD-injective module is  $\Sigma$ -RD-injective is a finite product of artinian valuation rings or finite rings. When  $R$  is a commutative local ring of maximal ideal  $P$ , it is shown that  $R/P$  is RD-flat if and only if every proper finitely generated ideal is contained in a proper principal ideal. If  $R$  satisfies the ascending chain condition on principal ideals then  $R/P$  is RD-flat if and only if  $R$  is a valuation ring. When  $R$  is commutative, necessary and sufficient conditions are given for every artinian  $R$ -module to be RD-injective and every noetherian  $R$ -module to be RD-flat. In this case each artinian  $R$ -module is RD-flat and each noetherian  $R$ -module is RD-coflat.

In section 1 of this paper RD-flat modules and RD-coflat modules over associative rings with unity are studied. A right module  $M$  over a ring  $R$  is said to be RD-flat if it is a factor of an RD-projective module by a pure submodule and  $M$  is said to be RD-coflat if it is a pure submodule of an RD-injective module. In [16] G. Puninski, M. Prest and P. Rothmaler studied RD-rings, i.e the rings for which every RD-exact sequence of right modules is pure-exact. We prove that a ring  $R$  is RD if and only if every left  $R$ -module is RD-flat (resp. RD-coflat).

In section 2, for each commutative ring  $R$  whose principal ideals are p-flat, it is shown that the following conditions are equivalent:  $R$  is arithmetic, every divisible module is fp-injective, each p-flat module is flat.

By [13, Observation 3(4) and Theorem 6] every artinian module over a commutative ring is  $\Sigma$ -pure-injective. However, [7, example 4.6] is an example of simple module over a noetherian domain that fails to be RD-injective. In section 3 we study RD-injectivity and RD-flatness of simple modules over commutative rings. We prove that the family of proper principal ideals of a local ring  $R$  is direct if and only if  $R/P$  is RD-injective, where  $P$  is the maximal ideal of  $R$ . In this case every noetherian  $R$ -module is RD-coflat and each artinian  $R$ -module is RD-flat. When  $R$  satisfies the ascending chain condition on principal ideals, then  $R$  is a valuation ring if and only if  $R/P$  is RD-injective. However this is not generally true and we give examples of local rings  $R$  which are not valuation rings and such that  $R/P$  is RD-injective. It is also proved that every artinian module, over a commutative ring, is RD-injective if and only if for every simple module  $S$ ,  $S$  and the sum of all submodules of finite length of the injective hull of  $S$  are RD-injective. These conditions are equivalent to RD-flatness of each noetherian module.

In section 4 we study the RD-injectivity of  $R$ , when  $R$  is a commutative local ring. If  $R$  is artinian then  $R$  is an RD-injective module if and only if  $R$  is quasi-Frobenius.

In section 5, rings for which each right RD-coflat module is RD-injective are studied. For these rings every right RD-injective module is  $\Sigma$ -pure-injective and is

a direct sum of indecomposable modules. These rings are right artinian. It is shown that commutative rings satisfying these properties are finite products of artinian valuation rings or finite rings. To prove this result we determine the RD-injective hull of  $R/P$ , where  $R$  is a commutative local perfect ring of maximal ideal  $P$ .

All rings in this paper are associative with unity, and all modules are unital. A left  $R$ -module is said to be *cyclically presented* if it is of the form  $R/Rr$ , where  $r \in R$ . We say that a left module is *uniserial* if its set of submodules is totally ordered by inclusion. Recall that a commutative ring  $R$  is a *valuation ring* if it is uniserial as  $R$ -module and that  $R$  is *arithmetical* if  $R_P$  is a valuation ring for every maximal ideal  $P$ .

An exact sequence of left  $R$ -modules  $0 \rightarrow F \rightarrow E \rightarrow G \rightarrow 0$  is *pure-exact* if it remains exact when tensoring it with any right  $R$ -module. In this case we say that  $F$  is a *pure submodule* of  $E$ . When  $rE \cap F = rF$  for every  $r \in R$ , we say that  $F$  is an *RD-submodule* of  $E$  (relatively divisible) and that the sequence is *RD-exact*.

An  $R$ -module  $F$  is *pure-injective* (respectively *RD-injective*) if for every pure (respectively RD-) exact sequence  $0 \rightarrow N \rightarrow M \rightarrow L \rightarrow 0$  of  $R$ -modules, the following sequence  $0 \rightarrow \text{Hom}_R(L, F) \rightarrow \text{Hom}_R(M, F) \rightarrow \text{Hom}_R(N, F) \rightarrow 0$  is exact.

An  $R$ -module  $F$  is *pure-projective* (respectively *RD-projective*) if for every pure (respectively RD-) exact sequence  $0 \rightarrow N \rightarrow M \rightarrow L \rightarrow 0$  of  $R$ -modules, the following sequence  $0 \rightarrow \text{Hom}_R(F, N) \rightarrow \text{Hom}_R(F, M) \rightarrow \text{Hom}_R(F, L) \rightarrow 0$  is exact.

## 1. RD-FLATNESS AND RD-COFLATNESS.

We begin with some preliminary results. As in [9] we set  $M^b = \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$  the *character* module of  $M$ . The first assertion of the following proposition is similar to [18, Proposition 9.1]. We prove the second assertion in the same way by replacing respectively  $\mathbb{Z}$  and  $\mathbb{Q}/\mathbb{Z}$  with  $R$  and  $E$ .

**Proposition 1.1.** *Let  $R$  be a ring. We consider the following exact sequence  $(S)$ :  $0 \rightarrow N \rightarrow M \rightarrow L \rightarrow 0$ , of right  $R$ -modules. Then:*

- (1)  *$(S)$  is pure-exact if and only if the following exact sequence of left modules splits:  $0 \rightarrow L^b \rightarrow M^b \rightarrow N^b \rightarrow 0$ .*
- (2) *If  $R$  is commutative and  $E$  an injective  $R$ -cogenerator then  $(S)$  is pure-exact if and only if the following exact sequence splits:*

$$0 \rightarrow \text{Hom}_R(L, E) \rightarrow \text{Hom}_R(M, E) \rightarrow \text{Hom}_R(N, E) \rightarrow 0.$$

**Proposition 1.2.** *Let  $R$  be a ring. We consider the following exact sequence  $(S)$ :  $0 \rightarrow N \rightarrow M \rightarrow L \rightarrow 0$ , of right  $R$ -modules. Then:*

- (1)  *$(S)$  is RD-exact if and only if the following exact sequence of left modules is RD-exact:  $0 \rightarrow L^b \rightarrow M^b \rightarrow N^b \rightarrow 0$ .*
- (2) *If  $R$  is commutative and  $E$  an injective  $R$ -cogenerator then  $(S)$  is RD-exact if and only if the following exact sequence is RD-exact:*

$$0 \rightarrow \text{Hom}_R(L, E) \rightarrow \text{Hom}_R(M, E) \rightarrow \text{Hom}_R(N, E) \rightarrow 0.$$

**Proof.** we prove this proposition by using [19, Proposition 2] and the natural isomorphism  $(R/rR \otimes_R -)^b \simeq \text{Hom}_R(R/rR, -^b)$ .  $\square$

As in [18] we say that a right module  $M$  is *RD-flat* if for every RD-exact sequence of left modules  $0 \rightarrow H \rightarrow F \rightarrow L \rightarrow 0$ , the sequence

$$0 \rightarrow M \otimes_R H \rightarrow M \otimes_R F \rightarrow M \otimes_R L \rightarrow 0 \text{ is exact.}$$

The next proposition and its proof is similar to that we know for flat modules. In particular, to prove that  $1 \Rightarrow 5$ , we do a similar proof as in [14, Theorem I.1.2].

**Proposition 1.3.** *Let  $R$  be a ring and  $M$  a right module. Then the following assertions are equivalent.*

- (1)  $M$  is *RD-flat*
- (2)  $M^b$  is *RD-injective*
- (3)  $M \simeq P/Q$ , where  $P$  is *RD-projective* and  $Q$  is a pure submodule of  $P$
- (4) Every RD-exact sequence  $0 \rightarrow Q \rightarrow P \rightarrow M \rightarrow 0$  is pure exact
- (5)  $M$  is direct limit of finite direct sums of cyclically presented modules.

When  $R$  is commutative and  $E$  an injective cogenerator, these conditions are equivalent to the following conditions:

- (2')  $\text{Hom}_R(M, E)$  is *RD-injective*.
- (6)  $M_P$  is *RD-flat* over  $R_P$  for each maximal ideal  $P$ .

Moreover, each direct limit of right *RD-flat* modules is *RD-flat*.

We easily deduce the following proposition.

**Proposition 1.4.** *A right  $R$ -module is *RD-projective* if and only if it is *RD-flat* and pure-projective.*

We say that a right  $R$ -module  $M$  is *RD-coflat* if every RD-exact sequence  $0 \rightarrow M \rightarrow P \rightarrow Q \rightarrow 0$  is pure exact. Then we get the dual of proposition 1.4.

**Proposition 1.5.** *Let  $R$  be a ring. Then:*

- (1) *A right  $R$ -module is *RD-injective* if and only if it is *RD-coflat* and pure-injective.*
- (2) *Every pure submodule of a right *RD-coflat* module is coflat too.*
- (3) *A right  $R$ -module  $M$  is *RD-injective* if and only if it is *RD-coflat* and satisfies the following condition: for any family of submodules  $(N_i)_{i \in I}$ , where  $N_i = (Mr_i : s_i)$ , for some elements  $r_i$  and  $s_i$  of  $R$ , and any family  $(x_i)_{i \in I}$  of elements of  $M$ , if the sets  $x_i + N_i$  have the finite intersection property, then their total intersection is non-empty.*

**Proof.** The first assertion is obvious.

Let  $E$  be a right *RD-coflat* module,  $F$  a right module and  $M$  a pure submodule of  $E$  which is an *RD-submodule* of  $F$ . Let  $H$  be the module defined by the following pushout diagram:

$$\begin{array}{ccc} M & \rightarrow & E \\ \downarrow & & \downarrow \\ F & \rightarrow & H \end{array}$$

It is easy to prove that  $E$  is an *RD-submodule* of  $H$ . It successively follows that  $E$  is a pure submodule of  $H$ ,  $M$  a pure submodule of  $H$  and  $M$  a pure submodule of  $F$ . This prove the second assertion.

For the last assertion, we apply [19, Theorem 2] and we do the same proof as in [19, Theorem 4].  $\square$

*Pure-essential extension, RD-essential extension, pure-injective hull and RD-injective hull* are defined as in [19] or [9, chapter XIII]. We deduce the following corollary. Observe that one gets a partial answer to [9, Problem 47].

**Corollary 1.6.** *Let  $R$  be a ring. Then:*

- *for any right module  $M$  the following conditions are equivalent:*
  - (1)  *$M$  is RD-coflat.*
  - (2)  *$M$  is a pure submodule of an RD-injective module.*
  - (3) *The RD-injective hull of  $M$  is its pure-injective hull.*
  - (4) *For each RD-exact sequence  $0 \rightarrow K \xrightarrow{u} L \xrightarrow{p} F \rightarrow 0$  of right modules, where  $F$  is finitely presented, the following sequence is exact:*

$$0 \rightarrow \text{Hom}_R(F, M) \rightarrow \text{Hom}_R(L, M) \rightarrow \text{Hom}_R(K, M) \rightarrow 0.$$
- *every direct product and every direct sum of RD-coflat right  $R$ -modules are RD-coflat.*

**Proof.** It is obvious that the three first conditions are equivalent.

$1 \Rightarrow 4$ . Let  $M \xrightarrow{v} \widehat{M}$  be the RD-injective hull of  $M$  and  $f : K \rightarrow M$  be a homomorphism. There exists a morphism  $g : L \rightarrow \widehat{M}$  such that  $v \circ f = g \circ u$ . Then, if  $q : \widehat{M} \rightarrow \widehat{M}/M$  is the natural map, there exists a morphism  $h : F \rightarrow \widehat{M}/M$  such that  $h \circ p = q \circ g$ . Since  $v$  is a pure monomorphism, there exists a morphism  $j : F \rightarrow \widehat{M}$  such that  $h = q \circ j$ . It is easy to verify that  $q \circ (g - j \circ p) = 0$ . It follows that there exists a morphism  $l : F \rightarrow M$  such that  $v \circ l = g - j \circ p$ . We easily get that  $v \circ f = v \circ l \circ u$ . Since  $v$  is a monomorphism, thus  $f = l \circ u$ .

$4 \Rightarrow 1$ . Let  $x_1, \dots, x_n \in \widehat{M}$ , and  $\{a_{i,j} \mid 1 \leq i \leq n, 1 \leq j \leq m\}$  be a family of elements of  $R$  such that  $\sum_{i=1}^n x_i a_{i,j} = y_j \in M$ , for each  $j$ ,  $1 \leq j \leq m$ . Let  $F$  be a finitely presented module generated by  $e_1, \dots, e_n$  with the relations  $\sum_{i=1}^n e_i a_{i,j} = 0$  for each  $j$ ,  $1 \leq j \leq m$  and  $h : F \rightarrow \widehat{M}/M$  be the morphism defined by  $h(e_i) = q(x_i)$ ,  $\forall i$ ,  $1 \leq i \leq n$ . We consider the following pushout diagram:

$$\begin{array}{ccc} L & \xrightarrow{p} & F \\ \downarrow & & \downarrow \\ \widehat{M} & \xrightarrow{q} & \widehat{M}/M \end{array}$$

Let  $g$  be the left vertical map. Then  $M$  is isomorphic to  $\ker p$ . It follows that  $M$  is an RD-submodule of  $L$ . Therefore there exists a morphism  $l : L \rightarrow M$  such that  $l(x) = x$ , for each  $x \in M$ . Let  $z_1, \dots, z_n$  be elements of  $L$  such that  $g(z_i) = x_i$  and  $p(z_i) = e_i$  for each  $i$ ,  $1 \leq i \leq n$ . Then we get that  $\sum_{i=1}^n l(z_i) a_{i,j} = y_j$  for each  $j$ ,  $1 \leq j \leq m$ . Hence  $M$  is a pure submodule of  $\widehat{M}$ .

Let  $(M_i)_{i \in I}$  be a family of RD-coflat modules and for each  $i \in I$  let  $E_i$  be the RD-injective hull of  $M_i$ . Then  $E = \prod_{i \in I} E_i$  is RD-injective and  $M = \prod_{i \in I} M_i$  is a pure submodule of  $E$ . It follows that  $M$  is RD-coflat. Whence  $\bigoplus_{i \in I} M_i$  is also RD-coflat since it is a pure submodule of  $M$ .  $\square$

From these previous propositions and [16, Theorem 2.5] we deduce the following:

**Theorem 1.7.** *Let  $R$  be a ring. The following assertions are equivalent:*

- (1) *Every right  $R$ -module is RD-flat*
- (2) *Every left  $R$ -module is RD-coflat.*
- (3) *Every left pure-injective module is RD-injective*

- (4) Every right pure-projective module is RD-projective
- (5) Every right finitely presented module is a summand of a direct sum of cyclically presented modules.
- (6) Every RD-exact sequence of right modules is pure-exact.
- (7) The left-right symmetry of 1 – 6.

As in [16] we will say that  $R$  is an *RD-ring* if it satisfies the equivalent conditions of theorem 1.7. By [16, Proposition 4.5] a commutative ring  $R$  is RD if and only if it is an arithmetic ring.

## 2. DIVISIBILITY AND P-FLATNESS.

If  $a \in R$ ,  $\ell(a)$  (respectively  $r(a)$ ) is the left (respectively right) annihilator of  $a$ . As in [12] we say that a right  $R$ -module  $F$  is *divisible* (or *p-injective*) if for each  $x \in F$  and each  $a \in R$  such that  $r(a) \subseteq (0 : x)$  there exists  $y \in F$  such that  $x = ya$ . Recall that a right module  $M$  is *fp-injective* (or *absolutely pure*) if it is a pure submodule of each overmodule. Then it is easy to check that every fp-injective right  $R$ -module is divisible and RD-coflat. The first assertion of the following proposition is proved in [8, Corollary 2.3] in the commutative case.

**Proposition 2.1.** *The following assertions hold.*

- (1) A right  $R$ -module is injective if and only if it is divisible and RD-injective.
- (2) A right  $R$ -module is fp-injective if and only if it is divisible and RD-coflat.

**Proof.** Only “if” requires a proof. Let  $F$  be a right module and  $E$  its injective hull. If  $F$  is divisible then  $F$  is an RD-submodule of  $E$  by [12, Proposition 3]. Moreover, the inclusion map  $F \rightarrow E$  splits (resp. is pure) if  $F$  is RD-injective (resp. RD-coflat). Hence  $F$  is injective (resp. fp-injective).  $\square$

We say that a right  $R$ -module  $F$  is *p-flat* if for each  $x \in F$  and each  $a \in R$ , the equality  $xa = 0$  implies  $x \in F\ell(a)$ . In [12]  $F$  is called *torsion-free*. The following proposition is the dual of proposition 2.1

**Proposition 2.2.** *The following assertions hold.*

- (1) A right  $R$ -module is projective if and only if it is p-flat and RD-projective.
- (2) A right  $R$ -module is flat if and only if it is p-flat and RD-flat.
- (3) Each p-flat cyclic right module is flat.

**Proof.** It is obvious that every flat right  $R$ -module is RD-flat. It is p-flat by [12, Proposition 3]. Conversely let  $0 \rightarrow K \rightarrow G \rightarrow F \rightarrow 0$  be an exact sequence of right modules. If  $F$  is p-flat then this sequence is RD-exact by [12, Proposition 3]. Moreover, this sequence splits (resp. is pure-exact) if  $F$  is RD-projective (resp. RD-flat). Hence  $F$  is projective (resp. flat).

If  $F$  is a cyclic right module generated by  $x$  and  $A$  a left ideal then each element of  $F \otimes_R A$  is of the form  $x \otimes a$  for some  $a \in A$ . If  $xa = 0$  then there exists  $c \in \ell(a)$  such that  $x = xc$ . It follows that  $x \otimes a = x \otimes ca = 0$ .  $\square$

The following proposition is a generalization of [9, Theorem IV.9.10 and Proposition IX.3.4].

**Proposition 2.3.** *Let  $R$  be a ring. Consider the following conditions:*

- (d) Every divisible right  $R$ -module is fp-injective.
- (t) Every p-flat left  $R$ -module is flat.

Then:

- (1) (d) implies (t) and the converse holds if  $r(a)$  is finitely generated for each  $a \in R$ .
- (2) If  $R$  is RD, then  $R$  satisfies condition (d) and the converse holds if  $R$  is a commutative ring whose principal ideals are p-flat.
- (3) If  $R$  is RD, then  $R$  satisfies condition (t) and the converse holds if  $R$  is a commutative ring whose principal ideals are p-flat.

The following lemmas are needed to prove this proposition. The first is well known: see [9, Lemma XIII.2.3]. Here we give a different proof.

**Lemma 2.4.** *The following assertions hold for a ring  $R$ .*

- Every right  $R$ -module  $M$  is isomorphic to a pure submodule of  $(M^b)^b$ .
- If  $R$  is commutative and  $E$  an injective  $R$ -cogenerator then  $M$  is isomorphic to a pure submodule of  $\text{Hom}_R(\text{Hom}_R(M, E), E)$ .

**Proof.** Let  $\Phi_M : M \rightarrow (M^b)^b$  be the natural map. It is easy to verify that  $\text{Hom}_{\mathbb{Z}}(\Phi_M, \mathbb{Q}/\mathbb{Z}) \circ \Phi_{M^b} = 1_{M^b}$ . Since the epimorphism  $\text{Hom}_{\mathbb{Z}}(\Phi_M, \mathbb{Q}/\mathbb{Z})$  splits, we deduce that  $\Phi_M$  is a pure monomorphism by proposition 1.1.  $\square$

**Lemma 2.5.** *Let  $R$  be a ring and  $F$  a left module. Then:*

- (1)  $F$  is p-flat if and only if  $F^b$  is divisible
- (2) If  $F^b$  is p-flat then  $F$  is divisible and the converse holds if  $\ell(a)$  is finitely generated for each  $a \in R$ .

**Proof.** Let  $(S): 0 \rightarrow K \rightarrow L \rightarrow F \rightarrow 0$  be an exact sequence where  $L$  is a free module. Then  $L^b$  is injective. By [12, Proposition 3],  $(S)$  is RD-exact if and only if  $F$  is p-flat, and  $(S)^b$  is RD-exact if and only if  $F^b$  is divisible. We conclude by proposition 1.2.

If  $F^b$  is p-flat then  $(F^b)^b$  is divisible. Thus  $F$  is divisible by lemma 2.4. Conversely we have the following commutative diagram:

$$\begin{array}{ccc} F^b \otimes_R Ra & \rightarrow & F^b \otimes_R R \\ \downarrow & & \downarrow \\ 0 \rightarrow (\text{Hom}_R(Ra, F))^b & \rightarrow & (\text{Hom}_R(R, F))^b \end{array}$$

where the vertical maps are natural homomorphisms. These homomorphisms are bijective because  $Ra$  and  $R$  are finitely presented. Since  $F$  is divisible then the lower sequence is exact. It follows that  $F^b$  is p-flat.  $\square$

**Proof of proposition 2.3.** (1) Let  $F$  be a p-flat left  $R$ -module. Then  $F^b$  is fp-injective. By [19, Proposition 7] this module is pure-injective. It follows that it is injective, whence  $F$  is flat. Conversely if  $F$  is a divisible right  $R$ -module, then we get that  $F^b$  is flat. It follows that  $(F^b)^b$  is injective. Then  $F$  is fp-injective by lemma 2.4.

(2) The first assertion holds by proposition 2.1 or [16, Lemma 2.16]. Conversely every ideal  $A$  is p-flat. It follows that  $A^b$  is injective, whence  $A$  is flat.

(3) The first assertion holds by proposition 2.2 or [16, Lemma 2.16]. The second is obvious.  $\square$

**Remark 2.6.** Let  $R$  be a ring whose principal left ideals are p-flat. If  $R$  satisfies (t) or (d) then we deduce that every left (and right) ideal is flat. When  $R$  is not

commutative then  $R$  is not necessarily RD: [16, example 2.19(a)] is an example of hereditary ring which is not RD.

If  $R$  is commutative then every principal ideal is p-flat if and only if  $R_P$  is an integer domain for each maximal ideal  $P$ .

### 3. RD-INJECTIVE ARTINIAN MODULES.

The aim of this section is to characterize commutative rings for which every artinian module is RD-injective. Throughout this section all rings are commutative. Recall that a family  $\mathcal{F}$  of submodules of an  $R$ -module  $M$  is *direct* if,  $\forall (U, V) \in \mathcal{F}^2$  there exists  $W \in \mathcal{F}$  such that  $U + V \subseteq W$ . We will prove the two following theorems.

**Theorem 3.1.** *Let  $R$  be a ring. Then the following conditions are equivalent:*

- (1) *Every simple  $R$ -module is RD-flat.*
- (2) *Every simple  $R$ -module is RD-injective.*
- (3) *Every  $R$ -module of finite length is RD-flat.*
- (4) *Every  $R$ -module of finite length is RD-injective.*
- (5) *Every artinian  $R$ -module is RD-flat.*
- (6) *Every noetherian  $R$ -module is RD-coflat.*
- (7) *For each maximal ideal  $P$  the family of proper principal ideals of  $R_P$  is direct.*

Moreover, if  $R$  satisfies these conditions, then every artinian module is a (finite) direct sum of uniserial modules and every noetherian module is a direct sum of 2-generated submodules. However the converse is not true.

**Proof.** It is obvious that  $5 \Rightarrow 3 \Rightarrow 1$  and  $6 \Rightarrow 4 \Rightarrow 2$ .

$1 \Leftrightarrow 2$ . We set  $E = \prod_{P \in \text{Max}(R)} E_R(R/P)$  where  $\text{Max}(R)$  is the set of maximal ideals of  $R$ . Then  $E$  is an injective cogenerator and for each simple module  $S$ ,  $S \simeq \text{Hom}_R(S, E)$ . We conclude by proposition 1.3.

$3 \Rightarrow 5$  since each artinian module is direct limit of modules of finite length.  $\square$

Let  $R$  be a ring. For every maximal ideal  $P$  we denote by  $L(P)$  the sum of all submodules of finite length of  $E(R/P)$  and we set  $J(P) = \bigcap_{n \in \mathbb{N}} (PR_P)^n$ .

**Theorem 3.2.** *Let  $R$  be a ring. Then the following conditions are equivalent:*

- (1) *Every artinian  $R$ -module is RD-injective.*
- (2) *Every noetherian  $R$ -module is RD-flat.*
- (3)  *$R/P$  and  $L(P)$  are RD-injective for every maximal ideal  $P$ .*
- (4) *For each maximal ideal  $P$ , the family of proper principal ideals of  $R_P$  and the family of principal ideals of  $R_P$  contained in  $J(P)$  are direct.*

Moreover, if  $P = P^2$  for each maximal ideal  $P$ , then these conditions are equivalent to conditions of theorem 3.1. In this case, every noetherian module and every artinian module is semi-simple.

To prove this theorem and complete the proof of theorem 3.1 we study our problem in the local case.

In the sequel we assume that  $R$  be a local ring. We denote  $P$  its maximal ideal,  $E$  the injective hull of  $R/P$  and  $J = \bigcap_{n \in \mathbb{N}} P^n$ . Let  $A$  be a proper ideal of  $R$  and  $A^*$  the set of nonzero principal ideals contained in  $A$ . For every  $r^* \in A^*$  let  $\phi_{r^*} : R/r^* \rightarrow R/A$  be the homomorphism defined by  $\phi_{r^*}(1 + r^*) = 1 + A$  and



$\phi : \bigoplus_{r^* \in A^*} R/r^* \rightarrow R/A$  be the homomorphism defined by the family  $(\phi_{r^*})_{r^* \in A^*}$ . We put  $E[A] = \{e \in E \mid A \subseteq (0 : e)\}$ . Recall that  $\text{Hom}_R(R/A, E) \simeq E[A]$ . The two following lemmas hold.

**Lemma 3.3.** *Then  $\ker \phi$  is an RD-submodule of  $\bigoplus_{r^* \in A^*} R/r^*$ .*

**Proof.** Let  $s^*$  be a proper principal ideal of  $R$  and  $f : R/s^* \rightarrow R/A$  be a nonzero homomorphism. We have  $f(1 + s^*) = a + A$  where  $a \in R \setminus A$ . Then  $as^* \subseteq A$ . Let  $g : R/s^* \rightarrow \bigoplus_{r^* \in A^*} R/r^*$  be the composition of  $\alpha : R/s^* \rightarrow R/as^*$ , defined by  $\alpha(1 + s^*) = a + as^*$ , with the inclusion map  $R/as^* \rightarrow \bigoplus_{r^* \in A^*} R/r^*$ . Then  $f = \phi \circ g$ . We conclude by [19, Proposition 2].  $\square$

**Lemma 3.4.** *Let  $R$  be a local ring, and  $A$  a proper ideal. Then:*

- (1)  *$E[A]$  is RD-injective if and only if  $R/A$  is RD-flat*
- (2)  *$R/A$  is RD-flat if and only if  $A^*$  is direct.*
- (3) *If  $A$  is finitely generated then  $E[A]$  is RD-injective if and only if  $A$  is principal.*

**Proof.** 1 is an immediate consequence of proposition 1.3.

Assume that  $R/A$  is RD-flat. By proposition 1.3  $\ker \phi$  is a pure submodule of  $\bigoplus_{r^* \in A^*} R/r^*$ . Let  $I \subseteq A$  be a finitely generated ideal and  $f : R/I \rightarrow R/A$  be the homomorphism defined by  $f(1 + I) = 1 + A$ . By [19, Proposition 3] there exists an homomorphism  $g : R/I \rightarrow \bigoplus_{r^* \in A^*} R/r^*$  such that  $f = \phi \circ g$ . There exists a finite family  $F$  of  $A^*$  such that  $g(1 + I) = \sum_{r^* \in F} a_{r^*} + r^*$ . We deduce that  $1 + A = \sum_{r^* \in F} a_{r^*}(1 + A)$ . Since  $R$  is local there exists  $r^* \in F$  such that  $a_{r^*}$  is a unit of  $R$ . Consequently  $I = (0 : 1 + I) \subseteq (0 : a_{r^*} + r^*) = r^*$ .

Conversely if  $R/A = \varinjlim_{r^* \in A^*} R/r^*$  then  $R/A$  is RD-flat by proposition 1.3.  $\square$

The next theorem characterizes local rings for which every simple module is RD-injective (and RD-flat). It is an immediate consequence of lemma 3.4.

**Theorem 3.5.** *Let  $R$  be a local ring. Then  $R/P$  is RD-flat if and only if  $P^*$  is direct.*

The following corollaries will be useful to provide examples of rings for which every simple module is RD-flat.

**Corollary 3.6.** *Let  $R$  be a local ring. Consider the two following conditions:*

- (1)  *$R/P$  is RD-flat.*
- (2) *There exists an ideal  $I \subset P$  such that  $R/I$  is a valuation ring and for each  $r \in R \setminus I$ ,  $I \subset Rr$ .*

*Then  $2 \Rightarrow 1$ . When  $P$  is finitely generated the two conditions are equivalent.*

**Proof.** First we assume that  $R/P$  is RD-flat and  $P$  is finitely generated. By theorem 3.5  $P$  is principal. We put  $P = Rp$  and  $I = J$ . Let  $a \in P \setminus J$ . There exists  $n \in \mathbb{N}$  such that  $a \notin Rp^n$ . We may assume that  $n$  is minimal. Then  $a = bp^{n-1}$  and since  $a \notin Rp^n$ ,  $b$  is a unit of  $R$ . Hence  $I \subset Rp^{n-1} = Ra$  and  $R/I$  is a valuation ring. Conversely, if  $a$  and  $b$  are elements of  $P \setminus I$  it is easy to prove that  $Ra \subseteq Rb$  or  $Rb \subseteq Ra$ . By theorem 3.5  $R/P$  is RD-flat.  $\square$

**Corollary 3.7.** *Let  $R$  be a local ring. Assume that  $R$  satisfies the ascending chain condition on principal ideals. Then  $R$  is a valuation ring if and only if  $R/P$  is RD-injective.*

**Proof.** Assume that  $R/P$  is RD-injective. We claim that there exists only one maximal principal ideal  $pR$  and that  $pR = P$ . If there exist at least two maximal principal ideals  $p^*$  and  $q^*$  then  $p^* + q^*$  is strictly contained in a proper principal ideal. Consequently we obtain a contradiction. If  $J \ni a \neq 0$  then, by induction, we get a strictly ascending chain of ideals  $(Rc_n)_{n \in \mathbb{N}^*}$  such that  $a = pc_1$  and  $c_n = pc_{n+1}, \forall n \in \mathbb{N}^*$ . Hence  $J = 0$ . We complete the proof by using corollary 3.6.  $\square$

We give two examples of local rings  $R$  which are not valuation rings and such that  $R/P$  is RD-injective.

**Example 3.8.** Let  $D$  a valuation domain,  $Q$  its field of fractions,  $H = Q^2$ .

We put  $R = \left\{ \begin{pmatrix} d & h \\ 0 & d \end{pmatrix} \mid d \in D, h \in H \right\}$  the trivial extension of  $D$  by  $H$ .

If  $I = \left\{ \begin{pmatrix} 0 & h \\ 0 & 0 \end{pmatrix} \mid h \in H \right\}$  then  $R/I \simeq D$ . By using that  $H$  is a divisible  $D$ -module, it is easy to prove that  $I \subset Ra$  for every  $a \in R \setminus I$ .

**Example 3.9.** Let  $K$  be a field and  $T$  the factor ring of the polynomial ring  $K[X_{p,n} \mid p, n \in \mathbb{N}]$  modulo the ideal generated by  $\{X_{p,n} - X_{p,n+1}X_{p+1,n} \mid p, n \in \mathbb{N}\}$ . We denote  $x_{p,n}$  the image of  $X_{p,n}$  in  $T$  and  $P$  the maximal ideal of  $T$  generated by  $\{x_{p,n} \mid p, n \in \mathbb{N}\}$ . We put  $R = T_P$ . Let  $p, q, m, n \in \mathbb{N}$ . Clearly  $x_{p,n} \in Rx_{p+q,n} \subset Rx_{p+q,n+m}$  and  $x_{q,m} \in Rx_{p+q,m} \subset Rx_{p+q,n+m}$ . If  $a \in R$  then there exists  $b \in T$  and  $s \in T \setminus P$  such that  $a = \frac{b}{s}$ . Then  $b$  is a linear combination with coefficients in  $K$  of finite products of elements of  $\{x_{p,n} \mid p, n \in \mathbb{N}\}$ . From above we deduce that  $a \in Rx_{p,n}$  for some  $(p, n) \in \mathbb{N}^2$ . It follows from this property that every proper finitely generated ideal is contained in a proper principal ideal.

If we prove that  $Rx_{p+1,n}$  and  $Rx_{p,n+1}$  are not comparable for every  $(p, n) \in \mathbb{N}^2$  then  $R$  doesn't satisfy the condition 2 of corollary 3.6. Else there exists  $p, n \in \mathbb{N}$  such that  $x_{p,n} \notin I$ . It follows that  $x_{p+1,n}$  and  $x_{p,n+1}$  are not elements of  $I$ , whence  $Rx_{p+1,n}$  and  $Rx_{p,n+1}$  are comparable. We get a contradiction.

For every  $\ell \in \mathbb{N}$ , let  $T_\ell = K[X_{p,\ell-p} \mid 0 \leq p \leq \ell]$ ,  $\varphi_\ell : T_\ell \rightarrow T_{\ell+1}$  be the homomorphism defined by  $\varphi_\ell(X_{p,\ell-p}) = X_{p+1,\ell-p}X_{p,\ell-p+1}, \forall p, 0 \leq p \leq \ell$  and  $P_\ell$  be the ideal of  $T_\ell$  generated by  $\{X_{p,\ell-p} \mid 0 \leq p \leq \ell\}$ . Clearly  $T$  is the direct limit of  $(T_\ell, \varphi_\ell)_{\ell \in \mathbb{N}}$ . Now assume that  $x_{p,n+1} \in Rx_{p+1,n}$ . Then there exist  $s \in T \setminus P$  and  $c \in T$  such that  $sx_{p,n+1} = cx_{p+1,n}$ . There exists an integer  $\ell \geq p+n+1$  such that  $c$  and  $s$  are elements of  $T_\ell$ . The following formulæ hold:  $x_{p,n+1} = \prod_{j=0}^{\ell-n-p-1} X_{p+j,\ell-p-j}^{\binom{\ell-n-p-1}{j}}$  and  $x_{p+1,n} = \prod_{j=0}^{j=\ell-n-p-1} X_{p+j+1,\ell-p-j-1}^{\binom{\ell-n-p-1}{j}}$ . Clearly  $x_{p,n+1}$  and  $x_{p+1,n}$  are two different monomials of  $T_\ell$ . We have  $s = k+t$  and  $c = a+b$  where  $k, a \in K, t, b \in P_\ell$  and  $k \neq 0$ . We get the following equality:  $ax_{p+1,n} - kx_{p,n+1} + bx_{p+1,n} - tx_{p,n+1} = 0$ . It is obvious that the degree (in  $T_\ell$ ) of  $x_{p+1,n}$  and  $x_{p,n+1}$  is less than the degree of each monomial of  $bx_{p+1,n}$  and  $tx_{p,n+1}$ . It follows that  $k = a = 0$ . Hence we get a contradiction. In the same way we prove that  $x_{p+1,n} \notin Rx_{p,n+1}$ .

Now we study the RD-injectivity of artinian modules over local rings.

**Theorem 3.10.** *The following conditions are equivalent for a local ring  $R$ :*

- (1) *Every artinian  $R$ -module is RD-injective.*
- (2)  *$R/P$  and  $R/J$  are RD-flat.*
- (3)  *$R/P$  and  $L(P)$  are RD-injective.*

(4)  $P^*$  and  $J^*$  are direct.

**Proof.** Assume that  $P$  is not finitely generated and  $R/P$  is RD-injective. Let  $a \in P$ . Then there exists  $b \in P \setminus Ra$ . We deduce that  $Ra \subset Ra + Rb \subseteq Rc$  for some  $c \in P$ . It follows that  $a = cd$  for some  $d \in P$ , whence  $P = P^2$ . In this case every artinian module is semi-simple of finite length and  $L(P)$  is simple.

Assume that  $P$  is finitely generated and  $R/P$  is RD-injective. By corollary 3.6 and its proof,  $P = pR$  for some  $p \in P$ , and  $R' = R/J$  is a noetherian valuation ring. Let  $M$  be an indecomposable artinian  $R$ -module. For every  $x \in M$ ,  $Rx$  is a module of finite length. It follows that there exists  $n \in \mathbb{N}$  such that  $p^n \in (0 : x)$ . We deduce that  $M$  is an  $R'$ -module. If  $U = E_{R'}(M)$  then  $U \simeq E_{R'}(R/P)^m = E[J]^m$  for some integer  $m$ . We put  $U = U_1 \oplus \cdots \oplus U_m$  where  $U_j \simeq E[J]$  for each  $j$ ,  $1 \leq j \leq m$ . Thus  $M \cap U_j \neq 0$  for every  $j$ ,  $1 \leq j \leq m$ . By [9, Proposition XII.2.1] there exists  $i$ ,  $1 \leq i \leq m$ , such that  $M \cap U_i$  is a pure  $R'$ -submodule of  $M$ . It follows that  $M \cap U_i$  is a summand of  $M$ . Hence  $M$  is isomorphic to a submodule of  $E[J]$  which is uniserial by [10, Theorem]. If  $M \neq E[J]$  there exists a positive integer  $n$  such that  $Rp^n = \text{ann}(M)$ . In this case we easily deduce that  $M$  is RD-injective. If  $M = E[J]$ , by proposition 1.3  $M$  is RD-injective if and only if  $R/J$  is RD-flat. Let us observe that  $E[J] = L(P)$ . The proof is now complete.  $\square$

The following corollary will be useful to provide examples of rings for which every artinian module is RD-injective.

**Corollary 3.11.** *Let  $R$  be a local ring. Assume that each artinian  $R$ -module is RD-injective and  $P$  is finitely generated. Suppose that  $J \neq J^2$ . Let  $I$  be the inverse image of  $I' = \bigcap_{n \in \mathbb{N}} (JR_J)^n$  by the natural map  $h : R \rightarrow R_J$ . Then  $R/I$  is a discrete valuation ring and for every  $r \in R \setminus I$ ,  $I \subset rR$ . Moreover, if  $J$  is nilpotent, then  $R$  is a discrete valuation ring.*

**Proof.** Let us observe that  $J \neq 0$  and it is a prime ideal. By corollary 3.6  $R/J$  is a discrete rank one valuation domain and  $J \subset rR$  for each  $r \in R \setminus J$ . Let  $q \in J \setminus J^2$  and  $r \in J$ . Then  $Rq + Rr \subseteq Rc$  for some  $c \in J \setminus J^2$ . There exist  $s \in R \setminus J$  and  $t \in R$  such that  $q = sc$  and  $r = tc$ . It follows that  $\frac{r}{1} = \frac{tq}{s}$ . Consequently  $JR_J$  is principal over  $R_J$  and  $R_J/I'$  is a valuation ring by corollary 3.6. We may assume that  $I = 0$ ,  $I' = 0$  and  $h$  is a monomorphism. For every  $0 \neq a \in J$  there exist a unit  $u$  and two integers  $m \geq 1$  and  $\alpha \geq 0$  such that  $\frac{a}{1} = \frac{uq^m}{p^\alpha}$ . There exists  $c \in J \setminus J^2$  such that  $q = cp^\alpha$ . It follows that  $a = ucq^{m-1}$ . If  $0 \neq b \in J$ , in the same way there exist two integers  $n \geq 1$  and  $\beta \geq 0$ , a unit  $v$  and  $d \in J \setminus J^2$  such that  $b = vdq^{n-1}$  and  $q = dp^\beta$ . Let  $\gamma = \max(\alpha, \beta)$ . There exist  $c'$  and  $d'$  in  $J \setminus J^2$  such that  $c = c'p^{\gamma-\alpha}$  and  $d = d'p^{\gamma-\beta}$ . It follows that  $q = c'p^\gamma = d'p^\gamma$ . Since  $h$  is a monomorphism, we deduce that  $c' = d'$ . Assume that  $m = n$ . It follows that  $b \in Ra$  if  $\alpha \geq \beta$ . Now assume that  $n > m$ . If  $\beta \leq \alpha$  then  $b \in Ra$ . If  $\beta > \alpha$ , let  $q' \in J$  such that  $q = q'p^{\beta-\alpha}$ . Then  $b = vp^{\gamma-\alpha}c'q'q^{n-2}$  and  $b = vu^{-1}q'q^{n-m-1}a$ .

Now suppose that  $J$  is nilpotent. Let  $n$  be the least integer such that  $(JR_J)^n = 0$ . We easily deduce that  $J^{n+1} = 0$ . We claim that  $J^n = \ker h$ . This is obvious if  $n = 1$ . Assume that  $n > 1$ . The inclusion  $J^n \subseteq \ker h$  is easy. Conversely let  $0 \neq r \in \ker h$ . There exists  $c \in J \setminus J^2$  such that  $Rr + Rq \subseteq Rc$ . The following equality  $r = cd$  holds for some  $d \in R$ . Since  $n > 1$ ,  $d \in J$ . If  $d \in \ker h$  then there exists  $s \in R \setminus J$  such that  $sd = 0$ . But there exists  $c' \in J$  such that  $c = sc'$ . It

follows that  $r = dsc' = 0$ . Hence a contradiction. Consequently  $d \notin J^n$ . Let  $m$  be the greatest integer such that  $d \in J^m$ . Then  $m < n$ ,  $\frac{d}{1} = \frac{q^m}{s'}$  and  $\frac{c}{1} = \frac{q}{t}$  for some  $s', t \in R \setminus J$ . It follows that  $\frac{r}{1} = \frac{q^{m+1}}{s't}$ . We deduce that  $m = n - 1$  and  $r \in J^n$ . Let  $a$  and  $b$  be two nonzero elements of  $J$ . Then there exists  $c \in J$  such that  $Ra + Rb + Rq = Rc$ . It follows that  $a = ca'$  and  $b = cb'$  for some  $a'$  and  $c'$  in  $R$ . Since  $a \neq 0$  and  $b \neq 0$ ,  $a'$  and  $b'$  are not in  $J^n$ . From the first part of the proof we deduce that  $R/J^n$  is a valuation ring, so there exist  $r \in R$  and  $s \in J^n$  such that  $b' = ra' + s$ . It follows that  $b = ra$ .  $\square$

**Example 3.12** (Examples). If, in our example 3.8,  $D$  is a discrete valuation domain of Krull dimension  $\geq 2$ , then every artinian  $R$ -module is RD-injective. But this property is not satisfied if  $D$  is a discrete rank one valuation domain. However, if  $R$  is the ring defined in our example 3.9, then every artinian  $R$ -module is RD-injective.

Let  $R$  be the ring defined by the following pullback diagram of ring maps:

$$\begin{array}{ccc} R & \rightarrow & T \\ \downarrow & & \downarrow \\ V & \rightarrow & K \end{array}$$

where  $V$  is a discrete rank one valuation domain,  $K$  its field of fractions and  $T$  a local ring of residual class field  $K$ . Then  $R$  is local and its maximal ideal  $P$  is the inverse image of the maximal ideal of  $V$ . Clearly  $R/P$  is RD-injective. If  $J = R \cap Q$ , where  $Q$  is the maximal ideal of  $T$ , then  $T = R_J$  and  $V = R/J$ . It is easy to prove that  $R/J$  is RD-flat if and only if  $T/Q$  is RD-flat over  $T$ .

We complete the proofs of theorem 3.1 and theorem 3.2.

**Proof of theorem 3.1.**

1  $\Leftrightarrow$  7 by theorem 3.5.

Assume that  $PR_P$  is not finitely generated over  $R_P$  for every maximal ideal  $P$ . If the equivalent conditions 1 and 2 are satisfied, then  $P^2R_P = PR_P$  and  $P = P^2$  for every maximal ideal  $P$ . It follows that each artinian  $R$ -module is semi-simple of finite length. Hence each artinian module is RD-flat and RD-injective. Let  $M$  be a noetherian  $R$ -module and  $R' = R/\text{ann}(M)$ . Then the equality  $P^2R_P = PR_P$  implies that  $R'_P = R/P$ . It follows that  $R'$  and  $M$  are semi-simple.

Now we assume that there exists a maximal ideal  $P$  such that  $P \neq P^2$ .

1  $\Rightarrow$  3 and 2  $\Rightarrow$  4. Let  $M$  be a module of finite length. By [2, théorème p.368] there exists a finite family  $F$  of maximal ideals such that  $M \simeq \prod_{P \in F} M_P$ . Consequently we may assume that  $M$  is indecomposable,  $R$  is local and its maximal ideal  $P = pR$ . There exists an integer  $n$  such that  $p^n$  annihilates  $M$ . Then it is easy to show that  $M$  is RD-flat and RD-injective.

5  $\Rightarrow$  6. Let  $M$  be a noetherian module. By [6, lemme 1.3] the diagonal map  $M \rightarrow \prod_{P \in \text{Max}(R)} M_P$  is a pure monomorphism. For every maximal ideal  $P$  we put  $F_P(M) = \text{Hom}_{R_P}(\text{Hom}_{R_P}(M_P, E(R/P)), E(R/P))$ . On the other hand, by lemma 2.4,  $M_P$  is a pure submodule of  $F_P(M)$ . It follows that  $M$  is a pure submodule of  $\prod_{P \in \text{Max}(R)} F_P(M)$ . It is enough to prove that  $F_P(M)$  is RD-injective. Consequently we may assume that  $R$  is local,  $P$  is its maximal ideal and  $E = E(R/P)$ . Let  $J = \bigcap_{n \in \mathbb{N}} R p^n$ ,  $E' = E[J]$  and  $R' = R/J$ . Thus  $E'$  and  $\text{Hom}_R(M, E) \simeq \text{Hom}_{R'}(M, E')$  are artinian. It follows that  $\text{Hom}_R(\text{Hom}_R(M, E), E)$  is RD-injective.

Let  $M$  be an indecomposable artinian module. By [2, Théorème p.368], there exists a maximal ideal  $P$  such that  $M$  is an  $R_P$ -module and every  $R$ -submodule of  $M$  is also an  $R_P$ -submodule. As in the proof of theorem 3.10 we prove that  $M$  is uniserial.

If  $M$  is a noetherian module then  $R' = R/\text{ann}(M)$  is a noetherian RD-ring. Consequently  $R'$  is a finite product of Dedekind domains and artinian valuation rings. It follows that  $M$  is a direct sum of 2-generated submodules.  $\square$

The following example shows that the converse of the last assertion of theorem 3.1 is not true.

**Example 3.13.** Let  $K$  be a field and for every  $n \in \mathbb{N}$ , let  $R_n$  be the localization of the polynomial ring  $K[X_n, Y_n]$  at the maximal ideal generated by  $\{X_n, Y_n\}$  and  $P_n$  the maximal ideal of  $R_n$ . For each  $n \in \mathbb{N}$ , let  $\delta_n : R_n \rightarrow R_{n+1}$  be the ring homomorphism defined by  $\delta_n(X_n) = X_{n+1}^2$  and  $\delta_n(Y_n) = Y_{n+1}^2$ . Let  $R$  be the direct limit of the system  $(R_n, \delta_n)_{n \in \mathbb{N}}$ . Then  $R$  is local and its maximal ideal  $P$  is the direct limit of  $(P_n)_{n \in \mathbb{N}}$ . It is obvious that  $P^2 = P$ . Hence every artinian  $R$ -module is semi-simple of finite length. Let  $r \in R$  such that  $X_0$  and  $Y_0$  are in  $Rr$ . We may assume that there exist  $n \in \mathbb{N}$ ,  $s, t \in R_n$  such that  $X_n^{2^n} = rs$  and  $Y_n^{2^n} = rt$ . It follows that  $tX_n^{2^n} = sY_n^{2^n}$ . Since  $R_n$  is a unique factorization domain there exists  $u \in R_n$  such that  $s = uX_n^{2^n}$  and  $t = uY_n^{2^n}$ . We deduce that  $(1 - ru)X_n^{2^n} = 0$ . Hence  $r \notin P$  since  $R$  is local. Consequently  $R$  doesn't satisfy the equivalent conditions of theorem 3.1.

**Proof of theorem 3.2.**

$3 \Leftrightarrow 1$ . Let  $M$  be an artinian module. By [2, Théorème p.368] we may assume that  $R$  is local. Hence it is an immediate consequence of theorem 3.10.

$1 \Rightarrow 2$ . By proposition 1.3 RD-flatness is a local property, so we may assume that  $R$  is local. Since  $R/P$  is RD-flat,  $R/J$  is noetherian and  $E[J] = E_{R/J}(R/P)$  is artinian. Hence, if  $M$  is a noetherian module then

$$\text{Hom}_R(M, E_R(R/P)) = \text{Hom}_{R/J}(M, E[J]) \text{ is artinian.}$$

We conclude by proposition 1.3.

$2 \Rightarrow 3$ . We may assume that  $R$  is local. Since  $R/P$  is RD-flat, we may suppose that  $P$  is principal. By corollary 3.6,  $R/J$  is noetherian. By proposition 1.3 it follows that  $L(P) = E[J]$  is RD-injective.

$3 \Leftrightarrow 4$  by theorem 3.10.  $\square$

#### 4. ON THE RD-INJECTIVITY OF $R$ .

In this section, we study the RD-injectivity of the module  $R$  when  $R$  is a commutative ring. Some examples of artinian rings  $R$ , such that  $R$  is not an RD-injective module, are known: [5, Example 1] and [15, Example 1]. When  $R$  is local we use the same notations as in section 3. We get the following:

**Proposition 4.1.** *Let  $R$  be a local ring. Consider the following conditions:*

- (1)  $R \simeq E$ .
- (2)  $R$  is a quasi-Frobenius ring.
- (3)  $R$  is a divisible module.
- (4)  $E$  is RD-flat.
- (5)  $E$  is RD-projective.

- (6)  $E$  is  $p$ -flat.
- (7)  $R$  is an RD-injective module.

Then:

- The conditions 1 and 5 are equivalent.
- If  $R$  is linearly compact, condition 3 is equivalent to condition 6 and condition 4 is equivalent to condition 7.
- If  $R$  is linearly compact and if  $R$  is an essential extension of a semi-simple module, the conditions 1 and 3 are equivalent.
- All conditions are equivalent if  $R$  is artinian.

**Proof.**  $1 \Rightarrow 5$  is obvious.

$5 \Rightarrow 1$ . Since  $R$  is local,  $E$  is cyclically presented by [19, Corollary 2]. It follows that  $E \simeq R$  because  $E$  is faithful.

If  $R$  is linearly compact then  $R \simeq \text{Hom}_R(E, E)$  by [1, Corollary 10]. It follows that  $7 \Leftrightarrow 4$  by proposition 1.3 and  $3 \Leftrightarrow 6$  by lemma 2.5.

$3 \Leftrightarrow 1$ . Only  $3 \Rightarrow 1$  requires a proof. For each  $a \in R$ ,  $Ra = (0 : (0 : a))$ . Hence the socle of  $R$  is simple. By [1, Theorem 7]  $R$  is self-injective.

It is known that  $2 \Leftrightarrow 1$  if  $R$  is artinian. The implication  $1 \Rightarrow 7$  always holds.

$4 \Rightarrow 5$  because  $E$  is finitely presented if  $R$  is artinian. This completes the proof.  $\square$

From [7, Example 4.3], it is easy to prove that a local complete noetherian ring  $R$  is a rank one discrete valuation domain if and only if  $R$  is an RD-injective module. We can extend this result to the following:

**Proposition 4.2.** *Let  $R$  be a coherent ring of finite global weak dimension. Then  $R$  is semihereditary if and only if  $R$  is an RD-coflat module.*

**Proof.** Only “if” requires a proof. Let  $n$  be the global weak homological dimension of  $R$ . By [4, Proposition 3.3]  $R_P$  is an integral domain for every maximal ideal  $P$ . It follows that each submodule of a free module is  $p$ -flat. If  $n > 1$  then there exists an exact sequence:

$$0 \longrightarrow F_n \xrightarrow{u_n} F_{n-1} \xrightarrow{u_{n-1}} F_{n-2} \dots \dots F_1 \xrightarrow{u_1} F_0,$$

where  $F_i$  is a free module of finite rank, for every  $i$ ,  $0 \leq i \leq n$ , and such that  $\text{Im } u_{n-1}$  is not projective. Since  $\text{Im } u_{n-1}$  is  $p$ -flat,  $F_n$  is an RD-submodule of  $F_{n-1}$ . The RD-coflatness of  $F_n$  implies that  $\text{Im } u_{n-1}$  is projective. From this contradiction we deduce that  $n \leq 1$ , whence  $R$  is semihereditary.  $\square$

## 5. RINGS WHOSE RD-INJECTIVE RIGHT MODULES ARE $\Sigma$ -RD-INJECTIVE.

In this section we will prove the following theorem 5.1. In the sequel, for every right  $R$ -module  $M$ , we denote its RD-injective hull by  $\widehat{M}$ .

As in [13, Observation 3(2)], if  $N$  is a subgroup of a right module  $M$  over a ring  $R$ , we say that  $N$  is a *finite matrix subgroup* if it is the kernel of the following map  $M \rightarrow M \otimes_R X$ , defined by  $m \rightarrow m \otimes x$ , where  $X$  is a finitely presented left module,  $x \in X$  and  $m \in M$ . For instance, for any  $r, s \in R$ ,  $(Mr : s)$  is a finite matrix subgroup: it is the kernel of the following map:  $M \rightarrow M \otimes_R R/Rr$  defined by  $m \rightarrow m \otimes s$ . If  $s$  is a unit then  $(Mr : s) = Mr$ .

Recall that a right  $R$ -module  $M$  is  $\Sigma$ -pure-injective if  $M^{(I)}$  is pure-injective for each index set  $I$ . A ring  $R$  is said to be *right perfect* if every flat right  $R$ -module is projective.

**Theorem 5.1.** *Let  $R$  be a ring. Consider the following conditions:*

- (1) *Every RD-coflat right  $R$ -module is RD-injective.*
- (2) *Each direct sum of right RD-injective modules is RD-injective.*
- (3) *Every RD-injective right  $R$ -module is  $\Sigma$ -pure-injective.*
- (4) *Every RD-injective right  $R$ -module is a direct sum of indecomposable submodules.*
- (5)  *$R$  is right artinian and RD-injective hulls of finitely generated right modules are finite direct sums of indecomposable submodules.*
- (6)  *$R$  is a finite product of artinian valuation rings or finite rings.*

*Then:*

- *Conditions 1, 2, 3 and 4 are equivalent and imply condition 5.*
- *When  $R$  is commutative, the six conditions are equivalent.*

**Proof.**  $1 \Rightarrow 2$  follows from corollary 1.6.

$2 \Rightarrow 3$  is obvious.

$3 \Rightarrow 1$ . Let  $M$  be a right RD-coflat module. Then  $\widehat{M}$  is  $\Sigma$ -pure-injective. We conclude by [13, Corollary 8].

$4 \Leftrightarrow 3$  follows from [13, Proposition 9 and Theorem 10].

$2 \Rightarrow 5$ . From proposition 2.1 we deduce that each direct sum of right injective modules is injective. Hence  $R$  is right noetherian. Since  $\widehat{R}$  is  $\Sigma$ -pure-injective then it satisfies the descending chain condition on its finite matrix subgroups by [13, Observation 3(4) and Theorem 6]. Consequently the family  $(\widehat{R}r)_{r \in R}$  satisfies the descending chain condition. Since  $Rr = \widehat{R}r \cap R$  for each  $r \in R$ , it follows that  $R$  verifies this descending chain condition on principal right ideals. Hence  $R$  is left perfect by [3, Theorem P]. By [17, Théorème 5 p.130]  $R$  also satisfies the descending chain condition on finitely generated right ideals. Hence  $R$  is right artinian.

Let  $M$  be a finitely generated right  $R$ -module. Since  $2 \Rightarrow 4$ ,  $\widehat{M} = \bigoplus_{i \in I} M_i$  where  $M_i$  is indecomposable for each  $i \in I$ . Then there is a finite subset  $J$  of  $I$  such that  $M \subseteq \bigoplus_{i \in J} M_i$ . Since  $M$  is relatively divisible in  $\bigoplus_{i \in J} M_i$ ,  $M \cap \bigoplus_{i \in I \setminus J} M_i = 0$  and  $\bigoplus_{i \in J} M_i \simeq \widehat{M} / \bigoplus_{i \in I \setminus J} M_i$  we conclude that  $\bigoplus_{i \in I \setminus J} M_i = 0$  and  $I = J$ .  $\square$

To prove the second assertion of this theorem some preliminary results are needed.

**Proposition 5.2.** *Let  $R$  be a finite ring. Then every RD-coflat right (or left) module is RD-injective.*

**Proof.** For every right  $R$ -module  $M$  the family of subgroups which are finite intersections of subgroups of the form  $(Mr : s)$ , where  $r, s \in R$ , is finite. We conclude by proposition 1.5.  $\square$

Now we consider a commutative local ring  $R$  satisfying the ascending chain condition on principal ideals. If  $A$  is a proper ideal we denote  $\mathcal{A}^*$  the subset of maximal elements of  $A^*$  and  $S = R/P$ . Let  $\psi : \bigoplus_{r^* \in \mathcal{A}^*} R/r^* \rightarrow R/A$  be the homomorphism defined by the family  $(\phi_{r^*})_{r^* \in \mathcal{A}^*}$ . We put  $\Psi = \text{Hom}_R(\psi, E)$ . It is easy to check that  $\Psi$  is the diagonal homomorphism induced by the inclusion maps

$E[A] \rightarrow E[r^*]$ . Then the following lemma holds and we prove the first assertion as in lemma 3.3. The second is an immediate consequence of proposition 1.3.

**Lemma 5.3.** *Then:*

- (1)  $\ker \psi$  is an RD-submodule of  $\bigoplus_{r^* \in \mathcal{A}^*} R/r^*$ .
- (2)  $\Psi(E[A])$  is an RD-submodule of  $\prod_{r^* \in \mathcal{A}^*} E[r^*]$ .

Since its maximal ideal is T-nilpotent, it is easy to prove that every commutative local perfect ring satisfies the ascending chain condition on principal ideals.

**Lemma 5.4.** *Let  $R$  be a commutative local perfect ring and  $A$  a proper ideal. Then  $\widehat{E[A]} \simeq \prod_{r^* \in \mathcal{A}^*} E[r^*]$ .*

**Proof.** We put  $G(A) = \prod_{r^* \in \mathcal{A}^*} E[r^*]$ . If  $N$  is a submodule of  $G(A)$  such that  $N \cap \Psi(E[A]) = 0$  and the image of  $\Psi(E[A])$  relatively divisible in  $G(A)/N$ , then for every  $0 \neq y \in N$ ,  $Ry \cap \Psi(E[A]) = 0$  and the image of  $\Psi(E[A])$  is relatively divisible in  $G(A)/Ry$ . Consequently it is sufficient to prove that for every  $y \in G(A)$  such that  $Ry \cap \Psi(E[A]) = 0$ , the image of  $\Psi(E[A])$  is not relatively divisible in  $G(A)/Ry$ . Let  $0 \neq y = (y_{r^*})_{r^* \in \mathcal{A}^*}$  such that  $Ry \cap \Psi(E[A]) = 0$ . Let  $F$  be the submodule of  $E$  generated by  $\{y_{r^*} \mid r^* \in \mathcal{A}^*\}$ ,  $B = \text{ann}(F)$  and  $t \in R$  such that  $t + B$  generates a minimal nonzero submodule of  $R/B$ . It follows that  $tF = S$ . Hence there exists  $s^* \in \mathcal{A}^*$  such that  $ty_{s^*} \neq 0$ . We set  $e = ty_{s^*}$  and  $z_{s^*} = 0$ . If  $s^* \neq r^* \in \mathcal{A}^*$  and  $s^* = Rs$  then  $s \notin r^*$ . It follows that  $(r^* : s) \subseteq P \subseteq (0 : e - ty_{r^*})$ . Since  $E[r^*]$  is injective over  $R/r^*$  there exists  $z_{r^*} \in E[r^*]$  such that  $sz_{r^*} = e - ty_{r^*}$ . We put  $z = (z_{r^*})_{r^* \in \mathcal{A}^*}$ . Then we get the following equality:  $sz + ty = \Psi(e)$ . Since  $Ry \cap \Psi(E[A]) = 0$  we have  $sz \neq 0$ . If  $e = sx$  for some  $x \in E[A]$  then  $e = 0$  since  $s \in A$ . It follows that the image of  $\Psi(E[A])$  is not relatively divisible in  $G(A)/Ry$ .  $\square$

**Remark 5.5.** We know that  $E[A]$  is indecomposable and it is a module of finite length if  $R$  is a commutative local artinian ring. If  $A$  is not principal then  $\widehat{E[A]}$  is decomposable, and if  $\mathcal{A}^*$  is not finite then  $\widehat{E[A]}$  is neither artinian nor noetherian.

**Remark 5.6.** Let  $\alpha : F \rightarrow M$  be an epimorphism of right  $R$ -modules where  $F$  is RD-projective. Let  $K = \ker \alpha$ . We say that  $\alpha : F \rightarrow M$  is an *RD-projective cover* of  $M$  if  $K$  is an RD-pure submodule of  $F$  and  $F$  is the only submodule  $N$  of  $F$  which verifies  $K + N = F$  and  $K \cap N$  is relatively divisible in  $N$ . In a similar way we define the *pure-projective cover* of  $M$ . If  $R$  is a commutative local perfect ring, then lemma 5.4 implies that, for each ideal  $A$ ,  $\psi : \bigoplus_{r^* \in \mathcal{A}^*} R/r^* \rightarrow R/A$  is an RD-projective cover of  $R/A$ .

Now we prove the last assertion of theorem 5.1.

**Proof of theorem 5.1.**

5  $\Rightarrow$  6.  $R$  is a finite product of local rings. We may assume that  $R$  is local. There exists a finite family of indecomposable RD-injective modules  $(S_i)_{1 \leq i \leq n}$  such that  $\widehat{S} = \bigoplus_{i=1}^n S_i$ . By [13, Propositin 9(3)] the endomorphism ring of each indecomposable pure-injective module is local. For every  $r^* \in \mathcal{P}^*$ ,  $E[r^*]$  is a summand of  $\widehat{S}$  by lemma 5.4. Then there exists  $i$ ,  $1 \leq i \leq n$  such that  $E[r^*] \simeq S_i$  by Krull-Schmidt theorem. Whence  $\mathcal{P}^*$  is a finite set. This implies that the set  $\mathcal{P}'^*$  of maximal principal proper ideals of  $R/P^2$  is finite. Clearly  $\mathcal{P}'^*$  is the set of vector lines of the vector space  $P/P^2$  over  $R/P$ . This last set is finite if and only if  $P$  is principal or  $R/P$  is a finite field.



6  $\Rightarrow$  1. We may assume that  $R$  is local. If  $R$  is an artinian valuation ring then each  $R$ -module is RD-injective by [11, Theorem 4.3]. We conclude by proposition 5.2 if  $R$  is finite.  $\square$

For rings whose right RD-flat modules are RD-projective one easily gets the following partial result:

**Proposition 5.7.** *Let  $R$  be a ring. Then  $R$  is right perfect if every RD-flat right  $R$ -module is RD-projective.*

**Proof.** We prove that each flat right  $R$ -module is projective.  $\square$

#### REFERENCES

- [1] P. N. Anh, Morita duality for commutative rings. *Comm. algebra* **18(6)** (1990), 1781–1788.
- [2] B. Ballet, Topologies linéaires et modules artiniens. *J. Algebra* **41** (1976), 365–397.
- [3] H. Bass, Finitistic dimension and a homological generalization of semi-primary rings. *Trans. Amer. Math. Soc.* **95** (1960), 466–488.
- [4] J. Bertin, Anneaux cohérents de dimension homologique finie. Colloque Algèbre commut., Rennes 1972, Publ. Sem. Math. Univ. Rennes, No.16, (1972).
- [5] J.W. Brewer and D.L. Costa, Contracted ideals and purity for ring extensions. *Proc. Amer. Math. Soc.* **53(2)** (1975), 271–276.
- [6] F. Couchot, Exemples d’anneaux auto  $fp$ -injectifs. *Comm. Algebra* **10(4)** (1982), 339–360.
- [7] F. Couchot, Modules with RD-composition series over a commutative ring. *Comm. Algebra* **31(7)** (2003), 3171–3194.
- [8] A. Facchini, Relative Injectivity and Pure-injective Modules over Prüfer Rings. *J. Algebra* **110** (1987), 380–406.
- [9] L. Fuchs and L. Salce, Modules over Non-Noetherian Domains, Mathematical Surveys and Monographs, vol. 84, American Mathematical Society, Providence, 2001.
- [10] D.T. Gill, Almost maximal valuation rings. *J. London Math. Soc.* **4** (1971), 140–146.
- [11] P. Griffith, On the Decomposition of Modules and Generalized Left Uniserial Rings. *Math. Ann.* **184** (1970), 300–308.
- [12] A. Hattori, A foundation of torsion theory for modules over general rings. *Nagoya Math. J.* **17** (1960), 147–158
- [13] B. Huisgen-Zimmermann, Purity, algebraic compactness, direct sum decompositions, and representation type. (English) Krause, Henning (ed.) et al., Infinite length modules. Proceedings of the conference, Bielefeld, Germany, September 7-11, 1998. Basel: Birkhuser. Trends in Mathematics. 331-367 (2000).
- [14] D. Lazard, Autour de la platitude. *Bull. Soc. Math. Fr.* **97** (1969), 81–128.
- [15] G. Puninski, The pure-injective and RD-injective hulls of a ring. *Math. Notes* **52(6)** (1992), 1224–1230.
- [16] G. Puninski, , M. Prest and P. Rothmaler, Rings described by various purities. *Comm. Algebra* **27(5)** (1999), 2127–2162.
- [17] G. Renault, Algèbre non commutative. Collection “Varia Mathematica”, Gauthier-Villars, 1975.
- [18] B. Stenström, Pure submodules. *Arkiv for Matematik* **7.L** (1967), 159–171.
- [19] R.B. Warfield, Purity and algebraic compactness for modules. *Pac. J. Math.* **28(3)** (1961), 689–719.