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FINITELY PRESENTED MODULES OVER SEMIHEREDITARY RINGS

FRANÇOIS COUCHOT

ABSTRACT. One proves that each almost local-global semihereditary ring R has the stacked base property and is almost Bézout. More precisely, if M is a finitely presented module, its torsion part tM is a direct sum of cyclic modules where the family of annihilators is an ascending chain of invertible ideals. These ideals are invariants of M . Moreover M/tM is a projective module which is isomorphic to a direct sum of finitely generated ideals. These ideals allow to define a finitely generated ideal whose isomorphic class is an invariant of M . The idempotents and the positive integers defined by the rank of M/tM are invariants of M too. It follows that each semihereditary ring of Krull-dimension one or of finite character, in particular each hereditary ring, enjoys the stacked base property. These results were already proved for Prüfer domains by Brewer, Katz, Klinger, Levy and Ullery. It is also shown that every semihereditary Bézout ring of countable character is an elementary divisor ring.

It is well-known for a long time that every Dedekind domain satisfies the *stacked bases property* (or the Simultaneous Basis Property). See [8]. The definitions will be given below. More recently this result was extended to every Prüfer domain of finite character [10] or of Krull dimension one [2] and more generally to each almost local-global Prüfer domain [2].

The aim of this paper is to show that every almost local-global semihereditary ring, and in particular every hereditary ring, has the stacked bases property too. This extension is possible principally because every semihereditary ring enjoys the following properties:

- every finitely presented module is the direct sum of its torsion submodule with a projective module,
- the annihilator of each finitely generated ideal is generated by an idempotent,
- and each faithful finitely generated ideal contains a nonzerodivisor.

So, the only difference with the domain case is that we have to manage nontrivial idempotents. Except this difference, we do as in section 4 of chapter V of [5].

All rings in this paper are unitary and commutative. We say that a ring R is of *finite (resp. countable) character* if every non-zero-divisor is contained in finitely (resp. countably) many maximal ideals. We say that R is *local-global* if each polynomial over R in finitely many indeterminates which admits unit values locally, admits unit values. A ring R is said to be *almost local-global* if for every faithful principal ideal I , R/I is local-global.

Let R be a semihereditary ring. If for each finitely presented module M and for each presentation $0 \rightarrow H \rightarrow F \rightarrow M \rightarrow 0$, where F is a finitely generated free module, there exist finitely generated ideals J_1, \dots, J_n and invertible ideals

$I_1 \supseteq \cdots \supseteq I_m$ such that $F \simeq J_1 \oplus \cdots \oplus J_n$ and $H \simeq I_1 J_1 \oplus \cdots \oplus I_m J_m$ we say that R has the *stacked bases property*. We will prove the two following theorems:

Theorem 1. *Every almost local-global semihereditary ring has the stacked bases property.*

Theorem 2. *Let R be an almost local-global semihereditary ring and M a finitely presented module. Then M decomposes as:*

$$M \simeq \left(\bigoplus_{1 \leq j \leq m} R/I_j \right) \oplus \left(\bigoplus_{1 \leq i \leq p} \left(\bigoplus_{1 \leq k \leq n_i} J_{i,k} e_i \right) \right)$$

where $I_1 \supseteq \cdots \supseteq I_m$ are proper invertible ideals, $\{e_1, \dots, e_p\}$ is a family of orthogonal idempotents, $\{n_1, \dots, n_p\}$ a strictly increasing sequence of integers > 0 and for each i , $1 \leq i \leq p$, $J_{i,1}, \dots, J_{i,n_i}$ are arbitrary invertible ideals of Re_i . The invariants of M are:

- (a) the ideals I_1, \dots, I_m ,
- (b) the idempotents e_1, \dots, e_p ,
- (c) the integers n_1, \dots, n_p and
- (d) the isomorphic class of $\bigoplus_{1 \leq i \leq p} (\prod_{1 \leq k \leq n_i} J_{i,k})$.

(R satisfies the Invariant Factor Theorem)

From these two theorems we deduce the two following corollaries:

Corollary 3. *Let R be a semihereditary ring. Assume that R satisfies one of the following conditions:*

- (a) R is of finite character
- (b) R has Krull-dimension ≤ 1 .

Then R has the stacked bases property and every finitely presented module M has a decomposition as in theorem 2.

Proof. Let $I = Rs$ where s is a non-zerodivisor. Then s doesn't belong to any minimal prime ideal. Thus R/I is either semilocal or zero-Krull-dimensional. So, it is local-global by [11, Proposition p.455]. \square

Corollary 4. *Let R be a hereditary ring. Then R has the stacked bases property and every finitely presented module M has a decomposition as in theorem 2.*

Proof. R is both one-Krull-dimensional and of finite character. \square

To prove theorem 1 and theorem 2 some preliminary results are needed. It is obvious that each semihereditary ring is a *pp-ring* (or a *Baer ring*), i.e. a ring for which every principal ideal is projective.

Lemma 5. *Let R be a pp-ring and Q its ring of quotients. Then the following assertions hold:*

- (1) For each finitely generated ideal A , $(0 : A)$ is generated by an idempotent.
- (2) every faithful finitely generated ideal contains a non-zerodivisor.
- (3) Q is absolutely flat.

Proof. The first assertion is well known if A is principal. Let e_1 and e_2 be two idempotents such that $(0 : A_i) = Re_i$, where A_i is an ideal, for $i = 1, 2$. Then $(0 : A_1 + A_2) = Re_1 \cap Re_2 = Re_1e_2$.

From the first assertion we deduce that for each $a \in R$ there exists an idempotent $e \in R$ such that $ae = 0$ and $Ra + Re$ is a faithful ideal. Then the two last assertions follows from [12, proposition 9]. \square

Proposition 6. *Let R be a semihereditary ring, M a finitely presented R -module and tM its torsion submodule. Then $M \simeq tM \oplus M/tM$ and M/tM is projective.*

Proof. Let Q the ring of fractions of R . By lemma 5, Q is absolutely flat. It follows that $M/tM \otimes_R Q \simeq M \otimes_R Q$ is a projective Q -module and M/tM is flat. By [3, proposition 2.3], M/tM is projective. \square

For an element $x \in R^n$, the *content* $c(x)$ is defined as the ideal generated by the coordinates of x . If M is a submodule of R^n , then $c(M)$ will mean the ideal generated by $c(x)$ with $x \in M$. If X is a matrix over R then $c(X)$ is the ideal generated by its entries.

We say that a ring R has the *UCS-property* (*unit content summand*) if, for each $n > 0$ the following holds: every finitely generated submodule M of R^n with unit content (i.e, $c(M) = R$) contains a rank one projective summand of R^n .

The following theorem is a generalization of [5, Theorem V.4.7] and we adapt proof of [2, Theorem 3] to show it.

Theorem 7. *Let R be an almost local-global pp-ring. Then R enjoys the UCS-property.*

Proof. Let X be a $n \times m$ -matrix over R such that $c(X) = R$. Let $M (\subseteq R^n)$ be the column space of X . There exist $x_1, \dots, x_p \in M$ such that $c(x_1) + \dots + c(x_p) = R$. For each j , $1 \leq j \leq p$, let ϵ_j be the idempotent which verifies $(0 : c(x_j)) = R(1 - \epsilon_j)$. Then for each j , $1 \leq j \leq p$, there exists $a_j \in c(x_j)$ such that $a_1\epsilon_1 + \dots + a_p\epsilon_p = 1$. By induction it is possible to get an orthogonal family of idempotents $(e_j)_{1 \leq j \leq p}$ and a family $(b_j)_{1 \leq j \leq p}$ such that $b_1e_1 + \dots + b_pe_p = 1$ and $e_j \in R\epsilon_j$, $\forall j$, $1 \leq j \leq p$. After removing the indexes j for which $b_je_j = 0$, we may assume that $b_je_j \neq 0$, $\forall j$, $1 \leq j \leq p$. Then $b_je_j = e_j$, $c(x_j)e_j$ is faithful over Re_j and $c(e_jX) = c(X)e_j = Re_j$. If we prove that there exists a rank one summand U_j of e_jR^n contained in e_jM for each j , $1 \leq j \leq p$, then $U_1 \oplus \dots \oplus U_p$ is a rank one summand of R^n contained in M . Consequently we may assume that there is only one idempotent equal to 1 and $x \in M$ such that $c(x)$ is a faithful ideal. Then $R^* = R/c(x)$ is a local-global ring. By [5, Lemma V.4.5] there is a y^* in the image of M such that $c(y^*) = R^*$. Therefore, for any preimage $y \in M$ of y^* , $c(x) + c(y) = R$ holds.

Let N be the submodule of M generated by x and y . The 2×2 -minors of the matrix whose columns are x and y generate an ideal I in R . Let e be the idempotent which verifies $(0 : I) = R(1 - e)$. Since $I(1 - e) = 0$, then $(1 - e)N$ is of rank one over $R(1 - e)$, and in view of $c((1 - e)x) + c((1 - e)y) = R(1 - e)$, it is a summand of $(1 - e)R^n$ contained in $(1 - e)M$. Since I is faithful over Re then $R^* = Re/I (\simeq R/(I + R(1 - e)))$ is local-global. By [5, Lemma V.4.5], $c(x^* + a^*y^*) = R^*$ for some $a^* \in R^*$. For a preimage $a \in R$ of a^* , we have $c(e(x + ay)) + I = Re$. Suppose that $c(e(x + ay)) \subseteq P$ for some maximal ideal P of Re . Then $e(x + ay)$ vanishes in $(Re/P)^n$, thus all the 2×2 -minors generators of I are contained in P , i.e., $I \subseteq P$. This is an obvious contradiction to $c(e(x + ay)) + I = Re$.

Hence $c(e(x+ay)) = Re$, and it follows that the submodule T generated by $e(x+ay)$ is a rank one summand of eR^n contained in eM . We get that $(1-e)N + T$ is a rank one summand of R^n contained in M . \square

The following theorem is a generalization of [1, Theorem 6] and we do a similar proof. For Prüfer domains we can also see [5, Theorem V.4.8].

Theorem 8. *A semihereditary ring R has the UCS-property if and only if it has the stacked bases property.*

Proof. To show that “stacked bases property” implies “UCS-property” we do as for Prüfer domains: see [5, Theorem V.4.8].

Assume that R has the UCS-property and let H be a finitely generated submodule of R^n ($n \geq 1$). Let ϵ_1 be the idempotent such that $(0 : c(H)) = R(1 - \epsilon_1)$. Then $H = \epsilon_1 H$ and $J_1 = c(H) \oplus R(1 - \epsilon_1)$ is invertible. Thus $J_1^{-1}H$ is a finitely generated submodule of $\epsilon_1 R^n$ and $c(J_1^{-1}H) = R\epsilon_1$. So, by the UCS-property, $J_1^{-1}H$ contains a rank one summand U_1 of $\epsilon_1 R^n$, i.e., $R^n = U_1 \oplus N_1$ for some N_1 . Hence $J_1^{-1}H = U_1 \oplus H_1$ with $H_1 = N_1 \cap J_1^{-1}H$ and $H = J_1 U_1 \oplus J_1 H_1$. If the second summand is not 0, let ϵ_2 be the idempotent such that $(0 : c(H_1)) = R(1 - \epsilon_2)$ and $J_2 = c(H_1) \oplus R(1 - \epsilon_2)$. Then $\epsilon_2 \in R\epsilon_1$ and $J_2^{-1}H_1$ contains a rank one summand U_2 of $\epsilon_2 R^n$. Let s be a non-zero-divisor such that $sJ_2^{-1} \subseteq R$. If $x \in J_2^{-1}H_1$ then $x = u + y$, $u \in U_1$, $y \in N_1$ and $sx \in H_1$. It follows that $su = 0$ whence $u = 0$. Hence $J_2^{-1}H_1 \subseteq N_1$ and U_2 is a summand of N_1 . We obtain $R^n = U_1 \oplus U_2 \oplus N_2$ for some N_2 . Hence $H = J_1 U_1 \oplus J_1 J_2 U_2 \oplus J_1 J_2 H_2$ where $H_2 = N_2 \cap J_2^{-1}H_1$. Repetition yields $J_1 J_2 \dots J_m H_m = 0$ for $m = \max\{\text{rank}(H_P) \mid P \in \text{Spec}(R)\} \leq n$, hence

$$R^n = U_1 \oplus \dots \oplus U_m \oplus N_m \quad \text{and} \quad H = J_1 U_1 \oplus \dots \oplus J_1 \dots J_m U_m$$

for rank one projective summand U_i of $\epsilon_i R^n$, where $(\epsilon_i)_{1 \leq i \leq m}$ is a family of nonzero idempotents such that $\epsilon_{i+1} \in R\epsilon_i$ for every i , $1 \leq i \leq m-1$. Here N_m is projective, so it is isomorphic to a direct sum of ideals. We thus have stacked bases for R^n and its submodule H . \square

Now it is obvious that theorem 1 is an immediate consequence of theorem 7 and 8.

We say that a ring R is *almost Bézout* if for each faithful principal ideal A , R/A is Bézout (i.e., finitely generated ideals are principal). The following proposition 9 was proved by Kaplansky[8] for Dedekind domains and we do a similar proof. One can also see [5, Proposition V.4.10].

Proposition 9. *Let R be an almost Bézout pp-ring. Suppose*

$$U = I_1 \oplus \dots \oplus I_n, \quad V = J_1 \oplus \dots \oplus J_m$$

are decompositions of finitely generated projective R -modules U, V into direct sum of invertible ideals. Then a necessary and sufficient condition for $U \simeq V$ is that

$$n = m \quad \text{and} \quad I_1 \dots I_n \simeq J_1 \dots J_m.$$

Proof. We prove that the condition is necessary as for integral domains.

For sufficiency we can also do the same proof since every invertible ideal I contains a non-zero-divisor a . It follows that $aI^{-1}J$ is faithful for each invertible ideal J . Therefore $aI^{-1}/aI^{-1}J$ is a principal ideal of $R/aI^{-1}J$ and generated by $b+aI^{-1}J$. Then $bR + aI^{-1}J = aI^{-1}$, thus $ba^{-1}I + J = R$. We get that $I \oplus J \simeq R \oplus IJ$. \square

The following lemma is needed to prove theorem 2.

Lemma 10. *Let R be an almost local-global semihereditary ring. Then:*

- (1) R is an almost Bézout ring.
- (2) For every invertible ideals $I, J, I/JI \simeq R/J$.

Proof. Let J be a faithful principal ideal of R . Every finitely generated ideal of R/J is of the form I/J where I is an invertible ideal of R . It follows that I/JI is a rank one projective R/J -module. By [11, Theorem p.457] or [5, Proposition V.4.4] I/JI is cyclic. Thus R/J is a Bézout ring: it is also a consequence of theorem 11.

Since JI contains a faithful principal ideal, I/JI is cyclic. From the obvious equality $I(JI : I) = IJ$, we deduce that $(JI : I) = J$. \square

Now we can prove theorem 2.

Proof of theorem 2.

Suppose that $M \simeq R^n/H$. By theorem 8 and its proof we have

$$R^n = U_1 \oplus \cdots \oplus U_m \oplus N_m \quad \text{and} \quad H = J_1 U_1 \oplus \cdots \oplus J_1 \cdots J_m U_m$$

Moreover there is a family of idempotents $(\epsilon_i)_{1 \leq i \leq m}$ such that U_i is isomorphic to an invertible ideal of $R\epsilon_i$ and $\epsilon_{i+1} \in R\epsilon_i$. We set $I_k = J_1 \cdots J_k \epsilon_k \oplus R(1 - \epsilon_k)$ for each k , $1 \leq k \leq m$. By lemma 10 $U_i/I_i U_i \simeq R\epsilon_i/I_i \epsilon_i \simeq R/I_i$ for each i , $1 \leq i \leq m$. By using [14, proposition 1.10] it is possible to get that $tM \simeq R/I_1 \oplus \cdots \oplus R/I_m$ with $I_1 \supseteq \cdots \supseteq I_m$. By [7, Theorem 9.3], I_1, \dots, I_m are invariants of tM and M .

Since $N_m \simeq M/tM$ is a finitely generated projective module there exists an orthogonal family of idempotents $(e_i)_{1 \leq i \leq p}$ such that $e_i N_m$ has a constant rank n_i over $R e_i$. Consequently $e_i N_m \simeq J_{i,1} \oplus \cdots \oplus J_{i,n_i}$ where $J_{i,k}$ is an invertible ideal of $R e_i$ for each k , $1 \leq k \leq n_i$. By proposition 9 the isomorphic class of $J_{i,1} \cdots J_{i,n_i}$ is an invariant of $e_i N_m$. We conclude that the isomorphic class of $\bigoplus_{1 \leq i \leq p} (\prod_{1 \leq k \leq n_i} J_{i,k})$ is an invariant of M . \square

It is also possible to deduce theorem 2 from the following:

Theorem 11. *Let R be an arithmetic local-global ring. Then R is an elementary divisor ring. Moreover, for each $a, b \in R$, there exist $d, a', b', c \in R$ such that $a = a'd$, $b = b'd$ and $a' + cb'$ is a unit of R .*

Proof. Since every finitely generated ideal is locally principal R is Bézout by [4, Corollary 2.7]. Let $a, b \in R$. Then there exist $a', b', d \in R$ such that $a = a'd, b = b'd$ and $Ra + Rb = Rd$. Consider the following polynomial $a' + b'T$. If P is a maximal ideal, then we have $aR_P = dR_P$ or $bR_P = dR_P$. So, either a' is a unit of R_P and $a' + b'r$ is a unit of R_P for each $r \in PR_P$, or b' is a unit of R_P and $a' + b'(1 - a'/b')$ is a unit of R_P . We conclude that the last assertion holds. Now let $a, b, c \in R$ such that $Ra + Rb + Rc = R$. We set $Rb + Rc = Rd$. Let b', c', s and q such that $b = b'd, c = c'd$ and $b' + c'q$ and $a + sd$ are units. Then $(b' + c'q)(a + sd) = (b' + c'q)a + s(b + qc)$ is a unit. We conclude by [6, Theorem 6]. \square

Another proof of theorem 2. Let M be a finitely presented R -module. There exists a non-zerodivisor $s \in R$ which annihilates tM . So, tM is a direct sum of cyclic modules since R/Rs is an elementary divisor ring by theorem 11. \square

Now we give a generalization of [5, Theorem III.6.5].

Theorem 12. *Every semihereditary Bézout ring of countable character is an elementary divisor ring.*

Proof. By [7, Theorem 5.1] and [9, Theorem 3.1] it is sufficient to prove that every 1 by 2, 2 by 1 and 2 by 2 matrix is equivalent to a diagonal matrix. By [9, theorem 2.4] R is Hermite. So it is enough to prove that the following matrix A is equivalent to a diagonal matrix:

$$A = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}.$$

Let e the idempotent such that $(0 : ac) = R(1 - e)$. Thus $A = eA + (1 - e)A$. Since $(1 - e)ac = 0$ we show, as in the proof of [13, the proposition], that there exist two invertible matrices P_1 and Q_1 and a diagonal matrix D_1 with entries in $R(1 - e)$ such that $P_1(1 - e)AQ_1 = D_1$. Since $Re/Rac \simeq R/R(ac + 1 - e)$ and $(ac + 1 - e)$ is a non-zero-divisor, ac is contained in countably many maximal ideals of Re . We prove, as in the proof of [5, Theorem III.6.5], that there exist two invertible matrices P_2 and Q_2 and a diagonal matrix D_2 with entries in Re such that $P_2eAQ_2 = D_2$. We set $P = (1 - e)P_1 + eP_2$, $Q = (1 - e)Q_1 + eQ_2$ and $D = (1 - e)D_1 + eD_2$. Then P and Q are invertible, D is diagonal and $D = PAQ$. \square

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