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Asymptotical behaviour of the presence probability in branching random walks and fragmentations

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Summary

For a subcritical Galton-Watson process (ζ_n) , it is well known that under an $X \log X$ condition, the quotient $P(\zeta_n > 0)/E\zeta_n$ has a finite positive limit. There is an analogous result for a (one-dimensional) supercritical branching random walk: when a is in the so-called subcritical speed area, the probability of presence around na in the n -th generation is asymptotically proportional to the corresponding expectation. In [13] this result was stated under a natural $X \log X$ assumption on the offspring point process and a (unnatural) condition on the offspring mean. Here we prove that the result holds without this latter condition, in particular we allow an infinite mean and a dimension $d \geq 1$ for the state-space. As a consequence the result holds also for homogeneous fragmentations as defined in [5], using the method of discrete-time skeletons; this completes the proof of Theorem 4 in [7]. Finally, an application to conditioning on the presence allows to meet again the probability tilting and the so-called additive martingale.

Key words. Fragmentation, branching random walk, large deviations, time-discretization, probability tilting.

A.M.S. Classification. 60 J 25, 60 F 10.

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1 Introduction

The common feature of many branching models consists in exponential growth. Let us recall some basic facts about a Galton-Watson process ζ_n with finite mean started from $\zeta_0 = 1$. We have $\mathbb{E}(\zeta_n) = \mathbb{E}(\zeta_1)^n$ and

- if $\mathbb{E}(\zeta_1) > 1$ and $\mathbb{E}(\zeta_1 \log_+ \zeta_1) < \infty$, then $\lim_n \zeta_n / \mathbb{E}(\zeta_n) = W > 0$ a.s. conditionally on non-extinction.

- if $\mathbb{E}(\zeta_1) < 1$ and $\mathbb{E}(\zeta_1 \log_+ \zeta_1) < \infty$, then,

$$(1) \quad \lim_n \frac{\mathbb{P}(\zeta_n \geq 1)}{\mathbb{E}(\zeta_n)} = K > 0,$$

and there is a representation formula for the constant K (see [1]). Moreover the condition $\mathbb{E}(\zeta_1 \log_+ \zeta_1) < \infty$ is in some sense necessary.

In a discrete-time branching random walk (BRW), the initial ancestor is at the origin in \mathbb{R}^d and the positions of its children form a point process Z . Each of these children has children in the same way: the positions of each family relative to its parent is an independent copy of Z . Let Z_n denote the point process in \mathbb{R}^d formed by the n -th generation.

The intensity of Z is the Radon measure ρ defined by

$$(2) \quad \mathbb{E} \left(\int_{\mathbb{R}^d} f(x) Z(dx) \right) = \int_{\mathbb{R}^d} f(x) \rho(dx).$$

and the intensity of Z_n is the n -th convolution product ρ^{*n} . We assume $1 < \rho(\mathbb{R}^d) \leq +\infty$ (supercriticality) and set

$$(3) \quad \widehat{Z}(\theta) := \int_{\mathbb{R}^d} e^{\theta \cdot x} Z(dx) \quad \text{and} \quad \Lambda(\theta) := \log \mathbb{E}(\widehat{Z}(\theta)) = \log \int_{\mathbb{R}^d} e^{\theta \cdot x} \rho(dx).$$

We shall also assume that $\Theta := \text{int} \{ \theta : \Lambda(\theta) < \infty \} \neq \emptyset$.

The asymptotic behaviour as $n \rightarrow \infty$ of the random measure Z_n , and more precisely, estimates of $Z_n(na + I)$, where $a = \nabla \Lambda(\theta)$ and I is some fixed bounded set, have raised a considerable interest. This behaviour depends on the value of θ . The quantity

$$\Lambda^*(a) = \theta \cdot \nabla \Lambda(\theta) - \Lambda(\theta)$$

plays the role of the negative of the logarithm of the mean reproduction in the Galton-Watson process.

In the range of supercriticality, i.e. for those $a \in \mathbb{R}^d$ such that $\Lambda^*(a) < 0$, the exponential rate of growth of $Z_n(na + I)$ is $-\Lambda^*(a)$ (see [8]) and a precise first order estimate is given in [9] under an $X \log X$ type condition on $\widehat{Z}(\theta)$.

In the range of subcriticality, i.e. for those $a \in \mathbb{R}^d$ such that $\Lambda^*(a) > 0$, a precise analog of (1) i.e.

$$(4) \quad \lim_n \frac{\mathbb{P}(Z_n(na + I) \geq 1)}{\mathbb{E}Z_n(na + I)} = K(a) > 0$$

was proved in [13] under certain conditions which force in particular the finiteness of $\rho(\mathbb{R}^d)$, and unfortunately this restricts the range of its applications. Our main purpose here is to show that the same estimate holds in fact under much weaker conditions. Let us recall that the method is based on a representation formula for the presence probability by means of an auxiliary random walk. It is a discrete version of the Feynman-Kac formula used in probabilistic representations of solution of reaction-diffusion equations ([2], [10], [12]). Then a careful study of large deviations for functionals of this walk is needed, and our improvement takes place there.

Fragmentations, as defined in [5], may be viewed as an extension of continuous-time branching random walks, but with a possible infinite offspring mean and infinite rate of branching. In [7], we succeeded to extend the results of the supercritical range to fragmentations, checking that the $X \log X$ assumption is automatically satisfied for the discrete-time skeleton. Moreover we proved that, in the subcritical range, the result can be deduced from its analogous version for BRW, by the method of discrete skeleton.

In Section 2 we state the result for BRW with a detailed proof in subsection 2.2. In Section 3 we extend the result to fragmentations, referring to Theorem 4 of [7]. In subsection 3.3 we show that conditioning on $\{Z_t(at+I) \geq 1\}$ and letting $t \rightarrow \infty$ provides the tilted probability on the fragmentation process, already found in [7].

2 Branching Random Walks

2.1 Notation and representation formula

To study spatial (Markov) branching processes, it is classical to use the probability generating functional. We define its action on a Borel function g satisfying $0 \leq g \leq 1$ and such that $1 - g$ has a compact support by the function

$$F[g](x) = \mathbb{E} \exp \int_{\mathbb{R}^d} \log g(x+y) Z(dy) , \quad x \in \mathbb{R}^d .$$

Its first moment is the linear operator acting on Borel functions with compact support by

$$M[g](x) = \mathbb{E} \int g(x+y) Z(dy) , \quad x \in \mathbb{R}^d ,$$

(see [1] chap. V.1 and V.2). Set $Z_n = \sum_i \delta_{z_i^n}$. Thanks to the Markov property, for every $n \geq 1$, the non linear operator u_n defined by

$$(5) \quad u_n[f](x) = 1 - \mathbb{E} \left(\prod_i (1 - f(x + z_i^n)) \right) ,$$

satisfies $u_n[f] := 1 - F^n[1 - f]$. Its linear counterpart, v_n defined by

$$(6) \quad v_n[f](x) = \mathbb{E} \left(\int_{\mathbb{R}^d} f(x+y) Z_n(dy) \right) = \int_{\mathbb{R}^d} f(x+z) \rho^{*n}(dz)$$

satisfies $v_n[f] = M^n[f]$. We set $u_0[f] = v_0[f] = f$. When $f = \mathbf{1}_J$ is the indicator function of a Borel set J , then

$$(7) \quad u_n[\mathbf{1}_J](x) = \mathbb{P}(Z_n(J - x) \geq 1) \quad , \quad v_n[\mathbf{1}_J](x) = \mathbb{E}(Z_n(J - x)) .$$

To study the propagation, let us introduce tilted (probability) measures¹. For $\theta \in \Theta$ and $a = \nabla\Lambda(\theta)$, let $(S_n := \xi_1 + \dots + \xi_n, n \in \mathbb{N})$ be a random walk with step distribution ρ_θ defined by

$$(8) \quad \int_{\mathbb{R}^d} g(x) \rho_\theta(dx) = \int_{\mathbb{R}^d} g(x - a) e^{\theta \cdot x - \Lambda(\theta)} \rho(dx) .$$

The distribution ρ_θ is centered, its covariance matrix is Hess $\Lambda(\theta)$ and we set $\sigma_\theta^2 = \det \text{Hess}\Lambda(\theta)$. In the sequel, notation such as \mathbb{E}_θ will refer to expectations with respect to this random walk.

From (6) and (8) we have the representation:

$$(9) \quad v_n[f](x) = e^{-n\Lambda^*(a)} \mathbb{E}_\theta \left[e^{-\theta \cdot S_n} f(x + S_n + na) \right] .$$

Let us define the auxiliary function f_θ by

$$(10) \quad f_\theta(y) = e^{-\theta \cdot y} f(y) .$$

We have

$$(11) \quad v_n[f](-na + c) = e^{-n\Lambda^*(a)} e^{\theta \cdot c} \mathbb{E}_\theta f_\theta(S_n + c)$$

so that the local central limit theorem entails that if f_θ is directly Riemann integrable (DRI), then for every $c \in \mathbb{R}^d$

$$(12) \quad \lim_n \sigma_\theta (2\pi n)^{d/2} e^{n\Lambda^*(a)} v_n[f](-an + c) = e^{\theta \cdot c} \int_{\mathbb{R}^d} f_\theta(y) dy .$$

To study the ratio between $u_n[f]$ and $v_n[f]$ we need the following representation formula (see [13], formulas 4.2 and 4.3).

Lemma 1 ([13]) *Let $(P^{\cdot x}, x \in \mathbb{R}^d)$ be the family of reduced Palm distributions defined by the disintegration formula:*

$$(13) \quad \mathbb{E} \left(\int_{\mathbb{R}^d} F(x, Z - \delta_x) Z(dx) \right) = \int_{\mathbb{R}^d} \rho(dx) E^{\cdot x}(F(x, Z)) .$$

and let for $r \geq 1$, $y, s \in \mathbb{R}^d$

$$(14) \quad H_r[f](y, s) := E^{\cdot y} \left(\int_0^1 \exp \langle Z, \log(1 - \beta u_{r-1}[f](s + \cdot)) \rangle d\beta \right) ,$$

Then

$$u_n[f](x) = e^{-n\Lambda^*(a)} \mathbb{E}_\theta \left\{ e^{-\theta \cdot S_n} f(x + S_n + na) \prod_{1 \leq r \leq n} H_r[f](a + \xi_r, x + S_n - S_r + (n - r)a) \right\} .$$

We are now able to state the main result of this section.

¹For simplicity we will assume that ρ is not lattice and that its support is not contained in any lower-dimensional hyperplane.

2.2 Main result

Theorem 2 *Let $\theta \in \Theta$ satisfy $\theta \cdot \Lambda'(\theta) - \Lambda(\theta) > 0$ and $\mathbb{E} \left(\widehat{Z}(\theta) \log^{1+\varepsilon} \left(1 + \widehat{Z}(\theta) \right) \right) < \infty$ for some $\varepsilon > 0$, and set $a = \nabla \Lambda(\theta)$. Let f be a function with compact support, Riemann integrable, satisfying $0 \leq f \leq 1$ and $\int_{\mathbb{R}^d} f(y) dy > 0$. Then for every $c \in \mathbb{R}^d$*

$$(15) \quad \lim_{n \rightarrow \infty} \frac{u_n[f](-an + c)}{v_n[f](-an + c)} = \frac{\int_{\mathbb{R}^d} e^{-\theta \cdot y} f(y) G[f](\theta, y) dy}{\int_{\mathbb{R}^d} e^{-\theta \cdot y} f(y) dy}$$

where

$$G[f](\theta, y) = \mathbb{E}_\theta \left\{ \prod_{r < \infty} H_r[f](a + \xi_r, y - ar - S_r) \right\} > 0.$$

(One can see similar formulas for a branching diffusion in [10] Th. 1, [12] Prop. 1), [2] Th.3. 3, [14] section 2.3).

Proof: In [13], the proof was in two parts. The first part was devoted to establish a lemma (4.1 p.33) and the second part consisted in checking conditions of the lemma. For easier reading we give here a complete proof, taking the main ideas of [13], although the essential modifications take place in the part “end of the proof” below.

Let us first remark that if we set $\tau_c f := f(c + \cdot)$ then τ_c commutes with u_n and v_n . Moreover $H_r[\tau_c f](y, s) = H_r[f](y, s + c)$, so that $G[\tau_c f](\theta, y) = G[f](\theta, y + c)$ and it is enough to prove (15) for $c = 0$.

Since f is bounded and has a compact support, there is some $b \in \mathbb{R}$ such that $f(x) = 0$ for $\theta \cdot x < b$. Since $f \leq 1$ we have

$$(16) \quad f_\theta(x) \leq e^{-b} \mathbf{1}_{\theta \cdot x \geq b}.$$

From Lemma 1 we deduce

$$(17) \quad u_n[f](-na) = e^{-n\Lambda^*(a)} \mathbb{E}_\theta \left\{ f_\theta(S_n) \prod_{1 \leq r \leq n} H_r(a + \xi_r, S_n - S_r - ra) \right\}.$$

Let us define

$$I_n^j := \sigma_\theta (2\pi n)^{d/2} \mathbb{E}_\theta \left\{ f_\theta(S_n) \prod_{r=1}^j H_r(a + \xi_r, S_n - S_r - ar) \right\} \quad \text{for } 1 \leq j \leq n$$

$$I_\infty^j := \int_{\mathbb{R}^d} f_\theta(s) \mathbb{E}_\theta \left\{ \prod_{r=1}^j H_r(a + \xi_r, s - S_r - ar) \right\} ds \quad \text{for } 1 \leq j.$$

Taking into account (9) and (12), the proof of (15) can be reduced to showing that

$$(18) \quad \lim_n I_n^n = I_\infty^\infty.$$

A conditioning on ξ_{j+1}, \dots, ξ_n gives $I_n^j = \sigma_\theta (2\pi n)^{d/2} \mathbb{E}_\theta [g_j(S_{n-j})]$ where

$$g_j(s) := \mathbb{E}_\theta \left\{ f_\theta(s + S_j) \prod_{k=1}^j H_k(a + \xi_k, s + S_j - S_k - ka) \right\}.$$

We check that g_j is DRI, so that the local central limit theorem yields

$$(19) \quad \lim_n I_n^j = \int_{\mathbb{R}^d} g_j(s) ds = I_\infty^j,$$

where the last equality comes from Fubini's theorem. Since I_∞^j decreases to I_∞^∞ as $j \rightarrow \infty$, it is enough to prove

$$(20) \quad \lim_j \limsup_{n>j} (I_n^j - I_n^n) = 0.$$

Let us remark that we have no control on the uniformity in j of (19). If we search for a convenient upperbound for $I_n^j - I_n^n$, the difficulty comes from the term S_r in $H_r(a + \xi_r, S_n - S_r - ar)$. Actually we will give a lowerbound of H_r after restricting the space.

Let $\eta \in (0, \Lambda^*(a))$. For $i < k$, let $A_k^i := \{\theta \cdot S_\ell \leq -\ell\eta \text{ for some } \ell \in (i+1, k)\}$. Since $\theta \in \Theta$, there exists $\phi > 0$ such that $\int_{\mathbb{R}^d} e^{-\phi\theta \cdot x} \rho_\theta(dx) < \infty$, hence we can find C_1 and C_2 such that for every i and $k > i$

$$(21) \quad \mathbb{P}_\theta(A_k^i) \leq C_1 e^{-C_2 i}.$$

From now, $C_k, k \geq 1$ denote strictly positive finite constants depending on θ, a, b, η but not on other quantities.

First step: Restriction of the space

$$(22) \quad \begin{aligned} 0 < \frac{I_n^j - I_n^n}{\sigma_\theta(2\pi n)^{d/2}} &\leq \mathbb{E}_\theta[f_\theta(S_n)(1 - \prod_j^n)] \leq \\ &\leq \mathbb{E}_\theta[f_\theta(S_n); A_n^j] + \mathbb{E}_\theta[f_\theta(S_n)(1 - \prod_j^n); (A_n^j)^c] \end{aligned}$$

Introducing an intermediate integer $n_1 \in (j, n)$ we have (see (16))

$$(23) \quad \mathbb{E}_\theta[f_\theta(S_n); A_n^j] \leq e^{-b} \mathbb{P}_\theta(A_{n_1}^{n_1}) + \mathbb{E}_\theta[f_\theta(S_n); A_{n_1}^j]$$

and, conditioning on $S_k, k \leq n_1$,

$$\mathbb{E}_\theta[f_\theta(S_n); A_{n_1}^j] = \mathbb{E}_\theta\left[\int_{\mathbb{R}^d} f_\theta(S_{n_1} + s) \rho_\theta^{*(n-n_1)}(ds); A_{n_1}^j\right].$$

Setting, for $q \geq 1$

$$(24) \quad \Delta_q := \sup_s \left| \sigma_\theta(2\pi q)^{d/2} \mathbb{E}_\theta f_\theta(s + S_q) - \int_{\mathbb{R}^d} f_\theta(x) \exp\left(-\frac{\|x-s\|^2}{2q\sigma_\theta^2}\right) dx \right|$$

we get

$$\sigma_\theta[2\pi(n-n_1)]^{d/2} \int_{\mathbb{R}^d} f_\theta(S_{n_1} + s) \rho_\theta^{*(n-n_1)}(ds) \leq \Delta_{n-n_1} + \int_{\mathbb{R}^d} f_\theta(x) dx$$

hence

$$(25) \quad \sigma_\theta [2\pi n]^{d/2} \mathbb{E}_\theta [f_\theta(S_n); A_{n_1}^j] \leq \left(1 - \frac{n_1}{n}\right)^{-d/2} \left\{ \Delta_{n-n_1} + \int_{\mathbb{R}^d} f_\theta(x) dx \right\} P_\theta(A_{n_1}^j).$$

Fix j . Choosing n_1 such that $n^{d/2} e^{-C_5 n_1} \rightarrow 0$, we get from (23), (21), (25) and the local central limit theorem ([15] Lemma 2) :

$$(26) \quad \limsup_{n > j} \sigma_\theta (2\pi n)^{d/2} \mathbb{E}_\theta [f_\theta(S_n); A_n^j] \leq C_1 e^{-C_2 j}.$$

Second step : lowerbound for H_r

We start with

$$(27) \quad u_n[f](x) \leq \min(v_n[f](x), 1) \quad , \quad x \in \mathbb{R}^d, \quad n \geq 0.$$

Since $f(x) \leq e^{\theta \cdot x - b}$ we have

$$(28) \quad v_n[f](x) \leq e^{n\Lambda(\theta)} e^{\theta \cdot x - b} \quad , \quad x \in \mathbb{R}^d, \quad n \geq 0.$$

Fix y satisfying $\theta \cdot y \geq -r\eta$. Applying (28) with $x = -y - ar + z + z_i$ we see that

$$(29) \quad v_{r-1}[f](-y - ar + z + z_i) \leq e^{-r[\Lambda^*(a) - \eta] + \theta \cdot (z + z_i) - b - \Lambda(\theta)}$$

for every $r \geq 1$ and $i \geq 1$. Since $\Lambda(\theta) < \infty$, the random variable $\widehat{Z}(\theta)$ defined by (3) is \mathbb{P} -a.s. finite, hence

$$(30) \quad \log \widehat{Z}(\theta) \geq \sup_i \{\theta \cdot z_i\} =: \tau(Z).$$

Let $B_r := \{Z : \tau(Z) < r[\Lambda^*(a) - \eta] + \Lambda(\theta) - \theta \cdot z + b\}$. For $Z \in B_r$, the bound in (29) is less than or equal to 1 so that (27) gives

$$u_{r-1}[f](-y - ar + z + z_i) \leq e^{-r[\Lambda^*(a) - \eta] + \theta \cdot (z + z_i) - b - \Lambda(\theta)} \leq 1$$

and then (when $\theta \cdot y \geq -r\eta$)

$$(31) \quad H_r(x, -y - ar + z) \geq \mathcal{H}_r(x, z) := E^{!x} \left(\mathbf{1}_{B_r} \int_0^1 e^{-w(r, Z, \beta, z)} d\beta \right)$$

where

$$(32) \quad w(r, Z, \beta, z) := - \int_{\mathbb{R}^d} \log(1 - \beta e^{-r[\Lambda^*(a) - \eta] + \theta \cdot (z + \zeta) - b - \Lambda(\theta)}) Z(d\zeta).$$

End of the proof

Coming back to the second term of (22) and using (31) we get

$$(33) \quad \begin{aligned} \mathbb{E}_\theta \left[f_\theta(S_n) \left(1 - \prod_j^n \right) ; (A_n^j)^c \right] &\leq \mathbb{E}_\theta \left[f_\theta(S_n) \sum_{k=j}^n (1 - \mathcal{H}_k(a + \xi_k, S_n)) \right] \\ &= \mathbb{E}_\theta \int_{\mathbb{R}^d} f_\theta(S_{n-1} + x) \left[\sum_{k=j}^n (1 - \mathcal{H}_k(a + x, S_{n-1} + x)) \right] \rho_\theta(dx) \end{aligned}$$

We want to bound $\sum_{r=j}^n (1 - \mathcal{H}_r(x, z))$. Adding up, from (31) and (32)

$$(34) \quad \begin{aligned} \sum_j^n [1 - \mathcal{H}_r(x, z)] &\leq E^{!x} \left(\sum_j^n \mathbf{1}_{B_r^c}(Z) \right) + E^{!x} \left(\int_0^1 \left(\sum_j^n \mathbf{1}_{B_r} (1 - e^{-w(r, Z, \beta, z)}) \right) d\beta \right) \\ &:= J_1(x, z) + J_2(x, z). \end{aligned}$$

From the definition of B_r we have

$$\sum_j^n \mathbf{1}_{B_r^c}(Z) \leq \left(\frac{\tau(Z) + \theta \cdot z - b - \Lambda(\theta)}{\Lambda^*(a) - \eta} - j + 1 \right)^+,$$

which, from inequality (30), gives by integration

$$(35) \quad J_1(x, z) \leq E^{!x} \left(\left(\frac{\log \widehat{Z}(\theta) + \theta \cdot z - b - \Lambda(\theta)}{\Lambda^*(a) - \eta} - j + 1 \right)^+ \right)$$

It remains to give an upperbound for $J_2(x, z)$. The function $r \mapsto w(r, Z, \beta, z)$ is decreasing in r , so by the classical sum-integral comparison, we obtain, for $Z \in B_j$

$$\sum_j^n [1 - e^{-w(r, Z, \beta, z)}] \leq \int_s^\infty [1 - e^{-w(r, Z, \beta, z)}] dr \leq C_3 \log[1 + w(s, Z, \beta, z)]$$

where the last inequality comes from $-\frac{\partial w}{\partial r} \geq w[\Lambda^*(a) - \eta]$.

The inequality²:

$$\int_0^1 \log \left[1 - \sum_i \log(1 - \beta a_i) \right] d\beta \leq \log \left(1 + \sum_i a_i \right)$$

for $0 \leq a_i \leq 1, i = 1, \dots$ gives

$$(36) \quad J_2(x, z) \leq C_4 E^{!x} \left(\log(1 + \widehat{Z}(\theta) e^{-j[\Lambda^*(a) - \eta] + \theta \cdot z - b - \Lambda(\theta)}) \right).$$

Combining (35) and (36) gives

$$(37) \quad J_1(x, z) + J_2(x, z) \leq C_5 E^{!x} \left(\log(1 + \widehat{Z}(\theta) e^{-j[\Lambda^*(a) - \eta] + \theta \cdot z - b - \Lambda(\theta)}) \right).$$

Setting

$$A(x, z) = E^{!x} \left([\log 1 + \widehat{Z}(\theta) e^{\theta \cdot z - b - \Lambda(\theta)}]^{1+\epsilon} \right)$$

and applying inequality (A2) p.38 of [13] we see that for every $\epsilon > 0$, the right hand side of (37) is bounded by $C_3 j^{-\epsilon} A(x, z)$ so that, from (34),

$$\sum_j^n [1 - \mathcal{H}_r(x, z)] \leq C_6 j^{-\epsilon} A(x, z).$$

²It can be proved using concavity of logarithm and Jensen's inequality.

Thanks to (33) that entails

$$\mathbb{E}_\theta [f_\theta(S_n)(1 - \prod_j^n); (A_n^j)^c] \leq C_6 j^{-\epsilon} \mathbb{E}_\theta (B(S_{n-1}))$$

where

$$B(s) := \int_{\mathbb{R}^d} f_\theta(x+s) A(a+x, s+x) \rho_\theta(dx).$$

It should be clear that the function B has bounded variation. By Fubini's theorem and the disintegration formula (13), (recalling that f has a compact support included in $\{y : \theta \cdot y \geq b\}$) we get

$$\int_{\mathbb{R}^d} B(s) ds \leq C_5 \mathbb{E} \left(\int_{e^{-\Lambda(\theta)}}^\infty v^{-2} [\log 1 + \widehat{Z}(\theta)v]^{1+\epsilon} \widehat{Z}(\theta) dv \right).$$

Since

$$\log(1 + \widehat{Z}(\theta)v) \leq \log(1+v) + \log_+ \widehat{Z}(\theta)$$

and by convexity

$$[\log(1 + \widehat{Z}(\theta)v)]^{1+\epsilon} \leq C_7 ([\log(1+v)]^{1+\epsilon} + [\log_+ \widehat{Z}(\theta)]^{1+\epsilon}),$$

we see that B is integrable under the assumptions of Theorem 2.

Invoking again the local central limit theorem,

$$\lim_n \sigma_\theta (2\pi n)^{d/2} \mathbb{E}_\theta (B(S_{n-1})) = \int_{\mathbb{R}^d} B(s) ds,$$

so that from (33)

$$(38) \quad \limsup_{n>j} \sigma_\theta (2\pi n)^{d/2} \mathbb{E}_\theta [f_\theta(S_n)(1 - \prod_j^n); (A_n^j)^c] \leq C_3 j^{-\epsilon} \int_{\mathbb{R}^d} B(s) ds.$$

Taking into account (26) and (38), we see that (20) holds, which entails (18) and completes the proof of (15). \blacksquare

3 Fragmentations

3.1 Notations and main result

We follow the notations of [5], [3], [6] and [7] and refer to these papers for details. We work with the space of numerical sequences

$$\mathcal{S} := \left\{ \mathbf{s} = (s_1, \dots) : s_1 \geq s_2 \geq \dots \geq 0 \text{ and } \sum_1^\infty s_i \leq 1 \right\},$$

which should be thought as the set of ranked masses of the fragments resulting from the split of some object with unit total mass. We consider a family of Feller processes $X = (X_t, t \geq 0)$ with values in \mathcal{S} and càdlàg paths. For every $a \in [0, 1]$, we let \mathbb{P}_a denote the law of X with initial distribution $(a, 0, \dots)$ (i.e. the process starts from a single fragment with mass a). We say that X is a (ranked) *homogeneous fragmentation* if the following two properties hold:

- (Homogeneity property) For every $a \in [0, 1]$, the law of aX under \mathbb{P}_1 is \mathbb{P}_a .
- (Fragmentation property) For every $\mathbf{s} = (s_1, \dots) \in \mathcal{S}$, the process started from $X(0) = \mathbf{s}$ can be obtained as follows. Consider $X^{(1)}, \dots$ a sequence of independent processes with respective laws \mathbb{P}_{s_1}, \dots , and for every $t \geq 0$, let $\hat{X}(t)$ be the random sequence obtained by ranking in decreasing order the terms of the random sequences $X^{(1)}(t), \dots$. Then \hat{X} has the law of X started from \mathbf{s} . Here we consider homogeneous fragmentations with no erosion ([6]), so that the distribution of X can be characterized by a measure ν on \mathcal{S} , called the *dislocation measure*. Informally ν specifies the rates at which a unit mass splits. It has to fulfil the conditions $\nu(\{(1, 0, \dots)\}) = 0$ and

$$(39) \quad \int_{\mathcal{S}} (1 - s_1) \nu(d\mathbf{s}) < \infty.$$

We shall assume that

$$(40) \quad \nu \left(\left\{ \mathbf{s} \in \mathcal{S} : \sum_{i=1}^{\infty} s_i < 1 \right\} \right) = 0,$$

which means that no mass is lost when a sudden dislocation occurs, and more precisely, entails that the total mass is a conserved quantity for the fragmentation process (i.e. $\sum_{i=1}^{\infty} X_i(t) = 1$ for all $t \geq 0$, \mathbb{P}_1 -a.s.). Moreover we also exclude the trivial case when $\nu \equiv 0$.

Given a real number $r > 0$, we say that a dislocation measure ν is r -geometric if it is finite and is carried by the subspace of configurations $\mathbf{s} = (s_1, \dots) \in \mathcal{S}$ such that $s_i \in \{r^{-n}, n \in \mathbb{N}\}$. This holds if and only if $\mathbb{P}_1(X_i(t) \in \{r^{-n}, n \in \mathbb{N}\}) = 1$ for all $t \geq 0$. We say that ν is *non-geometric* if it is not r -geometric for any $r > 0$.

The empirical measure of the logarithms of the fragments

$$(41) \quad Z^{(t)}(dy) := \sum_{i=1}^{\infty} \delta_{\log X_i(t)}, \quad t \geq 0$$

can be viewed as the generalization of a branching process in continuous time, with a possible infinite offspring mean and infinite rate of branching, corresponding to $\nu(\mathcal{S}) = \infty$. If we define

$$(42) \quad \Phi(p) := \int_{\mathcal{S}} \left(1 - \sum_i x_i^{p+1} \right) \nu(d\mathbf{x}), \quad p > \underline{p}$$

where

$$\underline{p} := \inf \left\{ p \in \mathbb{R} : \int_{\mathcal{S}} \sum_{i=2}^{\infty} s_i^{p+1} \nu(d\mathbf{s}) < \infty \right\},$$

then

$$(43) \quad \mathbb{E} \left(\int_{\mathbb{R}^d} e^{(p+1)y} Z^{(t)}(dy) \right) = \mathbb{E} \left(\sum_i X_i(t)^{p+1} \right) = \exp(-t\Phi(p))$$

and

$$(44) \quad M(p, t) := \int_{\mathbb{R}} e^{(p+1)y+t\Phi(p)} Z^{(t)}(dy) = e^{t\Phi(p)} \sum_i X_i^{p+1}(t), \quad t \geq 0$$

is the so-called additive martingale. The function Φ is concave, analytic and increasing. It is the Laplace exponent of a subordinator (χ_t)

$$(45) \quad \exp(-t\Phi(p)) = \mathbb{E} \exp(-p\chi_t)$$

(see [5] for details). If \bar{p} denotes the unique solution of the equation

$$\Phi(q) = (q+1)\Phi'(q), \quad q > \underline{p},$$

we have

$$\begin{aligned} \Phi(q) - (q+1)\Phi'(q) &< 0 \quad \text{for } \underline{p} < q < \bar{p} \\ \Phi(q) - (q+1)\Phi'(q) &> 0 \quad \text{for } p > \bar{p}. \end{aligned}$$

Since $\log X_1(t)$ (logarithm of the maximal size) grows as $t \rightarrow \infty$ like $-t\Phi'(\bar{p})$, we say that $\{a = -\Phi'(p); \underline{p} < p < \bar{p}\}$ is the supercritical range and $\{a = -\Phi'(p); \bar{p} < p\}$ the subcritical range.

Let us fix $\alpha < \beta$ and $p > \bar{p}$. Here we are interested in the asymptotic behaviour of

$$(46) \quad \begin{aligned} U(t, x) &:= \mathbb{P}(Z^{(t)}([\alpha + x, \beta + x]) \geq 1) \\ V(t, x) &:= \mathbb{E} Z^{(t)}([\alpha + x, \beta + x]), \end{aligned}$$

for $x = -t\Phi'(p)$ and $t \rightarrow \infty$.

The following theorem is proved in [7] using time-discretization and taking for granted Theorem 2 of the present paper.

Theorem 3 ([7]) *Assume that the dislocation measure ν is non-geometric.*

(i) *If $p > \underline{p}$, we have*

$$\lim_{t \rightarrow \infty} \sqrt{t} e^{-t((p+1)\Phi'(p) - \Phi(p))} V(t, -t\Phi'(p)) = \frac{1}{\sqrt{2\pi|\Phi''(p)|}} (p+1)^{-1} (e^{-(p+1)\alpha} - e^{-(p+1)\beta}).$$

(ii) *If $p > \bar{p}$, there exists a positive finite constant K_p such that*

$$\lim_{t \rightarrow \infty} \frac{U(t, -t\Phi'(p))}{V(t, -t\Phi'(p))} = K_p.$$

Remark Actually, for fixed $c \in \mathbb{R}$, we have

$$(47) \quad \lim_{t \rightarrow \infty} \frac{V(t, -t\Phi'(p) + c)}{V(t, -t\Phi'(p))} = e^{(p+1)c}, \quad \lim_{t \rightarrow \infty} \frac{U(t, -t\Phi'(p) + c)}{V(t, -t\Phi'(p) + c)} = K_p$$

(exactly as in (15)).

3.2 Comment

In [7] it was shown that K_p may be obtained as the constant coming from any discrete skeleton (i.e. for any arbitrary choice of the time mesh). Actually, a careful analysis may give a representation formula for the constants in terms of an underlying Lévy process instead of a (skeleton) random walk. We state it without proof not to overburden the paper. We have

$$(48) \quad \lim_{t \rightarrow \infty} \frac{U(t, -at)}{V(t, -at)} = \frac{\int_{\alpha}^{\beta} e^{-(p+1)y} G(p, y) dy}{\int_{\alpha}^{\beta} e^{-(p+1)y} dy}$$

where $G(p, y)$ is an expression that we explain now.

In the spirit of Section 2, we define

$$U_t[f](x) = 1 - \mathbb{E} \left(\prod_i (1 - f(x + \log X_i(t))) \right), \quad V_t[f](x) = \mathbb{E} \left(\sum_i f(x + \log X_i(t)) \right)$$

so that $U(t, x) = U_t[\mathbf{1}_{[\alpha, \beta]}](-x)$. The random walk $(S_n, n \geq 1)$ under the law \mathbb{P}_{θ} in Section 2 is now replaced by a certain Lévy process $(\hat{\zeta}_t)$.

Recall that Φ is the Laplace exponent of the Lévy process (χ_t) (see (45)). Let L be its characteristic measure so that

$$\Phi(\lambda) = \int_{]0, \infty[} (1 - e^{-\lambda x}) L(dx).$$

Then $(\hat{\zeta}_t, t \geq 0)$ is the dual of the Lévy process whose characteristic measure is $\tilde{L}_p(d\Delta) = e^{-p\Delta} L(d\Delta)$ and drift coefficient a , so that its mean expectation is 0 (see [4]). We have in particular

$$(49) \quad V_t[f](-at + c) = e^{(t(p+1)\Phi'(p) - \Phi(p))} e^{(p+1)c} \mathbb{E} f_{p+1}(\hat{\zeta}_t + c).$$

To study U_t we need more definitions. Let \mathbf{x}_* be a ‘size biased pick’ from the sequence \mathbf{x} , i.e. a random variable with values in $]0, 1[$ such that for every $\mathbf{x} \in \mathcal{S}$ and $i \in \mathbb{N}$

$$\mathbb{P}(\mathbf{x}_* = x_i \mid \mathbf{x} = (x_1, \dots)) = x_i$$

or in other words $\mathbb{E}(g(\mathbf{x}_*) \mid \mathbf{x}) = \langle \mathbf{x}, \bar{g} \rangle$, where $\bar{g}(y) = yg(y)$. At last we denote by $\mathbf{x}^!$ the random sequence obtained by removing the size biased pick \mathbf{x}_* from the sequence \mathbf{x} .

Let m be the ‘distribution’ under ν of the size biased pick, i.e.

$$\int_{(0,1)} g(x) m(dx) = \int_{\mathcal{S}} \nu(d\mathbf{x}) \mathbb{E}(g(\mathbf{x}_*) \mid \mathbf{x}).$$

We check easily that $\int_{(0,1)} (1-x) m(dx) < \infty$. For $x \in (0, 1)$, we denote by $\nu^{!x}$ the probability measure defined on \mathcal{S} by the disintegration

$$(50) \quad \int_{\mathcal{S}} \nu(d\mathbf{x}) \mathbb{E}(H(\mathbf{x}_*, \mathbf{x}^!)) = \int_{]0,1[} m(dx) \int_{\mathcal{S}} \nu^{!x}(d\mathbf{x}) H(x, \mathbf{x}).$$

In words, $\nu^{!x}$ is the conditional law under ν of $\mathbf{x}^!$ knowing $\mathbf{x}_* = x$. This definition is similar to (13) in the BRW framework.

With that notation we may define

$$(51) \quad H_r(\xi, z) = \int_0^1 \int_{\mathcal{S}} \exp \left(\sum_{i=1}^{\infty} \log(1 - \beta U_r(z + \log x_i)) \right) \nu^{!e^\xi}(d\mathbf{x}) d\beta,$$

and the function $G(p, y)$ in (48) is

$$G(p, y) = \mathbb{E} \left\{ \prod_{r < \infty, \Delta \hat{\zeta}_r \neq 0} H_r(\Delta \hat{\zeta}_r, y - ar - \hat{\zeta}_r) \right\}.$$

where $\Delta \hat{\zeta}_r = \hat{\zeta}_r - \hat{\zeta}_r^-$.

3.3 The tilted probability in the subcritical range

This section is an easy extension to fragmentations of the corresponding result for branching Brownian motion [10] section 6, or branching random walk [13] p. 30-31 but for the sake of completeness we give the complete proof. For fixed a in the subcritical range we show that conditioning on the presence of fragments of size e^{-at+1} at time t gives in the limit $t \rightarrow \infty$ the tilted probability $\mathbb{P}^{(p)}$ for the measure valued process $(Z_s, s \geq 0)$. This tilted probability, described in [7] Section 3.3, is the h-transform of the probability by the martingale $M(p, \cdot)$. This result is in the spirit of many results on conditioning branching spatial processes and superprocesses and related to the notion of immortal particle due to Evans (see an extensive bibliography in [11]).

Let $(\mathcal{F}(s), s \geq 0)$ be the natural filtration defined by $\mathcal{F}(s) := \sigma(Z_r, r \leq s)$ and $\mathcal{F} = \wedge \mathcal{F}_s$.

Proposition 4 *Let us fix $\alpha < \beta$, $p > \bar{p}$ and $a = -\Phi'(p)$. Then, for every $A \in \mathcal{F}$ we have:*

$$\lim_{t \rightarrow \infty} \mathbb{P}(A \mid Z_t([at + \alpha, at + \beta] \geq 1)) = \mathbb{P}^{(p)}(A)$$

where, for every $s > 0$,

$$\mathbb{P}^{(p)}|_{\mathcal{F}(s)} = M(p, s)\mathbb{P}|_{\mathcal{F}_s}.$$

Proof: Set $J = [\alpha, \beta]$. Fix $s > 0$ and $G_s \in \mathcal{F}_s$. We have for $t > 0$

$$(52) \quad \mathbb{P}(G_s \mid Z_{t+s}(a(t+s) + J) \geq 1) = \mathbb{E} \left[G_s \frac{\mathbb{P}[Z_{t+s}(a(t+s) + J) \geq 1 \mid \mathcal{F}_s]}{\mathbb{P}[Z_{t+s}(a(t+s) + J) \geq 1]} \right]$$

From (46)

$$\mathbb{P}[Z_{t+s}(a(t+s) + J) \geq 1] = U_{t+s}(-a(t+s))$$

and, by the fragmentation property

$$(53) \quad \mathbb{P}[Z_{t+s}(a(t+s) + J) \geq 1 \mid \mathcal{F}_s] = 1 - \prod_j (1 - A_j(t))$$

where $A_j(t) := U_t(\log X_j(s) - a(t+s))$.

To apply the results of the above section, we set $r_t := \sigma_p \sqrt{2\pi t} e^{t\Lambda^*(a)}$ (which tends to infinity with t) and $K'_p := \int_\alpha^\beta e^{-(p+1)y} dy$. From Theorem 3 ii)

$$(54) \quad \lim_t r_t U_{t+s}(-a(t+s)) = K_p K'_p e^{-s\Lambda^*(a)}.$$

To handle the expression (53), we apply Theorem 3 i) for every j :

$$\lim_t r_t A_j(t) = K_p K'_p X_j(s)^{p+1} e^{-(p+1)as},$$

but we need a uniform bound. From (49) and the local central limit theorem, there exists $\epsilon_t \rightarrow 0$ such that, for every $c \in \mathbb{R}$

$$r_t V_t(-at + c) \leq K'_p (1 + \epsilon_t) e^{(p+1)c}.$$

This yields:

$$r_t A_j(t) \leq r_t V_t(\log X_j(s) - a(t+s)) \leq K'_p (1 + \epsilon_t) X_j(s)^{p+1} e^{-(p+1)as}.$$

By dominated convergence, we have a.s.

$$\lim_t r_t \sum_j A_j(t) = \sum_j X_j(s)^{p+1} e^{-(p+1)as} = K_p K'_p M(s, p) e^{-s\Lambda^*(a)}$$

and since

$$\sum_j A_j(t) - \left(\sum_j A_j(t) \right)^2 \leq 1 - \prod_j (1 - A_j(t)) \leq \sum_j A_j(t),$$

we get

$$\lim_t \frac{\mathbb{P}[Z_{t+s}(a(t+s) + J) \geq 1 \mid \mathcal{F}_s]}{\mathbb{P}[Z_{t+s}(a(t+s) + J) \geq 1]} = \lim_t \frac{r_t \sum_j A_j(t)}{r_t U_{t+s}(-a(t+s))} = M(s, p)$$

a.s.. Invoking again the dominated convergence theorem allows to conclude from (52)

$$\lim_t \mathbb{P}(G_s \mid Z_{t+s}(a(t+s) + J) \geq 1) = \mathbb{E}[G_s M(p, s)]$$

which ends the proof of the proposition. ■

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