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## MODULES WITH RD-COMPOSITION SERIES OVER A COMMUTATIVE RING

FRANÇOIS COUCHOT

ABSTRACT. If  $R$  is a commutative ring, then we prove that every finitely generated  $R$ -module has a pure-composition series with indecomposable cyclic factors and any two such series are isomorphic if and only if  $R$  is a Bézout ring and a CF-ring. When  $R$  is a such ring, the length of a pure-composition series of a finitely generated  $R$ -module  $M$  is compared with its Goldie dimension and we prove that these numbers are equal if and only if  $M$  is a direct sum of cyclic modules. We also give an example of an artinian module over a noetherian domain, which has an RD- composition series with uniserial factors. Finally we prove that every pure-injective  $R$ -module is RD-injective if and only if  $R$  is an arithmetic ring.

In this paper, for a commutative ring  $R$ , we study the following properties:

- (1) Every finitely generated  $R$ -module  $M$  has a finite chain of RD-submodules with cyclic factors.
- (2) Every finitely generated  $R$ -module  $M$  has a finite chain of RD-submodules with indecomposable cyclic factors.
- (3)  $R$  satisfies (2) and any two chains of RD-submodules of  $M$ , with indecomposable cyclic factors, are isomorphic.

In [1], L.Salce and P.Zanardo proved that every valuation ring satisfies (3), and in [2] C.Naudé showed that each h-local and Bézout domain also satisfies (3).

In section 1, we give definitions and some preliminary results. It is proved that every ring that satisfies (1), is a Kaplansky ring (or an elementary divisor ring), but we don't know if the converse holds.

In section 2, we state that a ring  $R$  satisfies (3) if and only if  $R$  is a Bézout ring and a CF-ring ([3]). We show that  $R$  satisfies (2), when  $R$  is a semilocal arithmetic ring or an h-semilocal Bézout domain, and for every finitely generated  $R$ -module  $M$ , we prove that any chain of RD-submodules of  $M$  has the same length which is equal to the number of terms of a finite sequence of prime ideals, associated to  $M$ . However, we give two examples of semilocal arithmetic rings that don't satisfy (3).

In section 3, when  $R$  is a ring that verifies (3), the length  $l(M)$  of each chain of RD-submodules, with indecomposable cyclic factors, is compared with  $g(M)$ , the Goldie dimension of  $M$ . We show that  $g(M) \leq l(M)$  and that  $M$  contains a direct sum of  $g(M)$  nonzero indecomposable cyclic submodules which is an essential RD-submodule of  $M$ . Hence it follows that  $M$  is a direct sum of cyclic submodules if and only if  $g(M) = l(M)$ . These results were proved in [1], when  $R$  is a valuation ring and in [2], when  $R$  is an h-local Bézout domain.

Finally, in section 4, we give an example of an artinian module, over a noetherian domain, that has a finite chain of RD-submodules, with uniserial factors. We also give explicit examples of pure-injective modules that fail to be RD-injective, over

noetherian domains. We prove that every pure-injective  $R$ -module is RD-injective if and only if  $R$  is an arithmetic ring. When  $R$  is a domain, this result was proved in [4], by C.G Naudé, G. Naudé and L.M. Pretorius.

## 1. PRELIMINARIES

All rings in this paper are commutative with unity, and all modules are unital. We recall that a module, over a ring  $R$ , is *uniserial* if its set of submodules is totally ordered by inclusion. A ring  $R$  is a *valuation ring* if  $R$  is a uniserial module, and  $R$  is *arithmetic* if  $R_P$  is a valuation ring for every maximal ideal  $P$ . We say that  $R$  is a *Bézout ring* if every finitely generated ideal is principal and  $R$  is a *Kaplansky ring* (or *elementary divisor ring*) if every finitely presented  $R$ -module is a finite direct sum of cyclic presented modules (see [5] and [6]).

An exact sequence of  $R$ -modules  $: 0 \rightarrow F \rightarrow E \rightarrow G \rightarrow 0$  is *pure-exact* if it remains exact when tensoring it with any  $R$ -module. In this case we say that  $F$  is a *pure submodule* of  $E$ . When  $rE \cap F = rF$  for every  $r \in R$ , we say that  $F$  is an *RD-submodule* of  $E$  (relatively divisible) and that the sequence is *RD-exact*. Then every pure submodule is an RD-submodule, and the equivalence holds if and only if  $R$  is an arithmetic ring ([7] and [8]). We say that an  $R$ -module  $M$  has a *pure-composition series* (respectively an *RD-composition series*) if there exists a finite chain  $\{0\} = M_0 \subset M_1 \subset \dots \subset M_n = M$  of pure (respectively RD-) submodules. If we denote  $A_i = \text{ann}(M_i/M_{i-1})$ , then  $(A_i)_{1 \leq i \leq n}$  is called the *annihilator sequence* of the pure-composition series. We say that the annihilator sequence is *increasing* if  $A_1 \subseteq A_2 \subseteq \dots \subseteq A_n$ , and it is *totally ordered* if  $A_i$  and  $A_j$  are comparable  $\forall i, \forall j, 1 \leq i, j \leq n$ .

An  $R$ -module  $F$  is *pure-projective* (respectively *RD-projective*) if for every pure (respectively RD-) exact sequence:  $0 \rightarrow N \rightarrow M \rightarrow L \rightarrow 0$ , of  $R$ -modules, the following sequence:  $0 \rightarrow \text{Hom}_R(F, N) \rightarrow \text{Hom}_R(F, M) \rightarrow \text{Hom}_R(F, L) \rightarrow 0$ , is exact.

An  $R$ -module  $F$  is *pure-injective* (respectively *RD-injective*) if for every pure (respectively RD-) exact sequence:  $0 \rightarrow N \rightarrow M \rightarrow L \rightarrow 0$  of  $R$ -modules, the following sequence:  $0 \rightarrow \text{Hom}_R(L, F) \rightarrow \text{Hom}_R(M, F) \rightarrow \text{Hom}_R(N, F) \rightarrow 0$ , is exact. Then we have the following proposition.

**Proposition 1.1.** *Let  $R$  be a ring. The following conditions are equivalent:*

- (1) *Every finitely generated module has a pure-composition series with cyclic factors.*
- (2) *Every finitely generated module has an RD-composition series with cyclic factors.*

*Moreover, if  $R$  satisfies one of these conditions,  $R$  is a Kaplansky ring.*

**Proof.** If  $R$  satisfies (1), by the same proof as in [2, Theorem 2.3 and Corollary 2.4], we show that  $R$  is a Kaplansky ring.

Since every pure submodule is an RD-submodule, it follows that (1)  $\Rightarrow$  (2).

If  $R$  satisfies (2) and if we prove that  $R$  is an arithmetic ring, by [7, Theorem 3]  $R$  satisfies (1).

For every maximal ideal  $P$ , we must prove that  $R_P$  is a valuation ring. Clearly  $R_P$  also satisfies (2). We may assume that  $R$  is local and that  $P$  is its maximal ideal. If  $R$  is not a valuation ring, then, as in the proof of [8, Theorem 2], we may suppose that there exist  $a$  and  $b$  in  $P$  such that  $Ra \cap Rb = \{0\}$  and  $\text{ann}(a) =$

$\text{ann}(b) = P$ . Moreover, there exists an indecomposable  $R$ -module  $M$ , with two generators  $x$  and  $y$ , satisfying the relation  $bx - ay = 0$ . Since  $R$  satisfies (2), there exists an RD-composition series:  $\{0\} = M_0 \subset M_1 \subset \cdots \subset M_n = M$ , such that  $M_i/M_{i-1} = R(z_i + M_{i-1})$  for each  $i$ ,  $1 \leq i \leq n$ . For every  $i$ , there exist  $s_i$  and  $t_i$  in  $R$  such that  $z_i = s_i x + t_i y$ . Clearly  $M$  is generated by  $\{z_i \mid 1 \leq i \leq n\}$ . Since  $\dim_{R/P}(M/PM) = 2$ , there exist at least 2 indexes  $j$  and  $k$ ,  $1 \leq j < k \leq n$  such that  $(s_j, t_j) \notin P \times P$  and  $(s_k, t_k) \notin P \times P$ . Let  $j$  be the smallest index such that  $(s_j, t_j) \notin P \times P$ . Then  $\dim_{R/P}(M_j/PM_j) = 1$ , whence  $M_j = Rz_j$ . We may assume that there exists  $z \in M$ ,  $z = x + ty$ , where  $t \in R$ , such that  $Rz$  is an RD-submodule of  $M$ . Then  $M/Rz = R(y + Rz)$  and if  $I = \text{ann}(M/Rz)$ ,  $I = \{r \mid ry \in Rz\}$ . Let  $r \in I$ . There exists  $c \in R$  such that  $ry = c(x + ty)$ . Since  $cx + (tc - r)y = 0$ , it follows that  $\exists u \in R$  such that  $c = ub$  and  $r - tc = ua$ , whence  $r = u(a + tb)$ . Therefore  $I \subseteq R(a + tb) \subset Ra \oplus Rb$  which is semi-simple. We deduce that  $I = Rd$  where  $d = 0$  or  $d = a + tb$ . We get that  $M/Rz \simeq R/Rd$  is RD-projective. Consequently  $Rz$  is a direct summand of  $M$ , whence a contradiction.  $\square$

We don't know if the converse of the second assertion of this proposition holds. The following propositions state that some classes of arithmetic rings satisfy the equivalent conditions of the Proposition 1.1, and therefore these rings are Kaplansky.

We begin with the class of semi-local arithmetic rings, and we already know that these rings are Kaplansky by [6, Corollary 2.3].

If  $M$  is a finitely generated  $R$ -module, we denote  $\mu(M)$  the minimal number of generators of  $M$ .

**Proposition 1.2.** *Let  $R$  be a semilocal arithmetic ring. Then every finitely generated  $R$ -module  $M$  has a pure-composition series with cyclic factors and an increasing annihilator sequence.*

**Proof.** Let  $P_1, \dots, P_n$  be the maximal ideals of  $R$ , and  $J$  its Jacobson radical. As in the proof of [9, Theorem 15], we state that, for every  $k$ ,  $1 \leq k \leq n$ , there exists  $x_k \in M$  such that, for each  $y \in x_k + P_k M_{P_k}$ ,  $R_{P_k} y$  is a pure submodule of  $M_{P_k}$ , and  $\text{ann}_{R_{P_k}}(R_{P_k} y) = \text{ann}_{R_{P_k}}(M_{P_k})$ . Moreover  $\mu_{R_{P_k}}(M_{P_k}/R_{P_k} y) = \mu_{R_{P_k}}(M_{P_k}) - 1$  if  $M_{P_k} \neq \{0\}$ . When  $M_{P_k} = \{0\}$  we put  $x_k = 0$ . By the Chinese remainder Theorem, we get  $M/JM \simeq \prod_{k=1}^n (M/P_k M) \simeq \prod_{k=1}^n (M_{P_k}/P_k M_{P_k})$ . Consequently there exists  $x \in M$  such that  $x \equiv x_k \pmod{P_k M_{P_k}}$ ,  $\forall k$ ,  $1 \leq k \leq n$ . We deduce that  $Rx$  is a pure submodule of  $M$ , that  $\text{ann}(Rx) = \text{ann } M$  and that  $\mu(M/Rx) = \mu(M) - 1$ . We complete the proof by induction on  $m = \mu(M)$ .  $\square$

A domain  $R$  is said to be *h-semilocal* if  $R/I$  is semilocal for every nonzero ideal  $I$ , and  $R$  is said to be *h-local* if, in addition,  $R/P$  is local for every nonzero prime ideal  $P$ , [10].

**Proposition 1.3.** *Let  $R$  be a Bézout ring with a unique minimal prime ideal  $Q$ . We assume that  $Q$  is a uniserial module and  $R/Q$  an h-semilocal domain. Then every finitely generated  $R$ -module has a pure-composition series with cyclic factors and an increasing annihilator sequence.*

**Proof.** We may assume that  $R$  is not semilocal. If  $Q \neq \{0\}$ , then from [11, Lemma 17 and 18] there exists only one maximal ideal  $P$  such that  $QR_P \neq \{0\}$ , every ideal contained in  $Q$  is comparable to every ideal of  $R$ ,  $Q^2 = \{0\}$  and  $Q$  is a

torsion divisible  $R/Q$ -module. Let  $M$  be a finitely generated module. Since  $R/Q$  is a Bézout domain,  $M/QM \simeq F \oplus T$ , where  $T$  is a torsion  $R/Q$ -module and  $F$  a free  $R/Q$ -module of finite rank. Let  $m$  be this rank. We prove this proposition by induction on  $m$ .

If  $m = 0$ , then for every  $x \in M$ , there exists  $s \in R \setminus Q$  such that  $sx \in QM$ . Since  $Q$  is uniserial, there exists  $t \in Q$  and  $y \in M$  such that  $sx = ty$ . Since  $Q$  is a torsion  $R/Q$ -module there exists  $r \notin Q$  such that  $rt = 0$ . Since  $M$  is finitely generated, we get that  $Q \subset \text{ann}(M)$ , and that  $R/\text{ann}(M)$  is semilocal. From the previous proposition we deduce the result.

Now suppose  $m \geq 1$ . Let  $\{x_1 + QM, \dots, x_m + QM\}$  be a basis of  $F$ , and  $\{y_1 + QM, \dots, y_p + QM\}$  a spanning set of  $T$ . Since  $Q$  is the nilradical of  $R$ , then, by Nakayama Lemma  $\{x_1, \dots, x_m\} \cup \{y_1, \dots, y_p\}$  generates  $M$ . Let  $H = \{j \mid 1 \leq j \leq m, \text{ann}(x_j) = \text{ann}(M)\}$ . Since  $P$  is the only maximal ideal such that  $QR_P \neq \{0\}$ , then  $H = \{j \mid 1 \leq j \leq m, \text{ann}_{R_P}(R_P x_j) = \text{ann}_{R_P}(M_P)\}$  also. From [9, Theorem 15], it follows that there exists  $j \in H$  such that  $R_P x_j$  is a pure-submodule of  $M_P$ . Let  $N$  be a maximal ideal,  $N \neq P$ . Then  $M_N \simeq (M/QM)_N = F_N \oplus T_N$ . Consequently  $R_N x_j$  is a direct summand of  $M_N$ . We deduce that  $Rx_j$  is a pure-submodule of  $M$  and if  $M' = M/Rx_j$ , then  $M'/QM' \simeq F' \oplus T$  where  $F'$  is a free  $R/Q$ -module of rank  $(m - 1)$ . From the induction hypothesis it follows that  $M$  has a pure-composition series with cyclic factors and an increasing annihilator sequence.  $\square$

**Proposition 1.4.** *Let  $R$  be an arithmetical ring and  $M$  be a finitely generated  $R$ -module. If  $M$  has a pure-composition series with cyclic factors and an increasing annihilator sequence, then  $\mu(M)$  is the length of this pure-composition series.*

**Proof.** Let  $\{0\} = M_0 \subset M_1 \subset M_2 \cdots \subset M_{n-1} \subset M_n = M$  be a pure-composition series such that  $A_i = \text{ann}(M_i/M_{i-1}) \subseteq A_{i+1}$  for every  $i$ ,  $1 \leq i \leq n-1$ , and such that  $M_i/M_{i-1} = R(x_i + M_{i-1})$  for every  $i$ ,  $1 \leq i \leq n$ . Then, clearly  $\{x_i \mid 1 \leq i \leq n\}$  generates  $M$ . Therefore  $n \geq \mu(M)$ . Since  $M/M_{n-1} \neq \{0\}$ , there exists a maximal ideal  $P$  such that  $(M/M_{n-1})_P \neq \{0\}$ . Consequently  $(M_i/M_{i-1})_P \neq \{0\}$  for every  $i$ ,  $1 \leq i \leq n$ . Then  $M_P$  has a pure-composition series of length  $n$ . From [1, Lemma 1.4] we deduce that  $n = \mu_{R_P}(M_P)$ . But  $\mu_{R_P}(M_P) \leq \mu(M)$  and consequently  $n = \mu(M)$ .  $\square$

Now, as in [1] or [12], we establish the isomorphy of any two pure-composition series, with cyclic factors and totally ordered annihilator sequences, i.e. the existence of a bijection between the two sets of cyclic factors, such that corresponding factors are isomorphic. We use similar lemmas, with the same proofs, and we get the following theorem:

**Theorem 1.5.** *Let  $R$  be an arithmetic ring. Then any two pure-composition series of a finitely generated  $R$ -module, with cyclic factors and totally ordered annihilator sequences, are isomorphic.*

**Proof.** See the proof of [12, Theorem 1.6, p. 177].  $\square$

## 2. PURE-COMPOSITION SERIES WITH INDECOMPOSABLE CYCLIC FACTORS

We follow T.S. Shores and R. Wiegand [3], by defining a *canonical form* for an  $R$ -module  $M$  to be a decomposition  $M \simeq R/I_1 \oplus R/I_2 \oplus \cdots \oplus R/I_n$ , where  $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \neq R$ , and by calling a ring  $R$  a *CF-ring* if every direct sum of

finitely many cyclic modules has a canonical form. Now, following P. Vámos [13], we say that  $R$  is a *torch ring* if the following conditions are satisfied :

- (1)  $R$  is a nonlocal arithmetic ring.
- (2)  $R$  has a unique minimal prime ideal  $Q$  which is a nonzero uniserial  $R$ -module.
- (3)  $R/Q$  is an h-local domain.

We will say that  $R$  is a *semi-torch ring* if  $R$  satisfies the conditions (1), (2) and (3'):  $R/Q$  is an h-semilocal domain.

By [3, Theorem 3.12], a CF-ring is a finite product of indecomposable arithmetic rings. If  $R$  is an indecomposable CF-ring, then  $R$  is a valuation ring, or an h-local domain, or a torch ring.

We will say that  $R$  is a *semi-CF-ring* if  $R$  is a finite product of indecomposable arithmetic rings, where each indecomposable factor ring is semilocal, or an h-semilocal domain, or a semi-torch ring.

In the sequel, we will call a ring  $R$  to be a *PCS-ring* if every finitely generated module has a pure-composition series with indecomposable cyclic factors and if any two such pure-composition series are isomorphic.

We will state the following theorem which is one of the main results of this paper.

**Theorem 2.1.** *Let  $R$  be a ring. Then the following assertions are equivalent :*

- (1)  $R$  is a Bézout ring and a CF-ring.
- (2)  $R$  is a PCS-ring.

Some preliminary results are needed to prove this theorem. First we will state some results on finitely generated modules over a Bézout semi-CF-ring  $R$ . Recall that a module  $M$  has *Goldie dimension*  $n$  if it has an independent set of uniform submodules  $M_1, \dots, M_n$  such that  $M_1 \oplus \dots \oplus M_n$  is essential in  $M$ . We will denote  $g(M)$  the Goldie dimension of  $M$ . Then  $g(M) = n$  if and only if the injective hull  $E(M)$  of  $M$  is a direct sum of  $n$  indecomposable injective modules.

**Proposition 2.2.** *Let  $R$  be a Bézout semi-CF-ring. Then every finitely generated  $R$ -module has a pure-composition series with indecomposable cyclic factors and a finite Goldie dimension.*

**Proof.** Let  $M$  be a finitely generated  $R$ -module. We put  $R = \prod_{j=1}^m R_j$ , where  $R_j$  is indecomposable. Then  $M \simeq \bigoplus_{j=1}^m M_j$ , where  $M_j = R_j \otimes_R M$ . From Propositions 1.2 and 1.3,  $M_j$  has a pure-composition series:

$$0 \subset M_{1,j} \subset \dots \subset M_{n_j,j} = M_j,$$

with cyclic factors. Then the chain (c):

$$\begin{aligned} 0 \subset M_{1,1} \subset \dots \subset M_{n_1,1} = M_1 \subset M_1 \oplus M_{1,2} \subset \dots \subset M_1 \oplus M_2 \subset \dots \\ \dots \subset M_1 \oplus \dots \oplus M_{m-1} \oplus M_{1,m} \subset \dots \subset M_1 \oplus \dots \oplus M_m = M \end{aligned}$$

is a pure-composition series of  $M$ , with cyclic factors. Let  $N$  be a factor of this pure-composition series. Then  $N$  is a module over an indecomposable factor ring  $R'$  of  $R$ . When  $R'$  is a domain and  $\text{ann}_{R'}(N) = \{0\}$ , or when  $R'$  is a semitorch ring and  $\text{ann}_{R'}(N) \subseteq Q$ , where  $Q$  is the minimal prime of  $R'$ ,  $N$  is indecomposable. In the other cases,  $R/\text{ann}(N)$  is semilocal. Since each semilocal arithmetic ring has only finitely many minimal prime ideals which are pairwise comaximal, it follows that  $N$  is a finite direct sum of indecomposable cyclic modules. Consequently, we deduce from (c) a pure-composition series of  $M$ , with indecomposable cyclic factors.

It is sufficient to prove that  $M_j$  has a finite Goldie dimension for every  $j$ ,  $1 \leq j \leq m$ . We may assume that  $R$  is indecomposable. First, we suppose that  $M$  is cyclic. When  $R$  is a domain and  $\text{ann}(M) = \{0\}$ , or when  $R$  is a semitorch ring and  $\text{ann}(M) \subseteq Q$ , where  $Q$  is the minimal prime of  $R$ , we have  $g(M) = 1$ . In the other cases  $R/\text{ann}(M)$  is a semilocal ring. We may assume that  $R$  is semilocal. Let  $(S_i)_{i \in I}$  be a family of independant submodules of  $M$  such that  $\bigoplus_{i \in I} S_i$  is essential in  $M$ . For every maximal ideal  $P$ ,  $M_P$  is a uniserial  $R_P$ -module and consequently there is at most one index  $i$  in  $I$  such that  $(S_i)_P \neq \{0\}$ . It follows that  $I$  is a finite set. Hence  $M$  has a finite Goldie dimension. In the general case, let  $N$  be a submodule of  $M$  such that  $\mu(N) = \mu(M) - 1$  and  $M/N$  is cyclic. The inclusion map  $N \rightarrow E(N)$  can be extended to  $w : M \rightarrow E(N)$ . Let  $f : M \rightarrow E(N) \oplus E(M/N)$  defined by  $f(x) = (w(x), x + N)$ , for each  $x \in M$ . It is easy to verify that  $f$  is a monomorphism. It follows that  $g(M) \leq g(N) + g(M/N)$ . By induction on  $\mu(M)$ , we complete the proof.  $\square$

If  $R$  is a Bézout semi-CF-ring and  $N$  an indecomposable cyclic  $R$ -module, then  $J = \text{rad}(\text{ann}(N)) = \{r \in R \mid \exists p \in \mathbb{N}, \text{ such that } r^p \in \text{ann}(N)\}$  is the minimal prime ideal over  $\text{ann}(N)$ . For every finitely generated module  $M$ , from each pure-composition series of  $M$ , with indecomposable cyclic factors, we can associate a sequence of prime ideals. Then we have the following proposition:

**Proposition 2.3.** *Let  $R$  be Bézout semi-CF-ring, and  $M$  a finitely generated module. We consider two pure-composition series of  $M$  with indecomposable cyclic factors:*

$$(s): \{0\} = M_0 \subset M_1 \subset M_2 \dots \dots \subset M_{n-1} \subset M_n = M.$$

$$(s'): \{0\} = M'_0 \subset M'_1 \subset M'_2 \dots \dots \subset M'_{n'-1} \subset M'_{n'} = M.$$

We denote  $A_i = \text{ann}(M_i/M_{i-1})$ ,  $A'_i = \text{ann}(M'_i/M'_{i-1})$ ,  $J_i = \text{rad}(A_i)$ ,  $J'_i = \text{rad}(A'_i)$ ,  $S = (J_i)_{1 \leq i \leq n}$  and  $S' = (J'_i)_{1 \leq i \leq n'}$ .

Then  $n = n'$ , and there exists  $\sigma$  in the symmetric group  $S_n$  such that  $J_{\sigma(i)} = J'_i$  for every  $i$ ,  $1 \leq i \leq n$ .

**Proof.** Let  $C$  be a maximal totally ordered subsequence of  $S$  and  $P$  a maximal ideal of  $R$  containing every term of  $C$ . For each  $i$  such that  $J_i \not\subseteq P$  we have  $(M_i)_P = (M_{i-1})_P$ , and for each  $i$  such that  $J_i \subseteq P$  we have  $(M_i)_P \supset (M_{i-1})_P$ . We deduce from (s) a pure-composition series of  $M_P$ , with cyclic factors, whose the length is equal to the number of terms of  $C$ . By [12, Theorem 1.6, p. 177], the pure-composition series of  $M_P$ , deduced from (s) and (s') are isomorphic, and this conclusion holds for each maximal ideal  $P$  containing all terms of  $C$ . Consequently there is a maximal totally ordered subsequence  $C'$  of  $S'$ , which has the same terms as  $C$ . Conversely, the same conclusion holds for every maximal totally ordered subsequence of  $S'$ . Therefore  $S$  and  $S'$  have the same terms, eventually with different indices.  $\square$

We will say that a sequence of ideals  $(A_i)_{1 \leq i \leq n}$  is *almost totally ordered* if  $A_i$  and  $A_j$  are either comparable, or comaximal,  $\forall i, \forall j$ ,  $1 \leq i, j \leq n$ , and that this sequence is *almost increasing* if either  $A_i \subseteq A_j$ , or  $A_i$  and  $A_j$  are comaximal,  $\forall i, \forall j$ ,  $1 \leq i < j \leq n$ . Let us observe that every sequence of prime ideals of an arithmetic ring is almost totally ordered.

In the sequel, if  $R$  is a Bézout semi-CF-ring and  $M$  a finitely generated  $R$ -module, we denote  $\ell(M)$  the length of every pure-composition series of  $M$ , with indecomposable cyclic factors.

**Proposition 2.4.** *Let  $R$  be a Bézout semi-CF-ring, and  $M$  a finitely generated  $R$ -module. Then the following assertions are true:*

- (1) *From every pure-composition series of  $M$ , with indecomposable cyclic factors, we deduce a pure-composition series with an almost increasing annihilator sequence.*
- (2) *Any two pure-composition series of  $M$ , with indecomposable cyclic factors and almost totally ordered annihilator sequences, are isomorphic.*

**Proof.**

(1) We consider the following pure-composition series of  $M$ , with indecomposable cyclic factors,

$$(s) : 0 = M_0 \subset M_1 \subset M_2 \dots \dots \subset M_{n-1} \subset M_n = M,$$

where  $M_i/M_{i-1} = R(x_i + M_{i-1})$ .

We denote  $A_i = \text{ann}(M_i/M_{i-1})$  and  $S$  the sequence of prime ideals associated to  $M$ . We put  $S = (J_i)_{1 \leq i \leq n}$ .

We claim that, after a possible permutation of indices, we may assume that  $S$  is almost increasing. To prove this, we induct on  $n$ . If  $n = 1$ , it is obvious. Suppose  $n > 1$ . Let  $C$  be a maximal totally ordered subsequence of  $S$ , and  $S'$  the subsequence of all terms of  $S$  which are not terms of  $C$ . If  $C = S$  the claim is obvious. We may assume that  $C \neq S$ . Clearly, if  $J_i$  is a term of  $C$  and  $J_k$  a term of  $S'$ , then  $J_i \subset J_k$  or  $J_i$  and  $J_k$  are comaximal. Since the induction hypothesis can be applied to  $S'$ , then it is possible to get an almost increasing sequence, if we begin to index the terms of  $C$  and we end with the terms of  $S'$ .

We will prove, by induction on  $n = \ell(M)$ , that we can get a pure-composition series (t) of  $M$ , with indecomposable cyclic factors:  $\{0\} \subset N_1 \subset \dots \subset N_{n-1} \subset M$ , with an almost increasing annihilator sequence  $(B_i)_{1 \leq i \leq n}$  and such that  $\text{rad}(B_i) = J_i$ , for each  $i$ ,  $1 \leq i \leq n$ .

If  $n = 1$ , this is obvious. We assume that  $n > 1$ . Let  $k$  be the smallest index such that  $\text{rad}(A_k) = J_n$ . If  $k < n$ , one of the five following cases holds:

(a):  $\text{rad}(A_{k+1})$  and  $J_n$  are comaximal. It follows that  $A_k$  and  $A_{k+1}$  are also comaximal. Therefore, for every maximal ideal  $P$ , we have  $(M_{k+1}/M_k)_P = \{0\}$  or  $(M_k/M_{k-1})_P = \{0\}$ . Hence  $M_{k+1}/M_{k-1}$  is cyclic and its annihilator is  $A_k \cap A_{k+1}$ . Then  $M_{k+1}/M_{k-1} \simeq R/(A_k \cap A_{k+1}) \simeq R/A_k \oplus R/A_{k+1}$ . Hence  $M_{k+1}/M_{k-1} = R(x'_k + M_{k-1}) \oplus R(x'_{k+1} + M_{k-1})$ , where  $\text{ann}(x'_k + M_{k-1}) = A_k$  and  $\text{ann}(x'_{k+1} + M_{k-1}) = A_{k+1}$ . Therefore  $M_k$  can be replaced with  $M'_k = M_{k-1} + Rx'_{k+1}$ , where  $\text{ann}(M'_k/M_{k-1}) = A_{k+1}$  and  $A_k = \text{ann}(M_{k+1}/M'_k)$ . Then we replace  $k$  with  $(k+1)$  for the next step.

(b):  $\text{rad}(A_{k+1}) \subset J_n$ . By applying [3, Lemma 3.1] to  $R/(A_k \cap A_{k+1})$ , we get that  $A_{k+1} \subset A_k$ . With the same proof as in [12, Lemma 1.3 p.172], we state that  $M_{k+1}/M_{k-1} = M_k/M_{k-1} \oplus R(x_{k+1} + M_{k-1})$ . Therefore  $M_k$  can be replaced with  $M'_k = M_{k-1} + Rx_{k+1}$ , where  $\text{ann}(M'_k/M_{k-1}) = A_{k+1} \subset A_k = \text{ann}(M_{k+1}/M'_k)$ . Then we replace  $k$  with  $(k+1)$  for the next step.

(c):  $\text{rad}(A_{k+1}) = J_n$  and  $A_k \subsetneq A_{k+1}$ . Then we replace  $k$  with  $(k+1)$  for the next step.

(d):  $\text{rad}(A_{k+1}) = J_n$  and  $A_{k+1} \subset A_k$ . We do as in the case (b).



(e):  $\text{rad}(A_{k+1}) = J_n$  and  $A_{k+1}$  is not comparable with  $A_k$ . By applying Proposition 1.2 or 1.3 to  $M_{k+1}/M_{k-1}$ , it follows that there exists a pure submodule  $M'_k$  of  $M_{k+1}$ , containing  $M_{k-1}$ , such that  $M'_k/M_{k-1}$  and  $M_{k+1}/M'_k$  are cyclic and  $\text{ann}(M'_k/M_{k-1}) = A_k \cap A_{k+1}$ . For every maximal ideal  $P$ ,  $\{(A_k)_P, (A_{k+1})_P\} = \{(A_k \cap A_{k+1})_P, (A_k + A_{k+1})_P\}$ . By using [12, Theorem 1.6, p.177], it follows that the annihilator of  $M_{k+1}/M'_k$  is  $A_k + A_{k+1}$ . Therefore  $M_k$  can be replaced with  $M'_k$ . Then we replace  $k$  with  $(k+1)$  for the next step.

After  $(n-k)$  similar steps, we get from (s), a pure-composition series (s') of  $M$ ,  $\{0\} \subset M'_1 \cdots \subset M'_{n-1} \subset M'_n = M$ , with the annihilator sequence  $(A'_i)_{1 \leq i \leq n}$ , and such that  $\text{rad}(A'_n) = J_n$ . Let us observe that, if  $\text{rad}(A'_k) = J_n$ , then  $A'_k \subseteq A'_n$ , for each  $k$ ,  $1 \leq k < n$ . Hence, either  $A'_i \subseteq A'_n$ , or  $A'_i$  and  $A'_n$  are comaximal, for each  $i$ ,  $1 \leq i < n$ .

Now  $\ell(M'_{n-1}) = n-1$  and  $(J_i)_{1 \leq i < n}$  is the sequence of prime ideals associated to  $M'_{n-1}$ . Moreover, (s') induces a pure-composition series (s'') of  $M'_{n-1}$ , with the annihilator sequence  $(A'_i)_{1 \leq i < n}$ . From the induction hypothesis, we get from (s''), a pure-composition series (t') of  $M'_{n-1}$ ,  $\{0\} \subset N_1 \subset \cdots \subset N_{n-1} = M'_{n-1}$ , with an almost increasing annihilator sequence  $(B_i)_{1 \leq i < n}$  such that  $J_i = \text{rad}(B_i)$ , for every  $i$ ,  $1 \leq i < n$ . We put  $B_n = A'_n$ . From the above observation, it follows that  $(B_i)_{1 \leq i \leq n}$  is an almost increasing sequence.

Let us observe that the case (e) is not possible if  $(A_i)_{1 \leq i \leq n}$  is almost totally ordered. In this case, (s) and (t) are isomorphic.

(2) We consider two pure-composition series of  $M$ , with indecomposable cyclic factors and almost totally ordered annihilator sequences:

$$(s): \{0\} = M_0 \subset M_1 \subset M_2 \cdots \subset M_{n-1} \subset M_n = M.$$

$$(s'): \{0\} = M'_0 \subset M'_1 \subset M'_2 \cdots \subset M'_{n-1} \subset M'_n = M.$$

We denote  $A_i = \text{ann}(M_i/M_{i-1})$  and  $A'_i = \text{ann}(M'_i/M'_{i-1})$ . From (1) we may assume that the annihilator sequences are almost increasing and that  $\text{rad}(A_i) = \text{rad}(A'_i) = J_i$ ,  $\forall i$ ,  $1 \leq i \leq n$ . Let  $P$  be a maximal ideal of  $R$ . If  $J_i \subseteq P$ , then by [12, Theorem 1.6, p.117],  $(A_i)_P = (A'_i)_P$ . If  $J_i \not\subseteq P$ , then  $(A_i)_P = (A'_i)_P = R_P$ . Therefore  $A_i = A'_i$ , for every  $i$ ,  $1 \leq i \leq n$ .  $\square$

Now, we can prove the implication (1)  $\Rightarrow$  (2) of Theorem 2.1.

**Proof of Theorem 2.1.** (1)  $\Rightarrow$  (2). By [3, Corollary 3.9], every pure-composition series of a finitely generated  $R$ -module, with indecomposable cyclic factors, has an almost totally ordered annihilator sequence. Hence the result follows from Proposition 2.2 and Proposition 2.4.  $\square$

The following propositions are needed to prove the implication (2)  $\Rightarrow$  (1) of Theorem 2.1. Now we suppose that  $R$  is a PCS-ring and if  $M$  is a finitely generated module, we denote  $\ell(M)$  the length of every pure-composition series of  $M$  with indecomposable cyclic factors.

**Proposition 2.5.** *Let  $R$  be a PCS-ring. The following assertions are true:*

- (1) *For each indecomposable cyclic module  $N$ ,  $g(N) = 1$*
- (2) *For every finitely generated  $R$ -module  $M$ ,  $g(M) \leq \ell(M)$ .*

**Proof.**

- (1) Since  $N \simeq R/\text{ann}(N)$ , we can replace  $R$  with  $R/\text{ann}(N)$  and assume that  $R$  is an indecomposable module. Let  $I$  and  $J$  be ideals of  $R$  such that  $I \cap J = \{0\}$ . Then  $R/I \oplus R/J$  is a finite direct sum of indecomposable

cyclic modules. If  $I \neq \{0\}$  and  $J \neq \{0\}$  then every annihilator of each summand of  $R/I \oplus R/J$  is a nonzero ideal. On the other hand we consider the following exact sequence :

(i)  $0 \rightarrow R \xrightarrow{\sigma} R/I \oplus R/J \xrightarrow{\varphi} R/(I+J) \rightarrow 0$  such that  $\sigma(r) = (r+I, r+J)$  and  $\varphi(r+I, t+J) = (r-t) + (I+J)$ . Let  $P$  be a maximal ideal of  $R$ . Then we may assume that  $I_P \subseteq J_P$ , and, if  $r(x+I_P, y+J_P) = (z+I_P, z+J_P)$ , it is obvious that  $r(x+I_P, x+J_P) = (z+I_P, z+J_P)$ . Hence  $R_P$  is isomorphic to a pure  $R_P$ -submodule of  $(R/I \oplus R/J)_P$ . We deduce that (i) is a pure exact sequence, and from a pure-composition series of  $R/(I+J)$ , we get a pure-composition series of  $R/I \oplus R/J$  such that the first annihilator is  $\{0\}$ . Consequently, if  $I \neq \{0\}$  and  $J \neq \{0\}$ ,  $R/I \oplus R/J$  has two pure-composition series which are not isomorphic. Since  $R$  is a PCS-ring, we deduce that  $R$  is a uniform module and that  $g(R) = 1$ .

- (2) By induction on  $n = \ell(M)$ . If  $n = 1$  the result follows from (1). Now suppose  $n > 1$ . Let  $\{0\} \subset M_1 \subset \cdots \subset M_{n-1} \subset M_n = M$  be a pure-composition series of  $M$ , with indecomposable cyclic factors. As in the proof of Proposition 2.2, we get that  $g(M) \leq g(M_{n-1}) + 1$ , because  $g(M/M_{n-1}) = 1$ . Since  $\ell(M_{n-1}) = n - 1$ , it follows from the induction hypothesis that  $g(M) \leq n$ .  $\square$

**Proposition 2.6.** *Let  $R$  be a PCS-ring. Then  $R$  has only finitely many minimal prime ideals.*

**Proof.** We may assume that  $R$  is a reduced ring. If  $S$  denotes the total ring of quotients of  $R$ , then by [14, Proposition 2, p. 106],  $S$  is a Von Neumann regular ring. By Proposition 2.5,  $g(R)$  is finite. We deduce from [14, Proposition 2, p. 103] that  $S$  is semi-local. Consequently  $S$  is a semi-simple ring, i.e.  $S = \prod_{i=1}^n K_i$ , where  $K_i$  is a field for every  $i$ ,  $1 \leq i \leq n$ . If  $u : R \rightarrow S$  is the natural map, and,  $\forall i$ ,  $1 \leq i \leq n$ ,  $p_i : S \rightarrow K_i$  the canonical epimorphism, we denote  $P_i = \ker(p_i \circ u)$ . Then  $P_i$  is a prime ideal and  $\bigcap_{i=1}^n P_i = \{0\}$ . We deduce that  $\{P_i \mid 1 \leq i \leq n\}$  is the set of minimal prime ideals of  $R$ .  $\square$

**Proposition 2.7.** *Let  $R$  be a PCS-ring with a unique minimal prime ideal  $Q$ . Then  $Q$  is a uniserial module.*

**Proof.** If there exist  $a$  and  $b$  in  $Q$  such that  $a \notin Rb$  and  $b \notin Ra$ , then  $R/(Ra \cap Rb)$  is indecomposable with  $g(R/(Ra \cap Rb)) \geq 2$ . From Proposition 2.5, we get a contradiction.  $\square$

**Proposition 2.8.** *Let  $R$  be a PCS-ring. Then every nonminimal prime ideal is contained in only one maximal ideal.*

**Proof.** We do a similar proof as in [11, Lemma 10]. By Proposition 2.6,  $R$  has only finitely many minimal prime ideals. Since  $R$  is arithmetic, these minimal prime ideals are pairwise comaximal and consequently  $R$  is a finite product of indecomposable PCS-rings with a unique minimal prime. Hence we may assume that  $R$  has only one minimal prime ideal  $Q$ . Let  $P$  be a prime ideal such that  $Q \subset P$  and  $a \in P \setminus Q$ . Then, from [3, Lemma 3.1], it follows that  $Q \subset Ra$ . We may also assume that  $P$  is minimal over  $Ra$ . Suppose that  $P \subseteq M_1 \cap M_2$  where  $M_1$  and  $M_2$  are two distinct maximal ideals of  $R$ . Let  $x_1$  and  $x_2$  in  $R$ ,  $x_i \in M_j \setminus M_i$ ,  $i \neq j$ , such that  $x_1 + x_2 = 1$ . If  $S = R \setminus (M_1 \cup M_2)$ , we denote  $R' = S^{-1}R$ . Then  $ax_i \notin aM_iR'$ ,  $\forall i$ ,  $i = 1$  or  $2$ . Else,  $ax_i = \frac{am_i}{s}$  where  $m_i \in M_i$  and  $s \in S$ . Hence

there exists  $t \in S$  such that  $a(tx_i - tm_i) = 0$ . It follows that  $(tx_i - tm_i) \in Q$ , and since  $tm_i \in M_i$ , we get that  $tx_i \in M_i$ . But  $t, s$  and  $x_i$  are not in  $M_i$ , hence we get a contradiction. Let  $I = aM_1 \cap aM_2$ . Then  $\frac{ax_i}{1} \notin R'I$ . Now we consider  $R_1 = R/I$ . Then  $R_1$  is also a finite product of indecomposable rings with a unique minimal prime ideal. Hence we may assume that  $P_1 = P/I$  is the only minimal prime ideal of  $R_1$ . Then, if  $S_1 = R_1 \setminus (M_1/I \cup M_2/I)$  we have  $S_1^{-1}R_1 \simeq R'/R'I$ . Consequently  $ax_i + I \neq I$ ,  $\forall i, i \in \{1, 2\}$ . Since  $R$  is a Bézout ring and  $x_1 + x_2 = 1$ ,  $Rax_1 \cap Rax_2 = Rax_1x_2$ . We deduce that  $R_1(ax_1 + I) \cap R_1(ax_2 + I) = \{0\}$ . This contradicts that  $P_1$  is uniserial by Proposition 2.7.  $\square$

Now, we complete the proof of Theorem 2.1.

**Proof of Theorem 2.1.** (2)  $\Rightarrow$  (1). By Proposition 1.1,  $R$  is a Bézout ring. We deduce from Propositions 2.6, 2.7 and 2.8 and from [3, Theorem 3.12] that  $R$  is also a CF-ring.  $\square$

It is easy to get a semilocal arithmetic ring  $R$  which is not a PCS-ring.

**Example 2.9.** Let  $D$  a semilocal and h-local Bézout domain,  $F$  its classical field of quotients,  $P$  and  $Q$  two distinct maximal ideals of  $D$ ,  $E = (F/PD_P) \oplus (F/QD_Q)$  and  $R = \left\{ \begin{pmatrix} d & e \\ 0 & d \end{pmatrix} \mid d \in D, e \in E \right\}$  the trivial extension of  $D$  by  $E$ . Then  $R$  is a semilocal arithmetic ring with a unique minimal prime ideal which is not uniserial. By Proposition 2.7,  $R$  is not a PCS-ring.

**Example 2.10.** Let  $R$  be the Kaplansky domain of [15, example 1]. We may assume that  $R$  is semilocal and nonlocal. Then its Jacobson radical is a nonzero prime ideal contained in every maximal ideal. Since  $R$  is not h-local,  $R$  is not a PCS-ring.

The following questions can be proposed :

Let  $R$  be a ring such that every finitely generated  $R$ -module has a pure-composition series with indecomposable cyclic factors.

- (1) Is  $g(M)$  finite, for every finitely generated  $R$ -module  $M$ ?
- (2) Is  $\text{Minspec}(R)$  a finite set ?  $\text{Minspec}(R)$  denotes the set of minimal prime ideals of  $R$ .
- (3) For every finitely generated  $R$ -module  $M$ , are the lengths of any two pure-composition series of  $M$ , with indecomposable cyclic factors, equal?

Let us observe that, if the answer to the question (1) is positive, then  $\text{Minspec}(R)$  is finite (see the proof of Proposition 2.6), and if  $\text{Minspec}(R/I)$  is finite, for every proper ideal  $I$ , then the question (3) also has a positive answer (see the proof of Proposition 2.3.)

When  $\text{Minspec}(R)$  is compact in its Zariski topology, we can answer to the question (2).

**Proposition 2.11.** *Let  $R$  be an arithmetic ring. We suppose that  $\text{Minspec}(R)$  is compact and every finitely generated module has a pure-composition series with indecomposable cyclic factors.*

*Then  $\text{Minspec}(R)$  is a finite set.*

**Proof.** Since  $R$  has a pure-composition series with indecomposable cyclic factors,  $R$  is a finite direct product of indecomposable Kaplansky rings. We may

assume that  $R$  is semi-prime and indecomposable. Since  $R_P$  is a valuation domain for every maximal ideal  $P$ , by [16, Proposition 10],  $R$  is semi-hereditary. Let  $r \in R$ ,  $r \neq 0$ . Since  $\text{ann}(r)$  is generated by an idempotent,  $\text{ann}(r) = \{0\}$ . Hence  $R$  is a domain.  $\square$

We can answer to these questions when  $R$  is a Von Neumann regular ring. Then every exact short sequence of  $R$ -modules is pure. It follows that every finitely generated  $R$ -module has a pure-composition series with cyclic factors. However, we have the following result.

**Proposition 2.12.** *Let  $R$  be a Von Neumann regular ring. Then the following statements are true.*

- (1) *A finitely generated  $R$ -module has a pure-composition series with indecomposable cyclic factors if and only if it is a semi-simple module.*
- (2) *Every finitely generated  $R$ -module has a pure-composition series with indecomposable cyclic factors if and only if  $R$  is a semi-simple ring.*

**Proof.** Clearly (1)  $\Rightarrow$  (2). We can also deduce (2) from Proposition 2.11. (1) is an immediate consequence of the following Theorem.  $\square$

**Theorem 2.13.** *Let  $R$  be a ring. Then the following assertions are equivalent :*

- (1)  *$R$  is a Von Neumann regular ring.*
- (2) *Every indecomposable  $R$ -module is simple.*

**Proof.** (2)  $\Rightarrow$  (1). For every simple  $R$ -module  $S$ ,  $E_R(S) \simeq S$ . Since every simple  $R$ -module is injective, we deduce that  $R$  is Von Neumann regular.

(1)  $\Rightarrow$  (2). Let  $M$  be an indecomposable  $R$ -module,  $M \neq \{0\}$ . Then there exists a maximal ideal  $P$  such that  $M_P \neq \{0\}$ . Let  $a \in P$ . Then there exists an idempotent  $e$  such that  $Ra = Re$ . Since  $M_P \neq \{0\}$  and  $(1 - e) \notin P$ , we have  $(1 - e)M \neq \{0\}$ . We deduce that  $eM = \{0\}$  since  $M$  is indecomposable. Consequently  $M$  is an  $R/P$ -module. Since  $R/P$  is a field,  $M$  is a simple module.  $\square$

By using this theorem, we can also answer to the following question proposed by R. Wiegand in [17] : are the following assertions equivalent for a commutative ring  $R$  and an integer  $n \geq 1$  ?

- (1) *Every finitely generated  $R$ -module is a finite direct sum of submodules generated by at most  $n$  elements.*
- (2)  *$\mu(M) \leq n$ , for every indecomposable finitely generated  $R$ -module  $M$ .*

If  $R$  is a Von Neumann regular ring, then  $R$  satisfies the second assertion by Theorem 2.13, with  $n = 1$ . But if  $R$  satisfies the first assertion, then  $R$  is semi-simple by [18, Corollary 21.7] . Consequently the answer to Wiegand's question is negative.

### 3. THE GOLDIE DIMENSION

In this section we will prove the following theorem.

**Theorem 3.1.** *Let  $R$  be a PCS-ring. Then a finitely generated  $R$ -module  $M$  is a direct sum of cyclic submodules if and only if  $g(M) = \ell(M)$ .*

When  $R$  is a valuation ring this theorem was proved by L. Salce and P. Zanardo [1, Corollary 3.5], and when  $R$  is an h-local Bézout domain, it was proved by C. Naudé [2, Theorem 2.2]. Some preliminary results are needed to prove this theorem.

**Lemma 3.2.** *Let  $R$  be an arithmetic ring,  $\{0\} \subset M_1 \subset \cdots \subset M_{n-1} \subset M_n = M$  a pure-composition series of a finitely generated  $R$ -module  $M$ , with  $M_i/M_{i-1} = R(x_i + M_{i-1})$  for all  $i$ ,  $1 \leq i \leq n$ , and with an increasing annihilator sequence  $(A_i)_{1 \leq i \leq n}$ . Let  $c_1, \dots, c_n$  in  $R$  such that  $\sum_{i=1}^n c_i x_i = 0$ . Then  $c_i \in A_n$ , for every  $i$ ,  $1 \leq i \leq n$ .*

**Proof.** We prove this lemma by induction on  $n$ . If  $n = 1$  it is obvious. If  $n > 1$ , then  $c_n \in A_n$ . Moreover, since  $M_{n-1}$  is a pure submodule of  $M$ , there exist  $d_1, \dots, d_{n-1}$  in  $R$  such that  $c_n x_n = c_n(\sum_{i=1}^{n-1} d_i x_i) = -\sum_{i=1}^{n-1} c_i x_i$ . We deduce the following equality  $\sum_{i=1}^{n-1} (c_n d_i + c_i) x_i = 0$ . From the induction hypothesis it follows that  $(c_n d_i + c_i) \in A_{n-1}$  for every  $i$ ,  $1 \leq i \leq n-1$ . Since  $A_{n-1} \subseteq A_n$  and  $c_n \in A_n$ , we deduce that  $c_i \in A_n$ ,  $\forall i$ ,  $1 \leq i \leq n$ .  $\square$

**Lemma 3.3.** *Let  $R$  be a Bézout and a torch ring,  $Q$  its minimal prime ideal and  $P$  the only maximal ideal such that  $Q_P \neq \{0\}$ . Let  $M$  be a finitely generated  $R$ -module,  $\{0\} \subset M_1 \subset \cdots \subset M_n = M$  a pure-composition series, with  $M_i/M_{i-1} = R(x_i + M_{i-1})$  for all  $i$ ,  $1 \leq i \leq n$ , and with an increasing annihilator sequence  $(A_i)_{1 \leq i \leq n}$ . We assume that  $A_n \subseteq Q$ . Let  $M' = M_{n-1}$ .*

*If  $M'$  is not essential in  $M$ , then  $M$  has a non-zero cyclic summand.*

**Proof.** Let  $y \in M \setminus M'$  such that  $Ry \cap M' = \{0\}$ . Then  $y = \sum_{i=1}^n a_i x_i$ , where  $a_i \in R$  for all  $i$ ,  $1 \leq i \leq n$ . By Propositions 1.3 and 1.1  $R$  is a Kaplansky ring, hence there exist  $a, b_1, \dots, b_n$  in  $R$  such that  $a_i = ab_i$  for every  $i$ ,  $1 \leq i \leq n$ , and  $\sum_{i=1}^n Rb_i = R$ . We put  $z = \sum_{i=1}^n b_i x_i$ .

We claim that  $M/QM$  is a free  $R/Q$ -module with basis  $\{x_i + QM \mid 1 \leq i \leq n\}$ . Let  $c_1, \dots, c_n \in R$  such that  $\sum_{i=1}^n c_i x_i \in QM$ . There exist  $q_1, \dots, q_n$  in  $Q$  such that  $\sum_{i=1}^n (c_i - q_i) x_i = 0$ . From Lemma 3.2, since  $A_n \subseteq Q$ , it follows that  $c_i \in Q$ ,  $\forall i$ ,  $1 \leq i \leq n$ .

Now we claim that  $R_P y \neq \{0\}$ . Else there exists  $s \notin P$  such that  $sy = 0$ . From Lemma 3.2, it follows that  $a_i \in Q$ ,  $\forall i$ ,  $1 \leq i \leq n$ , whence  $y \in QM$ . If  $P'$  is a maximal ideal,  $P' \neq P$ , then  $M_{P'} \simeq (M/QM)_{P'}$  and consequently  $R_{P'} y = \{0\}$ . Since  $Ry \neq \{0\}$ , we get a contradiction.

Since  $R_P y \neq \{0\}$ ,  $R_P y \cap M'_P = \{0\}$  and  $R_P$  is a valuation ring, it follows that  $R_P z \cap M'_P = \{0\}$ .

Since  $R$  is a Kaplansky ring, there exists an invertible  $n \times n$  matrix  $\Lambda$  such that  $[b_1, \dots, b_n] \Lambda = [1, 0, \dots, 0]$ .

Then  $z = [b_1, \dots, b_n][x_1, \dots, x_n]^t = [b_1, \dots, b_n] \Lambda \Lambda^{-1}[x_1, \dots, x_n]^t$ .

Hence  $z = [1, 0, \dots, 0] \Lambda^{-1}[x_1, \dots, x_n]^t$ . We put  $[z_1, \dots, z_n]^t = \Lambda^{-1}[x_1, \dots, x_n]^t$ . Thus  $z = z_1$  and  $\{z_1, \dots, z_n\}$  generates  $M$ . Let  $N$  be the submodule of  $M$  generated by  $\{z_2, \dots, z_n\}$ . Then  $\{z_i + QM \mid 1 \leq i \leq n\}$  is a basis of  $M/QM$ , and for every maximal ideal  $P'$ ,  $P' \neq P$ , we have  $M_{P'} = (Rz_1)_{P'} \oplus N_{P'}$ .

We put  $\Lambda = (\lambda_{i,k})_{1 \leq i,k \leq n}$ . Then, since  $R_P$  is a valuation ring, there exists  $j$ ,  $1 \leq j \leq n$ , such that  $\lambda_{j,1}$  and  $b_j$  are units of  $R_P$  and such that  $\Lambda_{j,1}$  is an invertible matrix, where  $\Lambda_{j,1}$  is the  $(n-1) \times (n-1)$  matrix obtained from  $\Lambda$ , by deleting its first column and its  $j$ -th row.

Moreover  $\Lambda_{j,1}[z_2, \dots, z_n]^t =$

$$[x_1 - \lambda_{1,1} z_1, \dots, x_{j-1} - \lambda_{j-1,1} z_1, x_{j+1} - \lambda_{j+1,1} z_1, \dots, x_n - \lambda_{n,1} z_1]^t$$

and  $\{x_i - \lambda_{i,1}z_1 \mid 1 \leq i \leq n, i \neq j\}$  also generates  $N_P$ . Since  $b_j$  is a unit of  $R_P$ , as in the proof of [12, Lemma 2.1, p. 179], we get that  $R_P z_1 \cap \left(\sum_{\substack{i=1 \\ i \neq j}}^n R_P x_i\right) = \{0\}$ .

Let  $x \in (Rz_1)_P \cap N_P$ . Then there exist  $c$  and  $c_i$  in  $R_P$ , for every  $i$ ,  $1 \leq i \leq n$ ,  $i \neq j$ , such that  $x = cz_1 = \sum_{\substack{i=1 \\ i \neq j}}^n c_i(x_i - \lambda_{i,1}z_1)$ . From above, we deduce that

$$\left(c + \sum_{\substack{i=1 \\ i \neq j}}^n c_i \lambda_{i,1}\right)z_1 = \sum_{\substack{i=1 \\ i \neq j}}^n c_i x_i = 0. \text{ From Lemma 3.2, it follows that } c_i \in (A_n)_P$$

for every  $i$ ,  $1 \leq i \leq n$ ,  $i \neq j$ . On the other hand, if we set  $d = c + \sum_{\substack{i=1 \\ i \neq j}}^n c_i \lambda_{i,1}$ ,

then  $dz_1 = 0$  implies that  $db_i \in (A_n)_P$ ,  $\forall i$ ,  $1 \leq i \leq n$ . Since  $b_j$  is a unit of  $R_P$ , it follows that  $d \in (A_n)_P$ , whence  $c \in (A_n)_P$ . Consequently  $cz_1 \in M'_P$ . Since  $M'_P \cap R_P z_1 = \{0\}$ , we also get that  $R_P z_1 \oplus N_P = M_P$ .  $\square$

**Lemma 3.4.** *Let  $R$  be a Bézout and a torch ring and  $Q$  its minimal prime ideal. Let  $M$  be a finitely generated module and  $\{0\} \subset M_1 \subset M_2 \subset \cdots \subset M_n = M$  a pure-composition series with  $M_i/M_{i-1} = R(x_i + M_{i-1})$  for all  $i$ ,  $1 \leq i \leq n$ , and with an increasing annihilator sequence  $(A_i)_{1 \leq i \leq n}$ . We assume that  $Q \subset A_n$ . If  $M/M_{n-1} = R(x'_n + M_{n-1}) \oplus R(x''_n + M_{n-1})$ , where  $R(x'_n + M_{n-1})$  is indecomposable, we set  $M' = M_{n-1} + Rx''_n$ .*

*If  $M'$  is not essential in  $M$ , then  $M$  has a nonzero cyclic summand.*

**Proof.** There exists  $y \in M \setminus M'$  such that  $M' \cap Ry = \{0\}$ . Let  $A' = \text{ann}(M/M')$ . Then  $A_n \subseteq A'$  and there exists only one maximal ideal  $P$  such that  $A' \subseteq P$ . Therefore, if  $P'$  is a maximal ideal,  $P' \neq P$ , we have  $(M/M')_{P'} = \{0\}$ . It follows that there exists  $s \notin P'$  such that  $sx'_n \in M'$ . We also get that  $sy \in M'$  and since  $M' \cap Ry = \{0\}$ , we have  $sy = 0$ . Consequently  $R_{P'}y = \{0\}$  and  $M_{P'} = M_{P'}$  for every maximal ideal  $P' \neq P$ .

There exist  $a_1, \dots, a_n$  in  $R$  such that  $y = \sum_{i=1}^n a_i x_i$ . As in the previous lemma, there exist  $a, b_1, \dots, b_n$  such that  $a_i = ab_i$  for every  $i$ ,  $1 \leq i \leq n$ , and such that  $\sum_{i=1}^n Rb_i = R$ . Then  $a \notin A'$ , else  $y \in M'$ . Since  $a \notin Q$  and  $R/Q$  is an  $h$ -local Bézout domain, it follows that  $R/Ra = \bigoplus_{j=1}^m R/Rc_j$ , where  $R/Rc_j$  is indecomposable for every  $j$ ,  $1 \leq j \leq m$ . We denote  $P_j$  the only maximal ideal of  $R$  such that  $Rc_j \subseteq P_j$ , for each  $j$ ,  $1 \leq j \leq m$ .

If  $P \neq P_j$  for each  $j$ ,  $1 \leq j \leq m$ , then  $a$  is a unit of  $R_P$ . In this case we set  $z = y$  and  $d_i = a_i$ , for every  $i$ ,  $1 \leq i \leq n$ .

If there exists  $j$ ,  $1 \leq j \leq m$ , such that  $P = P_j$ , then we put  $c = c_j$ . There exists  $d \in R$  such that  $a = cd$ , and  $d$  is a unit of  $R_P$ . We set  $d_i = b_i d$ , for every  $i$ ,  $1 \leq i \leq n$ , and we put  $z = \sum_{i=1}^n d_i x_i$ . For every maximal ideal  $P' \neq P$ ,  $c$  is a unit of  $R_{P'}$ , whence  $R_{P'}z = R_{P'}y = \{0\}$ . Since  $M'_P \cap R_P y = \{0\}$  and  $R_P$  is a valuation ring, it follows that  $R_P z \cap M'_P = \{0\}$ .

In the two cases, there exists  $j$ ,  $1 \leq j \leq n$ , such that  $d_j$  is a unit of  $R_P$ . Let  $\ell$  be the greatest index such that  $A_\ell \subseteq Q$ . We claim that  $\ell < j$ . Else there exist  $t \in PR_P$ , and  $d'_{\ell+1}, \dots, d'_n \in R_P$  such that  $d_i = td'_i$  for every  $i$ ,  $\ell < i \leq n$ . Let  $s \in R_P A' \setminus R_P Q$ . Then  $sz = 0$ . We get the following equality :  $st(\sum_{i=\ell+1}^n d'_i x_i) =$

$-s(\sum_{i=1}^{\ell} d_i x_i)$ . Since  $(M_{\ell})_P$  is a pure-submodule of  $M_P$ , there exist  $d'_1, \dots, d'_\ell$  in  $R_P$  such that  $\sum_{i=1}^{\ell} s(td'_i + d_i)x_i = 0$ . From Lemma 3.2 it follows that  $s(td'_i + d_i) \in R_P A_{\ell} \subseteq R_P Q$ , for every  $i$ ,  $1 \leq i \leq \ell$ . Since  $s \notin R_P Q$  and  $t \in PR_P$ , we deduce that  $d_i \in PR_P$  for every  $i$ ,  $1 \leq i \leq \ell$ . Hence we get a contradiction.

Since  $Q \subset A_j \subseteq A_n$ , then  $M_j/M_{j-1} = R(x'_j + M_{j-1}) \oplus R(x''_j + M_{j-1})$ , such that  $(M_j/M_{j-1})_P = R_P(x'_j + (M_{j-1})_P)$ ,  $R_P(x''_j + (M_{j-1})_P) = \{0\}$  and  $R_{P'}(x'_j + (M_{j-1})_{P'}) = \{0\}$  for every maximal ideal  $P' \neq P$ .

Let  $N = \sum_{\substack{i=1 \\ i \neq j}}^n R x_i + R x''_j$ . Then  $N_P = \sum_{\substack{i=1 \\ i \neq j}}^n R_P x_i$  and  $N_{P'} = M_{P'}$  for every maximal ideal  $P' \neq P$ . As in the proof of [12, Lemma 2.1, p. 179], we state that  $N_P \oplus R_P z = M_P$ . Consequently we get  $M = N \oplus Rz$ .  $\square$

**Proposition 3.5.** *Let  $R$  be a Bézout and a torch ring. Then every finitely  $R$ -module  $M$  contains a pure and essential submodule  $B$  which is a direct sum of  $g(M)$  cyclic modules.*

**Proof.** We induct on  $m = \ell(M)$ . The case  $m = 1$  is obvious. Let  $m > 1$  and  $\{0\} \subset M_1 \cdots \subset M_n = M$  be a pure-composition series with cyclic factors and an increasing annihilator sequence  $(A_i)_{1 \leq i \leq n}$ . Let  $Q$  be the minimal prime of  $R$ . If  $A_n \subseteq Q$ , then  $n = m$  and we set  $M' = M_{n-1}$ . If  $Q \subset A_n$ , then  $M/M_{n-1} = R(x'_n + M_{n-1}) \oplus R(x''_n + M_{n-1})$  where  $R(x'_n + M_{n-1})$  is indecomposable ; in this case we set  $M' = M_{n-1} + R x''_n$ . In the two cases,  $\ell(M') = m - 1$ .

If  $M'$  is essential,  $M'$  has a pure and essential submodule  $B'$  which is a direct sum of  $g(M')$  non-zero indecomposable cyclic modules ; in this case, we are done by setting  $B = B'$ .

If  $M'$  is not essential, then by Lemma 3.3 or Lemma 3.4,  $M' = N \oplus Rz$ , for some  $z \in M$ ,  $z \neq 0$ , where  $Rz$  is indecomposable and  $N$  a submodule of  $M'$ . Then  $\ell(N) = m - 1$ . Thus  $N$  has a pure and essential submodule  $B''$  which is a direct sum of  $g(N)$  non-zero indecomposable cyclic submodules. We put  $B = B'' \oplus Rz$  to conclude the proof.  $\square$

Now we can prove the Theorem 3.1.

**Proof of Theorem 3.1.** By Proposition 2.5, the Goldie dimension of every cyclic indecomposable module is one. Consequently if  $M$  is a finite direct sum of cyclic modules, we have  $\ell(M) = g(M)$ .

Conversely let  $M$  be a finitely generated module such that  $\ell(M) = g(M)$ . By Proposition 2.6,  $R = \prod_{j=1}^m R_j$ , where  $R_j$  is an indecomposable PCS-ring for every  $j$ ,  $1 \leq j \leq m$ . We deduce that  $M \simeq \prod_{j=1}^m M_j$ , where  $M_j = R_j \otimes_R M$ . Then  $\ell(M) = \sum_{j=1}^m \ell(M_j)$  and  $g(M) = \sum_{j=1}^m g(M_j)$ . From Proposition 2.5, we deduce that  $\ell(M_j) = g(M_j)$  for every  $j$ ,  $1 \leq j \leq m$ . Consequently we may assume that  $R$  is indecomposable. When  $R$  is a valuation ring, the result was proved by L. Salce and P. Zanardo [1, Corollary 3.5], (or [12, Theorem 2.4, p. 180]), and when  $R$  is an  $h$ -local Bézout domain, it was proved by C. Naudé [2, Theorem 2.2.]. Hence we may assume that  $R$  is a Bézout and torch ring. We prove the result by using Proposition 3.5 and Proposition 2.5, with the same proof as in [12, Theorem 2.4, p. 180].  $\square$

Let  $R$  be a Bézout semi-CF-ring,  $M$  a finitely generated  $R$ -module and  $(s) : \{0\} = M_0 \subset M_1 \subset \cdots \subset M_{n-1} \subset M_n = M$ , a pure-composition series of  $M$ , with

indecomposable cyclic factors. We put  $g((s)) = \sum_{i=1}^n g(M_i/M_{i-1})$ . As in the proof of Proposition 2.5, we state that  $g(M) \leq g((s))$ .

However, if  $(s')$  is another pure-composition series of  $M$ , with indecomposable cyclic factors, we have not necessarily  $g((s)) = g((s'))$ . For instance:

**Example 3.6.** Let  $R$  be the ring defined in example 2.9. We put  $E_1 = F/PD_P$ ,  $E_2 = F/QD_Q$ ,  $A_i = \left\{ \begin{pmatrix} 0 & e \\ 0 & 0 \end{pmatrix} \mid e \in E_i \right\}$ , where  $i = 1$  or  $2$ , and  $J = A_1 + A_2$ . Then  $A_1 \cap A_2 = \{0\}$  and  $J$  is the minimal prime ideal of  $R$ . Let  $M = R/A_1 \oplus R/A_2$  and  $(s)$  be the pure-composition series with the annihilator sequence  $(A_1, A_2)$ . Then  $g(M) = g((s)) = 2$ . But  $M$  has a pure-composition series  $(s')$  whose factors are  $R$  and  $R/J$ . We have  $g((s')) = 3$ . Let us observe that  $M$  is not isomorphic to  $R \oplus R/J$ .

If  $M$  is a finitely generated module over a Bézout semi-CF-ring  $R$ , we denote  $\mathcal{S}(M)$  the set of all pure-composition series of  $M$ , with indecomposable cyclic factors, and we put  $h(M) = \inf\{g((s)) \mid (s) \in \mathcal{S}(M)\}$ . Then we have the following proposition:

**Proposition 3.7.** *Let  $R$  be a Bézout semi-CF-ring and  $M$  a finitely generated  $R$ -module. Then the following assertions are true:*

- (1)  $\mu(M) \leq \ell(M) \leq h(M)$ .
- (2)  $g(M) \leq h(M)$ .
- (3) *If  $M$  is a direct sum of cyclic modules, then  $g(M) = h(M)$ .*

We don't know if the converse of the third assertion holds.

#### 4. PURE-INJECTIVITY AND RD-INJECTIVITY

First, for every integer  $n \geq 2$ , we give an example of an artinian module  $M$  over a noetherian domain  $R$ , which has an RD-composition series of length  $n$ , with uniserial factors.

**Example 4.1.** Let  $K$  be an algebraically closed field of characteristic 0,  $K[X, Y]$  the polynomial ring in two variables  $X$  and  $Y$ , and  $f(X, Y) = Y^n - X^n(1+X)$ , where  $n \in \mathbb{N}$ ,  $n \geq 2$ . By considering that  $f(X, Y)$  is a polynomial in one variable  $Y$  with coefficients in  $K[X]$ , it follows from Eisenstein's criterion that  $f(X, Y)$  is irreducible.

Then  $R = \frac{K[X, Y]}{f(X, Y)K[X, Y]}$  is a domain. Let  $x$  and  $y$  be the images of  $X$  and  $Y$  in  $R$  by the natural map and  $P$  the maximal ideal of  $R$  generated by  $\{x, y\}$ . If  $\widehat{R}$  is the completion of  $R$  in its  $P$ -adic topology, then  $\widehat{R} \simeq \frac{K[[X, Y]]}{f(X, Y)K[[X, Y]]}$ . Since  $K$  has  $n$  distinct  $n$ -th roots of unity, by applying Hensel's Lemma to  $K[[X]]$ , we deduce that there exist  $u_1(X), \dots, u_n(X)$  in  $K[[X]]$  such that  $f(X, Y) = \prod_{i=1}^n (Y - Xu_i(X))$ . Let  $E = E_R(R/P)$ . Then by [19],  $\text{End}_R E = \widehat{R}$ ,  $E$  is also an injective  $\widehat{R}$ -module and every  $R$ -submodule of  $E$  is also an  $\widehat{R}$ -submodule. Moreover, there is a bijection between the set of ideals of  $\widehat{R}$  and the set of submodules of  $E$ . For every  $k$ ,  $1 \leq k \leq n$ ,  $Q_k = (y - xu_k(x))\widehat{R}$  is a minimal prime ideal of  $\widehat{R}$ , and  $Q_k \cap R = \{0\}$ .

For every  $k$ ,  $1 \leq k \leq n$ , let  $F_k = \{e \in E \mid Q_k \subseteq \text{ann}_{\widehat{R}}(e)\}$ . Then  $\widehat{R}/Q_k \simeq K[[Z]]$ , and consequently it is a discrete valuation domain. We deduce that  $F_k$  is a uniserial  $R$ -module. Since  $F_k$  is an injective  $\widehat{R}/Q_k$ -module, it follows that  $F_k$  is a divisible  $R$ -module for every  $k$ ,  $1 \leq k \leq n$ . We set  $E_k = \sum_{i=1}^k F_i$  for every



$k$ ,  $1 \leq k \leq n$ . Then  $\text{ann}_{\widehat{R}}(E_k) = \bigcap_{i=1}^k Q_i$  and consequently  $E_n = E$ . Moreover,  $E_k/E_{k-1} = (E_{k-1} + F_k)/E_{k-1} \simeq F_k/(F_k \cap E_{k-1})$ . We deduce that  $E_k/E_{k-1}$  is a uniserial  $R$ -module, and since  $R$  is a domain it follows that  $E_k/E_{k-1}$  is also divisible. Consequently  $E_k$  is divisible over  $R$ ,  $\forall k$ ,  $1 \leq k \leq n$ , and the following chain  $\{0\} \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = E$ , is an RD-composition series of  $E$ , of length  $n$ , with uniserial factors.

Now, we give some examples of pure-injective modules that fail to be RD-injective, over noetherian domains.

**Example 4.2.** Suppose that  $R$ ,  $F_k$  and  $E_k$ ,  $1 \leq k \leq n$ , are defined as in the example 4.1. Then, since  $F_k$  and  $E_k$  are artinian  $R$ -modules, they are also pure-injective by [7, Proposition 9]. Since  $E$  is indecomposable,  $F_k$  and  $E_k$  ( $k \leq n-1$ ) are not RD-injective.

Recall that an  $R$ -module  $M$  is *linearly compact* (in its discrete topology) if for every family of cosets  $\{x_i + M_i \mid i \in I\}$ , with the finite intersection property, has a non-void intersection.

We can also find a noetherian and linearly compact module  $M$  which is not RD-injective over a noetherian domain  $R$ .

**Example 4.3.** Let  $R$  be a complete local regular noetherian domain  $R$  with Krull dimension  $n \geq 2$  (for example,  $R = K[[X_1, \dots, X_n]]$ , where  $K$  is a field). Then there exists an exact sequence:

$$0 \longrightarrow F_n \xrightarrow{u_n} F_{n-1} \xrightarrow{u_{n-1}} F_{n-2} \cdots \cdots F_1 \xrightarrow{u_1} F_0,$$

where  $F_i$  is a free module of finite rank, for every  $i$ ,  $0 \leq i \leq n$ , and such that  $\text{Im } u_{n-1}$  is not a projective module. Since  $\text{Im } u_{n-1}$  is a torsion-free  $R$ -module, then  $F_n$  is an RD-submodule of  $F_{n-1}$ , which is not RD-injective. We deduce that  $R$  is a pure-injective module since it is a linearly compact module by [7, Proposition 9], that fails to be RD-injective.

It is proved in [4, Theorem 3.1] that a domain  $R$  is Prüfer, i.e. arithmetic, if and only if every pure-injective module is RD-injective. This result can be extended to every commutative ring.

**Theorem 4.4.** *Let  $R$  be a commutative ring. Then the following assertions are equivalent :*

- (1)  $R$  is an arithmetic ring.
- (2) Every RD-exact sequence of  $R$ -modules is pure-exact.
- (3) Every pure-injective  $R$ -module is RD-injective.

**Proof.** (1)  $\Rightarrow$  (2) follows from [8, Theorem 3]. (2)  $\Rightarrow$  (3) is obvious. (3)  $\Rightarrow$  (1) is an immediate consequence of the following proposition.  $\square$

**Proposition 4.5.** *Let  $R$  be a commutative ring. If  $R$  is not arithmetic, there exists a pure-injective module which is not RD-injective. More precisely, there exist a maximal ideal  $P$ , and two elements  $a$  and  $b$  in  $P$ , such that  $S = \{x \in E_R(R/P) \mid ax = 0 \text{ and } bx = 0\}$  is a pure-injective module which is not RD-injective.*

**Proof.** There exists a maximal ideal  $P$  such that  $R_P$  is not a valuation ring. Let  $a$  and  $b$  in  $P$  such that  $a \notin R_P b$  and  $b \notin R_P a$ . By [8, Theorem 2], there exists an indecomposable finitely presented  $R_P$ -module  $M$ , generated by two elements  $x_1$  and  $x_2$ , with the relation  $ax_1 = bx_2$ . By [7, Corollary 2],  $M$  is not an RD-projective  $R_P$ -module. Consequently there exists an RD-exact sequence  $0 \rightarrow L \rightarrow N \xrightarrow{\varphi} M \rightarrow 0$  which is not pure-exact, where  $N$  is an RD-projective  $R_P$ -module. We consider the following presentation of  $M : 0 \rightarrow R_P \xrightarrow{u} F \xrightarrow{v} M \rightarrow 0$ , where  $F$  is a free  $R_P$ -module with basis  $\{e_1, e_2\}$ , such that  $v(e_i) = x_i$ ,  $i = 1, 2$  and  $u(1) = ae_1 - be_2$ . Since  $F$  is a free module there exists  $\alpha : F \rightarrow N$  such that  $\varphi \circ \alpha = v$  and consequently  $\text{Im}(\alpha \circ u) \subseteq L$ . Then  $a\alpha(e_1) - b\alpha(e_2) \notin aL + bL$ . Else there exist  $y_1$  and  $y_2$  in  $L$  such that  $ay_1 - by_2 = a\alpha(e_1) - b\alpha(e_2)$ . We define  $\beta : F \rightarrow L$  by  $\beta(e_i) = y_i$ . Then  $(\alpha - \beta)(ae_1 - be_2) = 0$ , whence  $(\alpha - \beta)$  induces an homomorphism  $\gamma : M \rightarrow N$  such that  $\varphi \circ \gamma = 1_M$ . This is not possible.

Consequently if  $G = R_P/(aR_P + bR_P)$ , then  $G \otimes_{R_P} L \rightarrow G \otimes_{R_P} N$  is not injective. If  $E = E_R(R/P) \simeq E_{R_P}(R_P/PR_P)$ , then  $E$  is an injective cogenerator in the category of  $R_P$ -modules, whence the homomorphism  $\text{Hom}_{R_P}(G \otimes_{R_P} N, E) \rightarrow \text{Hom}_{R_P}(G \otimes_{R_P} L, E)$  is not surjective. If  $S$  is the  $R_P$ -module  $\text{Hom}_{R_P}(G, E)$ , then  $S$  is pure-injective by [7, Proposition 7], but the morphism  $\text{Hom}_{R_P}(N, S) \rightarrow \text{Hom}_{R_P}(L, S)$  is not surjective, so  $S$  is not RD-injective over  $R_P$ .

But  $S \simeq \text{Hom}_R(R/(aR + bR), E) \simeq \{x \in E \mid ax = 0 \text{ and } bx = 0\}$ , and  $S$  is also a pure-injective  $R$ -module that fails to be RD-injective over  $R$ .  $\square$

We deduce from this proposition the following example.

**Example 4.6.** Let  $K$  be a field and  $R = K[X, Y]$  the polynomial ring in two variables  $X$  and  $Y$ . If we take  $a = X - \alpha$ ,  $b = Y - \beta$ ,  $\alpha, \beta \in K$  and  $P$  the maximal ideal generated by  $a$  and  $b$ , then  $S$  is isomorphic to the simple  $R$ -module  $R/P$ . This is an example of a simple module which is not RD-injective over a domain  $R$ . Let us observe that  $S$  is finite if  $K$  is a finite field.

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