

# THE $\lambda$ - DIMENSION OF COMMUTATIVE ARITHMETIC RINGS

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ABSTRACT. It is shown that every commutative arithmetic ring  $R$  has  $\lambda$ -dimension  $\leq 3$ . An example of a commutative Kaplansky ring with  $\lambda$ -dimension 3 is given. Moreover, if  $R$  satisfies one of the following conditions, semi-local, semi-prime, self  $fp$ -injective, zero-Krull dimensional, CF or FSI then  $\lambda\text{-dim}(R) \leq 2$ . It is also shown that every zero-Krull dimensional commutative arithmetic ring is a Kaplansky ring and an adequate ring, that every Bézout ring with compact minimal prime spectrum is Hermite and that each Bézout fractionnally self  $fp$ -injective ring is a Kaplansky ring.

## 1. INTRODUCTION, DEFINITIONS AND PRELIMINARIES

All rings in this paper are commutative with unity and modules are unitary. Following P. Vámos [1], if  $\mathcal{P}$  is a ring property, we say that a ring  $R$  is *locally*  $\mathcal{P}$  if  $R_M$  has  $\mathcal{P}$  for every maximal ideal  $M$ , and  $R$  is *fractionnally*  $\mathcal{P}$  if the classical quotient ring  $Q(R/A)$  of  $R/A$  has  $\mathcal{P}$  for every proper ideal  $A$  of  $R$ .

An  $R$ -module  $E$  is said to be of *finite  $n$ -presentation* if there exists an exact sequence:

$$F_n \rightarrow F_{n-1} \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow E \rightarrow 0$$

with the  $F_i$ 's free  $R$ -modules of finite rank. We write  $\lambda_R(E) = \sup\{n \mid \text{there is a finite } n\text{-presentation of } E\}$ . If  $E$  is not finitely generated we also put  $\lambda_R(E) = -1$ .

The  $\lambda$ -dimension of a ring  $R$  ( $\lambda\text{-dim}(R)$ ) is the least integer  $n$  (or  $\infty$  if none such exists) such that  $\lambda_R(E) \geq n$  implies  $\lambda_R(E) = \infty$ . See [2, chapter 8]. Recall that  $R$  is noetherian if and only if  $\lambda\text{-dim}(R) = 0$  and  $R$  is coherent if and only if  $\lambda\text{-dim}(R) \leq 1$ .

This notion of  $\lambda$ -dimension of a ring was formulated in [2, chapter 8] to study the rings of polynomials or power series over a coherent ring.

In section 2 of this paper it is proved that every arithmetic ring has a  $\lambda$ -dimension  $\leq 3$ . We give an example of a Kaplansky ring whose the  $\lambda$ -dimension is exactly 3. However, if an arithmetic ring satisfies an additional property, (reduced, self  $fp$ -injective, semi-local, CF or fractionnally self-injective), its  $\lambda$ -dimension is at most 2.

In section 3 we study fractionnally self  $fp$ -injective rings. We prove that every reduced factor ring of a fractionnally self  $fp$ -injective ring is semihereditary. It is shown that each fractionnally self  $fp$ -injective ring which is Bézout is Kaplansky. To state this last result, we give a positive answer to a question of Henriksen by proving that any Bézout ring with compact minimal prime spectrum is Hermite.

An  $R$ -module  $E$  is said to be *uniserial* if the set of its submodules is totally ordered by inclusion. A ring  $R$  is a *valuation ring* if  $R$  is a uniserial module, and  $R$  is *arithmetic* if  $R$  is locally a valuation ring. A ring is a *Bézout ring* if every finitely generated ideal is principal. A ring  $R$  is an *Hermite ring* if  $R$  satisfies the following

property : for every  $(a, b) \in R^2$ , there exist  $d, a', b'$  in  $R$  such that  $a = da'$ ,  $b = db'$  and  $Ra' + Rb' = R$ . We say that  $R$  is a *Kaplansky ring* (or an *elementary divisor ring*) if for every matrix  $A$ , with entries in  $R$ , there exist a diagonal matrix  $D$  and invertible matrices  $P$  and  $Q$ , with entries in  $R$ , such that  $PAQ = D$ . Then we have the following implications :

Kaplansky ring  $\Rightarrow$  Hermite ring  $\Rightarrow$  Bézout ring  $\Rightarrow$  arithmetic ring ;

but these implications are not reversible [3] or [4].

Recall that  $R$  is a Kaplansky ring if and only if every finitely presented module is a finite direct sum of cyclic finitely presented modules ([5] and [6]). We say that  $R$  is an *adequate ring* if  $R$  is a Bézout ring satisfying the following property : for every  $(a, b) \in R^2$ ,  $a \neq 0$ , there exist  $r$  and  $s$  in  $R$  such that  $a = rs$ ,  $Rr + Rb = R$ , and if  $s'$  is a nonunit that divides  $s$ , then  $Rs' + Rb \neq R$ . An exact sequence  $0 \rightarrow F \rightarrow E \rightarrow G \rightarrow 0$  is *pure* if it remains exact when tensoring it with any  $R$ -module. In this case we say that  $F$  is a pure submodule of  $E$ . When  $R$  is an arithmetical ring then  $F$  is a pure submodule of  $E$  if and only if  $rF = rE \cap F$  for every  $r \in R$ , [7, Theorem 3].

The following proposition will be useful to provide us many examples in the second part of this paper.

**Proposition 1.1.** *Let  $I$  be an infinite set,  $D$  a valuation domain and  $N$  its maximal ideal. We consider*

$$S = \{f : I \rightarrow D \mid f \text{ constant except on a finite subset of } I\}$$

. Then the following statements are true.

- (1)  $S$  is a free  $D$ -module with basis  $\mathcal{B} = \{\mathbf{1}, \mathbf{e}_i \mid i \in I\}$  where  $\mathbf{1}(j) = 1$ , and  $\mathbf{e}_i(j) = \delta_{ij}$ , for every  $j \in I$ , and where  $\delta_{ij}$  is the Kronecker symbol.
- (2)  $M_0 = N\mathbf{1} + \sum_{i \in I} D\mathbf{e}_i$  and  $M_i = D(\mathbf{1} - \mathbf{e}_i) + N\mathbf{e}_i$ , for every  $i \in I$ , are the maximal ideals of  $S$ . Moreover,  $S_{M_0} \simeq D$  and  $S_{M_i} \simeq D$  for every  $i \in I$ . The Jacobson radical  $J(S) = SN = N\mathbf{1} + \sum_{i \in I} N\mathbf{e}_i$ .
- (3)  $S$  is a Kaplansky ring and an adequate ring.

**Proof.**

- (1) This assertion is obvious.
- (2) Let  $M$  be a maximal ideal of  $R$ . If  $\mathbf{e}_i \in M$ , for every  $i \in I$ , then  $M = M_0$ . The ideal  $E$  of  $S$  generated by  $\{\mathbf{e}_i \mid i \in I\}$  is a pure ideal of  $S$ , hence  $S/E$  is a flat  $S$ -module and  $S/E \simeq D$ . From this we deduce that  $S_{M_0} \simeq D$ . If there exists  $i \in I$  such that  $\mathbf{e}_i \notin M$ , then  $(\mathbf{1} - \mathbf{e}_i) \in M$  and we have  $M = M_i$ . Moreover,  $S/S(\mathbf{1} - \mathbf{e}_i)$  is a projective  $S$ -module and  $S/S(\mathbf{1} - \mathbf{e}_i) \simeq D$ . We deduce that  $S_{M_i} \simeq D$ . Now, it is easy to get that  $J(S) = N\mathbf{1} + \sum_{i \in I} N\mathbf{e}_i$ .
- (3) By using the basis  $\mathcal{B}$  of  $S$  over  $D$ , it is easy to prove that  $S$  is an Hermite ring and an adequate ring. From [8, Theorem 8] we deduce that  $S$  is a Kaplansky ring.  $\square$

An  $R$ -module  $E$  is *fp-injective* if  $\text{Ext}_R^1(F, E) = 0$  for any finitely presented  $R$ -module  $F$ , and  $R$  is *self fp-injective* if  $R$  is fp-injective as  $R$ -module. Recall that a valuation ring  $R$  is self fp-injective if and only if the set  $Z(R)$  of its zero divisors is its maximal ideal, [9, Theorem 2.8]. We recall that a module  $E$  is fp-injective if and only if it is a pure submodule of every overmodule.

We denote respectively  $\text{Spec}(R)$ ,  $\text{MaxSpec}(R)$  and  $\text{MinSpec}(R)$ , the space of prime ideals, maximal ideals, and minimal prime ideals of  $R$ , with the Zariski

topology. If  $X = \text{Spec}(R), \text{MaxSpec}(R)$  or  $\text{MinSpec}(R)$ , and  $A$  a subset of  $R$ , then we denote  $V(A) = \{P \in X \mid A \subseteq P\}$  and  $D(A) = \{P \in X \mid A \not\subseteq P\}$ .

Finally if  $E$  is an  $R$ -module,  $\text{flat-dim}(E)$  is the least integer  $n$  such that  $\text{Tor}_{n+1}^R(F, E) = 0$  for every  $R$ -module  $F$ , and  $\text{gl-w-dim}(R) = \sup\{\text{flat-dim}(E) \mid E \text{ } R\text{-module}\}$ .

## 2. THE $\lambda$ -DIMENSION

We begin with the more general result of this part.

**Theorem 2.1.** *Let  $R$  be an arithmetic ring. Then the following statements are true.*

- (1)  $\lambda\text{-dim}(R) \leq 3$ .
- (2) If  $R$  is a reduced ring then  $\lambda\text{-dim}(R) \leq 2$ .

**Proof.**

- (1) Let  $E$  be a module such that  $\lambda_R(E) \geq 3$ . We consider the following finite 3-presentation of  $E$ :

$$F_3 \xrightarrow{u_3} F_2 \xrightarrow{u_2} F_1 \xrightarrow{u_1} F_0 \rightarrow E \rightarrow 0.$$

We choose bases  $\mathcal{B}_0$  and  $\mathcal{B}_1$  of  $F_0$  and  $F_1$  respectively, and let  $A$  be the matrix associated with  $u_1$ , with respect to our given bases. Let  $M$  be a maximal ideal of  $R$ . By [7, Theorem 1]  $E_M$  is a direct sum of cyclic finitely presented  $R_M$ -modules. Therefore there exist a diagonal matrix  $D$  and two invertible matrices  $P$  and  $Q$ , with entries in  $R_M$  such that  $PAQ = D$ . It is not difficult to find  $t \in R \setminus M$ , such that  $P$  and  $Q$  are invertible matrices with entries in  $R_t$ ,  $D$  a diagonal matrix with entries in  $R$  such that  $PAQ = D$ . It follows that there exist  $a_1, \dots, a_n \in R$  such that  $E_t \simeq \bigoplus_{k=1}^n (R_t/a_k R_t)$ . Since  $\lambda_{R_t}(E_t) \geq 2$ , we deduce that  $(0 :_{R_t} a_k)$  is a finitely generated ideal of  $R_t$ , and there exists  $b_k \in R$ , such that  $(0 :_{R_M} a_k) = b_k R_M$ , for every  $k, 1 \leq k \leq n$ . By multiplying  $t$  with an element of  $R \setminus M$ , we may assume that  $(0 :_{R_t} a_k) = b_k R_t$  for every  $k, 1 \leq k \leq n$ . Now, since  $\lambda_{R_t}(E_t) \geq 3$ , by the same way, we get that there exists  $c_k \in R$ , such that  $(0 :_{R_t} b_k) = c_k R_t$  for every  $k, 1 \leq k \leq n$ . Then, the equality  $(0 :_{R_t} a_k) = b_k R_t$  implies that  $(0 :_{R_t} c_k) = b_k R_t$ , for every  $k, 1 \leq k \leq n$ . Hence  $\lambda_{R_t}(E_t) \geq 4$ .

If we denote  $U_M = D(t)$ , then  $(U_M)_{M \in \text{MaxSpec}(R)}$  is an open covering of  $\text{MaxSpec}(R)$ , and since this space is quasi-compact, a finite number of these open subsets cover  $\text{MaxSpec}(R)$ . Thus,  $\text{MaxSpec}(R) = \cup_{j=1}^m U_j$ , where  $U_j = D(t_j)$ . Let

$K = \ker(u_3)$ . Now, for every  $j, 1 \leq j \leq m$ ,  $K_{t_j}$  is a finitely generated  $R_{t_j}$ -module, hence there exists a finite subset  $G_j$  of  $K$  such that  $K_{t_j} = \sum_{g \in G_j} R_{t_j} g$ . Then  $K$  is generated by  $\cup_{j=1}^m G_j$  and we get that  $\lambda_R(E) \geq 4$ .

- (2) If  $R$  is reduced, then  $R_M$  is a valuation domain for every maximal ideal  $M$  of  $R$ . Consequently  $\text{gl-w-dim}(R) \leq 1$ , and from [2, Chapter 8] we deduce that  $\lambda\text{-dim}(R) \leq 2$ . We can also deduce this result from our following Corollary 2.13.  $\square$

The example 1.3b of [2] is a reduced arithmetic ring of  $\lambda$ -dimension 2. Now, to complete the proof of our Theorem 2.1, an example of arithmetic ring with  $\lambda$ -dimension 3 must be given.

**Example 2.2.** Let  $S$  be the ring defined in Proposition 1.1. We suppose that  $D$  has a nonzero and nonmaximal prime ideal  $J$ . Let  $a \in N \setminus J, b \in J, b \neq 0$  and  $A = Dab\mathbf{1} + \sum_{i \in I} J\mathbf{e}_i$ . We denote  $R = S/A$  and  $\bar{r} = r + A$  for every  $r \in S$ . Then  $R$  is a Kaplansky ring and also an adequate ring since  $A \subseteq J(S)$  by [6, Proposition 4.4]. Now it is easy to prove that  $(0 : \bar{a}\mathbf{1}) = R\bar{b}\mathbf{1}$  and  $(0 : \bar{b}\mathbf{1}) = R\bar{a}\mathbf{1} + \sum_{i \in I} R\bar{\mathbf{e}}_i$ . We deduce from this that  $\lambda_R(R/R\bar{a}\mathbf{1}) = 2$ . Hence  $\lambda\text{-dim}(R) = 3$ .

**Example 2.3.** Let  $S$  be the ring defined in Proposition 1.1,  $A = \sum_{i \in I} N\mathbf{e}_i$  and  $R = S/A$ . If  $M$  is a maximal ideal of  $S$ , we denote  $\bar{M} = M/A$ . Then it is easy to prove that  $R_{\bar{M}_0} \simeq D$  and  $R_{\bar{M}_i} \simeq D/N$  for every  $i \in I$ . Consequently  $R$  is a reduced ring, a Kaplansky ring and an adequate ring. For every  $a \in N, a \neq 0$ ,  $(0 : \bar{a}\mathbf{1}) = \sum_{i \in I} R\bar{\mathbf{e}}_i$  is not finitely generated. Then  $\lambda\text{-dim}(R) = 2$ . When  $D = \mathbb{Z}_2$ , the ring of 2-adics numbers, and  $I = \mathbb{N}$ , we obtain the example 1.3b of [2], if, in this example we replace  $\mathbb{Z}$  with  $\mathbb{Z}_2$ .

**Theorem 2.4.** *Let  $R$  be an arithmetic ring.*

*If  $R$  is self fp-injective then  $\lambda\text{-dim}(R) \leq 2$ .*

**Proof.** Let  $E$  be a module with  $\lambda_R(E) \geq 2$ . As in the proof of Theorem 2.1, for every  $M \in \text{MaxSpec}(R)$ , we can find  $t \in R \setminus M, a_1, \dots, a_n \in R, b_1, \dots, b_n \in R$  such that  $E_t \simeq \bigoplus_{k=1}^n (R_t/a_k R_t)$  and  $(0 :_{R_t} a_k) = b_k R_t$  for every  $k, 1 \leq k \leq n$ .

Since  $\lambda_R(E) \geq 2$ , then the canonical homomorphism  $(\text{Ext}_R^1(E, R))_t \rightarrow \text{Ext}_{R_t}^1(E_t, R_t)$  is an isomorphism. Thus  $\text{Ext}_{R_t}^1(E_t, R_t) = 0$  and  $\text{Ext}_{R_t}^1(R_t/a_k R_t, R_t) = 0$  for every  $k, 1 \leq k \leq n$ . From the following projective resolution of  $R_t/a_k R_t$  :  $R_t \xrightarrow{b_k} R_t \xrightarrow{a_k} R_t$ , we deduce that  $(0 :_{R_t} b_k) = a_k R_t$  for every  $k, 1 \leq k \leq n$ . Hence  $\lambda_{R_t}(E_t) \geq 3$ . Now, as in the proof of Theorem 2.1, we get that  $\lambda_R(E) \geq 3$ .  $\square$

When  $R$  is a reduced ring we have a more general result.

**Theorem 2.5.** *Let  $R$  be a reduced ring. Then  $R$  is self fp-injective if and only if  $R$  is a Von Neumann regular ring.*

**Proof.** Only necessity requires a proof. Since  $R$  is reduced,  $R$  is a subring of  $S = \prod_{P \in \text{MinSpec}(R)} Q(R/P)$ , and  $S$  is a Von Neumann regular ring. Hence, for every  $r \in R$ , there exists  $s \in S$  such that  $r^2 s = r$ . But, since  $R$  is self fp-injective,  $R$  is a pure submodule of  $S$ . Thus, there exists  $s' \in R$  such that  $r^2 s' = r$ .  $\square$

**Remark 2.6.** We can prove that for every  $n \in \mathbb{N}$ , there exists a self injective ring  $R$  such that  $\lambda\text{-dim}(R) = n$ . Let  $D$  be a local noetherian regular ring,  $N$  its maximal ideal and  $E$  the  $D$ -injective hull of  $D/N$ . If  $R = \left\{ \begin{pmatrix} d & x \\ 0 & d \end{pmatrix} \mid d \in D \text{ and } x \in E \right\}$  is the trivial extension of  $D$  by  $E$ , J.E. Roos proved that  $\lambda\text{-dim}(R) = n$  if and only if  $\text{Krull dim}(D) = n$  ([10, Theorem A']). If  $D$  is complete in its  $N$ -adic topology, then  $D$  is a linearly compact  $D$ -module, and since  $E$  is an artinian  $D$ -module,  $R$  is a linearly compact  $D$ -module. We deduce that  $R$  is a local linearly compact ring, and since  $R$  is an essential extension of a simple  $R$ -module, from [11, Theorem 7] it follows that  $R$  is a self injective ring. In the general case, we can prove that  $R$  is self fp-injective.

**Corollary 2.7.** *Let  $R$  be an arithmetic ring of Krull dimension 0.*

*Then  $\lambda\text{-dim}(R) \leq 2$ .*

**Proof.** For every  $M \in \text{MaxSpec}(R)$ , any element of  $MR_M$  is a zero divisor. From [9, Theorem 2.8], we deduce that  $R$  is locally self  $fp$  - injective, and from [9, Proposition 1.2] or [12, Corollary 8] that  $R$  is self  $fp$  - injective. The result is an immediate consequence of Theorem 2.4.  $\square$

Now, we give an example of a noncoherent Kaplansky ring  $R$  with Krull dimension 0, which is locally coherent.

**Example 2.8.** Let  $S$  be the ring defined in Proposition 1.1. We suppose that  $D$  is a valuation domain with Krull dimension one and its maximal ideal  $N$  is not finitely generated. We take  $I = \mathbb{N}^*$ . Let  $a \in N \setminus 0$  and  $(b_n)_{n \in \mathbb{N}}$  a sequence of nonzero elements of  $N$  such that  $b_{n+1} \notin Db_n$  for every  $n \in \mathbb{N}$ . We consider the ideal  $A = Dab_0\mathbf{1} + \sum_{n \in \mathbb{N}^*} Dab_n\mathbf{e}_n$  and the ring  $R = S/A$ . Then  $(0 :_R \overline{a\mathbf{1}}) = R\overline{b_0\mathbf{1}} + \sum_{n \in \mathbb{N}^*} R\overline{b_n\mathbf{e}_n}$ . Consequently  $R$  is a noncoherent ring with Krull dimension 0. But, for every  $n \in \mathbb{N}$ ,  $R_{\overline{M}_n} \simeq D/ab_nD$ . Thus  $R$  is locally coherent.

**Remark 2.9.** Let  $S$  be the ring defined in Proposition 1.1. We suppose that  $D = \mathbb{Z}_p$  (where  $p$  is a prime integer) the ring of  $p$ -adic numbers and  $I = \mathbb{N}^*$ . We consider  $A = \bigoplus_{n \in \mathbb{N}^*} Dp^n\mathbf{e}_n$  and  $R = S/A$ . Then  $R$  is isomorphic to the example of [9, p. 344]. This ring is a Kaplansky ring which is self  $fp$  - injective, but not locally self  $fp$  - injective.

The following proposition will be used to compute the  $\lambda$ -dimension of semi-local arithmetic rings and fractionnally self injective rings.

**Proposition 2.10.** *Let  $R$  be a valuation ring,  $M$  its maximal ideal,  $Z$  the subset of zero divisors of  $R$ . Then the following statements are true.*

- (1)  *$M$  is a flat module if and only if  $(0 : r)$  is not finitely generated for every  $r \in Z \setminus 0$ .*
- (2) *Let  $r$  and  $s$  in  $R$  such that  $rs \neq 0$ . Then:*
  - i)  *$(0 : rs) = ((0 : r) : s)$  and  $(0 : r) = s(0 : rs)$ .*
  - ii) *If  $(0 : r) \neq 0$ ,  $(0 : r)$  is finitely generated if and only if  $(0 : rs)$  is also.*

**Proof.**

- (1) Suppose that  $M$  is a flat module. Let  $r \in Z \setminus 0$  and  $s \in (0 : r)$ . Let  $\varphi : Rr \otimes M \rightarrow M$  be the homomorphism induced by the inclusion map  $Rr \rightarrow R$ . Then  $\varphi(r \otimes s) = 0$ . From [13, proposition 13 p.42] we deduce that there exist  $t \in (0 : r)$  and  $m \in M$  such that  $s = tm$ . Consequently  $Rs \subsetneq (0 : r)$ .

Conversely let  $s \in M$  and  $r \in R$  such that  $\varphi(r \otimes s) = rs = 0$ . If  $r \notin Z$  then  $s = 0$  and  $r \otimes s = 0$ . If  $r \in Z \setminus 0$ , since  $(0 : r)$  is not finitely generated, then there exist  $t \in (0 : r)$  and  $m \in M$  such that  $s = tm$ . Hence  $r \otimes s = rt \otimes m = 0$ .

- (2) i) It is easy to get the first equality and the inclusion  $s(0 : rs) \subseteq (0 : r)$ . Now, if  $t \in (0 : r)$ , then  $t \in Rs$  since  $rs \neq 0$ . We deduce that there exists  $c \in R$  such that  $t = cs$  and it is obvious that  $c \in (0 : rs)$ . Then ii) is a consequence of i).  $\square$

**Theorem 2.11.** *Let  $R$  be a valuation ring,  $M$  its maximal ideal and  $Z$  the subset of its zero divisors. Then the following statements are true.*

- (1) If  $Z = 0$  then  $R$  is coherent.
- (2) If  $Z \neq 0$  and  $Z \neq M$ , then  $\lambda\text{-dim}(R) = 2$ .
- (3) If  $Z \neq 0$  and  $Z = M$ , then  $\lambda\text{-dim}(R) \leq 2$ .
- (4) If  $R$  is a domain or a noncoherent ring then  $M$  is a flat ideal and for any  $R$ -module  $E$  with  $\lambda_R(E) = \infty$ ,  $\text{flat-dim}(E) \leq 1$ .

**Proof.**

- (1) It is obvious.
- (2) We have  $M = \cup_{r \in M \setminus Z} Rr$ . Since  $M$  is a direct limit of free modules,  $M$  is flat. Let  $E$  be a module such that  $\lambda_R(E) \geq 2$ . Then we may assume that  $E = R/rR$ , where  $r \in R$ . Since  $M$  is flat, we deduce from Proposition 2.10 that  $r \notin Z$  if  $r \neq 0$ . We get successively that  $\text{flat-dim}(R/rR) \leq 1$ ,  $\lambda_R(R/rR) \geq 3$  and  $\lambda\text{-dim}(R) = 2$ .
- (3) Since  $Z = M$ ,  $R$  is self *fp*-injective by [9, Theorem 2.8]. By Theorem 2.4,  $\lambda\text{-dim}(R) \leq 2$ .
- (4) It remains to examine the case  $Z = M$ . Let  $r \in R$  such that  $\lambda_R(R/rR) = \infty$ . If  $R$  is not coherent, then for every  $s \in M \setminus 0$ ,  $(0 : s)$  is not finitely generated by Proposition 2.10(2). We deduce that  $M$  is flat and that  $r = 0$  or  $r$  is a unit. Hence  $R/rR$  is a free module.  $\square$

**Corollary 2.12.** *Let  $R$  be a semi-local arithmetic ring. Then  $\lambda\text{-dim}(R) \leq 2$  and  $R$  is coherent if and only if  $R$  is locally coherent.*

**Proof.** Since  $\text{MaxSpec}(R)$  is finite,  $S = \prod_{M \in \text{MaxSpec}(R)} R_M$  is a faithfully flat  $R$ -module. We deduce that  $\lambda\text{-dim}R \leq \lambda\text{-dim}S = \sup\{\lambda\text{-dim}R_M \mid M \in \text{MaxSpec}(R)\}$ .  $\square$

**Corollary 2.13.** *Let  $R$  be an arithmetic ring. We suppose that  $R_M$  is a domain or a noncoherent ring for every  $M \in \text{MaxSpec}(R)$ . Then  $\lambda\text{-dim}(R) \leq 2$ .*

**Proof.** Let  $E$  be an  $R$ -module with  $\lambda_R(E) \geq 2$ . From Theorem 2.11, we deduce that  $\text{flat-dim}E_M \leq 1$  for every  $M \in \text{MaxSpec}(R)$ . Hence  $\text{flat-dim}E \leq 1$ . We consider the following finite 2-presentation of  $E$  :

$$L_2 \xrightarrow{u_2} L_1 \xrightarrow{u_1} L_0 \xrightarrow{p} E \rightarrow 0$$

Then  $\ker(p)$  is a finitely presented flat  $R$ -module. We deduce successively that  $\ker(p)$ ,  $\ker(u_1)$  and  $\ker(u_2)$  are finitely generated projective  $R$ -modules. Hence  $\lambda_R(E) \geq 3$ .  $\square$

**Corollary 2.14.** *Let  $R$  be a valuation ring and  $A$  a nonzero proper ideal of  $R$ . Then the following statements are true.*

- (1) If  $A$  is prime then  $R/A$  is coherent.
- (2) If  $A$  is finitely generated, then  $R/A$  is coherent and self *fp*-injective.
- (3) If  $A$  is not prime and not finitely generated then  $\lambda\text{-dim}(R/A) = 2$ .

**Proof.**

- (1) It is obvious.
- (2) We have  $A = Ra$  for some  $a \in R$ . If  $r \notin Ra$ , then there exists  $s \notin Ra$  such that  $a = rs$ . Clearly  $Ra \subseteq (Ra : r)$ . Let  $c \in (Ra : r)$ . If  $cr = 0$  then  $c \in Ra$  since  $rs \neq 0$ . If  $cr \neq 0$ , then there exists  $d \in R$  such that  $cr = da = dsr$ . Hence  $r(c - ds) = 0$ . If  $ds = vc$  for some  $v \in R$ , we get that  $rc(1 - v) = 0$ . Since  $rc \neq 0$ ,  $v$  is a unit and we obtain that  $c \in Ra$ , and  $(Ra : r) = Ra$ .

- (3) Since  $A$  is not prime, there exist  $s$  and  $r \in R \setminus A$ , such that  $sr \in A$ . Hence  $A \subsetneq (A : r)$  and we prove easily that  $(A : r)$  and  $(A : r)/A$  are not finitely generated.  $\square$

**Proposition 2.15.** *Let  $R$  be an arithmetic ring and  $A$  a finitely generated proper ideal of  $R$  such that  $(0 : A) \subseteq J(R)$ , the Jacobson radical of  $R$ . Then  $R/A$  is a coherent and self  $fp$ -injective ring.*

**Proof.** Then, for every maximal ideal  $M$  of  $R$ ,  $AR_M$  is a nonzero finitely generated ideal of  $R_M$ . By Corollary 2.14,  $R_M/AR_M$  is self  $fp$ -injective. We deduce that  $R/A$  is self  $fp$ -injective.

Since in every arithmetic ring the intersection of two finitely generated ideals is a finitely generated ideal, [14, Corollary 1.11], it is sufficient to prove that  $(A : b)$  is finitely generated for every  $b \in R \setminus A$ . Let  $M$  be a maximal ideal of  $R$ . Then there exists  $a \in A$  such that  $AR_M = aR_M$ . Since  $A$  is finitely generated, there exists  $t \in R \setminus M$  such that  $AR_t = aR_t$ . Now, if  $b \in aR_M$ , then  $b = \frac{c}{s}a$  for some  $c \in R$  and  $s \in R \setminus M$ , and we get the equality  $t'sb = t'ca$  for some  $t' \in R \setminus M$ . We deduce that  $t's \in (A : b)$ , and  $(AR_{t't's} :_{R_{t't's}} b) = t's R_{t't's}$ . If  $b \notin aR_M$  then there exist  $c \in R$  and  $s \notin M$  such that  $a = \frac{c}{s}b$ . As in the proof of Corollary 2.14,  $(R_M a :_{R_M} b) = R_M c$ . For some  $t' \in R \setminus M$  we have  $t'sa = t'cb$ . We deduce that  $t'c \in (A : b)$  and

$(AR_{t't's} :_{R_{t't's}} b) = t'cR_{t't's}$ . Hence, for every  $M \in \text{MaxSpec}(R)$ , we may assume that there exist  $t_M \in R \setminus M$  and  $c_M \in (A : b)$  such that  $(AR_{t_M} :_{R_{t_M}} b) = c_M R_{t_M}$ . A finite number of open subsets  $D(t_M)$  cover  $\text{MaxSpec}(R)$ . Let  $D(t_1), \dots, D(t_n)$  be these open subsets and  $c_1, \dots, c_n \in (A : b)$  such that  $(AR_{t_k} :_{R_{t_k}} b) = c_k R_{t_k}$ , for every  $k$ ,  $1 \leq k \leq n$ . Then we get that  $\{c_k \mid 1 \leq k \leq n\}$  generates  $(A : b)$ .  $\square$

**Remark 2.16.** Let  $S$  be the ring defined in Proposition 1.1. We assume that  $D$  has a nonzero and nonmaximal prime ideal  $J$ . Let  $B = \sum_{i \in I} J e_i$ ,  $R' = R/B$ ,  $a \in N \setminus J$ ,  $b \in J \setminus 0$ ,  $a' = a\mathbf{1} + B$  and  $b' = b\mathbf{1} + B$ . Then, if  $R$  is the ring of the example 2.2, we have  $R = R'/a'b'R'$ . Since  $\lambda\text{-dim}(R) = 3$ ,  $R$  is not coherent and not self  $fp$ -injective. Consequently the assumption  $(0 : A) \subseteq J(R)$  cannot be omitted in the Proposition 2.15.

Following Vámos [1], we say that  $R$  is a *torch ring* if the following conditions are satisfied :

- (1)  $R$  is an arithmetical ring with at least two maximal ideals.
- (2)  $R$  has a unique minimal prime ideal  $P$  which is a nonzero uniserial module.

We follow T.S. Shores and R. Wiegand [14], by defining a *canonical form* for an  $R$ -module  $E$  to be a decomposition  $E \simeq R/I_1 \oplus R/I_2 \oplus \dots \oplus R/I_n$ , where  $I_1 \subseteq I_2 \subseteq \dots \subseteq I_n \neq R$ , and by calling a ring  $R$  a *CF-ring* if every direct sum of finitely many cyclic modules has a canonical form.

**Theorem 2.17.** *Let  $R$  be CF-ring. Then the following statements are true.*

- (1)  $\lambda\text{-dim}(R) \leq 2$ .
- (2)  $R$  is coherent if and only if  $R$  is locally coherent.

**Proof.** In [14, Theorem 3.12] it is proved that every CF-ring is arithmetic and a finite product of indecomposable CF-rings. If  $R$  is indecomposable then  $R$  is either

a domain (1), or a semi-local ring (2), or a torch ring (3). In the case (1)  $R$  is coherent, and the theorem is a consequence of Corollary 2.12 in the case (2). We may assume that  $R$  is a torch ring. Then, there is only one maximal ideal  $M$  such that  $P_M \neq \{0\}$ , and we have  $P^2 = 0$ . For every maximal ideal  $N \neq M$ ,  $R_N$  is a domain. Consequently, if  $R_M$  is not coherent we deduce from Corollary 2.13 that  $\lambda\text{-dim}R = 2$ . Now we assume that  $R_M$  is coherent. As in the previous proposition it is sufficient to prove that  $(0 : r)$  is finitely generated for any  $r \in R$ . Then we have  $(0 :_{R_M} r) = sR_M$  for some  $s \in R$ . Since the canonical homomorphism  $R \rightarrow R_M$  is monic,  $rs = 0$ .

If  $r \notin P$ , then  $s \in P$ . For every maximal ideal  $N$  of  $R$ ,  $N \neq M$ , we have  $rR_N \neq 0$  and  $sR_N = 0$ . Consequently  $(0 : r)_N = 0 = sR_N$ . We deduce that  $(0 : r) = Rs$ .

If  $r \in P$ , then  $s \notin P$  since  $P^2 = 0$ . Since  $R$  satisfies the condition iii) of [14, Theorem 3.10],  $V(s)$  is a finite subset of  $\text{MaxSpec}(R)$ . We denote  $V(s) = \{M, N_k \mid 1 \leq k \leq n\}$ . Since  $rR_{N_k} = 0$  then there exists  $s_k \notin N_k$  such that  $s_k r = 0$ , for every  $k$ ,  $1 \leq k \leq n$ . Let  $A$  be the ideal of  $R$  generated by  $\{s, s_k \mid 1 \leq k \leq n\}$ . Then for every maximal ideal  $N \neq M$  we have  $A_N = R_N = (0 :_{R_N} r) = (0 : r)_N$ . Hence we get that  $A = (0 : r)$ .  $\square$

In [1] Vámos proved that every fractionnally self-injective ring (FSI-ring) is a CF-ring. Consequently the following corollary holds.

**Corollary 2.18.** *Let  $R$  be a fractionnally self-injective ring. Then the following statements are true.*

- (1)  $\lambda\text{-dim}(R) \leq 2$ .
- (2)  $R$  is coherent if and only if  $R$  is locally coherent.

### 3. FRACTIONNALLY SELF $fp$ -INJECTIVE RINGS

First we give a generalization of results obtained in [1] on fractionnally self-injective rings.

**Theorem 3.1.** *Let  $R$  be a fractionnally self  $fp$ -injective ring. Then the following statements are true.*

- (1)  $R$  is an arithmetic ring.
- (2) For every proper ideal  $A$  of  $R$ ,  $\text{MinSpec}(R/A)$  is a compact space. Moreover if  $A$  is semi-prime then  $R/A$  is semihereditary.

**Proof.**

- (1) It is the main result of [12] (Theorem 1).
- (2)  $\text{MinSpec}(R/A)$  is homeomorphic to  $\text{MinSpec}(R/\text{rad}A)$ . We may assume that  $A$  is semi-prime. Then  $gl\text{-}w\text{-dim}(R/A) \leq 1$ . By Theorem 2.5,  $Q(R/A)$  is a Von Neumann regular ring. We deduce that  $R/A$  is semi-hereditary from [15, Theorem 5] and that  $\text{MinSpec}(R/A)$  is compact from [16, Proposition 10].  $\square$

**Remark 3.2.** If  $R$  is the ring of our example 2.3, then it is isomorphic to the ring of [12, Proposition 4] which is not fractionnally self  $fp$ -injective.

Now we give a positive answer to a question proposed by M. Henriksen, [17, p. 1382]. The following theorem is a generalization of [6, Theorem 2.4] and [17, Corollary 1.3].



**Theorem 3.3.** *Every Bézout ring  $R$  with compact minimal prime spectrum is Hermite.*

**Proof** Let  $a$  and  $b$  be in  $R$ . We may assume that  $a \neq 0$  and  $b \neq 0$ . Then there exist  $a', b', d, m$  and  $n$  in  $R$  such that  $a = da'$ ,  $b = db'$  and  $ma + nb = d$ . We denote  $c = ma' + nb'$ , and  $N$  the nilradical of  $R$ . We have  $(1 - c)d = 0$ . Since  $d \neq 0$  it follows that  $c \notin N$ .

First we suppose that  $(N : c) = N$ . Let  $a'', b'', d', m'$  and  $n'$  be in  $R$  such that  $a' = d'a''$ ,  $b' = d'b''$  and  $m'a' + n'b' = d'$ . Then  $c \in Rd'$  and consequently  $(N : d') = N$ . Since  $(1 - m'a'' - n'b'')d' = 0$ ,  $1 - m'a'' - n'b'' \in N$ . Hence  $m'a'' + n'b''$  is a unit and the following equalities hold :  $a = a''d'd$ ,  $b = b''d'd$  and  $Ra'' + Rb'' = R$ .

Now we suppose that  $(N : c) \neq N$ . Since  $\text{MinSpec}(R)$  is compact  $R' = R/N$  is a semi-hereditary ring. Denote  $\bar{r} = r + N$  for any  $r \in R$ . Then there exists an idempotent  $\bar{e}$  of  $R'$  such that  $(0 : \bar{c}) = R'(\bar{1} - \bar{e})$ . Since idempotents can be lifted modulo  $N$ , we may assume that  $e = e^2$ . We deduce that  $(N : c) = R(1 - e) + N$ , and  $(1 - e)c \in N$ . Let  $P \in D(1 - e)$ . Thus  $c \in P$  and  $(1 - c) \notin P$ . Consequently  $D(1 - e) \subseteq D(1 - c)$ , and since  $(1 - e)$  is an idempotent,  $(1 - e) \in R(1 - c)$ . But  $(1 - c)d = 0$  and therefore  $(1 - e)d = 0$  and  $ed = d$ . As in the proof of [6, Theorem 2.4] we denote  $a_1 = a'e$ ,  $b_1 = b'e + (1 - e)$ ,  $m_1 = me$  and  $n_1 = ne + (1 - e)$ . Then we get  $a = a_1d$ ,  $b = b_1d$  and  $m_1a + n_1b = d$ . Let  $c_1 = m_1a_1 + n_1b_1 = ce + (1 - e)$  and  $r \in (N : c_1)$ . We get that  $r(1 - e) \in N$  and  $rec \in N$ . Hence  $re \in (N : c) = R(1 - e) + N$ . Consequently  $re \in N$  and  $(N : c_1) = N$ . From the previous part of the proof we deduce that  $R$  is Hermite.  $\square$

**Corollary 3.4.** *Let  $R$  be a Bézout ring of Krull dimension at most one.*

*If  $\text{MinSpec}(R)$  is compact then  $R$  is a Kaplansky ring.*

**Proof.** Let  $N$  be the nilradical of  $R$ . By [18, Corollary p.213],  $R/N$  is a Kaplansky ring. From the previous theorem and [19, Theorem 3], the result follows.  $\square$

**Theorem 3.5.** *Let  $R$  be a fractionnally self fp-injective ring . If  $R$  is a Bézout ring then  $R$  is a Kaplansky ring.*

**Proof.** By Theorem 3.1 and Theorem 3.3  $R$  is Hermite. Hence by [8, Theorem 6], it is sufficient to prove that for all  $a, b, c \in R$  such that  $Ra + Rb + Rc = R$  there exist  $p$  and  $q \in R$  such that  $Rpa + R(pb + qc) = R$ . We put

$$A = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}.$$

By using the same terminology as in [6] let  $E$  be an  $R$ -module named by  $A$ . It is easy to check that  $E$  is an  $R/Rac$ -module. Let  $J = \text{rad}(Rac)$ . It follows that  $\bar{R} = R/J$  is semihereditary by Theorem 3.1. Thus  $\bar{E} = E/JE$  is named by

$$\bar{A} = \begin{pmatrix} \bar{a} & \bar{0} \\ \bar{b} & \bar{c} \end{pmatrix}.$$

Since  $\bar{R}$  is Hermite and  $\bar{a} \cdot \bar{c} = \bar{0}$  we show, as in the proof of [18, the Proposition], that there exist two invertible matrices  $P$  and  $Q$  and a diagonal matrix  $D$  with entries in  $\bar{R}$  such that  $P\bar{A}Q = D$ . We put

$$D = \begin{pmatrix} \bar{s} & \bar{0} \\ \bar{0} & \bar{t} \end{pmatrix}.$$

By [6, Theorem 3.1] we may assume that  $\bar{s}$  divides  $\bar{t}$ . The equality  $P^{-1}DQ^{-1} = \bar{A}$  implies that  $\bar{a}, \bar{b}, \bar{c} \in \bar{R}\bar{s}$ . It follows that  $\bar{s}$  is a unit. Hence  $\bar{E}$  is a cyclic  $\bar{R}$ -module. By Nakayama Lemma it follows that  $E$  is cyclic over  $R/Rac$ . Hence  $E$  is cyclic over  $R$  too. Now we do as at the end of the proof of [6, Theorem 3.8] to conclude.  $\square$

By [12, Theorem 6] every arithmetic ring of Krull dimension zero is fractionnally self  $fp$ -injective. By Theorem 3.5 every Bézout ring of Krull dimension zero is Kaplansky. However it is possible to prove a more general result.

**Theorem 3.6.** *Let  $R$  be an arithmetic ring of Krull dimension 0. Then  $R$  is a Kaplansky ring and an adequate ring.*

**Proof** First we prove that  $R$  is Hermite. Let  $a$  and  $b$  be in  $R$ . We denote  $U = \{M \in \text{Spec}(R) \mid aR_M \subseteq bR_M\}$  and  $F = \{M \in \text{Spec}(R) \mid aR_M \not\subseteq bR_M\}$ . Recall that  $\text{Spec}(R)$  is a (totally disconnected) Hausssdorf compact space, where  $D(A)$  is open and closed, for every finitely generated ideal  $A$  of  $R$ . Let  $M \in U$ . Then there exist  $c \in R$  and  $t \in R \setminus M$  such that  $\frac{a}{1} = \frac{cb}{t}$ . We deduce that there exists  $s \in R \setminus M$  such that  $s(ta - cb) = 0$ . Hence, for every  $Q \in D(st)$ ,  $R_Qa \subseteq R_Qb$ . Consequently  $U$  is open and  $F$  is closed. Now, let  $M \in F$ . Then  $bR_M \subsetneq aR_M$  and there exists  $t \in R \setminus M$  such that  $bR_Q \subseteq aR_Q$ , for every  $Q \in D(t)$ . If we denote  $W_M = D(t)$ , then  $F \subseteq \cup_{M \in F} W_M$ . Since  $F$  is compact,  $F$  is contained in a finite union  $W$  of these open and closed subsets of  $\text{Spec}(R)$ . Consequently, there exists an idempotent  $e$  of  $R$  such that  $F \subseteq W = D(e)$  and  $D(1 - e) \subseteq U$ . Since  $R_Mb \subseteq R_Ma$  for every  $M \in D(e)$ , there exists  $r \in R$  such that  $be = rae$ . There also exists  $s \in R$  such that  $a(1 - e) = sb(1 - e)$ . Now if we take  $d = ae + b(1 - e)$ ,  $a' = s(1 - e) + e$ ,  $b' = (1 - e) + re$ , then  $a = da'$ ,  $b = db'$  and  $ea' + (1 - e)b' = 1$ .

Now we prove that  $R$  is adequate. Let  $a$  and  $b$  be in  $R$ ,  $a \neq 0$ . There exists an idempotent  $e$  in  $R$  such that  $D(b) = D(e)$ . If we take  $r = (1 - e) + ae$  and  $s = a(1 - e) + e$ , then  $a = rs$  and  $Rr + Rb = R$ . Let  $s'$  be a nonunit in  $R$  that divides  $s$ . Then  $V(s') \subseteq V(b)$ . Hence  $Rb + Rs' \neq R$ . From [8, Theorem 8] we deduce that  $R$  is a Kaplansky ring.  $\square$

The following proposition gives an answer to a question of [6, p.233].

**Proposition 3.7.** *Let  $R$  be an arithmetic ring and  $J(R)$  its Jacobson radical. Then the following conditions are equivalent:*

- (1)  $R$  has a unique minimal prime ideal
- (2) For every  $d \notin J(R)$ ,  $(0 : d) \subseteq J(R)$

**Proof.** The implication  $1 \Rightarrow 2$  is easy.

$2 \Rightarrow 1$ . Suppose there are at least two minimal prime ideals  $I$  and  $J$ . Let  $a \in I \setminus J$  and  $P$  a maximal ideal containing  $I$ . Then  $IR_P$  is the nilradical of  $R_P$ . It follows that there exist  $s \in R \setminus P$  and a positive integer  $n$  such that  $sa^n = 0$ . Then  $s \in J \setminus I$ . Let  $Q$  be a maximal ideal containing  $J$ . There also exist  $t \in R \setminus Q$  and a positive integer  $m$  such that  $ts^m = 0$ . Since  $t \notin J(R)$ ,  $s^m \in (0 : t) \subseteq J(R)$ . But  $s \notin P$  implies that  $s^m \notin J(R)$ . Hence we get a contradiction.  $\square$

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