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► **To cite this version:**

Jean Farago. Power fluctuations in stochastic models of dissipative systems. *Physica A: Statistical Mechanics and its Applications*, 2004, 331, pp.69-89. hal-00002937

HAL Id: hal-00002937

<https://hal.science/hal-00002937>

Submitted on 24 Sep 2004

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Power fluctuations in stochastic models of dissipative systems.

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Abstract

We consider different models of stochastic dissipative equations and theoretically compute the probability distribution functions (actually the associated large deviation functions) of the *time averaged* injected power required to sustain a nontrivial stationary state. We discuss the results and in particular draw from our results some general features shared by these distributions in realistic dissipative systems.

PACS: 05.40.-a: Fluctuation phenomena, random processes, noise, and Brownian motion.

02.50.-r: Probability theory, stochastic processes, and statistics.

1 Introduction

Among out of equilibrium statistical systems, strongly dissipative mediums have a very particular status. Usually, statistical theories of non equilibrium systems deal with tools inspired or inherited from thermal equilibrium, a scheme that Onsager theory exemplifies especially well [1, 2]. That such a continuity may be possible in numerous cases is always due to the fact that genuine equilibrium keeps a certain relevance, thanks to a time decoupling which induces partial equilibrations [3]. Even in recent theories of nonequilibrium statistical mechanics aiming at describing systems as complicated as glassy materials, equilibrium concepts are generalized and successfully adapted, since again time separation is there at work [4].

In contrast, strongly dissipative systems have a full rest natural “fixed point” or equilibrium (something like a zero temperature situation). To avoid such a state and explore the richness of their dynamics, the system has to be fed continuously with energy, by means of an external action which holds it in a non trivial stationary state (if the injection mechanism has itself some stationarity properties). As a result, the statistical state (i.e. the stationary measure) reached is very far from any concept of (Boltzmann-like) equilibrium, and new approaches must be followed. For instance, granular matter looks more or less like a gas when sufficiently shaken; actually, the so-called “granular temperature” defined as the mean kinetic energy per particle, is a rather hazy notion, since its values are not unique when the system is not monodisperse [5, 6], in a clear violation of a basic requirement for a temperature— that is, the equilibration of temperatures of any pair of subsystems at equilibrium (or equivalently, the energy equipartition principle).

Therefore, these systems, sometimes termed “far from equilibrium”, do not belong to the traditional statistical physics (as far as we are concerned with relevant degrees of freedom: of course, the Navier-Stokes equation can be derived from local thermodynamical equilibrium considerations; however, a turbulent velocity field obeys a dissipative dynamical equation (NS, say) and its statistics has nothing to do with standard statistical physics) and consequently requires alternative approaches.

Some years ago, such an original approach was proposed [7], which tried to look at the problem of dissipative systems from a *global* and phenomenological point of view. In particular, it was recognized that the common feature of all dissipative systems is the energy evolution equation

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$\dot{E} = I - D$, which explicites and summarizes the physics of these systems from a “macroscopical” (or experimental) point of view : two channels of energy flow compete, one corresponding to the energy injected by the external operator (I), the other related to an inner dissipation term (D , easily recognized, since it is in general a volume term proportional to a dissipation coefficient). Of course, the physics is actually more complicated, since these energy flows induce complicated structurations of the local fields, which themselves determine *in fine* the statistical properties of I and D ; but it was thought that if something common to all dissipative systems could be once firmly asserted, this will be done necessarily using that kind of general evolution equation, which transcends all peculiar details of each experimental situation.

Experimental measurements were first performed in a turbulent von Karmán flow, and statistical properties of I were extracted and analysed in stationary regimes. Surprisingly enough, it was shown that the fluctuations of I are non gaussian (despite its *a priori* “extensive” (on the surface) nature), and decay abnormally slowly as Reynolds number increases (i.e. more slowly than a normal regression $\sim 1/\text{Re}^{1/2}$). The analysis of these experiments is still the subject of active research and debates [8], and the physical interpretation of the observed scaling is up to now not fully completed. Similar studies were afterwards made on various systems including granular gases, shell models of turbulence, self-organized critical systems, and led to interesting developments: for instance, a very convincing way to estimate the number of effective degrees of freedom in a shaken granular gas has been proposed [9].

From a theoretical point of view, exploring the properties of I and D is not easy, for the systems considered are characterized by invariant measures generally unknown. In a preceding paper [10], we studied a simple model (“zero-dimensional”) of dissipative system, where the statistical properties of I and D were calculable. More precisely, we considered the following stochastic equation

$$\ddot{x} + \gamma\dot{x} + V'(x) = \psi(t) \quad (1)$$

where ψ is a white noise and $V(x)$ a potential. We read this equation as the dynamical evolution of a single dissipative coordinate ($\ddot{x} + \gamma\dot{x} + V'(x) = 0$) subjected to an external forcing $\psi(t)$ (the stochasticity of the forcing mimics the chaoticity developed by realistic dissipative systems when driven vigorously enough out of equilibrium; this *ad hoc* choice allows exact computations; we verified that purely chaotic dissipative systems (like a periodically forced asynchronous pendulum) gives qualitatively the same results, as far as time-averaged observables are concerned (see below); however, some interesting studies and comparisons could be performed with deterministic chaotic systems).

The energy of the coordinate is naturally defined as $E = \dot{x}^2/2 + V(x)$, and is conserved in the absence of noise and dissipation. This energy obeys the evolution equation

$$\dot{E} = \psi\dot{x} - \gamma\dot{x}^2 \quad (2)$$

where the above-mentioned structure $\dot{E} = I - D$ is clearly apparent. The properties of instantaneous quantities of such systems are not complicated to compute, but we were interested in calculating the distribution of partially time-averaged quantities like

$$\varepsilon = \frac{1}{\tau} \int_t^{t+\tau} dt' I(t') \quad (3)$$

Initially, the focus on these time-averaged variables was motivated by the so-called Fluctuation Theorem [11, 12, 13], but beyond this particular debate, this partial time averaging presents a major advantage for analysis: it smoothes the short time scale dynamics, and gives the possibility to highlight phenomena occurring at slow time scales, broadening the analysis based only upon non dynamical considerations. As an example, the systems considered in [10] display a great variety of different dynamics, according to the form of the potential $V(x)$: confined, unconfined, activated, etc. . . Despite these great qualitative discrepancies, we showed that the probability density function of ε is asymptotically *the same* in the limit of large τ (to be precise: $\lim_{\tau \rightarrow \infty} \frac{1}{\tau} \log P(\varepsilon) = f(\varepsilon)$ with f independent of V ; this f is called the large deviation function associated to ε). In this result can be seen that peculiar details of the short time dynamics of energy have faded away through the averaging. We also remarked other interesting facts concerning the pdf of injected and dissipated

power (averaged during a time τ): the *large deviation function* of the injected power is curiously dependent on the initial conditions set of the system at the beginning of the averaging window: if the distribution of ε is measured, starting always with the *same* status for the system (referenced here with the initial conditions of position and velocity (x_0, v_0)), the *large deviation function* of the injected power is the same as the dissipated power; conversely, in the permanent regime, that is when measurements of ε are performed the (statistical) stationarity¹ being reached, rare energy fluctuations cause a *singularity* to arise in the large deviation function of the injected power (but not in the dissipation).

These unexpected properties of the statistics of ε rise immediately the question of their generality. In the models we studied in [10], we chose for $\psi(t)$ a Gaussian white noise (i.e. $\langle \psi(t)\psi(t') \rangle = \Gamma\delta(t-t')$), to make the computations as simplest as possible. The purpose of this paper is to study generalizations of the systems previously considered, and discuss whether the properties of ε yield new concepts, possibly useful for more realistic systems.

In a first part, we recall the results obtained in [10]. Then we present some generalizations of the simple model (coloured noise, non linear friction), and discuss the properties of the large deviation functions of ε associated with them. The details of the calculations are entirely postponed in appendices.

2 Injected and dissipated power in Langevin equation

In [10], we studied the system

$$\ddot{x} + \gamma\dot{x} + V'(x) = \psi(t) \quad (4)$$

$$\langle \psi(t)\psi(t') \rangle = 2\Gamma\delta(t-t') \quad (5)$$

which was thought as an intrinsic dissipative system (driven by $\ddot{x} + \gamma\dot{x} + V'(x) = 0$) driven away from equilibrium by a Gaussian white noise. Of course, it is not the common interpretation of a Langevin equation, which describes usually the thermalization of a particle in a fluid; in that context $\gamma\dot{x}$ and ψ are two different faces of the same action. There we abandoned that reference, and in particular made no citation of the Einstein relation $\Gamma = \gamma k_B T$. We studied the statistical properties of

$$\varepsilon = \frac{1}{\tau} \int_0^\tau dt' \dot{x}(t')\psi(t') \quad (6)$$

the injected power averaged over a finite time interval of length τ . This task is *a priori* complicated, for ε involves explicitly the dynamics. Nevertheless we succeeded and computed the distribution $Prob(\varepsilon)$ for any value of τ and *any potential* (in fact any potential non repulsive at ∞). The distributions are of course potential dependent, but an prominent feature emerged in the limit of large τ . We know that at large τ , the distribution obeys the large deviation theorem:

$$\exists f, \quad \lim_{\tau \rightarrow \infty} \frac{1}{\tau} \log P(\varepsilon) = f(\varepsilon) \quad (7)$$

f is called the large deviation function associated with ε . This property is sometimes uncorrectly noted $P(\varepsilon) \sim \exp(\tau f(\varepsilon))$. We demonstrated that this large deviation function is in fact *independent* of the potential $V(x)$, what is rather unexpected, since the underlying dynamics is on the contrary very sensitive to the form of the potential. We found that if the distribution of the initial energy (i.e. at $t = 0$) is bounded, the large deviation function has the expression

$$f(\varepsilon) = -\frac{\gamma}{4\tilde{\varepsilon}}(\tilde{\varepsilon} - 1)^2 \quad \text{where } \tilde{\varepsilon} = \varepsilon/\Gamma \quad (8)$$

(the maximum of this function corresponds to $\langle \varepsilon \rangle$ and is always Γ in this model since $\langle I \rangle = \langle D \rangle = \gamma\langle \dot{x}^2 \rangle$). Interestingly enough, this expression is no longer valid if the distribution of initial energy is

¹characterized by time-independent probability distributions

unbounded, which is the case in the permanent regime since $Prob(v^2) \propto \exp(-\Gamma v^2/2\gamma)/|v|$. In that case, a phenomenon analogous to a phase transition occurs and (8) is valid only above $\tilde{\varepsilon}_c = 1/3$. Below that value, the curve is replaced by a portion of a straight line:

$$f(\varepsilon) = \begin{cases} -\frac{\gamma}{4\tilde{\varepsilon}}(\tilde{\varepsilon} - 1)^2 & \text{if } \tilde{\varepsilon} \geq 1/3 \\ -\gamma(1 - 2\tilde{\varepsilon}) & \text{if } \tilde{\varepsilon} \leq 1/3 \end{cases} \quad (9)$$

(A physical explanation of this singularity is given in [10]). Finally we computed the large deviation associated with the dissipation $\delta = \frac{1}{\tau} \int_0^\tau dt \gamma \dot{x}^2(t)$ and found that it is equal to (8), $\tilde{\delta} = \delta/\Gamma$ replacing $\tilde{\varepsilon}$.

3 Free and harmonically bounded particle driven by coloured noise

We now consider a slightly different system. The dynamics is given by

$$\ddot{x} + \gamma \dot{x} + \omega^2 x = \psi(t) \quad (10)$$

$$\langle \psi(t)\psi(t') \rangle = \frac{\Gamma}{\tau_0} \exp\left(-\frac{|t-t'|}{\tau_0}\right)$$

It resembles the previous case (with $V(x) = \omega^2 x^2/2$), except that the external force acting on the system is no longer a Gaussian white noise, but a coloured noise, an Orstein-Uhlenbeck process. The first merit of this system is that henceforth, there is no possible confusion in the interpretation of our system: the situation described by (10) does not correspond anyhow to a thermalization process. Actually, a Langevin equation describing the thermalization of a particle in a thermal bath with finite correlation time (exponentially correlated), would have involved a frictional term

$$\int_0^t du (\Gamma/k_B T \tau_0) e^{-(t-u)/\tau_0} v(u) \quad (11)$$

instead of our single γv [1].

The second merit of the system is to provide us with one dimensionless quantity $\alpha = \gamma\tau_0$ (in the free case $\omega = 0$), which is a generalization of the previous case where only one typical energy can be constructed. It allows to test the reaction of a given system under sollicitations which differ only by a characteristic time.

The details of calculations are fully postponed in the first appendix. Here we discuss only the results obtained. The large deviation function $f(\varepsilon)$ has again generically two expressions depending on the general shape of the initial velocity distribution. If this distribution is bounded at $t = 0$, the large deviation function has the expression (with the definitions $\beta = \omega\tau_0$ and $\tilde{\varepsilon} = \varepsilon/\Gamma$)

$$f(\varepsilon) = \frac{1}{16\tau_0} \left(-\frac{\tilde{\varepsilon}}{\alpha}(\alpha^2 + 1 - 2\beta^2)\theta^2(\tilde{\varepsilon}) - 6\frac{\tilde{\varepsilon}}{\alpha}(\beta^2 + \frac{\alpha}{\tilde{\varepsilon}})\theta(\tilde{\varepsilon}) + (\alpha + 1)(8 + \frac{\tilde{\varepsilon}}{\alpha}(\alpha - 1)(\alpha^2 - 1 - 4\beta^2)) \right) \quad (12)$$

$$\text{where } \theta(\varepsilon) \text{ is the largest real root of } \theta^3 - (\alpha^2 + 1 - 2\beta^2)\theta - 2(\beta^2 + \alpha/\tilde{\varepsilon}) = 0$$

An alternative and useful expression of this function is also

$$f(\varepsilon) = g(\lambda) + \lambda\varepsilon \quad \text{with } \lambda \text{ such that } g'(\lambda) = -\varepsilon$$

$$g(\lambda) = \frac{1}{2\tau_0} \left(\alpha + 1 - \hat{\theta}(\lambda) \right) \quad \text{with } \hat{\theta}(\lambda) \text{ the largest positive root of} \quad (13)$$

$$(\theta^2 - \alpha^2 - 1 + 2\beta^2)^2 - 8\beta^2\theta - 4(\beta^4 + \alpha^2 - 2\beta^2 + 4\Gamma\lambda\gamma\tau_0^2) = 0$$

(remark that g is the Legendre transformation of f). If $\omega = 0$, the previous expressions are simplified; for example we get

$$g_{\beta=0}(\lambda) = \left(\alpha + 1 - \sqrt{\alpha^2 + 1 + 2\alpha\sqrt{1 + 4\Gamma\lambda/\gamma}} \right) / (2\tau_0), \quad (14)$$

and f could be as well explicitly computed, since the inversion of $g(\lambda)$ involves roots of a third degree polynomial. Incidentally it can be checked that (13) gives the correct limit for $\tau_0 \rightarrow 0$: owing to the fact that $\theta = 1 + O(\tau_0)$, the leading order yields

$$(\theta^2 - 1)^2 = 8\beta^2 + 4(\alpha^2 - 2\beta^2 + 4\Gamma\lambda\gamma\tau_0^2) \quad (15)$$

whence $g_{\tau_0=0}(\lambda) = \frac{\gamma}{2}(1 - \sqrt{1 + 4\Gamma\lambda/\gamma})$ whatever the value of β , as established in [10].

The typical shape of that function is given on the figure 1 and is typically asymmetric. The principal characteristics of these functions are the location and the curvature of the maximum, and the nature of the divergence at $\varepsilon \rightarrow 0$ and $\varepsilon \rightarrow \infty$. The maximum of f is obtained for the average injected power $\varepsilon = \bar{\varepsilon}$ (and correspondingly $\lambda = \bar{\lambda} = 0$ since $0 = [g'(\bar{\lambda}) + \bar{\varepsilon}]d\lambda/d\varepsilon + \bar{\lambda}$),

$$\bar{\varepsilon} = \frac{\Gamma}{\alpha + 1 + \beta^2} \quad (16)$$

It is interesting to note that the presence of a confining harmonic potential is always a hindrance to injection of energy into the system. Whether this fact is absolutely general whatever the form of the (confining) potential is still an open question.

The curvature of f near the maximum $\bar{\varepsilon}$ can also be explicitly computed, for it is easy to show that $f''(\bar{\varepsilon}) = -1/g''(0)$. We obtain

$$\sigma_I^2 \equiv 1/|2f''(\bar{\varepsilon})| = \frac{\Gamma^2}{\gamma} \frac{\alpha^2 + 3\alpha + 1 + \beta^2}{(\alpha + 1 + \beta^2)^3} \quad (17)$$

This quantity is relevant as one easily measured (numerically or experimentally) besides the mean value. Incidentally, we can form with $\bar{\varepsilon}$ and σ_I^2 a characteristic energy

$$T_{curv} \equiv \frac{\sigma_I^2}{\bar{\varepsilon}} = -\frac{g''(0)}{2g'(0)} = \frac{\Gamma}{\gamma} \left(1 + \frac{\alpha - \beta^2(1 + 2\alpha) - \beta^4}{(\alpha + 1 + \beta^2)^2} \right) \quad (18)$$

The form of $\frac{\gamma}{\Gamma}T_{curv}(\alpha, \beta)$ is plotted on figure 2. We shall see that this characteristic energy T_{curv} is in a certain sense an invariant of the energy flow process (see later).

Let us look now at asymptotic branches. The $\varepsilon \rightarrow 0$ branch is characterized by a $\varepsilon^{-1/3}$ divergence:

$$f(\varepsilon) \underset{\varepsilon \rightarrow 0}{\sim} -\frac{3\gamma}{4^{4/3}\alpha^{2/3}} \left(\frac{\Gamma}{\varepsilon} \right)^{1/3} \quad (19)$$

This exponent $-1/3$ is a novelty with respect to the ‘‘pure’’ Langevin case, where the exponent was -1 . This is not very intuitive, but follows closely the decreasing of $\bar{\varepsilon}$ with α : it is probably due to the fact that $I = \psi v$ is bounded as soon as $\alpha \neq 0$, which was not the case with a white noise; this important change in the shape of $f(\varepsilon)$ shows that the white noise limit is a singular case, even for the function $f(\varepsilon)$ which is a priori related to a time-integrated observable. Besides, it shows that the short time dynamical details continue to play a role in the large deviation function. . . We can just remark that this vicinity of $\varepsilon = 0$ becomes independent of β : we can thus conjecture that this exponent is potential independent and is controlled only by the injection and dissipation mechanisms.

The other asymptotics corresponds to $\varepsilon \rightarrow \infty$, and is much more softer than the $\varepsilon = 0$ one, that is a simple linear behaviour, witness of a singularity of $g(\lambda)$ occurring for a finite negative value $\lambda = \lambda_0$. The slope of the line is homogeneous to $(\text{energy})^{-1}$. In the pure Langevin case, $f(\varepsilon) = -\gamma\Gamma((\varepsilon/\Gamma) - 1)^2/4\varepsilon$ and the slope is simply $-\gamma/4\Gamma$. We are thus naturally led to define a characteristic energy T_{slope} by asserting that the slope of the ∞ divergence is given by $-1/4T_{slope}$. From (12),

$$T_{slope} = -4\alpha^2 \frac{\Gamma}{\gamma} \left(-(\alpha^2 + 1 - 2\beta^2)\zeta^2 - 6\zeta\beta^2 + (\alpha^2 - 1)(\alpha^2 - 1 - 4\beta^2) \right)^{-1} \quad (20)$$

$$\text{where } \zeta \text{ largest solution of } \zeta^3 - (\alpha^2 + 1 - 2\beta^2)\zeta - 2\beta^2 = 0 \quad (21)$$

The figure 3 gives a plot of $\frac{\gamma}{\Gamma}T_{slope}$ as a function of α and β . The behaviour of T_{curv} and T_{slope} is qualitatively quite similar, what is not so surprising: the typical form of $g(\lambda)$ is shown on figure 4. If we try for g (in the vicinity of $\lambda = 0$) the Ansatz $g(\lambda) \approx g(\lambda_0) - C^t \sqrt{\lambda - \lambda_0}$, we deduce that would this Ansatz be exact, we would have $T_{curv} = T_{slope}$. As this Ansatz is not so bad (cf. fig. 4), we deduce $T_{curv} \approx T_{slope}$. Thus, the large deviation function $f(\varepsilon)$ has its right asymptotic branch slightly constraint by the vicinity of its maximum, but this relationship remains however relatively weak (the relative difference does not exceed 45%). Physically, the energy T_{curv} is more interesting, since it is constructed with the top of the curve, i.e. with quantities which are easy to measure/compute, which is not the case with the asymptotic tails, associated to rare events.

3.1 Singularity of f in the permanent regime

From the preceding paragraph, it is clear that unlike the white noise case, the large deviation function of ε is sensitive to the short time dynamics and to the presence of a potential. Thus, the irrelevance of $V(x)$ demonstrated in [10] is specific to the white noise case and it is probably hopeless to seek in general for such simplifications. Nevertheless, something already observed in [10] is still present here and is probably extremely general: in the permanent regime –that is, when the initial conditions are not fixed, but sampled from the stationary distribution–, the shape is altered and a negative tail appears below a certain *positive* value ε_c . This tail is simply the straight line $g(\lambda_c) + \lambda_c \varepsilon$ where λ_c corresponds to ε_c . This effect corresponds to the fact that for small values of ε , the probability of these rare events is no longer dominated by the inner dynamics (inside the time interval $[0, \tau]$) but by rare and very energetic initial conditions (cf. [10] for details). This situation is due to the fact that I and D are intimately coupled to the energy which has a conservative character. As a result, this phenomenon must also occur in the models considered here, and more generally in any case where the stationary distribution of energy is unbounded. As an example, consider the case $\beta = 0$ (for the sake of simplicity). The computation of the so-called “fluctuation term” in $\langle e^{-\lambda \tau \varepsilon} \rangle$ provides a prefactor to $\exp(-\tau g(\lambda))$ which displays a cut for $\lambda > \lambda_c = 2\gamma(2\alpha^2 + 3\alpha + 1)/\Gamma$. The corresponding value for ε is

$$\varepsilon_c = \frac{\Gamma}{\sqrt{9 + 16\alpha^2 + 24\alpha}} \frac{1}{(\alpha^2 + 1 + 2\alpha\sqrt{9 + 16\alpha^2 + 24\alpha})^{3/2}} \quad (22)$$

and tends to $1/3$ when $\tau_0 \rightarrow 0$, as expected from [10] This singularity has an interesting physical meaning: it is intimately related to the three observables (E, I, D) and their mutual dependence and balance (the stationary distribution is a property of the dynamics). The disadvantage of that phenomenon is that it is located in a region of quite rare events, and corresponds to a second order singularity: as a result, it is not easily observable. One way to make the singularity observable could be to consider “very hot” and externally controlled initial conditions, where large initial energies be likely; this will certainly shift the location of the singularity. For instance, in the pure Langevin case $\alpha = \beta = 0$ it can be shown that it shifts the singularity from $\varepsilon/\Gamma = 1/3$ to $\varepsilon/\Gamma = 1/\sqrt{3 + 2\sqrt{2}} \approx 0.414$ (in the limit of flat initial condition). Thus, this shift would be anyway limited, but could be however sufficient to make the singularity measurable. In that case, precursors of this singularity could be observable at finite τ by plotting $\frac{1}{\tau} d^2(\log P)/d\varepsilon^2$: this function must display a steep jump at the singularity location.

3.2 Characteristic time of the energy flow

Above was defined a quantity related to the curvature at the top of $f(\varepsilon)$ as $\sigma_I^2 = 1/|2f''(\bar{\varepsilon})|$. In fact σ_I^2 is also given by the following expression:

$$\sigma_I^2 = \int_0^\infty dt [\langle I(t)I(0) \rangle_{st} - \langle I \rangle_{st}^2] \quad (23)$$

where $\langle \dots \rangle_{st}$ denotes the averaging in the (out-of-equilibrium) stationary regime. To demonstrate that, let us compute $\langle \varepsilon^2 \rangle - \langle \varepsilon \rangle^2$ in the large τ limit. We have

$$P(\varepsilon) = p(\varepsilon) \cdot \exp(\tau f(\varepsilon)) \sim p(\langle \varepsilon \rangle) \cdot \exp[\tau f''(\langle \varepsilon \rangle)(\varepsilon - \langle \varepsilon \rangle)^2/2] \quad (24)$$

where $p(\varepsilon)$ is the preexponential factor of the distribution. It is readily obtained $\langle \varepsilon^2 \rangle - \langle \varepsilon \rangle^2 \sim 1/|\tau f''(\langle \varepsilon \rangle)|$. Expanding the second moment, the result is established.

In all the stochastic models we considered so far, we always noticed that large deviations functions associated with energy injection and energy dissipation were intimately related, and even equal in the vicinity of the maximum. It is easy and instructive to demonstrate this property (which is always valid) directly from the expression of σ_I^2 (we recall that the stationarity implies $\langle I \rangle_{st} = \langle D \rangle_{st}$). Indeed, we have

$$\sigma_I^2 = \int_0^\infty dt \left[\langle (\dot{E}(t) + D(t))(\dot{E}(0) + D(0)) \rangle_{st} - \langle D \rangle_{st}^2 \right] \quad (25)$$

$$= \langle (E(\infty) - E(0))(\dot{E}(0) + D(0)) \rangle_{st} + \int_0^\infty dt \left[\langle D(t)(\dot{E}(0) + D(0)) \rangle_{st} - \langle D \rangle_{st}^2 \right] \quad (26)$$

$$= \langle E \rangle \langle D \rangle - \langle E(0)D(0) \rangle_{st} + \int_0^\infty dt \left[\langle D(0)(\dot{E}(-t) + D(t)) \rangle_{st} - \langle D \rangle_{st}^2 \right] \quad (27)$$

$$= \int_0^\infty dt [\langle D(t)D(0) \rangle_{st} - \langle D \rangle_{st}^2] = \sigma_D^2 \quad (28)$$

where we made use only of time translation invariance, assumed in a stationary regime. This property allows us to define some quantities naturally associated with the energy dynamics. We already saw T_{curv} which thus can be computed from the fluctuations of the dissipation term, but the very physical interpretation of this energy is not clear up to now. Similarly, we define a characteristic time τ_e associated with the energy dynamics,

$$\tau_e = \frac{\sigma_I^2}{\langle I \rangle_{st}^2} = \frac{\sigma_D^2}{\langle D \rangle_{st}^2} \quad (29)$$

which presents also the remarkable property to be symmetric with respect to the injection and the dissipation. These considerations are extremely general and make use only of the $\dot{E} = I - D$ structure and the time translation invariance. Thus *the equation (29) is always true* in the stationary regime, and τ_e is a correlation time attached to the full energy flow process. Usually, we define the correlation time of an observable as $\tau_X = [\int_0^\infty dt \langle X(t)X(0) - \langle X \rangle^2 \rangle] / \langle X^2 - \langle X \rangle^2 \rangle$. Thus, τ_e is more or less a bare time whence τ_I and τ_D are constructed as

$$\tau_I = \frac{\langle I \rangle^2}{\langle I^2 \rangle - \langle I \rangle^2} \tau_e \quad (30)$$

(and a similar one with D). These relations are interesting, since if we imagine that τ_e is fixed (or takes it as the natural time unit), we see that at the injection level, the fluctuations level is enslaved by the characteristic time of the injection mechanism. At the other side, this balance exists with a dissipation time which is possibly quite different from the former; in that case the fluctuations are again constrained to adjust correspondingly. This simple reasoning of course does not take into account the fact that the ‘‘reference’’ time τ_e is itself a property of the established regime.

As an example, let us look again at our toy-model. Noticing that $\tau_e = T_{curv}/\bar{\varepsilon}$, we get

$$\tau_e = \frac{1}{\gamma} \frac{\alpha^2 + 3\alpha + 1 + \beta^2}{\alpha + 1 + \beta^2} \quad (31)$$

This result is independent of Γ the strength of the excitation, a fact obvious in our case, since no dimensionless number can be made with Γ . But for more complicated models a Γ dependence is to be expected *a priori*. We notice also the effect of the confining potential: the oscillation time ω only prevents the correlation time τ_e to diverge when τ_0 is large. This is obviously due to the unavoidable oscillation at frequencies near ω in the response function of x .

4 Anharmonic potentials

Unlike the pure Langevin model, the addition of an anharmonic confining potential makes the computation of $f(\varepsilon)$ untractable. Actually the mapping to an effective dynamics (as exposed

in appendix) is no longer feasible, due to a proliferation of new types of terms in the action. However, the previous analysis will be qualitatively the same, and the pdf of ε has typically the same asymmetric shape, with a right tail driven by a term $\exp(-\tau\varepsilon/T_{slope})$ (this term is in general multiplied by a prefactor $\sim \varepsilon^{-\nu}$ where ν is potential dependent [10]): figure 5(a) shows the quantity $\frac{1}{\tau} \log P(\varepsilon)$ for a model $\alpha = \beta = 1$, $\Gamma = 1$ and a double well potential $V(x) = 0.25 * x^2 - x^4$. We see that the convergence of $\log(P)/\tau$ to $f(\varepsilon)$ is quite slow, due to correction terms of order $\log(\tau)/\tau$. The other part of the figure shows the comparison of two pdf for the same value of (τ, α) , but for two different potentials (harmonic $\beta = 1$ and double-well). It clearly shows a strong dependence of $\log(P)$ (and therefore f) with respect to the potential. The positive tail is less pronounced in the nonlinear case, which is a bit surprising, since the parameter was chosen such that the curvature in the bottom of the double well is identical to the harmonic curvature. Naively, as in a double well φ^4 potential the frequency decreases with the energy we would expect a scenario similar to an effective harmonic potential with a lower well frequency, and thus, an enlargement of the rare positive events. This is not observed, thus demonstrating that probably rare energetic events correspond to scenarios where the steep external branches of the potential are often explored by the particle – very energetic oscillations have *in fine* an increasing oscillation. Finally, a second order singularity in f is also expected in the permanent regime, although difficult to observe, as easily seen in figure 5 (a): for $\tau = 50$ (whereas the typical dynamical time are of order 1), the difference $\frac{1}{\tau} \log(P) - f$ is not negligible (cf the maximum of the curve), and several millions of statistic steps are however unable to sample the vicinity of $\varepsilon = 0$. To conclude, let us mention that a variational approach could be performed on these nonlinear models: using $\langle e^X \rangle \geq e^{\langle X \rangle}$, we could compute for each λ the best harmonic potential describing the dynamics.

5 Nonlinear friction

There is another way to generalize the original Langevin equation, namely to change the dissipation term γv to a nonlinear term $\varphi(v)$, where φ is an odd function, positive for positive v . It is worth looking at these models, since we can that way test the influence of the dissipation term on the shape of $P(\varepsilon)$. As an example, let us consider the simplest case

$$\dot{v} + \varphi(v) = \psi(t) \quad \text{with} \quad \langle \psi(t)\psi(t') \rangle = 2\Gamma\delta(t-t') \quad (32)$$

It is shown in the appendix that the function $g(\lambda)$ can be expressed in terms of the lowest energy of a Schrödinger equation:

$$g(\lambda)/Ga = - \text{Min}_{\langle \zeta | \zeta \rangle = 1} \left\langle \zeta \left| -\frac{\partial^2}{\partial v^2} + \frac{1}{4\Gamma^2} (\varphi^2(v) - 2\Gamma\varphi'(v) + 4\Gamma\lambda v\varphi(v)) \right| \zeta \right\rangle \quad (33)$$

whence we get

$$f(\varepsilon)/\Gamma = - \text{Max}_{\substack{\langle \zeta | v\varphi | \zeta \rangle = \varepsilon\Gamma \\ \langle \zeta | \zeta \rangle = 1}} \left\langle \zeta \left| -\frac{\partial^2}{\partial v^2} + \frac{1}{4\Gamma^2} (\varphi^2(v) - 2\Gamma\varphi'(v)) \right| \zeta \right\rangle \quad (34)$$

Thus, the large deviation function comes in this case from an extremalization principle, but this fact is specific to the situation considered. The inspection of (33) is instructive, for we see immediately that if $\varphi(v)$ diverges faster than v , $g(\lambda)$ has no longer a singularity at finite λ . As a result, the right asymptotic tail of $f(\varepsilon)$ is no longer a straight line: the precise form of this asymptotics depends on the dissipation efficiency. And we find obviously that if $\varphi(v)/v \rightarrow \infty$, the right tail of $f(\varepsilon)$ goes faster to $-\infty$. To be more quantitative, if $\varphi(v) \sim v^\nu$ for large v (with $\nu > 1$), we can show by considering the effective potential in (33) that $f(\varepsilon) \sim -\varepsilon^{2\nu/(\nu+1)}$. Does it affect the left asymptotics as well? To answer this question, the $\lambda \rightarrow +\infty$ limit must be investigated. The effective potential is deeply changed by this change of sign and is now single well and is equivalent to $\lambda v\varphi(v)$. As a result we expect a minimum energy scaling like $\lambda^{1/2}$ [14] (provided $\varphi(v) \sim v$ near zero). This leads to a divergence ε^{-1} near zero, whatever the precise form of $\varphi(v)$. However the condition $\varphi \propto v$ near $v = 0$ is crucial, otherwise the left tail of f is affected: if for instance $\varphi \sim v^3$, the zero point energy of the Schrödinger potential scales like $\lambda^{1/3}$ and we find $f(\varepsilon) \sim \varepsilon^{-1/2}$.

This example is quite interesting, since it shows that the dissipation mechanism can affect deeply the general shape of $f(\varepsilon)$, and particularly the form of the asymptotic tails. In particular, the behaviour of $\varphi(v)$ for small velocities dictates the form of $f(\varepsilon)$ near $\varepsilon = 0$, whereas correspondingly the large v behaviour of φ is responsible for the large ε value. Thus, for realistic situations, a study of the asymptotic tails of the ε distribution could provide an estimation of an effective (in average) dissipation/energy relation: in a turbulent flow for instance, the greater the energy present in the system, the greater the dissipation; we argue that the implicit relation between these observables is partly encoded in the asymptotic tails of $f(\varepsilon)$.

6 Conclusion

In this paper, we studied the injected power distributions of several stochastic models, in order to precise the results obtained in [10] and test their possible generality. This study showed that, in general, the large deviation functions remain sensitive to the details of the short-time dynamics of the system, despite the time averaging: this observation makes the Langevin-like models very peculiar, since in these cases the large deviation function is insensitive to the presence of a pinning potential. However, the common scenario of dissipative systems in a permanent regime makes all these to share some common characteristics: first, they are strongly asymmetric, which reflects the fact that we follow a positive conserved quantity, with a systematic dissipation; secondly, they display a second order singularity in the permanent regime which constructs a negative tail in the large deviation function associated to the injected power (this singularity is not present in the dissipation, since the dissipation is always positive). Third, the large deviation functions of injected and dissipated power have the same curvature at their maximum, due to the structure of the energy equation. We can even conjecture that in general the two large deviation functions are strictly equal in a vicinity of the maximum (it is easy to show it for the models presented here, but probably often true). From this equality a characteristic time (or a characteristic energy) naturally associated to the energy flow can be defined. It would be interesting to measure that time in an extended dissipative system, when the system is chaotic and generates itself its disorder and the noisy character of the energy injection (instead of being imposed by the operator): the typical time scale is in that case controlled by the dynamics of the system itself (for instance, what is the behaviour of τ_e near an instability threshold ?) . More generally, it would be worth now performing similar analysis to extended systems where non equilibrium stationary states can produce structurations of the system.

7 Appendix A: large deviation function for the non markovian system

We derive in this appendix the main results of section 3. We consider the injected power averaged over a time interval τ :

$$\varepsilon = \frac{1}{\tau} \int_0^\tau dt \psi(t) v(t) \quad (35)$$

for the model (10). As τ is finite, this observable remains a fluctuating quantity, and we examine its probability distribution function $\pi(\varepsilon)$. More precisely, we are interested in the computation of the so-called *large deviation function*, a function which emerges from the consideration of large values of τ . It can be proved that for large values of τ the probability of ε verifies

$$\log \pi(\varepsilon) \underset{\tau \rightarrow \infty}{\sim} \tau f(\varepsilon) \quad (36)$$

(this equivalence is sometimes boldly noted $\pi(\varepsilon) \sim e^{\tau f(\varepsilon)}$, understanding the prefactor). The function f is nothing but the large deviation function associated with ε . We concentrate ourselves on this sole quantity, since the prefactor is quite more involved to evaluate, and is physically much less interesting.

For convenience, we are going to compute two different types of pdf, namely $\pi_{\mathbf{w}_0}(\varepsilon)$ the probability of ε , *knowing that the initial conditions of the process are* $\mathbf{w}_0 = (x(0) = x_0, v(0) = v_0, \psi(0) = \psi_0)$, and $\pi(\varepsilon)$ the probability of ε in the permanent (stationary) regime. The latter is simply related to the former through

$$\pi(\varepsilon) = \int dx_0 dv_0 d\psi_0 P_{st}(x_0, v_0, \psi_0) \pi_{\mathbf{w}_0}(\varepsilon) \quad (37)$$

where P_{st} is the stationary distribution of the (correlated) variables (x, v, ψ) . As π and $\pi_{\mathbf{w}_0}$ are different, their large deviation functions are termed $f_{\mathbf{w}_0}$ and f respectively henceforth.

7.1 Path integral representation of the characteristic function

The characteristic function of $\pi_{\mathbf{w}_0}(\varepsilon)$ is defined by

$$\langle e^{-\lambda\tau\varepsilon} \rangle_{\mathbf{w}_0} = \int d\varepsilon \pi_{\mathbf{w}_0}(\varepsilon) \exp(-\lambda\tau\varepsilon) \quad (38)$$

As soon as this function is known, the original pdf can be retrieved via an inverse Fourier transform (in the complex λ variable). At the level of the large deviation function, equation (38) gives a rapid answer: there exists a function $g_{\mathbf{w}_0}(\lambda)$ such that

$$\langle e^{-\lambda\tau\varepsilon} \rangle_{\mathbf{w}_0} \underset{\tau \rightarrow \infty}{\sim} \exp[\tau g_{\mathbf{w}_0}(\lambda)] \quad (39)$$

(in the bold sense !), and this function is given by the Legendre transform of $f_{\mathbf{w}_0}$:

$$g_{\mathbf{w}_0}(\lambda) = f_{\mathbf{w}_0}(\varepsilon) - \lambda\varepsilon \quad (40)$$

$$f'_{\mathbf{w}_0}(\varepsilon) = \lambda \quad (41)$$

(the inversion is simply: $f_{\mathbf{w}_0} = g_{\mathbf{w}_0} + \lambda\varepsilon, g'_{\mathbf{w}_0}(\lambda) = -\varepsilon$). All that is a priori correct, but for a point: it is possible that the prefactor of $e^{\tau g}$ in the characteristic function diverges for a particular value of λ , and that therefore the function be not defined for certain values of λ , *whereas the function g remains defined in this range*. In that case, the rapid Legendre inversion above mentioned must be replaced by a careful analysis of the integral (38) (see [10] and below).

Let us give now a path integral representation of $\langle e^{-\lambda\tau\varepsilon} \rangle_{\mathbf{w}_0}$. It is quite easy to do that in our case, since the noise ψ is nothing but a Ornstein-Uhlenbeck, that is a process described by an ordinary Langevin equation:

$$\dot{\psi} + \tau_0^{-1}\psi = \zeta(t) \quad (42)$$

$$\langle \zeta(t)\zeta(t') \rangle = 2\frac{\Gamma}{\tau_0^2}\delta(t-t') \quad (43)$$

where ζ is a Gaussian white noise. The path-integral representation of the propagator of this process is known [10, 15, 16, 17] :

$$P(\psi_1, \tau | \psi_0, 0) = e^{\frac{\tau}{2\tau_0}} \times \int_{\psi(0)=\psi_0}^{\psi(\tau)=\psi_1} [\mathcal{D}\psi] \exp\left(-\frac{1}{4\Gamma\tau_0^{-2}} \int_0^\tau dt (\dot{\psi} + \tau_0^{-1}\psi)^2\right) \quad (44)$$

As a result, the statistical weight of a particular occurrence of the noise, knowing that it originates from ψ_0 at $t = 0$ is given by

$$e^{\frac{\tau}{2\tau_0}} \times \exp\left(-\frac{1}{4\Gamma} \int_0^\tau dt (\tau_0\dot{\psi} + \psi)^2\right) \quad (45)$$

Therefore, we get the probability of a particular occurrence of the process x (beginning at \mathbf{w}_0) as

$$\begin{aligned} Prob \left[[x(t)], t \in [0, \tau] \mid x(0) = x_0, v(0) = v_0, \dot{v}(0) = \psi_0 - \gamma v_0 - V'(x_0) \right] \\ = e^{\frac{\tau}{2\tau_0}(1+\alpha)} \times \exp\left(-\frac{1}{4\Gamma} \int_0^\tau dt [\tau_0(\ddot{v} + \gamma\dot{v} + vV''(x)) + (\dot{v} + \gamma v + V'(x))]^2\right) \quad (46) \end{aligned}$$

where we defined $\alpha = \gamma\tau_0$, and $V(x) = \omega^2 x^2/2$. It must be noted that the jacobian of the transformation $\psi \rightarrow x$ is equal to $\exp(\gamma\tau/2)$ and is taken into account in the preceding expression.

From now on, it is easy to deduce an expression for $\langle e^{-\lambda\tau\varepsilon} \rangle_{\mathbf{w}_0}$, since the distribution of individual paths is at hand:

$$\begin{aligned} \langle e^{-\lambda\tau\varepsilon} \rangle_{\mathbf{w}_0} &= e^{\frac{\tau}{2\tau_0}(1+\alpha)} \\ &\times \int_{\mathbf{w}(0)=\mathbf{w}_0} [\mathcal{D}x] \exp \left(-\frac{1}{4\Gamma} \int_0^\tau dt [\tau_0(\ddot{v} + \gamma\dot{v} + vV'') + (\dot{v} + \gamma v + V')]^2 - \lambda \int_0^\tau dt v(\dot{v} + \gamma v + V') \right) \end{aligned} \quad (47)$$

(with obviously $v(t) = \dot{x}(t)$). It is convenient to express explicitly the final values of ψ and (x, v) :

$$\begin{aligned} \langle e^{-\lambda\tau\varepsilon} \rangle_{\mathbf{w}_0} &= \int d\mathbf{w}_1 e^{\frac{\tau}{2\tau_0}(1+\alpha) - \lambda(E_1^2 - E_0^2)} \\ &\times \int_{\mathbf{w}(0)=\mathbf{w}_0}^{\mathbf{w}(\tau)=\mathbf{w}_1} [\mathcal{D}v] \exp \left(-\frac{1}{4\Gamma} \int_0^\tau dt [\tau_0(\ddot{v} + \gamma\dot{v} + vV''(x)) + (\dot{v} + \gamma v + V'(x))]^2 - \lambda\gamma \int_0^\tau dt v^2 \right) \end{aligned} \quad (48)$$

where $E \equiv \frac{1}{2}v^2 + V(x)$. The essential point of the derivation is now the possibility to find four constants $U, \hat{\gamma}, \hat{\tau}_0, \hat{\omega}$, such that the difference

$$\begin{aligned} \Delta &= \int_0^\tau dt ([\tau_0(\ddot{v} + \gamma\dot{v} + \omega^2 v) + (\dot{v} + \gamma v + \omega^2 x)]^2 + 4\Gamma\lambda\gamma v^2) \\ &\quad - U \int_0^\tau dt [\hat{\tau}_0(\ddot{v} + \hat{\gamma}\dot{v} + \hat{\omega}^2 v) + (\dot{v} + \hat{\gamma}v + \hat{\omega}^2 x)]^2 \end{aligned} \quad (49)$$

implies only boundary terms. If we choose $(U, \hat{\gamma}, \hat{\tau}_0, \hat{\omega})$ fulfilling (we define of course $\hat{\alpha} \equiv \hat{\tau}_0\hat{\gamma}$, and $\beta = \tau_0\omega, \hat{\beta} = \hat{\tau}_0\hat{\omega}$)

$$\tau_0^2 = U\hat{\tau}_0^2 \quad (50)$$

$$\alpha^2 + 1 - 2\beta^2 = U(\hat{\alpha}^2 + 1 - 2\hat{\beta}^2) \quad (51)$$

$$\frac{1}{\tau_0^2}(\beta^4 + \alpha^2 - 2\beta^2) + 4\Gamma\lambda\gamma = \frac{U}{\hat{\tau}_0^2}(\hat{\beta}^4 + \hat{\alpha}^2 - 2\hat{\beta}^2) \quad (52)$$

$$\frac{\beta^4}{\tau_0^4} = U\frac{\hat{\beta}^4}{\hat{\tau}_0^4}, \quad (53)$$

it completes the goal and we can write

$$\begin{aligned} \langle e^{-\lambda\tau\varepsilon} \rangle_{\mathbf{w}_0} &= \int d\mathbf{w}_1 e^{\frac{\tau}{2\tau_0}(1+\alpha) - \frac{\lambda}{2}(E_1^2 - E_0^2) - \frac{\Delta}{4\Gamma}} \\ &\times \int_{\mathbf{w}(0)=\mathbf{w}_0}^{\mathbf{w}(\tau)=\mathbf{w}_1} [\mathcal{D}x] \exp \left(-\frac{U}{4\Gamma} \int_0^\tau dt [\hat{\tau}_0(\ddot{v} + \hat{\gamma}\dot{v} + \hat{\omega}^2 v) + (\dot{v} + \hat{\gamma}v + \hat{\omega}^2 x)]^2 \right) \end{aligned} \quad (54)$$

The very reason we made this transformation is that the remaining path integral is closely related to the propagator of a “renormalized” process of type (10), where $(\Gamma, \tau_0, \gamma, \omega) \rightarrow (\Gamma/U, \hat{\tau}_0, \hat{\gamma}, \hat{\omega})$:

$$\int_{\mathbf{w}(0)=\mathbf{w}_0}^{\mathbf{w}(\tau)=\mathbf{w}_1} [\mathcal{D}x] \exp \left(-\frac{U}{4\Gamma} \int_0^\tau dt [\hat{\tau}_0(\ddot{v} + \hat{\gamma}\dot{v} + \hat{\omega}^2 v) + (\dot{v} + \hat{\gamma}v + \hat{\omega}^2 x)]^2 \right) = e^{-\frac{\tau}{2\tau_0}(1+\hat{\alpha})} \times \hat{P}(\mathbf{w}_1, \tau | \mathbf{w}_0, 0) \quad (55)$$

As a result,

$$\langle e^{-\lambda\tau\varepsilon} \rangle_{\mathbf{w}_0} = \exp \left(\tau \left[\frac{1+\alpha}{2\tau_0} - \frac{1+\hat{\alpha}}{2\hat{\tau}_0} \right] \right) \times \int d\mathbf{w}_1 e^{-\frac{\lambda}{2}(E_1^2 - E_0^2) - \frac{\Delta}{4\Gamma}} \times \hat{P}(\mathbf{w}_1, \tau | \mathbf{w}_0, 0) \quad (56)$$

The remaining integral has a nice behaviour for large τ , since $\widehat{P}(\mathbf{w}_1, \tau | \mathbf{w}_0, 0)$ tends to its equilibrium value $\widehat{P}_{st}(\mathbf{w}_1)$. Thus, we get the veritable equivalence

$$\langle e^{-\lambda\tau\varepsilon} \rangle_{\mathbf{w}_0} \sim \exp\left(\tau \left[\frac{1+\alpha}{2\tau_0} - \frac{1+\widehat{\alpha}}{2\widehat{\tau}_0} \right]\right) \times \int d\mathbf{w}_1 e^{-\frac{\lambda}{2}(E_1^2 - E_0^2) - \frac{\lambda}{4\Gamma}} \times \widehat{P}_{st}(\mathbf{w}_1) \quad (57)$$

whence we extract the function $g_{\mathbf{w}_0}$

$$g_{\mathbf{w}_0}(\lambda) = \frac{1}{2} \left(\frac{1+\alpha}{2\tau_0} - \frac{1+\widehat{\alpha}}{2\widehat{\tau}_0} \right) \quad (58)$$

It is worth noticing that the subscript \mathbf{w}_0 is useless, for the function $g_{\mathbf{w}_0}$ is independent of \mathbf{w}_0 ; we abandon henceforth it (for g) in the following.

Let us define $\widehat{\theta} = (1 + \widehat{\alpha})\tau_0/\widehat{\tau}_0$. We can verify that $\widehat{\theta}$ is the largest positive root of

$$(\theta^2 - \alpha^2 - 1 + 2\beta^2)^2 - 8\beta^2\theta - 4(\beta^4 + \alpha^2 - 2\beta^2 + 4\Gamma\lambda\gamma\tau_0^2) = 0 \quad (59)$$

when it exists; if not, $\langle e^{-\lambda\tau\varepsilon} \rangle$ diverges. It proves the result (13).

Similarly, We get the characteristic function of the process in the permanent regime by simply integrating over \mathbf{w}_0 :

$$\langle e^{-\lambda\tau\varepsilon} \rangle \sim \exp(\tau g(\lambda)) \times \int d\mathbf{w}_1 d\mathbf{w}_0 e^{-\frac{\lambda}{2}(E_1^2 - E_0^2) - \frac{\lambda}{4\Gamma}} \times \widehat{P}_{st}(\mathbf{w}_1) P_{st}(\mathbf{w}_0) \quad (60)$$

(Note that the stationary distributions P are not the same for \mathbf{w}_0 and \mathbf{w}_1 , since \widehat{P} refers to the renormalized process). We have to remark that the same function g is also shared by the stationary process. But it does *not* imply that the associated $f_{\mathbf{w}_0}$ and f functions will be the same.

7.2 Analytical properties of characteristic functions

To derive the large deviation functions f , it is required to look the analytical properties of the prefactor, since the latter can lessen the analyticity domain of the characteristic function with respect to that implied by the sole inspection of g .

A priori, the function g is analytical over the whole complex $\widetilde{\lambda}$ space, except on a cut located on (but in general not equal to) \mathbb{R}^- . This cut begins when the root $\widehat{\theta}$ disappears. In this paragraph, we considers only the free case $\omega = 0$, for the sake of simplicity. But the discussion remains valid for the bounded case as well.

Let us first consider the non stationary situation: here, for a fixed value of \mathbf{w}_0 , the prefactor is composed by functions as regular as g itself (g has a cut on $\lambda < \lambda_- = -\gamma/4\Gamma$) times an integral over \mathbf{w}_1 . Thus, limitations to analyticity could arise if for some values of λ , this integral no longer converges. A lengthy computation gives this integral (when converging) proportional to

$$\left[\left([\sqrt{1 + \alpha^2 + 2\alpha\eta} + \alpha]^2 - 1 \right) \sqrt{\sqrt{1 + \alpha^2 + 2\alpha\eta} - \alpha + 1} \right]^{-1} \quad (61)$$

(with $\eta = \sqrt{1 + 4\Gamma\lambda/\gamma}$) which again is as regular as g itself. It is concluded that for the non stationary case, the prefactor cannot hinder the Legendre inversion, and that $f_{\mathbf{w}_0}$ is really given by the inverse Legendre of g .

Quite different is the stationary case, since now two integrals have to converge. Of that over \mathbf{w}_1 we already proved the analyticity over $\mathbb{C} \setminus \mathbb{R}^-$. That implying \mathbf{w}_0 is completely different, and another lengthy computation gives it proportional to

$$\left(\sqrt{1 + \alpha^2 + 2\alpha\eta} + \alpha + 1 \right)^{-1} \left((\eta + 1)(3\alpha + 1 - \sqrt{1 + \alpha^2 + 2\alpha\eta}) \right)^{-1/2} \quad (62)$$

This time, a new cut appears, since for $\lambda > \lambda_c = 2(2\alpha^2 + 3\alpha + 1)\gamma/\Gamma$, the preceding expression is not defined, which expresses the fact that the integral over \mathbf{w}_0 is not defined. This extra cut has deep consequences on the large deviation function f .

7.3 Large deviation functions

For the nonstationary case, the usual rule applies, that is f is the Legendre transform of g :

$$\begin{aligned} f_{\mathbf{w}_0}(\varepsilon) &= g(\lambda) + \lambda\varepsilon \\ g'(\lambda) &= -\varepsilon \end{aligned} \quad (63)$$

For the stationary case, this expression is not always valid: (63) must give $\lambda < \lambda_c$ otherwise there is a problem. In that case, the inverse Laplace transform $\lambda \rightarrow \varepsilon$ must be lead carefully [10] but the result is quite simple: if no allowed λ fulfills (63), the large deviation function for the injection considered is simply $f(\varepsilon) = g(\lambda_c) + \lambda_c\varepsilon$, therefore a portion of a straight line. Thus, we have

$$\begin{aligned} f(\varepsilon) &= f_{\mathbf{w}_0}(\varepsilon) \quad \text{if } \varepsilon > \varepsilon_c \\ f(\varepsilon) &= g(\lambda_c) + \lambda_c\varepsilon \quad \text{if } \varepsilon < \varepsilon_c \end{aligned} \quad (64)$$

and ε_c defined as $g'(\lambda_c) = -\varepsilon_c$.

7.4 Nonlinear potentials

For a general potential $V(x)$, the expression (48) remains true. If V is not harmonic, the renormalization procedure is no more possible, since the action involves, besides boundary terms, the terms

$$\frac{1}{4\Gamma} \left(\tau_0^2 \ddot{v}^2 - \dot{v}^2 (\tau_0^2 V'' - \alpha^2 - 1) + V'^2 + v^2 (\tau_0^2 V''^2 + \gamma^2 - V'') - \frac{\gamma \tau_0^2}{2} v^3 V''' + \frac{\tau_0^2 v^4}{3} V^{(4)} \right) \quad (65)$$

Thus, there is a ‘‘proliferation’’ of new terms which cannot be balanced by a simple redefinition of the constants. In fact, it seems that the integrability of the problem is restricted to the linear case.

8 Appendix B: nonlinear friction

Here we sketch the computations pertaining to the section 5. From the dynamical equation (32),

$$\langle e^{-\lambda\tau\varepsilon} \rangle = \int [\mathcal{D}\psi] \exp \left(-\frac{1}{4\Gamma} \int_0^\tau (\dot{v} + \varphi(v))^2 - \lambda \int_0^\tau (\dot{v} + \varphi(v))v \right) \quad (66)$$

This integral is a sum over different realizations of ψ . To write it as an integral over realizations of v , care must be taken of the fact that the Jacobian of the transformation is $\frac{1}{2} \int_0^\tau \varphi'(v) dt$. Then

$$\langle e^{-\lambda\tau\varepsilon} \rangle_{v_0} = \int dv_1 e^{-\frac{\lambda}{2}(v_1^2 - v_0^2)} \int_{v(0)=v_0}^{v(\tau)=v_1} [\mathcal{D}v] \exp \left(-\frac{1}{4\Gamma} \int_0^\tau [(\dot{v} + \varphi(v))^2 - 2\Gamma\varphi'(v) + 4\Gamma\lambda v\varphi(v)] \right) \quad (67)$$

The idea is the same as for the linear friction cases: we seek for a function $\xi(v)$ and a constant Q such that for all v

$$\varphi(v)^2 - 2\Gamma\varphi'(v) + 4\Gamma\lambda v\varphi(v) = \xi^2(v) - 2\Gamma\xi'(v) + Q \quad (68)$$

with the ‘‘initial condition’’ $\xi(0) = 0$. This is a Riccati equation, and the standard transformation $\xi(v) = -2\Gamma\zeta'(v)/\zeta(v)$ leads to a Schrödinger equation:

$$-4\Gamma^2\zeta'' + [\varphi^2 - 2\Gamma\varphi' + 4\Gamma\lambda v\varphi]\zeta - Q\zeta = 0 \quad (69)$$

Owing to the fact that the potential $\varphi^2 - 2\Gamma\varphi' + 4\Gamma\lambda v\varphi$ is even, that ζ never vanishes (since ξ must be defined $\forall v$), we deduce that Q is necessarily the lowest eigenvalue of the Schrödinger problem and ζ the associated eigenfunction.

Following the trick of the previous Appendix, we easily deduce that $g(\lambda) = -Q(\lambda)/4\Gamma$, that is, using variational properties of Schrödinger eigenvalues:

$$g(\lambda)/\Gamma = - \operatorname{Min}_{\int \zeta^2=1} \int \left[\zeta'^2 + \frac{1}{4\Gamma^2}(\varphi^2 - 2\Gamma\varphi' + 4\Gamma\lambda v\varphi)\zeta^2 \right] dv \quad (70)$$

$$= - \operatorname{Min}_{\langle \zeta | \zeta \rangle=1} \left\langle \zeta \left| -\frac{\partial^2}{\partial v^2} + \frac{1}{4\Gamma^2}(\varphi^2 - 2\Gamma\varphi' + 4\Gamma\lambda v\varphi) \right| \zeta \right\rangle \quad (71)$$

To get f , we must take the Legendre inversion of this formula: $f(\varepsilon) = \min_{\lambda} [g(\lambda) + \lambda\varepsilon]$. As $g'(\lambda) = -\varepsilon$, a simple transformation yields:

$$f(\varepsilon)/\Gamma = - \operatorname{Max}_{\langle \zeta | v\varphi | \zeta \rangle / \langle \zeta | \zeta \rangle = \varepsilon} \left\langle \zeta \left| -\frac{\partial^2}{\partial v^2} + \frac{1}{4\Gamma^2}(\varphi^2 - 2\Gamma\varphi') \right| \zeta \right\rangle \quad (72)$$

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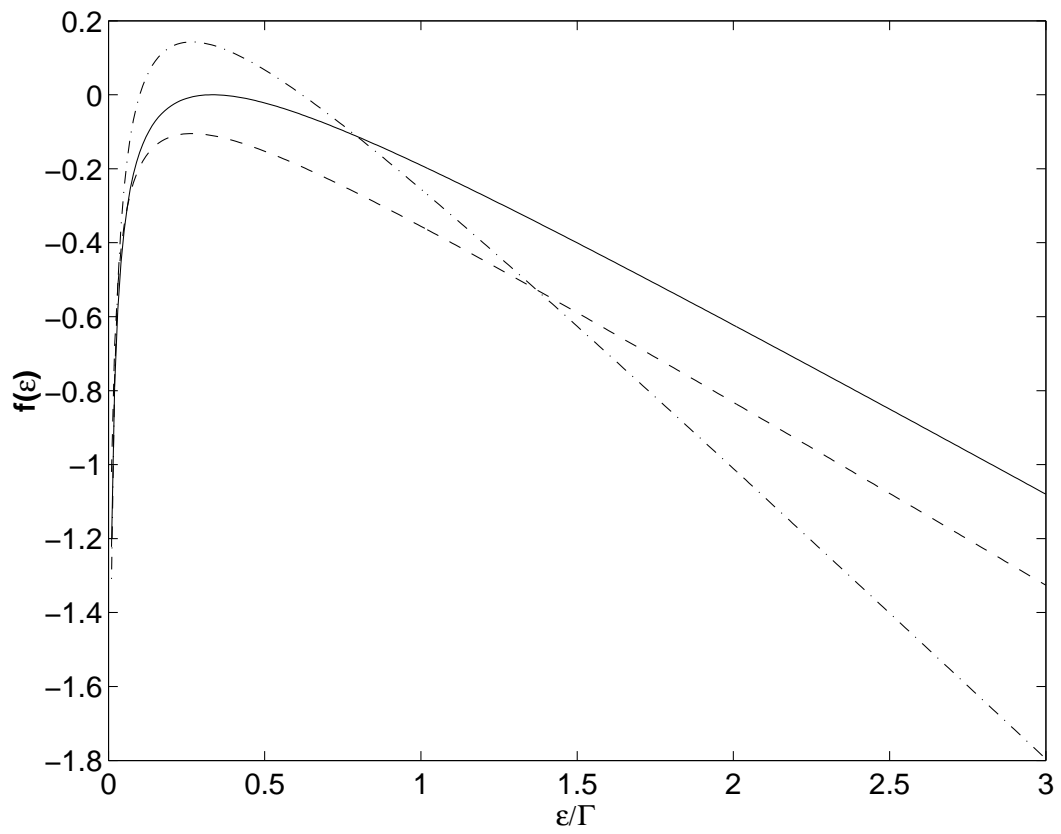


Figure 1: Typical shape of $f(\tilde{\varepsilon})$ for $\alpha = 0.5, 1, 2$ and $\beta^2 = (\alpha^2 + 1)/2$ (respectively dashed, plain, dash-dotted)

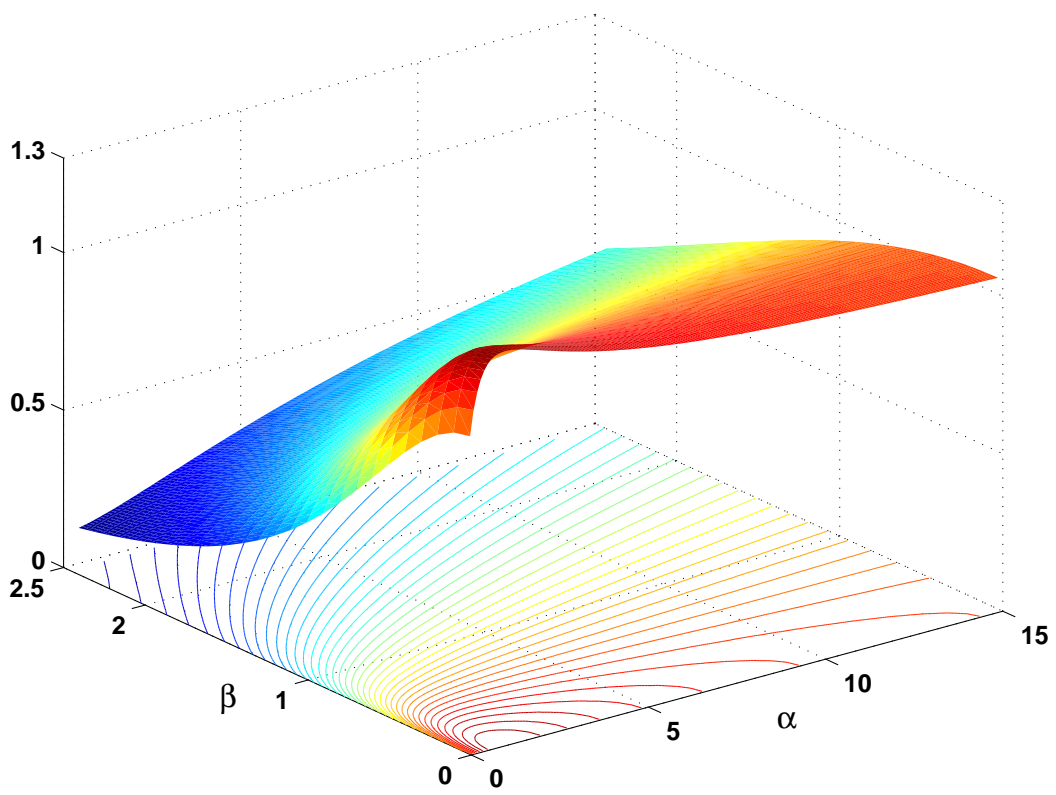


Figure 2: Surface plot of $T_{curv}(\alpha, \beta)$. We took $\Gamma/\gamma = 1$ for convenience.

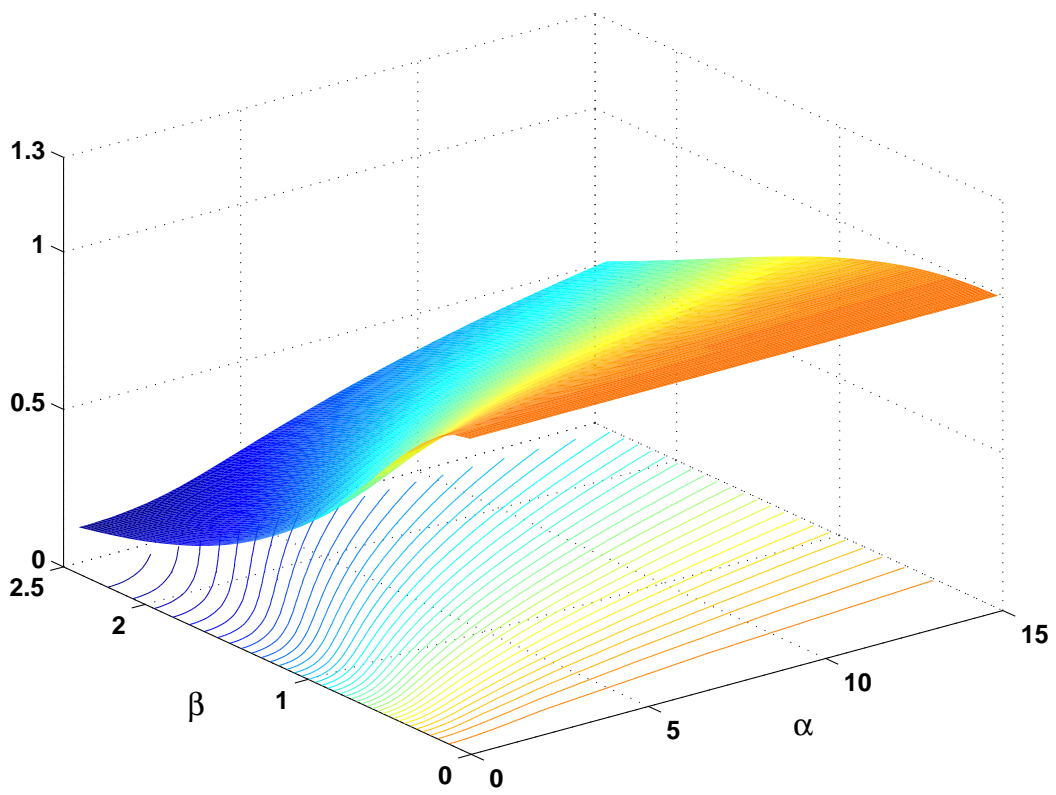


Figure 3: The energy T_{slope} associated with the $\varepsilon \rightarrow \infty$ divergence. We took $\Gamma/\gamma = 1$ for convenience.

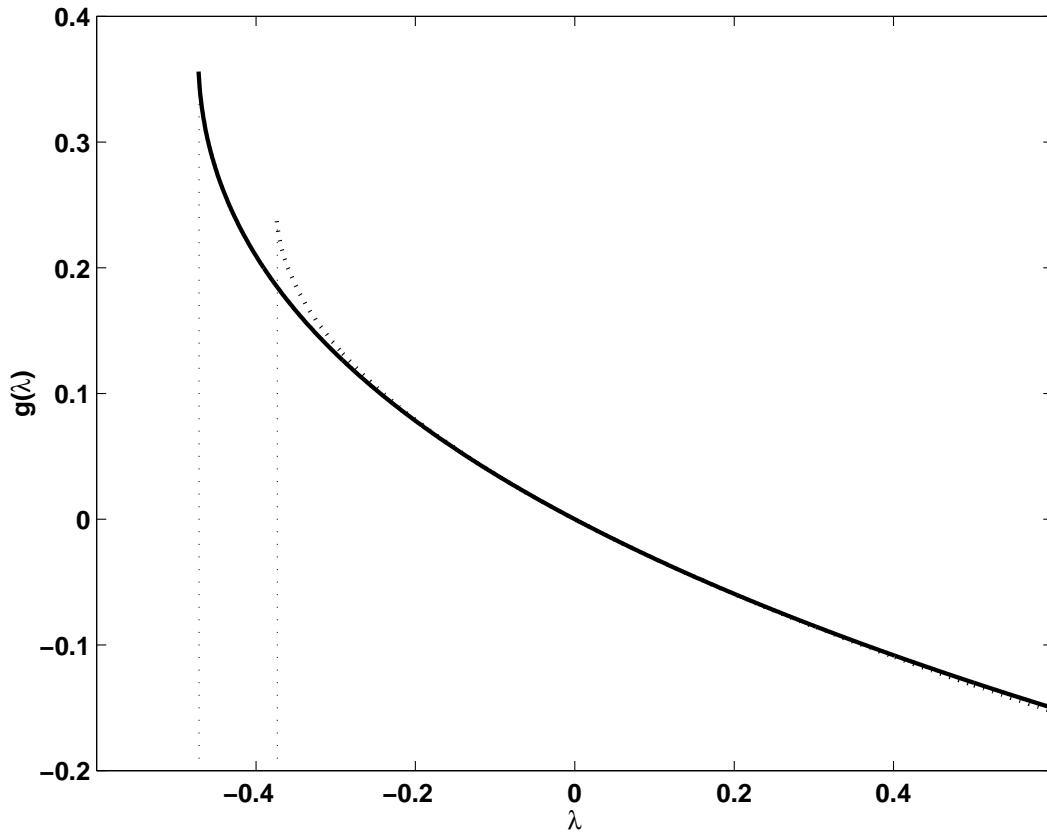


Figure 4: Exact function $g(\lambda)$ ($\alpha = \beta = 1$) together with the parabolic Ansatz (dots) based on T_{curv} (cf. text for details)

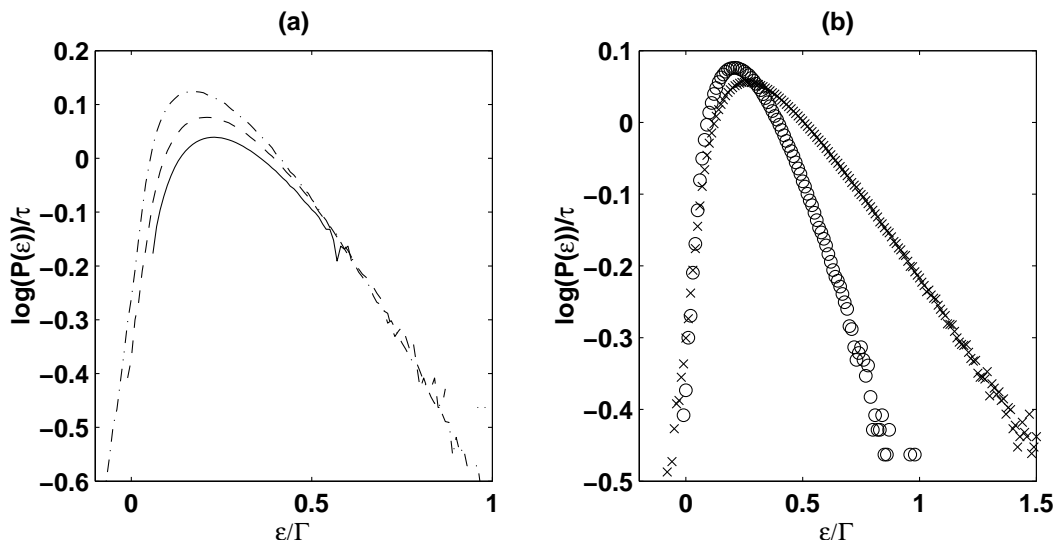


Figure 5: Typical shape of the distribution $P(\varepsilon)$ for a double well confining potential (See details in text). (a): $P(\varepsilon)$ for different values of τ ($\tau = 10$, dash-dotted, $\tau = 20$, dashed, $\tau = 50$, plain). (b): for $\tau = 20$, comparison between double well potential case (circles) and harmonic potential (crosses)