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# Modular representations of cyclotomic Hecke algebras of type $G(r, p, n)$

Gwenaëlle Genet\* and Nicolas Jacon†

## Abstract

We give a classification of the simple modules for the cyclotomic Hecke algebras over  $\mathbb{C}$  in the modular case. We use the unitriangular shape of the decomposition matrices of Ariki-Koike algebras and Clifford theory.

## 1 Introduction

Let  $r$ ,  $p$  and  $n$  be integers such that  $p$  divides  $r$ . The complex reflection group of type  $G(r, p, n)$  is defined to be the groups of  $n \times n$  permutation matrices such that the entries are either 0 or  $r^{\text{th}}$  roots of unity and the product of the nonzero entries is a  $\frac{r}{p}$ -root of unity.

In [2] and [5], Ariki and Broué-Malle have defined a Hecke algebra associated to each complex reflection group  $G(r, p, n)$ . It can be seen as a deformation of the algebra of the group  $G(r, p, n)$ . According to a conjecture of Broué and Malle, such algebras, called cyclotomic Hecke algebras, should occur as endomorphism algebras of the Lusztig induced character. The representation theory of cyclotomic Hecke algebras of type  $G(r, 1, n)$  (also known as Ariki-Koike algebras) is beginning to be well understood. They are cellular algebras, the simple modules have been classified in both semi-simple and modular cases and the decomposition matrices are known in characteristic 0 (see [21] for a survey of these results).

In [2], Ariki has shown that a Hecke algebra of type  $G(r, p, n)$  can be considered as the 0-component of a graded system for a Hecke algebra of type  $G(r, 1, n)$  with a special choice of parameters. As a consequence, in the semi-simple case, he has given a complete set of non isomorphic simple modules by using Clifford theory. In the modular case, partial results have been obtained by Hu in [17] (see also [16] and [18] where the case  $r = p = 2$  which corresponds to Hecke algebras of type  $D_n$  is studied). The main problem is that the restrictions of the simple modules for non semi-simple Ariki-Koike algebras are much more complicated to describe than for semi-simple ones.

The purpose of this paper is precisely to give a parametrization of the simple modules for the non semi-simple cyclotomic Hecke algebras of type  $G(r, p, n)$

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over  $\mathbb{C}$ . It comes from the “canonical basic set” introduced by Geck and Rouquier in [13] and extended by the second author in [20]. This set induces a parametrization of the simple modules of non semi-simple Ariki-Koike algebras by some FLOTW multipartitions (this kind of multipartitions has been defined by Foda, Leclerc, Okado, Thibon and Welsh in [9]). We proceed in an analogous way than in [14, Theorem 2.1], using the unitriangular shape of the decomposition matrices of Ariki-Koike algebras and Clifford theory.

## 2 Cyclotomic Hecke algebras, Clifford theory

### 2.A Cyclotomic Hecke algebras

Let  $r, p$  and  $n$  be integers such that  $p$  divides  $r$ . We denote  $d := \frac{r}{p}$ . Let  $R$  be an integral ring,  $q$  a unit of  $R$  and a sequence  $x = (x_1, \dots, x_d)$  of elements in  $R$ . Denote  $\mathfrak{H}_{r,p,n}^{q,x}(R)$ , the Hecke  $R$ -algebra associated to the complex reflection group  $G(r, p, n)$ , for the parameters  $q, x$ , defined by the following presentation:

$$\begin{aligned}
\text{generators : } & a_0, \dots, a_n, \\
\text{relations : } & (a_0 - x_1) \dots (a_0 - x_d) = 0 \\
& (a_i - q)(a_i + 1) = 0, \quad 1 \leq i \leq n, \\
& a_1 a_3 a_1 = a_3 a_1 a_3, \\
& a_i a_{i+1} a_i = a_{i+1} a_i a_{i+1}, \quad 2 \leq i \leq n-1, \\
& (a_1 a_2 a_3)^2 = (a_3 a_1 a_2)^2, \\
& a_1 a_i = a_i a_1, \quad 4 \leq i \leq n, \\
& a_i a_j = a_j a_i, \quad 2 \leq i < j \leq n, \quad j \geq i+2, \\
& a_0 a_1 a_2 = (q^{-1} a_1 a_2)^{2-r} a_2 a_0 a_1 + (q-1) \sum_{i=1}^{r-2} (q^{-1} a_1 a_2)^{1-k} a_0 a_1, \\
& \quad = a_1 a_2 a_0, \\
& a_0 a_i = a_i a_0, \quad 3 \leq i \leq n.
\end{aligned}$$

We have the following special cases:

- if  $r = p = 1$ ,  $\mathfrak{H}_{r,p,n}^{q,x}(R)$  is the Hecke algebra of type  $A_{n-1}$ ,
- if  $r = 2$  and  $p = 1$ ,  $\mathfrak{H}_{r,p,n}^{q,x}(R)$  is the Hecke algebra of type  $B_n$ ,
- if  $r = p = 2$ ,  $\mathfrak{H}_{r,p,n}^{q,x}(R)$  is the Hecke algebra of type  $D_n$ ,
- if  $p = 1$ ,  $\mathfrak{H}_{r,p,n}^{q,x}(R)$  is the Ariki-Koike algebra defined in [4].

Suppose the ring  $R$  contains a  $p^{\text{th}}$  root of unity  $\eta_p$  and a  $p^{\text{th}}$  root  $y_i$  of  $x_i$ , for each  $i \in [1, d]$ . We define a sequence  $Q = (Q_1, \dots, Q_r)$  of elements in  $R$  from the sequence  $x$ : for  $j \in [1, r]$ ,  $j = sp + t$  with  $s \in [0, d-1]$  and  $t \in [1, p]$ , let  $Q_j := y_{s+1} \eta_p^{t-1}$ . Then,  $\mathfrak{H}_{r,1,n}^{q,Q}(R)$  is the  $R$ -algebra defined by

$$\begin{aligned}
\text{generators : } & T_1, \dots, T_n, \\
\text{relations : } & (T_1^p - x_1) \dots (T_1^p - x_d) = 0, \\
& (T_i - q)(T_i + 1) = 0, \quad 2 \leq i \leq n, \\
& T_1 T_2 T_1 T_2 = T_2 T_1 T_2 T_1, \\
& T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad 2 \leq i \leq n-1, \\
& T_i T_j = T_j T_i, \quad 1 \leq i < j \leq n, \quad j \geq i+2.
\end{aligned}$$

We use [2, Proposition 1.6] and identify  $a_0$  with  $T_1^p$ ,  $a_1$  with  $T_1^{-1}T_2T_1$  and  $a_i$  with  $T_i$ , for  $i \in [2, n]$ , such that  $\mathfrak{H}_{r,p,n}^{q,x}(R)$  is considered as a subalgebra of  $\mathfrak{H}_{r,1,n}^{q,Q}(R)$ . Then by [15, §4.1], the Ariki-Koike algebra  $\mathfrak{H}_{r,1,n}^{q,Q}(R)$  is graded over the cyclotomic Hecke algebra  $\mathfrak{H}_{r,p,n}^{q,x}(R)$  by a cyclic group of order  $p$ :

$$\mathfrak{H}_{r,1,n}^{q,Q}(R) = \bigoplus_{i=0}^{p-1} T_1^i \mathfrak{H}_{r,p,n}^{q,x}(R).$$

Put  $\mathfrak{H} := \mathfrak{H}_{r,1,n}^{q,Q}(R)$  and  $\mathfrak{H}' := \mathfrak{H}_{r,p,n}^{q,x}(R)$ . We can define an automorphism  $g_{\mathfrak{H}'}^{\mathfrak{H}}$  of  $\mathfrak{H}'$  by

$$g_{\mathfrak{H}'}^{\mathfrak{H}}(h) := T_1^{-1}hT_1, \text{ for all } h \in \mathfrak{H}'.$$

We also have an automorphism  $f_{\mathfrak{H}'}^{\mathfrak{H}}$  of  $\mathfrak{H}$  which is defined on the grading by

$$f_{\mathfrak{H}'}^{\mathfrak{H}}(T_1^j h) = \eta_p^j T_1^j h, \text{ for } j \in [0, p-1], h \in \mathfrak{H}'.$$

The next theorem of Dipper and Mathas will be very useful in the following of this paper.

**Theorem 2.1** ([8]) *With the previous notations, assume that we have a partition of  $Q$ :*

$$Q = Q^1 \amalg Q^2 \amalg \dots \amalg Q^s$$

such that

$$f_{\Gamma}(q, Q) = \prod_{1 \leq \alpha < \beta \leq s} \prod_{\substack{Q_i \in Q^\alpha \\ Q_j \in Q^\beta}} \prod_{-n < a < n} (q^a Q_i - Q_j)$$

is a unit of  $R$ . Then,  $\mathfrak{H}_{r,1,n}^{q,Q}(R)$  is Morita-equivalent to the following algebra:

$$\bigoplus_{\substack{n_1, \dots, n_s > 0 \\ n_1 + \dots + n_s = n}} \mathfrak{H}_{r_1, 1, n_1}^{q, Q^1}(R) \otimes \dots \otimes \mathfrak{H}_{r_s, 1, n_s}^{q, Q^s}(R)$$

where for  $i \in [1, s]$ ,  $r_i = |Q^i|$ .

## 2.B Clifford theory

Let  $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_d$  and  $\mathbf{v}$  be indeterminates over  $\mathbb{C}$ . For  $i \in [1, d]$ , put

$$\mathbf{x}_i := \mathbf{y}_i^p$$

and for  $i \in [1, r]$ ,  $i = sp + t$  with  $s \in [0, d-1]$ ,  $t \in [1, p]$ ,  $\eta_p := \exp(\frac{2i\pi}{p})$ ,

$$\mathbf{Q}_i := \eta_p^{t-1} \mathbf{y}_{s+1}.$$

Form the sequences  $\mathbf{x} := (\mathbf{x}_1, \dots, \mathbf{x}_d)$  and  $\mathbf{Q} := (\mathbf{Q}_1, \dots, \mathbf{Q}_r)$ .

Let  $A := \mathbb{C}[\mathbf{x}_1, \mathbf{x}_1^{-1}, \dots, \mathbf{x}_d, \mathbf{x}_d^{-1}, \mathbf{v}, \mathbf{v}^{-1}]$ ,  $K := \mathbb{C}(\mathbf{x}_1, \dots, \mathbf{x}_d, \mathbf{v})$  its field of fractions.

Let  $\theta : A \rightarrow \mathbb{C}$  be a ring homomorphism such that  $\mathbb{C}$  is the field of fractions of  $\theta(A)$ .

We put for  $i \in [1, d]$ ,

$$x_i := \theta(\mathbf{x}_i),$$

for  $i \in [1, r]$ ,

$$Q_i := \theta(\mathbf{Q}_i)$$

and

$$v := \theta(\mathbf{v}).$$

Note that for  $i \in [1, r]$ ,  $i = sp + t$  with  $s \in [0, d - 1]$ ,  $t \in [1, p]$ , then  $Q_i := \eta_p^{t-1} y_{s+1}$ , for a complex number  $y_{s+1}$  such that  $y_{s+1}^p = x_{s+1}$ . We also form the sequences  $x := (x_1, \dots, x_d)$  and  $Q := (Q_1, \dots, Q_r)$ .

We will note for short  $\mathbf{H}$ , resp.  $H$ , for the Ariki-Koike algebras  $\mathfrak{H}_{r,1,n}^{\mathbf{v};\mathbf{Q}}(K)$ , resp.  $\mathfrak{H}_{r,1,n}^{v,Q}(\mathbb{C})$  and  $\mathbf{H}'$ , resp.  $H'$ , for the cyclotomic Hecke algebras  $\mathfrak{H}_{r,p,n}^{\mathbf{v};\mathbf{x}}(K)$ , resp.  $\mathfrak{H}_{r,p,n}^{v,x}(\mathbb{C})$ .

Since  $\mathbf{H}$ , resp.  $H$ , is graded over  $\mathbf{H}'$ , resp.  $H'$ , we will give some results which comes from Clifford theory.

Let  $\mathfrak{H}$  be the Ariki-Koike algebra  $\mathbf{H}$ , resp.  $H$ , and  $\mathfrak{H}'$  be the cyclotomic Hecke algebra  $\mathbf{H}'$ , resp.  $H'$ . In both cases, we will denote by  $\text{Res}$  the functor of restriction from the category of the  $\mathfrak{H}$ -modules to the category of the  $\mathfrak{H}'$ -modules and by  $\text{Ind}$  the functor  $\mathfrak{H} \otimes_{\mathfrak{H}'}$  – from the category of the  $\mathfrak{H}'$ -modules to the category of the  $\mathfrak{H}$ -modules.

Let  $V$  be a  $\mathfrak{H}$ -simple module and  $U$  be a simple submodule of  $\text{Res } V$ . Put

$$o_{f,V} := \min \{k \in \mathbb{N}_{>0} \mid f^k V \simeq V\}$$

where  $f := f_{\mathfrak{H}'}$ , and for  $i \in [0, o_{f,V} - 1]$ ,  $f^i V$  is the  $\mathfrak{H}$ -module with the same underlying space as  $V$  and the structure of module is given by composing the original action of  $\mathfrak{H}$  with  $f^i$ .

In the same way, let

$$o_{g,U} := \min \{k \in \mathbb{N}_{>0} \mid g^k U \simeq U\}$$

where  $g := g_{\mathfrak{H}'}$ , and for  $i \in [0, o_{g,U} - 1]$ ,  $g^i U$  is the  $\mathfrak{H}'$ -module with the same underlying space as  $U$  with the action of  $\mathfrak{H}'$  twisted by  $g^i$ .

Denote  $[X : Y]$  the multiplicity of a simple module  $Y$  in the semi-simple module  $X$ .

**Lemma 2.2** *Denote for short  $g := g_{\mathfrak{H}'}$  and  $f := f_{\mathfrak{H}'}$ . Every simple  $\mathfrak{H}'$ -module  $U$  appears as a direct summand of the restriction  $\text{Res } V$  of a simple  $\mathfrak{H}$ -module  $V$  with multiplicity-free. With such modules  $U$  and  $V$ , a conjugate  $g^i U$  of  $U$ ,  $i \in [0, o_{g,U} - 1]$ , occurs in the restriction of the conjugate  $f^i V$  of  $V$ , for all*

$j \in [0, o_{f,V} - 1]$  and the conjugates of  $V$  are the only simple  $\mathfrak{H}$ -modules whose restrictions contain such a  $g^i U$ . Besides

$$\text{Res } V \simeq \bigoplus_{i=0}^{o_{f,V}-1} f^i V, \quad (1)$$

$$\text{Ind } U \simeq \bigoplus_{i=0}^{o_{g,U}-1} g^i U, \quad (2)$$

$$p = o_{g,U} o_{f,V}. \quad (3)$$

*Proof:* It is easy to see that

$$\text{Res } V \simeq [\text{Res } V : U] \bigoplus_{i=0}^{o_{g,U}-1} g^i U, \quad (4)$$

(see [6, Theorem 11.1]). Using [14, Proposition 2.2], we get

$$\text{Ind } U \simeq [\text{Ind } U : V] \bigoplus_{i=0}^{o_{f,V}-1} f^i V$$

as well as

$$\frac{p}{o_{f,V} o_{g,U}} = [\text{Res } V : U][\text{Ind } U : V]. \quad (5)$$

As  $\text{Res } V$  is  $g$ -stable, [6, Proposition 11.14] implies that  $\text{End}_{\mathfrak{H}}(\text{Ind } \text{Res } V)$  is graded over  $\text{End}_{\mathfrak{H}'}(\text{Res } V)$  by a cyclic group of order  $p$ . By (4) and (5), the dimension of  $\text{End}_{\mathfrak{H}}(\text{Ind } \text{Res } V)$  over  $\mathbb{C}$  is  $\frac{p^2}{o_{f,V}}$  and by the grading over  $\text{End}_{\mathfrak{H}'}(\text{Res } V)$ , this dimension is equal to  $p[\text{Res } V : U]^2 o_{g,U}$ . By comparison with (5), it follows that

$$[\text{Res } V : U] = [\text{Ind } U : V]. \quad (6)$$

Now, by definition of  $o_{f,V}$ , for short  $\sigma$ , there exists an  $\mathfrak{H}$ -automorphism

$$t : V \rightarrow f^\sigma V.$$

So,  $t^\frac{\sigma}{\sigma}$  is an  $\mathfrak{H}$ -isomorphism. By Schur's lemma,  $t^\frac{\sigma}{\sigma}$  is a scalar. There exists a complex number  $c$  such that  $(\frac{t}{c})^\frac{\sigma}{\sigma} = \text{Id}_V$ . Then

$$\text{Res } V \simeq \bigoplus_{i=0}^{\frac{\sigma}{\sigma}-1} \text{Ker}\left(\frac{t}{c} - \eta_p^{\sigma i} \text{Id}_V\right).$$

It is easy to see that for each  $i \in [0, \frac{\sigma}{\sigma} - 1]$ ,  $\text{Ker}(\frac{t}{c} - \eta_p^{\sigma i} \text{Id}_V)$  is an  $\mathfrak{H}'$ -module isomorphic to  $g^i \text{Ker}(\frac{t}{c} - \text{Id}_V)$ . So the semi-simple module  $\text{Res } V$  decomposes into more than  $\frac{\sigma}{\sigma}$  simple modules and by (5) by less than  $\frac{\sigma}{\sigma}$  simple modules. Therefore  $\text{Res } V$  decomposes into exactly  $\frac{\sigma}{\sigma}$  simple modules and  $[\text{Ind } U : V] = 1$  so  $[\text{Res } V : U] = 1$  by (6).

As the module induced from a simple  $\mathfrak{H}'$ -module to  $\mathfrak{H}$  is semi-simple, the assertions of the Lemma are proved.  $\blacksquare$

## 2.C The semi-simple case

We keep the notations of the previous subsection.

We study the representation theory of the algebras  $\mathbf{H} := \mathfrak{H}_{r,1,n}^{\mathbf{v},\mathbf{Q}}(K)$  and  $\mathbf{H}' := \mathfrak{H}_{r,p,n}^{\mathbf{v},\mathbf{x}}(K)$ . By [1, Main Theorem],  $\mathbf{H}$  is a split semi-simple algebra. This implies that  $\mathbf{H}'$  is also split semi-simple.

A complete set of non isomorphic modules for  $\mathbf{H}$  has been given in [4] and [7]. Let  $\Pi_n^r$  be the set of  $r$ -partitions of  $n$ . We say that  $\lambda = (\lambda(i))_{i \in [1,r]}$  is a  $r$ -partition of size  $n$  if for every  $i$ ,  $\lambda(i)$  is a partition and  $\sum_{i=1}^r |\lambda(i)| = n$ . For each  $r$ -partition  $\lambda \in \Pi_n^r$ , we can associate an  $\mathfrak{H}_{r,1,n}^{\mathbf{v},\mathbf{Q}}(A)$ -module  $S^{\mathbf{v},\mathbf{Q}}(\lambda)$  called a ‘‘Specht module’’ that is free over  $A$ . Here, we use the definition of ‘‘classical’’ Specht module contrary to [7] where the dual Specht modules are defined. We have the following theorem.

**Theorem 2.3** ([4],[7]) *The following set is a complete set of non isomorphic absolutely irreducible  $\mathfrak{H}_{r,1,n}^{\mathbf{v},\mathbf{Q}}(K)$ -modules:*

$$\left\{ S_K^{\mathbf{v},\mathbf{Q}}(\lambda) := K \otimes_A S^{\mathbf{v},\mathbf{Q}}(\lambda) \mid \lambda \in \Pi_n^r \right\}.$$

For each  $\lambda \in \Pi_n^r$ , the set of standard tableaux of shape  $\lambda$  is a basis of the underlying  $\mathbb{C}$ -vector space of  $S_K^{\mathbf{v},\mathbf{Q}}(\lambda)$  (see [4]). Using [4, Propositions 3.16, 3.17] and some combinatorial properties, it is easy to get the

**Proposition 2.4** *Let  $\lambda \in \Pi_n^r$ . The map*

$$h_\lambda^{\mathbf{v},\mathbf{Q}} : {}^f_{\mathbf{H}'} S_K^{\mathbf{v},\mathbf{Q}}(\lambda) \rightarrow S_K^{\mathbf{v},\mathbf{Q}}(\varpi(\lambda)) \quad (7)$$

*that sends a standard  $\lambda$ -tableau  $\mathbb{T} = (\mathbb{T}_i)_{i \in [1,r]}$  to the standard  $\varpi(\lambda)$ -tableau  $\varpi(\mathbb{T}) := (\mathbb{T}_{\varpi^{-1}(i)})_{i \in [1,r]}$  is an  $\mathbf{H}$ -isomorphism where  $\varpi$  is the permutation of  $\mathfrak{S}_r$  defined by  $\mathbf{Q}_{\varpi(i)} := \eta_p \mathbf{Q}_i$ , for all  $i \in [1,r]$  and  $\varpi(\lambda) = (\lambda(\varpi^{-1}(1)), \dots, \lambda(\varpi^{-1}(r)))$ .*

Define

$$o_\lambda := \min \{k \in \mathbb{N}_{>0} \mid \varpi^k(\lambda) = \lambda\}.$$

It is clear that  $o_\lambda = o_{f, S_K^{\mathbf{v},\mathbf{Q}}(\lambda)}$  and

$$\text{Res } S_K^{\mathbf{v},\mathbf{Q}}(\lambda) \simeq \bigoplus_{i=0}^{\frac{o_\lambda}{\eta_p} - 1} \text{Ker}(h_\lambda^{o_\lambda} - \eta_p^{o_\lambda i} \text{Id}_{S_K^{\mathbf{v},\mathbf{Q}}(\lambda)}),$$

where  $h_\lambda^{o_\lambda} : f^{o_\lambda} S_K^{\mathbf{v},\mathbf{Q}}(\lambda) \rightarrow S_K^{\mathbf{v},\mathbf{Q}}(\lambda)$  sends a standard  $\lambda$ -tableau  $\mathbb{T}$  to the standard  $\lambda$ -tableau  $\varpi^{o_\lambda}(\mathbb{T})$ . For  $i \in [0, \frac{o_\lambda}{\eta_p} - 1]$ , put

$$S_K^{\mathbf{v},\mathbf{Q}}(\lambda, i) := \text{Ker}(h_\lambda^{o_\lambda} - \eta_p^{o_\lambda i} \text{Id}_{S_K^{\mathbf{v},\mathbf{Q}}(\lambda)}). \quad (8)$$

Let  $\mathcal{L}$  be a set of representatives of  $r$ -partitions for the action of the cyclic group generated by  $\varpi$  over  $\Pi_n^r$ . It follows from Lemma 2.2, the

**Proposition 2.5** *The set*

$$\{S_K^{\mathbf{v}, \mathbf{Q}}(\lambda, i) ; i \in [0, \frac{p}{o_\lambda} - 1], \lambda \in \mathcal{L}\}$$

*is a complete set of non isomorphic simple  $\mathbf{H}'$ -modules.*

If  $\mathbb{T}$  is a standard  $\lambda$ -tableau, then for  $j \in [0, \frac{p}{o_\lambda} - 1]$ , the element

$$(\mathbb{T}, j) := \sum_{i=0}^{\frac{p}{o_\lambda}-1} \eta_p^{-o_\lambda i j} \varpi^{o_\lambda i}(\mathbb{T})$$

belongs to the eigenspace  $\text{Ker}(h_\lambda^{o_\lambda} - \eta_p^{o_\lambda j} \text{Id}_{S_K^{\mathbf{v}, \mathbf{Q}}(\lambda)})$ . It comes from the fact that

$\mathbb{T} = \frac{o_\lambda}{p} \sum_{j=0}^{\frac{p}{o_\lambda}-1} (\mathbb{T}, j)$ , that for every  $j \in [0, \frac{p}{o_\lambda} - 1]$ , the eigenspace  $\text{Ker}(h_\lambda^{o_\lambda} - \eta_p^{o_\lambda j} \text{Id}_{S_K^{\mathbf{v}, \mathbf{Q}}(\lambda)})$  is generated by the  $(\mathbb{T}, j)$ ,  $\mathbb{T}$  a standard  $\lambda$ -tableau. So we have just described the simple  $\mathbf{H}'$ -modules, see also [2].

The following part is now concerned with the non semi-simple case: simple  $\mathfrak{H}_{r,1,n}^{q,Q}(\mathbb{C})$ -modules and  $\mathfrak{H}_{r,p,n}^{q,x}(\mathbb{C})$ -modules.

## 2.D The non semi-simple case

We keep the notations of §2.B.

### 2.D.1 Decomposition maps

The aim of this part is to define the decomposition maps for  $H := \mathfrak{H}_{r,1,n}^{q,Q}(\mathbb{C})$  and  $H' := \mathfrak{H}_{r,p,n}^{q,x}(\mathbb{C})$ . Recall that  $A := \mathbb{C}[\mathbf{x}_1, \mathbf{x}_1^{-1}, \dots, \mathbf{x}_d, \mathbf{x}_d^{-1}, \mathbf{v}, \mathbf{v}^{-1}]$ ,  $K := \mathbb{C}(\mathbf{x}_1, \dots, \mathbf{x}_d, \mathbf{v})$  and that  $\theta : A \rightarrow \mathbb{C}$  is a ring homomorphism.

Using [10], there exists a discrete valuation ring  $\mathcal{O}$  with maximal ideal  $I(\mathcal{O})$  such that  $A \subset \mathcal{O}$  and  $I(\mathcal{O}) \cap A = \ker(\theta)$ . Let  $k_0 := \mathcal{O}/I(\mathcal{O})$  be the residue field of  $\mathcal{O}$ . Denote by  $\bar{\cdot} : \mathcal{O} \rightarrow k_0$  the canonical map. We obtain the algebras  $\mathfrak{H}_{r,1,n}^{\bar{\mathbf{q}}, \bar{\mathbf{Q}}}(k_0)$  and  $\mathfrak{H}_{r,p,n}^{\bar{\mathbf{q}}, \bar{\mathbf{x}}}(k_0)$ . The field  $k_0$  can be considered as an extension of  $\mathbb{C}$  that is the quotient field of  $\theta(A)$ . By [12, Lemma 7.3.4], there is an isomorphism between the Grothendieck groups of finitely generated  $H$ -modules and  $\mathfrak{H}_{r,1,n}^{\bar{\mathbf{q}}, \bar{\mathbf{Q}}}(k_0)$ -modules (resp.  $H'$ -modules and  $\mathfrak{H}_{r,p,n}^{\bar{\mathbf{q}}, \bar{\mathbf{x}}}(k_0)$ -modules):

$$R_0(H) \simeq R_0(\mathfrak{H}_{r,1,n}^{\bar{\mathbf{q}}, \bar{\mathbf{Q}}}(k_0)) \quad \text{and} \quad R_0(H') \simeq R_0(\mathfrak{H}_{r,p,n}^{\bar{\mathbf{q}}, \bar{\mathbf{x}}}(k_0)).$$

As a consequence, we obtain well-defined decomposition maps by choosing  $\mathcal{O}$ -forms for the simple  $\mathfrak{H}_{r,1,n}^{\mathbf{q}, \mathbf{Q}}(K)$ -modules (resp.  $\mathfrak{H}_{r,p,n}^{\mathbf{q}, \mathbf{x}}(K)$ -modules) and reducing them modulo  $I(\mathcal{O})$ :

$$d^{r,1,n} : R_0(\mathbf{H}) \rightarrow R_0(H),$$



$$d^{r,p,n} : R_0(\mathbf{H}') \rightarrow R_0(H').$$

For a simple  $\mathbf{H}$ -module  $V$  and  $W$  a simple  $\mathbf{H}'$ -module, there exist non negative integers  $d_{V,M}^{(1)}$  with  $M$  a simple  $H$ -module and  $d_{W,N}^{(p)}$  with  $N$  a simple  $H'$ -module such that:

$$d^{r,1,n}([V]) = \sum_{M \in \text{Irr}(H)} d_{V,M}^{(1)}[M]$$

and

$$d^{r,p,n}([W]) = \sum_{N \in \text{Irr}(H')} d_{W,N}^{(p)}[N].$$

By using the same argument as in [11, Lemma 5.2] (see also [12, Theorem 7.4.3.c]), we obtain the following result.

**Proposition 2.6** *The following diagram commutes:*

$$\begin{array}{ccc} R_0(\mathbf{H}) & \xrightarrow{\text{Res}} & R_0(\mathbf{H}') \\ d^{r,1,n} \downarrow & & \downarrow d^{r,p,n} \\ R_0(H) & \xrightarrow{\text{Res}} & R_0(H') \end{array}$$

Moreover, for any simple  $\mathbf{H}$ -module  $V$  and any simple  $H$ -module  $M$ , we have

$$d_{V,M}^{(1)} = d_{f_{\mathbf{H}'}^H V, f_{H'}^H M}^{(1)},$$

and for any simple  $\mathbf{H}'$ -module  $W$  and any simple  $H'$ -module  $N$ , we have:

$$d_{W,N}^{(p)} = d_{g_{\mathbf{H}'}^H W, g_{H'}^H N}^{(p)}.$$

In the following, we will see that under some additionnal hypotheses, the above property leads to some interesting results about the decomposition map of cyclotomic Hecke algebras of type  $G(r, p, n)$  and about the simple  $H'$ -modules.

### 2.D.2 Simple modules of non semi-simple Ariki-Koike algebras

We will work under the following hypothesis. We assume that  $v$  is a primitive root of unity,

$$v := \eta_e = \exp\left(\frac{2i\pi}{e}\right),$$

for a positive integer  $e > 1$ . Let

$$f := \gcd(e, p), \quad e = fe', \quad p = fp',$$

then  $\eta_f = \eta_p^{p'} = \eta_e^{e'}$ .

By the definition of  $Q$  and Theorem 2.1, without lost of generality, we can suppose that there exist integers  $v_1 \leq \dots \leq v_\delta$  that belong to  $[0, e' - 1]$  such that  $Q$  consists of the complex numbers  $\eta_e^{v_i} \eta_p^j$ , where  $i$  ranges over  $[1, \delta]$  and  $j$  ranges over  $[0, p - 1]$ . Then  $r = \delta fp'$  and we can split  $Q$  into  $p'$  sets:

$$Q = Q^1 \amalg \dots \amalg Q^{p'},$$

with  $Q^j$  formed by the  $\eta_e^{v_i} \eta_p^{lp'+j}$ ,  $i \in [1, \delta]$  and  $l \in [0, f-1]$ . We consider  $Q^j$  as an ordered sequence

$$Q^j = (\eta_p^{j-1} \eta_e^{v_1}, \dots, \eta_p^{j-1} \eta_e^{v_\delta}, \eta_p^{p'+j-1} \eta_e^{v_1}, \dots, \eta_p^{p'+j-1} \eta_e^{v_\delta}, \dots, \eta_p^{(f-1)p'+j-1} \eta_e^{v_1}, \dots, \eta_p^{(f-1)p'+j-1} \eta_e^{v_\delta}). \quad (9)$$

*Remark:* First note that, for  $j \in [1, p']$ ,  $Q^j = \eta_p^{j-1} Q^1$  and

$$Q^1 = (\eta_e^{v_1}, \dots, \eta_e^{v_\delta}, \eta_e^{v_1+e'}, \dots, \eta_e^{v_\delta+e'}, \dots, \eta_e^{v_1+(f-1)e'}, \dots, \eta_e^{v_\delta+(f-1)e'}),$$

with  $v_1 \leq \dots \leq v_\delta \leq v_1 + e' \leq \dots \leq v_\delta + (f-1)e'$ .

Note also that the quotient of two elements that belong to  $Q^j$ , for  $j \in [1, p']$ , is a power of  $v = \eta_e$  while the quotient of an element of  $Q^j$  by an element of  $Q^l$ ,  $l \in [1, p']$ ,  $l \neq j$  is not such a power, by definition of  $f$ .

If we order the elements of  $Q$  by the ordered elements of  $Q^1$ , then the ordered elements of  $Q^2$ , etc, Theorem 2.1 implies that the Ariki-Koike algebra  $H$  is Morita equivalent to the algebra

$$\bigoplus_{\substack{n_1, \dots, n_{p'} \geq 0 \\ n_1 + \dots + n_{p'} = n}} \mathfrak{H}_{f\delta, 1, n_1}^{q, Q^1}(\mathbb{C}) \otimes \dots \otimes \mathfrak{H}_{f\delta, 1, n_{p'}}^{q, Q^{p'}}(\mathbb{C}).$$

So it is clear that  $H$  is also Morita equivalent to the algebra

$$\bigoplus_{\substack{n_1, \dots, n_{p'} \geq 0 \\ n_1 + \dots + n_{p'} = n}} \mathfrak{H}_{f\delta, 1, n_1}^{q, Q^1}(\mathbb{C}) \otimes \dots \otimes \mathfrak{H}_{f\delta, 1, n_{p'}}^{q, Q^1}(\mathbb{C}).$$

We order  $\mathbf{Q}$  and we split it into  $p'$  sets following what we have done for  $Q$ :

$$\mathbf{Q} = \mathbf{Q}^1 \amalg \dots \amalg \mathbf{Q}^{p'}.$$

### 2.D.3 Parametrization of simple $H$ -modules

We can now give a parametrization for the simple  $H$ -modules. By [3, Theorem 2.5], the simple modules of the Ariki-Koike algebra  $\mathfrak{H}_{f\delta, 1, n_i}^{q, Q^1}(\mathbb{C})$ ,  $i \in [1, p']$ , are the quotient modules  $D_{\mathbb{C}}^{v, Q^1}(\lambda) := S_{\mathbb{C}}^{v, \mathbf{Q}^1}(\lambda) / J(S_{\mathbb{C}}^{v, \mathbf{Q}^1}(\lambda))$  of the specialisations of the Specht modules  $S_{\mathbb{C}}^{v, \mathbf{Q}^1}(\lambda) := \mathbb{C} \otimes S^{v, \mathbf{Q}^1}(\lambda)$  by their radical of Jacobson, with  $\lambda$  a Kleshchev  $f\delta$ -partition (for the parameters  $Q^1$ ) of size  $n_i$ . So the simple modules of  $H$  are labelled by the  $p'$ -tuples of Kleshchev  $f\delta$ -partitions (associated to  $Q^1$ )  $(\lambda^1, \dots, \lambda^{p'})$  with  $\sum_{i=1}^{p'} |\lambda^i| = n$ . We will denote this set by  $\Lambda^0$  and for  $\lambda \in \Lambda^0$ ,  $D_{\mathbb{C}}^{v, Q}(\lambda)$  the corresponding simple  $H$ -module.

In [20], another parametrization for the simple  $\mathfrak{H}_{f\delta,1,n_i}^{q,Q^1}(\mathbb{C})$ -modules has been found by using Lusztig's  $a$ -function (see Theorem 2.7 below). Following this paper, we will associate to each  $r$ -partition an  $a$ -value. To do this, we first describe the  $a$ -value of a  $f\delta$ -partition  $\mu = (\mu(1), \mu(2), \dots, \mu(f\delta))$ .

For  $k = 1, \dots, \delta$ ,  $s = 1, \dots, f$  and  $j = (s-1)\delta + k$  we put  $w_j := v_k + (s-1)e'$  so that  $w_j$  is the power of the  $j^{\text{th}}$ -component of  $Q^1$ .

Then, for  $j := 1, \dots, f\delta$ , we define:

$$m^{(j)} := w_j - \frac{je}{f\delta} + e$$

and for  $s = 1, \dots, n$ , we put:

$$B_s^{(j)} := \mu(j)_s - s + n + m^{(j)}$$

where we use the convention that  $\mu(j)_p := 0$  if  $p$  is greater than the height of  $\mu(j)$ . For  $j = 1, \dots, f\delta$ , let  $B^{(j)} = (B_1^{(j)}, \dots, B_n^{(j)})$ . Then, we define:

$$a_{f\delta}^{(1)}(\mu) := \sum_{\substack{0 \leq i \leq j < f\delta \\ (a,b) \in B^{(i)} \times B^{(j)} \\ a > b \text{ if } i=j}} \min\{a, b\} - \sum_{\substack{1 \leq i, j \leq f\delta \\ a \in B^{(i)} \\ 1 \leq k \leq a}} \min\{k, m^{(j)}\} + g(n).$$

where  $g(n)$  is a rational number which only depends on the  $m^{(j)}$  and on  $n$  (the expression of  $g$  is given in [20]).

Let  $\lambda \in \Pi_n^r$ , write  $\lambda = (\lambda(1), \dots, \lambda(r))$  as  $\lambda = (\lambda^1, \dots, \lambda^{p'})$  where, for  $i \in [1, p']$ ,  $\lambda^i = (\lambda(f\delta(i-1) + 1), \dots, \lambda(f\delta(i-1) + f\delta))$ .

Finally, we define

$$a_r^{(1)}(\lambda) := \sum_{i=1}^{p'} a_{f\delta}^{(1)}(\lambda^i).$$

To each Specht module  $S_K^{\mathbf{v}, Q^1}(\lambda)$  of  $\mathfrak{H}_{f\delta,1,n'}^{q,Q^1}(K)$ ,  $n' \in [0, n]$  and  $\lambda$  an  $f\delta$ -partition, we also associate an  $a$ -value

$$a_{f\delta}^{(1)}(S_K^{\mathbf{v}, Q^1}(\lambda)) := a_{f\delta}^{(1)}(\lambda).$$

To each simple  $\mathfrak{H}_{f\delta,1,n'}^{q,Q^1}(\mathbb{C})$ -module  $D_{\mathbb{C}}^{v, Q^1}(\lambda)$ ,  $n' \in [0, n]$  and  $\lambda$  a Kleshchev  $f\delta$ -partition of size  $n'$ , we again associate an  $a$ -value

$$a_{f\delta}^{(1)}(D_{\mathbb{C}}^{v, Q^1}(\lambda)) := \min\{a_{f\delta}^{(1)}(\mu) \mid \mu \in \Pi_{n'}^{f\delta}, d_{S_K^{\mathbf{v}, Q^1}(\mu), D_{\mathbb{C}}^{v, Q^1}(\lambda)}^{(1)} \neq 0\}.$$

**Theorem 2.7 ([19, Theorem 2.3.8])** *Let  $\lambda$  be a Kleshchev  $f\delta$ -partition of  $n'$  for a non-negative integer  $n'$ . There exists a unique  $\mu := \kappa(\lambda)$  such that  $a_{f\delta}^{(1)}(D_{\mathbb{C}}^{v, Q^1}(\lambda)) = a_{f\delta}^{(1)}(\mu)$  and  $d_{S_K^{\mathbf{v}, Q^1}(\mu), D_{\mathbb{C}}^{v, Q^1}(\lambda)}^{(1)} \neq 0$ . The function  $\kappa$  is a bijection between the set of the Kleshchev  $f\delta$ -partitions of size  $n'$  and the set of the FLOTW  $f\delta$ -partitions of size  $n'$ .*

Recall what is a FLOTW  $f\delta$ -partition  $\lambda = (\lambda(1), \dots, \lambda(f\delta))$  of size  $n'$  associated to the parameters  $v$  and  $Q^1$ , see [9]. Denote each partition  $\lambda(i)$ ,  $i \in [1, f\delta]$ , as  $(\lambda(i)_1, \lambda(i)_2, \dots)$ . The multipartition  $\lambda$  satisfies

1. for  $i \in [1, f]$ ,  $j \in [1, \delta - 1]$ ,  $k$  positive integer,

$$\lambda((i-1)\delta + j)_k \geq \lambda((i-1)m + j + 1)_{k+v_{j+1}-v_j},$$

2. for  $i \in [1, f-1]$ ,  $k$  positive integer,

$$\lambda(i\delta)_k \geq \lambda(i\delta + 1)_{k+v_1+e'-v_\delta},$$

3. for  $k$  positive integer,

$$\lambda(f\delta)_k \geq \lambda(1)_{k+v_1+e'-v_\delta},$$

as well as

4. to each node of the diagram of  $\lambda$  which is located in the  $a^{\text{th}}$  row and the  $b^{\text{th}}$  column of the  $c^{\text{th}}$  partition of  $\lambda$  (for two positive integers  $a$  and  $b$  and  $1 \leq c \leq f\delta$ ), we associate its residue  $\eta_e^{b-a+v_c}$ . Then, for any positive integer  $k$ , the cardinality of the set of the residues associated to the nodes in both the  $k^{\text{th}}$  columns and the right rims of the Young diagrams of  $\lambda$  is not  $e$ .

In the same way, to each Specht module  $S_K^{\mathbf{v}, \mathbf{Q}}(\lambda)$  of  $\mathbf{H}$ ,  $\lambda \in \Pi_n^r$ ,  $\lambda = (\lambda^1, \dots, \lambda^{p'})$ , we can define  $a_r^{(1)}(S_K^{\mathbf{v}, \mathbf{Q}}(\lambda)) := a_r^{(1)}(\lambda)$  that is the sum  $\sum_{i=1}^{p'} a_{f\delta}^{(1)}(\lambda^i)$ .

To each simple  $H$ -module  $D_C^{v, Q}(\lambda)$ ,  $\lambda \in \Lambda^0$ , we associate the  $a$ -value

$$a_r^{(1)}(D_C^{v, Q}(\lambda)) := \min\{a_r^{(1)}(\mu) \mid \mu \in \Pi_n^r, d_{S_K^{\mathbf{v}, \mathbf{Q}}(\mu), D_C^{v, Q}(\lambda)}^{(1)} \neq 0\}. \quad (10)$$

With Theorem 2.1, it is clear that there exists a unique  $\mu := \kappa'(\lambda)$  such that  $a_r^{(1)}(D_C^{v, Q}(\lambda)) = a_r^{(1)}(\mu)$  and  $d_{S_K^{\mathbf{v}, \mathbf{Q}}(\mu), D_C^{v, Q}(\lambda)}^{(1)} \neq 0$ . In fact,

$$\kappa'(\lambda) = (\kappa(\lambda^1), \dots, \kappa(\lambda^{p'}))$$

if  $\lambda = (\lambda^1, \dots, \lambda^{p'})$ .

The function  $\kappa'$  is a bijection between  $\Lambda^0$  and  $\Lambda^1$  the set of the  $p'$ -tuples of FLOTW  $f\delta$ -partitions  $(\lambda^1, \dots, \lambda^{p'})$  of size  $\sum_{i=1}^{p'} |\lambda^i| = n$ .

#### 2.D.4 Preliminary results

In order to prove the main Theorem, we will need some preliminary results. First, recall that we have ordered  $\mathbf{Q}$  by the ordered elements of  $\mathbf{Q}^1$ , then by the ordered elements of  $\mathbf{Q}^2$  etc as in §2.D.2. Let  $\varpi$  be the permutation of  $\mathfrak{S}_r$  defined in Proposition 2.4 and let

$$\lambda := (\lambda^1[1], \dots, \lambda^1[f\delta], \lambda^2[1], \dots, \lambda^2[f\delta], \dots, \lambda^{p'}[1], \dots, \lambda^{p'}[f\delta])$$

be an  $r$ -partition. Hence, following the notations of the previous paragraph, we have  $\lambda^j[i] = \lambda(f\delta(j-1) + i)$ . Then, it is easy to verify that

- for  $i \in [1, (p' - 1)f\delta]$ , then  $\varpi(i) = i + f\delta$ ,
- for  $i \in [r - f\delta + 1, r - \delta]$ , then  $\varpi(i) = i - r + (f + 1)\delta$ ,
- for  $i \in [r - \delta + 1, r]$ , then  $\varpi(i) = i - r + \delta$ .

Then, by Proposition 2.4, we have:

$$S_K^{\mathbf{v}, \mathbf{Q}}(\lambda) \simeq S_K^{\mathbf{v}, \mathbf{Q}}(\varpi(\lambda))$$

where  $\varpi(\lambda) = (\lambda^{p'}[f\delta - \delta + 1], \dots, \lambda^{p'}[f\delta], \lambda^{p'}[1], \dots, \lambda^{p'}[f\delta - \delta], \lambda^1[1], \dots, \lambda^1[f\delta], \dots, \lambda^{p'-1}[1], \dots, \lambda^{p'-1}[f\delta])$ .

**Proposition 2.8** *Let  $\lambda \in \Pi_n^r$ , then  $a_r^{(1)}(S_K^{\mathbf{v}, \mathbf{Q}}(\lambda)) = a_r^{(1)}(f_{\mathbf{H}'}^{\mathbf{H}} S_K^{\mathbf{v}, \mathbf{Q}}(\lambda))$ . Besides, if  $\lambda \in \Lambda^0$ , then  $a_r^{(1)}(D_{\mathbb{C}}^{v, Q}(\lambda)) = a_r^{(1)}(f_{\mathbf{H}'}^{\mathbf{H}} D_{\mathbb{C}}^{v, Q}(\lambda))$ .*

*Proof:* By the definition of the  $a$ -value (see §2.D.3), we have:

$$a_r^{(1)}(S_K^{\mathbf{v}, \mathbf{Q}}(\lambda)) := \sum_{i=1}^{p'} a_{f\delta}^{(1)}(\lambda^i[1], \dots, \lambda^i[f\delta])$$

Thus, by using the above properties, it is sufficient to show that  $a_{f\delta}^{(1)}(\lambda^{p'}[1], \dots, \lambda^{p'}[f\delta]) = a_{f\delta}^{(1)}(\lambda^{p'}[f\delta - \delta + 1], \dots, \lambda^{p'}[f\delta], \lambda^{p'}[1], \dots, \lambda^{p'}[f\delta - \delta])$ .

Using the notations of §2.D.3, for  $i = 1, \dots, \delta$  and  $j = 1, \dots, f$  we write  $m^{(i)}[j] := m^{((j-1)\delta+i)}$ . Then, all we have to do is to prove that  $m^{(i)}[j]$  doesn't depend on  $j$ . We have  $m^{(i)}[j] = v_i - \frac{((j-1)\delta+i)e}{f\delta} + (j-1)e' = v_i - \frac{ie}{f}$  because  $e'f = e$ . Hence, we obtain the desired result.

With the definition (10), it is trivial that the previous result about Specht modules implies that, for any  $\lambda \in \Lambda^0$ ,  $a_r^{(1)}(D_{\mathbb{C}}^{v, Q}(\lambda)) = a_r^{(1)}(f_{\mathbf{H}'}^{\mathbf{H}} D_{\mathbb{C}}^{v, Q}(\lambda))$ .  $\blacksquare$

With Proposition 2.8 and Lemma 2.2, we can associate an  $a$ -value to each simple  $\mathbf{H}'$ -module  $U$ . If  $U$  appears in the restriction  $\text{Res } S_K^{\mathbf{v}, \mathbf{Q}}(\lambda)$ , for  $\lambda \in \Pi_n^r$ , put

$$a_r^{(p)}(U) := a_r^{(1)}(S_K^{\mathbf{v}, \mathbf{Q}}(\lambda)).$$

In the same way, we can associate an  $a$ -value to each simple  $H'$ -module  $W$ . If  $W$  appears in the restriction  $\text{Res } D_{\mathbb{C}}^{v, Q}(\lambda)$ , for  $\lambda \in \Lambda^0$ , then

$$a_r^{(p)}(W) := a_r^{(1)}(D_{\mathbb{C}}^{v, Q}(\lambda)).$$

Another useful proposition is the following one.

**Proposition 2.9** *Let  $\lambda \in \Lambda^1$ , then  $\varpi(\lambda) \in \Lambda^1$ , where  $\varpi$  is defined in the Proposition 2.4.*

*Proof:* Write  $\lambda \in \Lambda^1$  as  $\lambda = (\lambda^1, \dots, \lambda^{p'})$  where  $\lambda^i = (\lambda^i[1], \dots, \lambda^i[f\delta])$  is a FLOTW  $f\delta$ -partition for  $Q^1$ . We will see that  $\varpi(\lambda) \in \Lambda^1$  so the result will be proved. The  $r$ -partition  $\varpi(\lambda)$  is equals to  $(\lambda^{p'}[f\delta - \delta + 1], \dots, \lambda^{p'}[f\delta], \lambda^{p'}[1], \dots, \lambda^{p'}[f\delta - \delta], \lambda^1[1], \dots, \lambda^1[f\delta], \dots, \lambda^{p'-1}[1], \dots, \lambda^{p'-1}[f\delta])$ . It belongs to  $\Lambda^1$  if and only if  $(\lambda^{p'}[f\delta - \delta + 1], \dots, \lambda^{p'}[f\delta], \lambda^{p'}[1], \dots, \lambda^{p'}[f\delta - \delta])$  is a FLOTW  $f\delta$ -partition. It is easy to verify that the three first conditions hold. Now, the set of the residues associated to the nodes in both the  $k^{\text{th}}$  columns and the right rims of the Young diagrams of this last multipartition consists of the residues of  $\lambda^{p'}$  but multiplied by  $\eta_e^{e'}$ . So the fourth condition about FLOTW multipartitions hold as well as the Proposition.  $\blacksquare$

### 3 A parametrization of the simple $H'$ -modules

We keep the above notations. The aim of this section is to describe a parametrization of the simple  $H'$ -modules, with  $H' = \mathfrak{H}_{r,p,n}^{v,Q}(\mathbb{C})$ . The proof is highly inspired by [14, Theorem 2.1].

**Theorem 3.1** *For all simple  $H'$ -module  $N$ , there exists a simple  $\mathbf{H}'$ -module  $W_N$  such that*

$$d^{r,p,n}([W_N]) = [N] + \sum_{a_r^{(p)}(L) < a_r^{(p)}(N)} d_{W_N, L}[L],$$

and  $a_r^{(p)}(N) = a_r^{(p)}(W_N)$ .

Let  $\Lambda^{1'} := \{W_N \mid N \in \text{Irr}(H')\}$ . Then  $\Lambda^{1'}$  consists in the  $S_K^{\mathbf{v}, \mathbf{Q}}(\lambda, i)$ , with  $\lambda \in \Lambda^1 \cap \mathcal{L}$ ,  $i \in [0, \frac{p}{o_\lambda} - 1]$  with the notations of the Proposition 2.5 and the map  $N \in \text{Irr}(H') \mapsto W_N \in \Lambda^{1'}$  is bijective. So  $\Lambda^{1'}$  is a parametrization of the simple  $H'$ -modules.

*Proof :* Fix a simple  $H'$ -module  $N$ . By Lemma 2.2, there exists  $\lambda \in \Lambda^0$  such that  $N$  appears in the restriction  $\text{Res} D_{\mathbb{C}}^{v,Q}(\lambda)$ . We have

$$d^{r,1,n}([S_{\mathbb{C}}^{\mathbf{v}, \mathbf{Q}}(\kappa(\lambda))]) = [D_{\mathbb{C}}^{v,Q}(\lambda)] + \sum_{\substack{a_r^{(p)}(D_{\mathbb{C}}^{v,Q}(\mu)) < \\ a_r^{(p)}(D_{\mathbb{C}}^{v,Q}(\lambda))}} d_{S_K^{\mathbf{v}, \mathbf{Q}}(\kappa(\lambda)), D_{\mathbb{C}}^{v,Q}(\mu)}^{(1)} [D_{\mathbb{C}}^{v,Q}(\mu)], \quad (11)$$

and  $a_r^{(1)}(S_{\mathbb{C}}^{\mathbf{v}, \mathbf{Q}}(\kappa(\lambda))) = a_r^{(1)}(D_{\mathbb{C}}^{v,Q}(\lambda))$ . With the notations of (8) and Proposition 2.6, this equality implies that

$$\sum_{i=0}^{\frac{p}{o_{\kappa(\lambda)}} - 1} d^{r,p,n}([S_{\mathbb{C}}^{\mathbf{v}, \mathbf{Q}}(\kappa(\lambda), i)]) = \sum_{i=0}^{\frac{p}{o_{f_{H'}^H, D_{\mathbb{C}}^{v,Q}(\lambda)}} - 1} [(g_{H'}^H)^i N] + [N'], \quad (12)$$

where  $N'$  is a sum of  $H'$ -modules,  $D$ , with  $a$ -value  $a_r^{(p)}(D) < a_r^{(p)}(D_C^{v,Q}(\lambda))$  and, for all  $i, j$ ,  $a_r^{(p)}(S_C^{\mathbf{v},\mathbf{Q}}(\kappa(\lambda), i)) = a_r^{(p)}((g_{H'}^H)^i N)$ .

Denote by  $\tau(\lambda) \in \Lambda^0$ , the multipartition defined by  $D_C^{v,Q}(\tau(\lambda)) := f_{H'}^H D_C^{v,Q}(\lambda)$ . If we twist the action of  $H$  by  $f_{H'}^H$ , the equality (11) gives that  $\kappa(\tau(\lambda)) = \varpi(\kappa(\lambda))$  by Proposition 2.9 and by definition of  $\kappa$ . As this map is bijective, it is clear that

$$O_{f_{H'}^H, S_K^{\mathbf{v},\mathbf{Q}}(\kappa(\lambda))} = O_{f_{H'}^H, D_C^{v,Q}(\lambda)}$$

and therefore for all  $i \in [0, \frac{p}{o_{\kappa(\lambda)}} - 1]$ ,

$$O_{g_{H'}^H, S_K^{\mathbf{v},\mathbf{Q}}(\kappa(\lambda), i)} = O_{g_{H'}^H, N}$$

with the last equality of Lemma 2.2.

Recall that for  $i \in [0, \frac{p}{o_{\kappa(\lambda)}} - 1]$ ,  $S_K^{\mathbf{v},\mathbf{Q}}(\kappa(\lambda), i)$  is conjugate to  $S_K^{\mathbf{v},\mathbf{Q}}(\kappa(\lambda), 0)$  by  $(g_{H'}^H)^i$ . The equality (12) implies that there exists a unique  $i \in [0, \frac{p}{o_{\kappa(\lambda)}} - 1]$  such that

$$d^{r,p,n}([S_C^{\mathbf{v},\mathbf{Q}}(\kappa(\lambda), i)]) = [N] + [N''], \quad (13)$$

where  $N''$  is a sum of  $H'$ -modules,  $D$ , with  $a$ -value  $a_r^{(p)}(D) < a_r^{(p)}(N)$ . This construction makes clear all the statements of the Theorem.  $\blacksquare$

*Remark:* The above theorem is only concerns with the case where  $v$  is a primitive  $e^{\text{th}}$ -root of unity. If  $v$  isn't a root of unity, it is readily checked that an analogue of this theorem holds by replacing  $\Lambda^1$  by the set of Kleshchev multipartitions  $\Lambda^0$  at  $e = \infty$ . The proof of this result can be easily obtained following the outline of [14].

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